Kyoto U.
2002/4/10

Seiberg–Witten Curve for the E-String Theory

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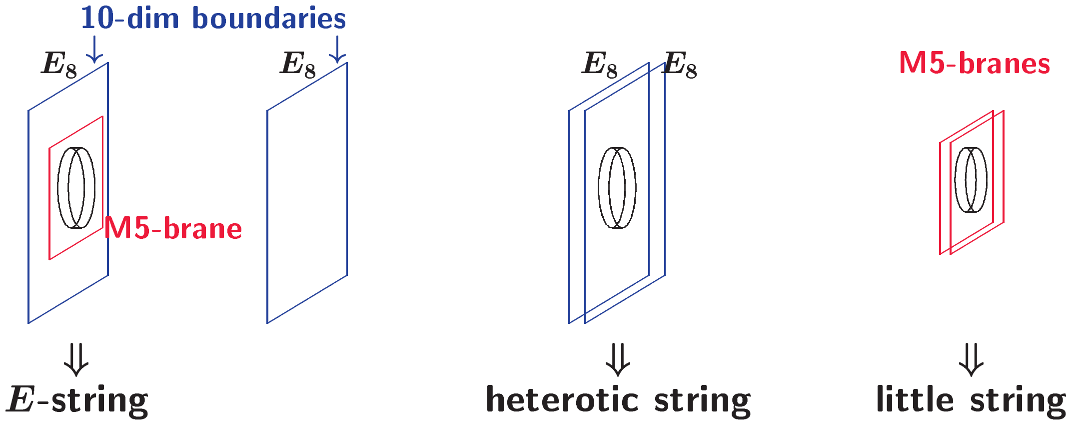
hep-th/0203025

1. What is E-String?

(Ganor–Hanany)

Heterotic M-theory (M on $S^1/\mathbb{Z}_2$)



- E-string is "$\frac{1}{2}$ heterotic string"

	E-string	heterotic
transverse space dimension	4	8
# of supercharges	8	16
symmetry	E_8	$E_8 \times E_8$
bosonic degrees of freedom	$(12, 4)$	$(24, 8)$

- 6-dim (1,0) non-critical string theory
- no gravity

String Theory Description

(1) Worldvolume of a M5-brane sitting on
a 10-dim boundary of heterotic M-theory

$$\Updownarrow$$

(2) Heterotic on $K3$ with a small E_8 instanton

$$\Updownarrow$$

(3) F-theory on elliptic CY$_3$($\supset \frac{1}{2}K3$)

IIB D3-brane $\curvearrowright$ P^1_{base} $\Rightarrow$ E-string

$\Downarrow$ (Compactify further on S^1)

(3') F-theory on (elliptic CY$_3$ $\times$ S^1)

$$\Updownarrow$$

(4) M-theory on elliptic CY$_3$

M2-brane $\curvearrowright$
2-cycle in $\frac{1}{2}K3$ $\Rightarrow$ wrapped E-string

M5-brane $\curvearrowright$ $\frac{1}{2}K3$ $\Rightarrow$ unwrapped E-string

($cf.$ M5-brane $\curvearrowright$ $K3$ $\Rightarrow$ heterotic string)

(Klemm–Mayr–Vafa)

2. How to Study Low Energy Dynamics?

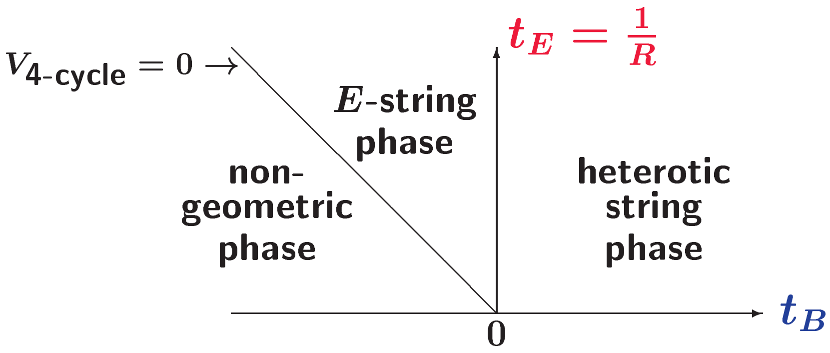
- Compactify on a circle

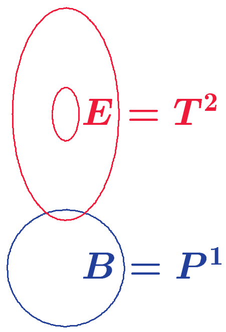
E-string theory on $R^5 \times S^1$

$\cap$

M-theory on elliptic CY_3

- Kähler moduli space of the $\mathrm{CY}_3 \quad \supset \frac{1}{2}K3$





In the E-string phase,
relevant lightest states: tensionless strings $\curvearrowright S^1$
$\Rightarrow$ BPS particles

