ODE/IM correspondence and its application to $\mathcal{N}=2$ SCFT

Hongfei Shu (Tokyo Institute of Technology)
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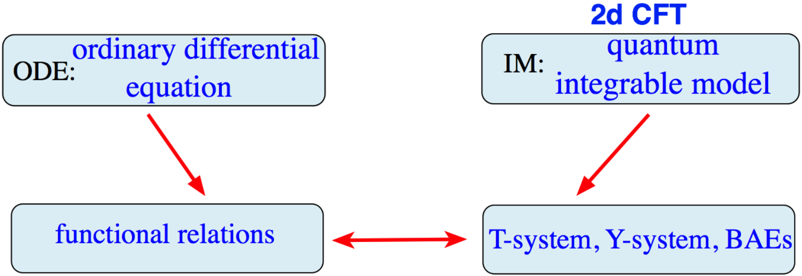
Part I: Introduction of ODE/IM correspondence

Part II: Its application to Argyres-Douglas theories

1. This is a correspondence between classical and quantum integrable model.
2. This correspondence is mainly based on the functional relations (T-/Y-system, BAEs, $\cdots$) that appear on both sides.



These functional relations have the same form and same asymptotic behavior(same boundary condition) $\Longrightarrow$ Same solution

ODE/IM correspondence

1. The first example was found in [Dorey-Tateo '98, BLZ'98].

ODE		IM (2d CFT)
$(-\frac{d^2}{dz^2} + \frac{\ell(\ell+1)}{z^2} + z^{2M} - E)\psi = 0$	$\Longleftrightarrow$	Six vertex model (continuum $\Downarrow$ limit) $c \leq 1$ 2d CFT
Energy E	$\Longleftrightarrow$	spectral parameter λ
Stokes mupliter $T(E)$	$\Longleftrightarrow$	transfer matrix $\mathbf{T}(\lambda)$
T-/Y-system	$\Longleftrightarrow$	**T**-/**Y**-system
relation of expansion coefficient	$\Longleftrightarrow$	Baxter's T-Q relation
$\vdots$	$\vdots$	$\vdots$

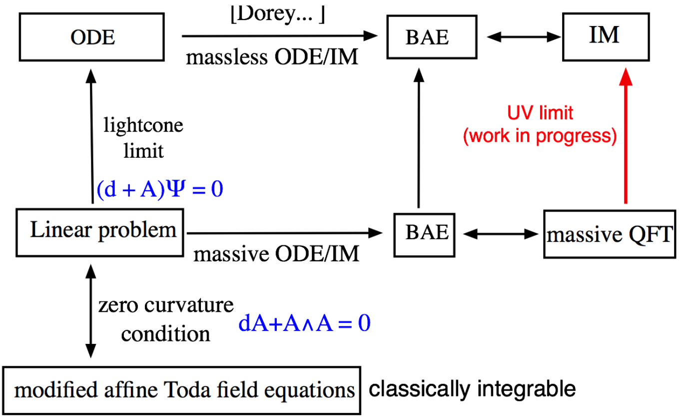
ODE/CFT correspondence

1. Sometimes we also call it as ODE/CFT correspondence.
2. The same BAEs and T-/Y-system also emerge from the study of integrable structure of $c = 1 - (\beta - \frac{1}{\beta})^2 \leq 1$ CFT.
3. In particular from the minimal model $M_{2,2N+3}$, one could construct the A_{2N}−type T-/Y-system[BLZ '94 ∼'96].

