On the Hawking effect

Ralf Schützhold

Institute for Theoretical Physics,
Dresden University of Technology,
Germany

Particle Physics Group,
Kyoto University,
12.07.2000

Curved space-time

Klein-Fock-Gordon equation

$$\Box\Phi = \frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\Phi\right) = 0$$

Inner product

$$(\psi|\phi) = i\int_\Sigma d\Sigma^\mu\ \psi^*\ \overleftrightarrow{\partial}_\mu\ \phi$$

with $\psi\ \overleftrightarrow{\partial}_\mu\ \phi = \psi\,\partial_\mu\phi - \phi\,\partial_\mu\psi$

Independent of hyper-surface Σ owing to Gauss' law

$$\oint_{\partial M_\Sigma} d\Sigma_\mu\ A^\mu = \int_{M_\Sigma} d^4x\ \sqrt{-g}\ \nabla_\mu\ A^\mu$$

Complete set of solutions F_I, F_J^*

$$\begin{aligned}(F_I|F_J) = -\,(F_I^*|F_J^*) &= \delta(I,J)\\ (F_I|F_J^*) &= 0\end{aligned}$$

Minkowski example

$$F_{\vec{k}} = |\vec{k}\,|^{-1/2}\ \exp(-i\,|\vec{k}\,|\,t + i\,\vec{k}\,\vec{r}\,)$$

Quantization

Globally hyperbolic (causal and complete) space-time

$\rightarrow$ Green functions $\Delta_{\mathrm{ret}}(\underline{x}, \underline{x}')$ and $\Delta_{\mathrm{adv}}(\underline{x}, \underline{x}')$

Canonical commutation relations

$$\left[\hat{\Phi}(\underline{x}), \hat{\Phi}(\underline{x}')\right] = \Delta_{\mathrm{ret}}(\underline{x}, \underline{x}') - \Delta_{\mathrm{adv}}(\underline{x}, \underline{x}')$$

Minkowski example $\Delta_{\mathrm{ret}}(\underline{x}, \underline{x}') \sim \Theta(x_0 - x_0')\,\delta\left[(\underline{x} - \underline{x}')^2\right]$

Creation- and annihilation operators

$$\hat{A}_I = \left(F_I | \hat{\Phi}\right)$$

with commutation relations

$$\begin{aligned} \left[\hat{A}_I^\dagger, \hat{A}_J\right] &= \delta(I, J) \\ \left[\hat{A}_I, \hat{A}_J\right] = \left[\hat{A}_I^\dagger, \hat{A}_J^\dagger\right] &= 0 \end{aligned}$$

Vacuum-definition

$$\forall_I \; : \; \hat{A}_I \, |0\rangle = 0$$

Depends on $\{F_I\}$!

Energy

$$\mathcal{A} = \int\limits_M d^4x \, \frac{\sqrt{-g}}{2} \, g^{\mu\nu} (\partial_\mu \Phi)(\partial_\nu \Phi) \, ,$$

Energy-momentum tensor

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{A}}{\delta g^{\mu\nu}} = (\partial_\mu \Phi)(\partial_\nu \Phi) - \frac{g_{\mu\nu}}{2} (\partial_\gamma \Phi)(\partial^\gamma \Phi)$$

At first glance no conservation law

$$\nabla_\mu T^\mu_\nu = \frac{1}{\sqrt{-g}} \, \partial_\mu \left(\sqrt{-g} \, T^\mu_\nu \right) - \frac{1}{2} \, T^{\alpha\beta} \, \partial_\nu \, g_{\alpha\beta} = 0$$

Time-like Killing vector ξ_μ

$$H = \int d\Sigma_\mu \, T^{\mu\nu} \, \xi_\nu$$

Conserved energy

$$\hat{H} = \sum\!\!\!\!\!\!\!\int\limits_I \frac{\omega_I}{2} \left(\hat{A}_I \hat{A}_I^\dagger + \hat{A}_I^\dagger \hat{A}_I \right)$$

Particle definition via diagonalization of Hamiltonian

Particle creation

$$\begin{aligned} F_I^{\mathrm{in}} &\neq F_I^{\mathrm{out}} \\ \to \hat{A}_I^{\mathrm{in}} &\neq \hat{A}_I^{\mathrm{out}} \\ \to |0, \mathrm{in}\rangle &\neq |0, \mathrm{out}\rangle \end{aligned}$$

Bogoliubov transformation

$$F_I^{\mathrm{out}} = \sum\!\!\!\!\!\!\!\int_J \alpha_{IJ}\, F_J^{\mathrm{in}} + \beta_{IJ}\, F_J^{\mathrm{in}\,*}$$

Bogoliubov coefficients

$$\alpha_{IJ} = \left(F_I^{\mathrm{out}} | F_J^{\mathrm{in}}\right) \qquad \beta_{IJ} = \left(F_I^{\mathrm{out}\,*} | F_J^{\mathrm{in}}\right)$$

In analogy

$$\hat{A}_I^{\mathrm{out}} = \sum\!\!\!\!\!\!\!\int_J \alpha_{IJ}\, \hat{A}_J^{\mathrm{in}} + \beta_{IJ}\, \hat{A}_J^{\mathrm{in}\,\dagger}$$