In the M-theory description,

$$\text{BPS state} \quad = \quad \begin{array}{l} \text{M2-brane} \curvearrowright \\ \text{holomorphic 2-cycle in } \frac{1}{2}K3 \end{array}$$

$\Rightarrow$ Counting of BPS states comes down to counting of holomorphic curves in $\frac{1}{2}K3$

Geometry of $\frac{1}{2}K3$

- $\frac{1}{2}K3$ ($=$ almost del Pezzo $\mathcal{B}_9$) is obtained by blowing up P^2 at 9 points
- Homology group $H_2(\frac{1}{2}K3, \mathbb{Z})$ is generated by

$$\begin{cases} e_0 & \leftrightarrow \text{ lines in } P^2 \\ e_j, \;\; j = 1, \ldots, 9 & \leftrightarrow \text{ nine blow-ups} \end{cases}$$

$$e_0 \cdot e_0 = +1, \quad e_j \cdot e_j = -1 \;\; (j = 1, \ldots, 9),$$
$$e_i \cdot e_j = 0 \;\; (0 \leq i \neq j \leq 9)$$

Thus $H_2(\frac{1}{2}K3, \mathbb{Z})$ is a Lorentzian self-dual lattice:

$$H_2(\tfrac{1}{2}K3, \mathbb{Z}) = \Gamma^{9,1}$$

By an unimodular change of basis,

$$= \Gamma^{1,1} \oplus \Gamma_8$$

$$\Gamma^{1,1} : \begin{cases} \text{base } [B] \\ \text{fiber } [E] \end{cases} \qquad \Gamma_8 : E_8 \text{ lattice}$$

- BPS spectrum

$$\left.\begin{matrix} n[B] \\ +k[E] \end{matrix}\right\} \text{2-cycle} \;\; \Leftrightarrow \;\; \left.\begin{matrix} \text{winding } n \\ \text{momentum } k \end{matrix}\right\} E\text{-string}$$

BPS state partition function

$$\mathcal{F}(t_0,\ldots,t_9) \equiv$$
$$\mathcal{F}_{classical} + \sum_{n_0,\ldots,n_9} N_{n_0,\ldots,n_9} \sum_{k=1}^{\infty} \frac{1}{k^3} e^{2\pi i k \vec{n}\cdot\vec{t}}$$

Kähler moduli:

$$\{t_0,\ldots,t_9\} = \{\underbrace{t_B, t_E}_{\Gamma^{1,1}}, \underbrace{m_1,\ldots,m_8}_{\Gamma_8}\}$$

For later convenience

$$\phi \equiv t_B, \;\; \tau \equiv t_E$$

- **Expand $\mathcal{F}$ as**

$$\mathcal{F}(\phi,\tau;m_i) = \mathcal{F}_{classical} + \sum_{n=1}^{\infty} Z_n(\tau;m_i)\, e^{2\pi i\, n\phi}$$

n-wound E-string partition function Z_n

$=$ partition function of M5 $\overset{n\text{-times}}{\curvearrowright}$ $(\frac{1}{2}K3 \times T^2)$

$=$ partition function of
$\mathcal{N}=4$ $U(n)$ topological Yang–Mills on $\frac{1}{2}K3$

3. $\mathcal{N}=4$ $U(n)$ Yang–Mills on $\frac{1}{2}K3$

- Symmetries

(1) $SL(2,\mathbb{Z})$ symmetry (Montonen–Olive duality)

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \quad \text{: gauge coupling}$$

$$\tau \leftrightarrow -\frac{1}{\tau} \quad \Leftrightarrow \quad \begin{array}{l}\text{electric states} \\ \leftrightarrow \text{magnetic states}\end{array}$$

(2) Double periodicity of parameters m_j:

$$m_i \to m_i + 1, \quad m_i \to m_i + \tau$$

(3) Weyl group symmetry of E_8

$\Rightarrow$ $Z_n(\tau; m_i)$ becomes a (quasi-) Jacobi form.