Remark and Generalization

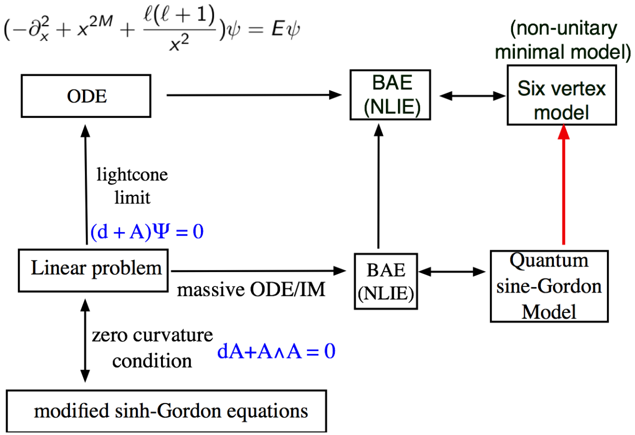
1. No conceptual explanation is found so far, even in the simplest case.
2. There is no systematic way to find the corresponding ODE from a given 2d CFT (Integrable model).
3. The ODE/IM was generalized to higher order ODE [Dorey-Dunning-Masoero-Suzuki-Tateo '07].

General ODE/IM correspondence (ODE/IQFT)



[Lukyanov-Zamolodchikov, Ito, Locke, HS]

Simplest case: Modified sinh Gordon equation



$$\partial_z \partial_{\bar{z}} \phi - e^{2\phi} + p(z)\bar{p}(z) e^{-2\phi} = 0 \qquad p(z) = z^{2M} - s^{2M}$$

$$A_z = \frac{1}{2} \partial_z \phi \sigma^3 - e^{\phi}[\sigma^+ e^{\phi} + \sigma^- p(z) e^{-\phi}]$$

Some applications of ODE/IM correspondence

ODE/IM correspondence is a powerful method. It has been applied to

- Scattering amplitude/null polygon Wilson loop: the area of minimal surface ending on Wilson loop in AdS [Alday-Maldacena, Ito $\cdots$].
- Quantization of the $O(3)$ nonlinear sigma model (Sausage model) [Bazhanov-Lukyanov-Kotousov '17]
- The BPS spectrum/quantum curve in Argyres-Douglas (AD) theory [Gaiotto '14, Ito-HS '17].

Motivation of this work

- Complete the dictionary of ODE/IM correspondence.
- Apply the ODE/IM correspondence (or integrability method) to $\mathcal{N} = 2$ gauge theory.
- Especially for the not well understand Argyres-Douglas theory.
- Find (another type) correspondence between $\mathcal{N} = 2$ gauge theory and 2d CFT.

Outline

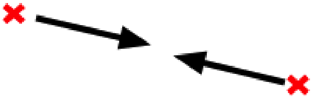
1. Introduction of Argyres-Douglas theory
2. Quantum curve of A_n Argyres-Douglas theory
3. Study the quantum curve via ODE/IM correspondence
4. Generalize to other AD theories
5. Conclusion and Outlook

Introduction of Argyres-Douglas theory

Introduction of Argyres-Douglas theory

Argyres-Douglas (AD) theory [Argyres-Douglas '95]

- Special points in the moduli space of $\mathcal{N} = 2$ gauge theories.
- Mutually non-local monopoles/dyons become massless.
- Strongly coupled $\mathcal{N} = 2$ superconformal field theory in 4D.
- The Lagrangian description is not known.



Properties of Argyres-Douglas theory

- Realized by degeneration of the Seiberg-Witten curve of $\mathcal{N}=2$ gauge theories: A_n, D_n, E_n Argyres-Douglas theory

$$\xi^2 = W_G^R(x, u_1, \cdots, u_n), \quad W_{A_n}^{n+1} = x^{n+1} - u_1 x^{n-1} - \cdots - u_{n-1}x - u_n$$

- We also label the A_n-type curve as (A_1, A_n)

$$A_1 : \ \xi^2, \qquad A_n : \ x^{n+1} + \cdots$$

- This is also generalized to (f, f') case, where $f, f' = ADE$.