Different vacua

$$\langle 0, \mathrm{in}|\, \hat{N}_I^{\mathrm{out}}\, |0, \mathrm{in}\rangle = \sum\!\!\!\!\!\!\!\int_J |\beta_{IJ}|^2 > 0$$

Black holes

Schwarzschild metric

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2\, d\Omega^2$$

Regge-Wheeler coordinate $r_* = r + 2M\ \ln(r/(2M) - 1)$

$$ds^2 = \left(1 - \frac{2M}{r}\right) (dt^2 - dr_*^2) - r^2\, d\Omega^2$$

Regular coordinate $r_\diamond$

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - dr_\diamond^2 - r^2\, d\Omega^2$$

In the vicinity of the horizon

$$F_{\omega\,\ell\,m}(t, r_*, \vartheta, \varphi) \sim \frac{\exp(-i\,\omega\,t \pm i\,\omega\,r_*)}{\sqrt{\omega}}\, Y_{\ell\,m}(\vartheta, \varphi)$$

Respectively

$$F_{\omega\,\ell\,m}(t, r_\diamond, \vartheta, \varphi) \sim \frac{\exp(-i\,\omega\,t)}{\sqrt{\omega}}\, (r_\diamond/M)^{\pm 4 i \omega M}\, Y_{\ell\,m}(\vartheta, \varphi)$$

Arbitrarily fast oscillating at horizon

Collapse

$$ds^2 = h(r_\diamond, t)dt^2 - dr_\diamond^2 - r^2(r_\diamond, t)d\Omega^2$$

$t \to -\infty$ Minkowski metric $h \approx 1$, $r = r_\diamond$

$t \to \infty$ Schwarzschild metric $h = 1 - 2M/r$

Initial modes

$$F^{\rm in}_{\omega\,\ell\,m}(t, r_\diamond, \vartheta, \varphi) \sim \frac{\exp(-i\,\omega\,t \pm i\,\omega\,r_\diamond)}{\sqrt{\omega}}\,Y_{\ell\,m}(\vartheta, \varphi)$$

Klein-Fock-Gordon equation

$$\frac{\partial}{\partial t}\,\frac{r^2}{\sqrt{h}}\,\frac{\partial}{\partial t}\,\Phi \;=\; \frac{\partial}{\partial r_\diamond}\,r^2\,\sqrt{h}\,\frac{\partial}{\partial r_\diamond}\,\Phi \;+ r^2\,\sqrt{h}\,\nabla^2_{\vartheta\varphi}\,\Phi$$

$\to$ freezing of dynamics at horizon

$\to$ sudden approximation

$$\begin{aligned} F^{\rm in}_{\omega\,\ell\,m}(t\uparrow 0, r_\diamond, \vartheta, \varphi) &= F^{\rm in}_{\omega\,\ell\,m}(t\downarrow 0, r_\diamond, \vartheta, \varphi) \\ \dot{F}^{\rm in}_{\omega\,\ell\,m}(t\uparrow 0, r_\diamond, \vartheta, \varphi) &= \frac{1}{\sqrt{h}}\,\dot{F}^{\rm in}_{\omega\,\ell\,m}(t\downarrow 0, r_\diamond, \vartheta, \varphi) \end{aligned}$$

Hawking effect

Overlap coefficient

$$\beta_{IJ} = i \int\limits_{\Sigma} d\Sigma^{\mu}\; F_I^{\text{in}}\; \overset{\leftrightarrow}{\partial}_{\mu}\; F_J^{\text{out}}$$

Angular integration

$$\beta_{\omega\omega'} \sim \int\limits_0^{\infty} \frac{dr_\diamond}{r_\diamond} \frac{\omega' - \omega\, r_\diamond/(4M)}{\sqrt{\omega\omega'}} \exp(\pm\, i\, \omega\, r_\diamond)\, (r_\diamond/M)^{\pm 4 i\omega M}$$

$\rightarrow$ complex Γ-functions

$$\beta_{\omega\omega'} \sim \sqrt{\frac{\omega'}{\omega}}\, \Gamma(\pm\, i\, \omega'\, 4M)(\pm\, i\, \omega\, 4M)^{\pm\, i\, \omega'\, 4M}$$

Reflection formula

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z \sin \pi z}$$

Thermal distribution

$$|\beta_{\omega\omega'}|^2 \sim \frac{\omega^{-1}}{\exp(8\pi\, M\, \omega') - 1}$$

$\rightarrow$ UV-divergence

Alternative collapse

$$ds^2 = h(r_*, t)\left(dt^2 - dr_*^2\right) - r^2(r_*, t)\, d\Omega^2$$

Same asymptotics: Minkowski and Schwarzschild

But different way of collapse !

$$\partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \Phi\right) \to \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2}\right) \Phi$$

for $r \to 2M$

Throughout solutions

$$\Phi \sim \exp(-i\,\omega\, t\, \pm\, i\,\omega\, r_*)$$

$\to$ no mixing of positive and negative frequencies

$\to$ no particle production !

$\to$ possible collapse-dependence of Hawking effect

Back-reaction problem

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = \kappa\, \langle \hat{T}_{\mu\nu} \rangle$$