- Partition function of the $U(1)$ theory

$$Z_1 = \frac{P(m_i;\tau)}{\eta^{12}}$$

where

$$P(m_i;\tau) = \Theta_{E_8}(m_i;\tau) = \frac{1}{2}\sum_{\ell=1}^{4}\prod_{j=1}^{8}\theta_\ell(m_j|\tau)$$

- Properties of higher Z_n

(1) Holomorphic anomaly

n M5's $\curvearrowright$ manifolds with $b_2^+ = 1$
cannot be separated in a holomorphic manner

$\Rightarrow$ $Z_n(\tau; m_i)$ also depends on $\bar{\tau}$

- Holomorphic anomaly enters in Z_n via

$$\widehat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi}\frac{1}{\mathrm{Im}(\tau)}$$

($E_2(\tau)$: Eisenstein function of weight 2)

- E_2 dependence is governed by a relation

$$\frac{\partial Z_n}{\partial E_2} = \frac{1}{24}\sum_{m=1}^{n-1} m(n-m) Z_m Z_{n-m}$$

(Minahan–Nemeschansky–Warner)

(2) Gap condition

Instantons with degree $k < n$ do not exist
in $U(n)$ gauge theory.

$\Rightarrow$ q-expansion of Z_n has the form:

$$n^3 q^{\frac{n}{2}} Z_n(\tau; m_i) = 1 + \mathcal{O}(q^n) \quad (q = e^{2\pi i \tau})$$

These conditions uniquely determine Z_n.

(Minahan–Nemeschansky–Vafa–Warner)

$$Z_1 = \frac{P(m_i;\tau)}{\eta^{12}}$$

where $\quad P(m_i;\tau) = \frac{1}{2}\sum_{\ell=1}^{4}\prod_{j=1}^{8}\theta_\ell(m_j|\tau),$

$$Z_2 = \frac{1}{\eta^{24}}\left[f_{20}(\tau)P(2m_i;2\tau) + f_{21}(\tau)P(m_i;\tfrac{\tau}{2}) + f_{21}(\tau+1)P(m_i;\tfrac{\tau+1}{2})\right] + \frac{1}{24}E_2 {Z_1}^2$$

where

$$f_{20}(\tau) = \frac{1}{24}\theta_3(\tau)^4\theta_4(\tau)^4\left(\theta_3(\tau)^4 + \theta_4(\tau)^4\right),$$

$$f_{21}(\tau) = -\frac{1}{384}\theta_3(\tau)^4\theta_2(\tau)^4\left(\theta_3(\tau)^4 + \theta_2(\tau)^4\right),$$

...

Calculation of Z_n becomes more and more laborious for higher n's.

4. Seiberg–Witten Formulation

- Compactify further on another circle

E-string theory on $R^4 \times T^2$

$\Rightarrow$ The low energy effective theory can be described by 4-dim $\mathcal{N}=2$ $U(1)$ gauge theory.

- Effective Lagrangian has the general form

$$\mathcal{L}_{eff} = \frac{1}{4\pi}\mathrm{Im}\left[\int d^4\theta \frac{\partial \mathcal{F}(A)}{\partial A}\bar{A} + \int d^2\theta \frac{1}{2}\frac{\partial^2 \mathcal{F}(A)}{\partial A^2} W^\alpha W_\alpha\right]$$

where the $\mathcal{N}=2$ vector multiplet is expressed by its $\mathcal{N}=1$ components

$$\begin{aligned} W_\alpha &= (A_\mu, \lambda) \quad \text{: vector multiplet} \\ A &= (\phi\ , \psi) \quad \text{: chiral multiplet} \end{aligned}$$

Components fields:

A_μ		gauge field
λ	ψ	Weyl spinor
ϕ		complex scalar

$\mathcal{L}_{eff}$ is governed by a holomorphic function $\mathcal{F}(A)$ called prepotential.