Some results of 4d/2d correspondence

[Beem-Lemos-Liendo-Peelaers-Rastelli-van Rees '13][Cordova-Shao '15]
[Xie-Yan-Yau '16] [Creutzig '17] [Fredrickson-Pei-Yan-Ye '17] $\cdots$

- From the superconformal algebra of $\mathcal{N}=2$ SCFT, one can construct certain 2d chiral algebra.
- $\mathcal{N}=2$ SCFT $\Longleftrightarrow$ 2d chiral algebra $\left(c_{4d}=-\frac{1}{12}c_{2d}\right)$
- Schur index of $\mathcal{N}=2$ SCFT$\Longleftrightarrow$vacuum character of 2d chiral algebra

AD theory	chiral algebra
(A_1, A_{2N})	$M_{2,2N+3}$ minimal model
(A_1, A_{2N-1})	$\mathcal{B}_{N+1}$ algebra

Table: Argyres-Douglas theories and corresponding chiral algebras

Study AD theory via ODE/CFT correspondence

SW curve for AD theory

$$\xi^2 = W_G^R(x, u_1, \cdots, u_n)$$

$\xi \to -i\epsilon\partial_x$

Nekrasov-Sahashvili limit of the Omega-background

quantum curve

$$\left(-\partial_x^2 + W_G^R(x, u_1, \cdots, u_n)\right)\psi(x) = 0$$

ODE/CFT correspondence

2d CFT (IM)

↕

2d chiral algebra via 4d/2d correspondnece

Quantum curve of A_n Argyres-Douglas theory

Omega background and quantum deformed SW curve

- N=2 SYM $\Longleftrightarrow$ classical integrable system
  (SW curve) $\Longleftrightarrow$ (spectral curve)
- We consider the Nekrasov-Sahashivili limit ($\epsilon_2 = 0, \epsilon_1 = \hbar$) in the omega background [Nekrasov-Sahashivili '09].
- Classical integrable system $\Longrightarrow$ the quantum integrable model with $\epsilon_1 = \hbar$.

- The SW differential of the AD theory $\lambda_{SW} = \xi dx$.
- This define a symplectic structure $d\lambda_{SW} \sim d\xi \wedge dx$ on (ξ, x) space.

Seiberg Witten curve $\Longrightarrow$ quantum curve ($\xi \to -\epsilon \frac{d}{dx}$)

$$\left(-\epsilon^2 \frac{d^2}{dx^2} + W_G^R(x, u_1, \ldots, u_n)\right) \psi(x) = 0$$

This is the standard procedure to obtain quantum curve form the Seiberg-Witten curve.

$$u_1 = \cdots = u_{n-1} = 0, \qquad x = \epsilon^{\frac{2}{n+3}} z, \ \ u_n = \epsilon^{\frac{2(n+1)}{n+3}} E$$

$$\left(-\frac{d^2}{dz^2} + z^{n+1} - E\right) \psi(x) = 0$$

Study quantum curve via ODE/CFT correspondence

$$\left(-\frac{d^2}{dz^2} + z^{n+1} - E\right)\psi = 0$$

More general polynomial is difficult in ODE/CFT.

Symanzik rotation of the ODE

- Under the rotation $(z, E) \to (bz, b^{n+1}E)$, the ODE becomes

$$\frac{1}{b^2}\left(-\frac{d^2}{dz^2} + b^{n+3}(z^{n+1} - E)\right)\psi(bz, b^{n+1}E) = 0$$

- If we choose $b = \omega = \exp(\frac{2\pi i}{n+3})$, the ODE is invariant.

- If $\psi(z, E)$ is a solution of the ODE, $\psi(\omega z, \omega^{n+1}E) = \psi(\omega z, \omega^{-2}E)$ is also the solution to the ODE.

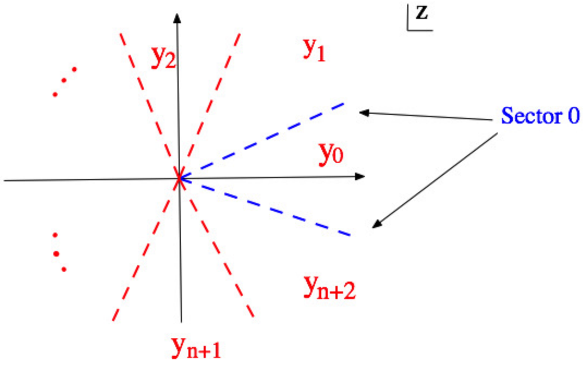
WKB approximation

- At large $|z|$, we solve the ODE by WKB approximation solution $y(z, E)$ around real positive axis:

$$y_0(z, E) \sim \frac{z^{-\frac{n+1}{4}}}{\sqrt{2i}} \exp\left(-\frac{z^{\frac{n+3}{2}}}{\frac{n+3}{2}}\right).$$

- This is the subdominant solution at large $|z|$ in sector $\mathcal{S}_0$, where $\mathcal{S}_k : |\arg z - \frac{2k\pi}{n+3}| < \frac{\pi}{n+3}$.