E-string partition function $\mathcal{F}(\phi, \tau; m_i)$

$=$ Prepotential of the low energy $U(1)$ theory
with a variable ϕ and parameters τ, m_i

Recall that $\mathcal{F}(\phi)$ has the expansion

$$\mathcal{F}(\phi) = \mathcal{F}_{classical} + \sum_{n=1}^{\infty} Z_n(\tau; m_i)\, e^{2\pi i\, n\phi}$$

$\{Z_n(\tau; m_i)\}$ form an infinite series of Jacobi forms.

A claim of Seiberg–Witten theory:
$\{Z_n(\tau; m_i)\}$ are finitely-generated by the use of Seiberg–Witten curve.

• How does Seiberg–Witten curve describe the prepotential?

The effective theory has one-dimensional moduli space spanned by VEV of the scalar field ϕ.

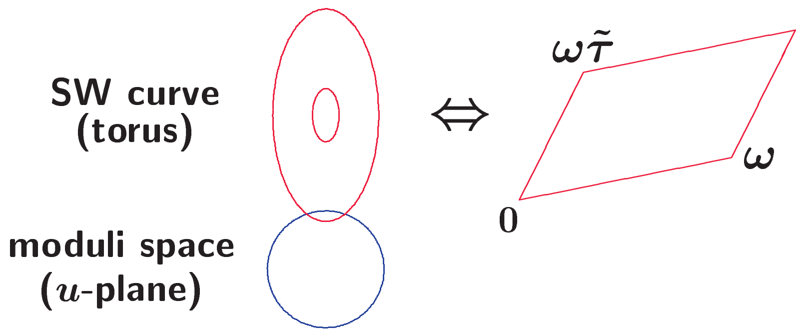
By expanding $\mathcal{L}_{eff}$ in terms of component fields, one finds the quantum gauge coupling $\tilde{\tau}$ to be

$$\tilde{\tau}(\phi) = \frac{1}{2}\frac{\partial^2 \mathcal{F}(\phi)}{\partial \phi^2}$$

Seiberg–Witten curve

$$y^2 = 4x^3 - f(u)x - g(u)$$

describes the ϕ dependence of $\tilde{\tau}$.



f, g are expressed in terms of $\tilde{\tau}, \omega$ by

$$f = \frac{1}{(2\pi\omega)^4}\frac{E_4(\tilde{\tau})}{12}, \quad g = \frac{1}{(2\pi\omega)^6}\frac{E_6(\tilde{\tau})}{216}.$$

i) Use these equations inversely:

$$\begin{cases} f(u) \\ g(u) \end{cases} \Rightarrow \begin{cases} \tilde{\tau}(u) \\ \omega(u) \end{cases}$$

ii) ϕ is given by

$$\phi(u) = \int \omega(u) du$$

iii) Eliminate the variable u:

$$\begin{cases} \phi(u) \\ \tilde{\tau}(u) \end{cases} \Rightarrow \tilde{\tau}(\phi)$$

- Seiberg–Witten curve for the E-string theory in a generic form:

$$y^2 = 4x^3 - f(u,\tau;m_i)x - g(u,\tau;m_i),$$

$$\begin{aligned} f(u,\tau;m_i) &= \tfrac{1}{12}E_4(\tau)u^4 + a_1(\tau;m_i)u^3 \\ &\quad + a_2(\tau;m_i)u^2 + a_3(\tau;m_i)u + a_4(\tau;m_i), \\ g(u,\tau;m_i) &= \tfrac{1}{216}E_6(\tau)u^6 \\ &\quad + b_1(\tau;m_i)u^5 + b_2(\tau;m_i)u^4 + b_3(\tau;m_i)u^3 \\ &\quad + b_4(\tau;m_i)u^2 + b_5(\tau;m_i)u + b_6(\tau;m_i). \end{aligned}$$

We choose a "gauge"

$$a_1(\tau;m_i) = 0.$$

Z_n are expressed in terms of curve coefficients by

$$\begin{aligned} Z_1 &= -\frac{1}{4}\frac{E_4}{\eta^{12}}b_1 \ , \\ Z_2 &= \frac{1}{8}\Bigg(-\frac{1}{24}E_6\,a_2 + \frac{1}{2}E_4\,b_2 \\ &\qquad + \frac{1}{24}\frac{E_4\,E_6}{\eta^{24}}{b_1}^2 + \frac{1}{48}\frac{E_2\,{E_4}^2}{\eta^{24}}{b_1}^2\Bigg) \ , \end{aligned}$$