T-function



- There are $n+3$ sectors in the z-plane (Stokes phenomenon).
- The subdominant solution at sector k : y_k

$$y_k(z,E) = \omega^{\frac{k}{2}} y_0(\omega^{-k} z, \omega^{2k} E) \qquad \text{(Sibuya Trick).}$$

- No singularity at $z=0$, $y_{n+3}(z) \propto y_0(e^{i2\pi}z) = y_0(z)$.

- Wronskian $W_{i,j} = y_i y_j' - y_j y_i' = \text{const} \implies (y_0, y_1)$: a set of basis .

$$y_k = -\frac{W_{1,k}}{W_{0,1}} y_0 + \frac{W_{0,k}}{W_{0,1}} y_1 = -T_{k-2}(e^{\frac{2(k+1)\pi i}{n+3}} E) y_0 + T_{k-1}(e^{\frac{2k\pi i}{n+3}} E) y_1$$

$$T_s(E) = \frac{W_{0,s+1}}{W_{0,1}} (\omega^{-(s+1)} E)$$

Plücker relation of the 2×2 determinants

$$W_{k_1,k_2} W_{k_3,k_4} = W_{k_1,k_4} W_{k_3,k_2} + W_{k_3,k_1} W_{k_4,k_2}$$

$\Longrightarrow$

Functional relations of T-function

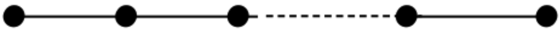
$$T_s(\omega E) T_s(\omega^{-1} E) = T_{s+1} T_{s-1} + 1$$

Y-system

Y-function $Y_s = T_{s-1} T_{s+1}$

$$Y_s(e^{\frac{2\pi i}{n+3}} E) Y_s(e^{-\frac{2\pi i}{n+3}} E) = (1 + Y_{s-1})(1 + Y_{s+1})$$

No singularity around $z = 0 \Longrightarrow T_{n+2} = \frac{W_{0,n+3}}{W_{0,1}}(\omega^{-(s+1)} E) = 0, \;\; Y_{n+1} = 0$
We get the A_n-type Y-system.



Corresponding 2d CFT

- We rewrite the Y-system in terms of (UV limit of) A_n TBA equation.

$$\log Y_k(\theta) = m_k L e^{\theta} - \sum_{j=1}^{n} \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \phi_{kj}(\theta-\theta') \log(1+\frac{1}{Y_j})(\theta'), \; E = e^{\frac{2(n+1)}{n+3}\theta}$$

- Then we obtain the ground state energy (effective central charge) of the corresponding 2d CFT

$$c_{eff} = \frac{n}{n+3} = c - 24\Delta_{min}.$$

For even n

This corresponds to the non-unitary minimal model $M_{2,n+3}$ with central charge and lowest conformal dimension Δ_{min} operator (as ground state)

$$c(n) = 1 - \frac{3(n+1)^2}{n+3}, \; \Delta_{min}(M) = \Delta_{1,1+\frac{n}{2}} = \frac{1-(n+1)^2}{8(n+3)},$$

- The same T-/Y-system and Bethe ansatz equations also appear in the study of $M_{2,2N+3}$ [BLZ '94 '96 '98].
- Both of them have the same asymptotic behavior $\Longrightarrow$ same solution $\Longrightarrow$ ODE/CFT.