$\cdots$

By calculating Z_n up to $n=8$
we determine all coefficient functions:

$$b_1 = -4\frac{P(m_i;\tau)}{E_4},$$

$$a_2 = \frac{1}{E_4\eta^{24}}[f_{a2,0}(\tau)P(2m_i;2\tau) + f_{a2,1}(\tau)P(m_i;\tfrac{\tau}{2}) + f_{a2,1}(\tau+1)P(m_i;\tfrac{\tau+1}{2})]$$

where
$$f_{a2,0}(\tau) = \tfrac{2}{3}\left(E_4(\tau) - 9\theta_2(\tau)^8\right),$$
$$f_{a2,1}(\tau) = \tfrac{1}{24}\left(E_4(\tau) - 9\theta_4(\tau)^8\right),$$

$$b_2 = \frac{1}{{E_4}^2\Delta}[f_{b2,0}(\tau)P(2m_i;2\tau) + f_{b2,1}(\tau)P(m_i;\tfrac{\tau}{2}) + f_{b2,1}(\tau+1)P(m_i;\tfrac{\tau+1}{2})]$$

where

$$f_{b2,0}(\tau) = \frac{1}{36}\left(\theta_3(\tau)^4 + \theta_4(\tau)^4\right) \times \left(E_4(\tau)^2 + 60E_4(\tau)\theta_2(\tau)^8 - 45\theta_2(\tau)^{16}\right),$$

$$f_{b2,1}(\tau) = -\frac{1}{576}\left(\theta_3(\tau)^4 + \theta_2(\tau)^4\right) \times \left(E_4(\tau)^2 + 60E_4(\tau)\theta_4(\tau)^8 - 45\theta_4(\tau)^{16}\right),$$

...

5. Symmetries and Flows of Curves

Coefficients a_j, b_j can be expressed in terms of affine E_8 Weyl-orbit characters at level-j:

$$b_1 = -\frac{4}{E_4}\widehat{w}^{(1)}\left[{}^{\;\;\;0}_{0000000}\right],$$

$$a_2 = \frac{1}{24E_4\Delta}\Big(\left(-16E_4 + 9{\theta_2}^8\right)\widehat{w}^{(2)}\left[{}^{\;\;\;0}_{1000000}\right] + 9\left({\theta_3}^8 - {\theta_4}^8\right)\widehat{w}^{(2)}\left[{}^{\;\;\;0}_{0000001}\right] - 135{\theta_2}^8\widehat{w}^{(2)}\left[{}^{\;\;\;0}_{0000000}\right]\Big),$$

$\cdots$

5-dim Seiberg–Witten curve can be obtained by taking the limit $\mathrm{Im}\tau \to \infty$ $(q \to 0)$. In this limit characters of affine E_8 reduce to those of finite E_8 algebra. Coefficient functions become

$$b_1 \to -4,$$

$$a_2 \to -\tfrac{2}{3}w\left[{}^{\;\;\;0}_{1000000}\right] + 12w\left[{}^{\;\;\;0}_{0000001}\right] - 1440,$$