CFT	ODE
spectral parameter λ	E
q(β)	$\omega(\frac{n+1}{2})$
$\mathbf{T}_j \|\Delta\rangle = T_j \|\Delta\rangle$	T_j-function
$\mathbf{Y}_j \|\Delta\rangle = Y_j \|\Delta\rangle$	Y_j-function

Table: Details of the correspondence

Comparing the 2d CFT and 2d chiral algebra

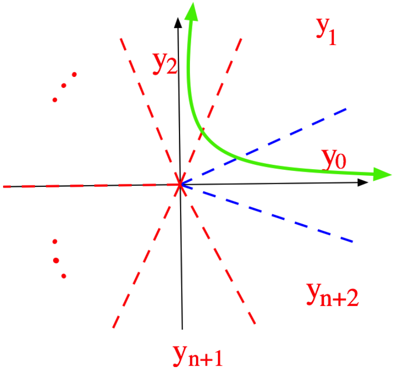
Let us compare the 2d IM and 2d chiral algebra

AD theory	2d CFT	2d chiral algebra
(A_1, A_{2N})	$M_{2,2N+3}$	$M_{2,2N+3}$ [Xie-Yan-Yau '16]

Table: Comparing the 2d CFT and 2d chiral algebra

$$\text{2d CFT} \Longleftrightarrow \text{2d chiral algebra}$$

Quantntization condition



$$\psi(z,E) \propto y_0, \quad z \in \mathcal{S}_0, \quad , \quad \psi(z,E) \propto y_{k+1}, \quad z \in \mathcal{S}_{k+1}$$

$$y_0 = \frac{W_{0,k+2}}{W_{k+1,k+2}} y_{k+1} + \frac{W_{k+1,0}}{W_{k+1,k+2}} y_{k+2}$$

$$\log Y_k(\omega^k E) = (2l+1)\pi i, \quad l : \text{integer}$$

- Y-function $\Longleftrightarrow$ quantum deformed period
  (period integral of Seiberg-Witten differential around certain contour)

- One can express the Bohr-Sommerfeld quantization condition of quantum deformed period by using TBA equation.

- In general case, to obtain the correction of Bohr-Sommerfeld quantization condition of the quantum deformed period, one has to consider the higher order WKB approximation about ϵ.

Remark

- For odd n, the corresponding 2d CFT is not minimal model $M_{2,n+3}$.
- There is no need to let 2d CFT equal to 2d chiral algebra. The underlying interpretation is still missing.

Generalize to other AD theories

If you can not solve it, generalize it!

Quantum curve of (A_n, A_m) AD theory

- We consider the simple form quantum curve of (A_n, A_m) AD theory

$$(A_n, A_m): \quad \left((-1)^n \frac{d^{n+1}}{dz^{n+1}} + z^{m+1} - E\right)\psi(z, E) = 0$$

- This is the A_n-type ODE in [Dorey-Dunning-Masoero-Suzuki-Tateo '07]

According to [Dorey-Dunning-Masoero-Suzuki-Tateo '07] and [Dorey-Dunning-Gliozzi-Tateo '07], these correspond to

AD theory	2d CFT	2d chiral algebra
(A_n, A_m)	$\frac{(A_n)_L \times (A_n)_1}{(A_n)_{L+1}}$	$\frac{(A_n)_L \times (A_n)_1}{(A_n)_{L+1}}$ [Xie-Yan-Yau '16]

Table: Comparing the 2d CFT and 2d chiral algebra

where $L = \frac{n+1}{m+1} - (n+1)$. We get

$$\text{2d CFT} \Longleftrightarrow \text{2d chiral algebra}$$

Another way to obtain the quantum curve

The conformal limit of the linear problem associated with the $A_n^{(1)}$ modified affine Toda field equations:

$$(\partial_z + A(z))\Psi(z) = 0$$

$$A(z) = p(z)\sqrt{n_0^\vee}E_{\alpha_0} + \sum_{i=1}^{n}\sqrt{n_i^\vee}E_{\alpha_i}$$

($n_i^\vee$: dual Coxeter labels, E_{α_i}: generators of $A_n^{(1)}$)

The first component of Ψ satisfy one $(n+1)-$th order ODE. Choosing $p(z) = z^{m+1} + \cdots \implies$ Quantum curves of the (A_n, A_m)-type AD theories

Generalize to D_n type ODE (quantum curve)

- For D_n type AD theory, we use the $2n \times 2n$ matrix representation of $A_{D_n}(z)$.
- We get the ODE (quantum curve)

$$D_n : \quad \left(\frac{d^{2n-1}}{dz^{2n-1}} - 2^{n-1} \sqrt{p} \frac{d}{dz} \sqrt{p} \right) \psi(z, E) = 0.$$

- This is the simple case of D-type ODE in [Dorey-Dunning-Masoero-Suzuki-Tateo '07].

Generalize to E_6 type ODE (quantum curve)

- For E_6 type AD theory, we use the 27×27 matrix representation.
- We get the ODE (quantum curve)

$$
\begin{aligned}
0 \;=\;& 6(p' + 3p\partial)\partial^{-9}(2p' + 3p\partial)\psi_1 + \frac{1}{\sqrt{3}}p^{(5)}\psi_1 + \frac{367\sqrt{3}}{24}p^{(4)}\psi_1^{(1)} \\
&+ \frac{21\sqrt{3}}{2}p^{(3)}\psi_1^{(2)} + \frac{39\sqrt{3}}{2}p^{(2)}\psi_1^{(3)} + \frac{75\sqrt{3}}{4}p^{(1)}\psi_1^{(4)} + \frac{15\sqrt{3}}{2}p\psi_1^{(5)} \\
&- \frac{1}{32\cdot 27}\psi_1^{(17)}.
\end{aligned}
$$

- This is not known in [Dorey-Dunning-Masoero-Suzuki-Tateo '07].

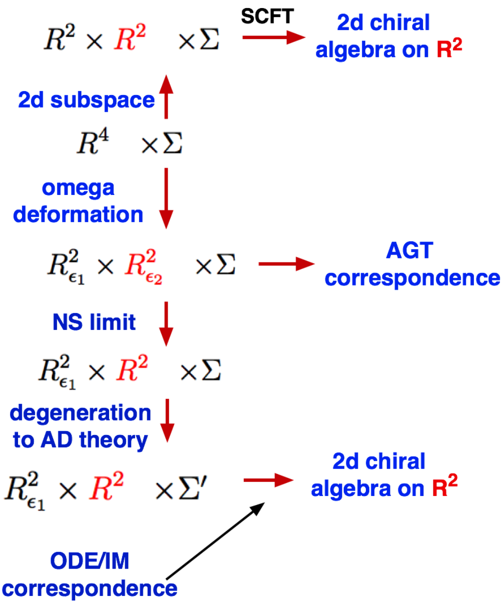
Conclusion and Outlook

- The ODE/CFT correspondence describes the relation between AD theory (quantum curve) and the related 2d chiral algebra obtained by 4d/2d correspondence.
- ODE/CFT is a powerful method to study AD theory.
- The underlying interpretation is still missing (for me).

- A_{2N+1} type (+flavor symmetry)
- D_n and E_n type
- ODE/CFT for E-type ODE
- Deformation by some operator $\rightarrow$ RG flow of AD theory.

Thanks for your attention !

One interpretation



Integrability structure in CFT ($c < 1$)

$$T(u) = -\frac{1}{24} + \sum_{n \in Z} L_n e^{-inu}, \quad T(u+2\pi) = T(u) \quad \text{on cylinder}$$

Integrals of Motion (IM= conserved charge) I_{2k-1}:

$$T_2(u) = T(u), \quad T_{2k}(u) =: T^{2k}(u) : + \text{terms with derivative}$$

$$\mathbf{I}_{2k-1} = \int_0^{2\pi} du T_{2k}(u) \quad (k = 1, 2, \cdots)$$

$$[\mathbf{I}_{2k-1}, \mathbf{I}_{2k'-1}] = 0 \qquad \text{[Sasaki-Yamada '88]}$$

In the "classical limit" $c \to -\infty$:

$$T(u) \to -\frac{c}{6} U(u), \quad [,] \to \frac{6\pi}{ic}\{,\}_P, \quad I_{2k-1} \to \text{"KdV series"} \, \mathbf{I}^{cl}_{2k-1}.$$

KdV equation: e.o.m.associated with $\mathbf{I}_3$ ("Hamiltonian").