$\cdots$

Five-dimensional E_8 Curve

$$
\begin{aligned}
y^2 \;=\;& 4x^3 \\
-\;& \Big(\tfrac{1}{12}u^4 + (-\tfrac{2}{3}w[{}^{0}_{1000000}] + 12w[{}^{0}_{0000001}] - 1440)u^2 \\
& +(-2w[{}^{1}_{0000000}] + 96w[{}^{0}_{1000000}] - 1152w[{}^{0}_{0000001}] + 103680)u \\
& +(\tfrac{4}{3}w[{}^{0}_{2000000}] - \tfrac{4}{3}w[{}^{0}_{0100000}] - \tfrac{8}{3}w[{}^{0}_{0000100}] - \tfrac{16}{3}w[{}^{0}_{1000001}] \\
& \quad +24w[{}^{0}_{0000002}] + \tfrac{328}{3}w[{}^{1}_{0000000}] + 48w[{}^{0}_{0000010}] \\
& \quad -\tfrac{8960}{3}w[{}^{0}_{1000000}] + 28128w[{}^{0}_{0000001}] - 2105280)\Big)x \\
-\;& \Big(\tfrac{1}{216}u^6 - 4u^5 + (-\tfrac{1}{18}w[{}^{0}_{1000000}] - 3w[{}^{0}_{0000001}] + 840)u^4 \\
& +(-\tfrac{1}{6}w[{}^{1}_{0000000}] - 4w[{}^{0}_{0000010}] \\
& \quad -8w[{}^{0}_{1000000}] + 528w[{}^{0}_{0000001}] - 79680)u^3 \\
& +(\tfrac{2}{9}w[{}^{0}_{2000000}] + \tfrac{1}{9}w[{}^{0}_{0100000}] - \tfrac{28}{9}w[{}^{0}_{0000100}] - \tfrac{152}{9}w[{}^{0}_{1000001}] \\
& \quad -92w[{}^{0}_{0000002}] - \tfrac{316}{9}w[{}^{1}_{0000000}] + 416w[{}^{0}_{0000010}] \\
& \quad +\tfrac{13088}{9}w[{}^{0}_{1000000}] - 35536w[{}^{0}_{0000001}] + 3911520)u^2 \\
& +(\tfrac{2}{3}w[{}^{1}_{1000000}] - \tfrac{2}{3}w[{}^{0}_{0001000}] - \tfrac{16}{3}w[{}^{0}_{1000010}] - \tfrac{74}{3}w[{}^{1}_{0000001}] \\
& \quad -96w[{}^{0}_{0000011}] - \tfrac{160}{3}w[{}^{0}_{2000000}] - \tfrac{280}{3}w[{}^{0}_{0100000}] + 80w[{}^{0}_{0000100}] \\
& \quad +1088w[{}^{0}_{1000001}] + 7872w[{}^{0}_{0000002}] + \tfrac{9680}{3}w[{}^{1}_{0000000}] - 15264w[{}^{0}_{0000010}] \\
& \quad -\tfrac{196448}{3}w[{}^{0}_{1000000}] + 1075776w[{}^{0}_{0000001}] - 97251840)u \\
& +(-\tfrac{8}{27}w[{}^{0}_{3000000}] + w[{}^{2}_{0000000}] + \tfrac{4}{9}w[{}^{0}_{1100000}] + \tfrac{2}{9}w[{}^{0}_{0010000}] \\
& \quad -\tfrac{16}{9}w[{}^{0}_{2000001}] - \tfrac{8}{3}w[{}^{1}_{0000010}] - 8w[{}^{0}_{0000020}] - \tfrac{88}{9}w[{}^{0}_{0100001}] \\
& \quad -\tfrac{112}{3}w[{}^{0}_{0000101}] - \tfrac{1000}{9}w[{}^{0}_{1000002}] - \tfrac{992}{3}w[{}^{0}_{0000003}] - 70w[{}^{1}_{1000000}] \\
& \quad -64w[{}^{0}_{0001000}] + \tfrac{112}{9}w[{}^{0}_{1000010}] + 636w[{}^{1}_{0000001}] + 3472w[{}^{0}_{0000011}] \\
& \quad +1584w[{}^{0}_{2000000}] + 3038w[{}^{0}_{0100000}] + \tfrac{7376}{9}w[{}^{0}_{0000100}] - 18480w[{}^{0}_{1000001}] \\
& \quad -176608w[{}^{0}_{0000002}] - \tfrac{196384}{3}w[{}^{1}_{0000000}] + 197968w[{}^{0}_{0000010}] \\
& \quad +936200w[{}^{0}_{1000000}] - 12291232w[{}^{0}_{0000001}] + 971250560)\Big)
\end{aligned}
$$

where

$$
w[{}^{\;\;\;\;\;\;n_2}_{n_1\,n_3\,n_4\,n_5\,n_6\,n_7\,n_8}](m_i) = \sum_{\vec{\alpha}\in \text{Weyl-orbit}} e^{i\vec{m}\cdot\vec{\alpha}}
$$

6. Discussions

- We have constructed the Seiberg–Witten curve for the 6-dim E-string theory.
- By applying the mirror symmetry technique one can generate BPS state partition function at any higher order.
- $\left.\begin{array}{l}\text{Lower-rank symmetry groups} \\ \text{5-dim and 4-dim}\end{array}\right\}$ SW curve

  can be easily obtained by suitable parameter tuning.

Future Problems

- Analysis of 6-dim non-critical strings with lower-rank symmetry groups
- Application of the technique to Little String Theory
- Finding an explicit expression of holomorphic sections of the elliptic fibration