"Classical case" ($c \to -\infty$)

Lax pair $\Longleftrightarrow$ e.o.m. associated to $\mathbf{I}^{cl}_{2n-1}$.
One can consider the Lax operator and their associated linear problem:

$$\pi_j[\mathcal{L}(u)] := \pi_j[\partial_u - \phi' H - \lambda(E+F)], \quad \mathcal{L}\Psi = 0$$

1. λ: spectral parameter, $-U(u) = (\phi')^2(u) + \phi''(u)$
2. $\pi_j[H, E, F]$ generators of $sl(2)$ with $2j+1$ rep.

Monodromy matrix: $\Psi(u)\mathbf{M}(\lambda) = \Psi(u+2\pi)$, Transfer matrix: $\mathbf{T}_j = \mathrm{tr}[\mathcal{M}_j]$

$$\{\mathbf{T}_j(\lambda), \mathbf{T}_{j'}(\lambda')\}_P = 0, \quad \log[\mathbf{T}_{\frac{1}{2}}] \sim \lambda[1 - \sum_{n=1}^{\infty} I^{cl}_{2n-1}\lambda^{-2n}], \ \lambda \to \infty$$

$\mathbf{T}_j$ is the generating function for I^{cl}_{2k-1}.

Free field rep (Feigin-Fuchs rep = quantum Miura transformation)

$$-\beta^2 T(u) := \varphi'(u) :^2 + (1-\beta^2)\varphi''(u) + \frac{\beta^2}{24},$$

$$\varphi(u) = iQ + iPu + \sum_{n\neq 0} \frac{a_{-n}}{n} e^{inu}.$$

where Q, P, a_n satisfy the Heisenberg algebra:

$$[Q, P] = \frac{i}{2}\beta^2, \quad [a_n, a_m] = \frac{n}{2}\beta^2 \delta_{n+m,0}, \quad [Q, a_n] = 0 = [P, a_n].$$

Fock space $\mathcal{F}_p \simeq \mathcal{V}_\Delta$

$$\mathcal{F}_p = \text{Span}\{a_{n_1} \cdots a_{n_k} |p\rangle\}, \quad a_{n>0} |p\rangle = 0, \quad P |p\rangle = p |p\rangle$$

$c = 13 - 6(\beta + \beta^{-1}), \quad \Delta = (\frac{p}{\beta})^2 - \frac{c-1}{24}, \quad L_0 |p\rangle = \Delta |p\rangle$

Quantum enveloping algebra $\mathcal{U}_q(sl(2))$:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = \frac{q^H - q^{-H}}{q - q^{-1}}, \quad q = e^{i\pi\beta^2}$$

Quantum counterparts of $\mathbf{L}-$matrices and monodromy matrix are given as

$$\mathbf{L}_j(\lambda) = \pi_j[e^{-i\pi PH}]\mathbf{M}_j(\lambda)$$

$$\mathbf{M}_j(\lambda) = \pi_j\left[e^{2i\pi PH} P\exp[\lambda\int_0^{2\pi} du(: e^{-2\varphi(u)} : q^{\frac{H}{2}}E + : e^{2\varphi(u)} : q^{-\frac{H}{2}}F)]\right]$$

This $L-$matrices satisfy the Yang-Baxter equation

$$\mathbf{R}_{jj'}(\lambda\mu^{-1})(\mathbf{L}_j(\lambda)\otimes 1)(1\otimes\mathbf{L}_{j'}(\mu)) = (1\otimes\mathbf{L}_{j'}(\mu))(\mathbf{L}_j(\lambda)\otimes 1)\mathbf{R}_{jj'}(\lambda\mu^{-1})$$

Transfer operator $\mathbf{T}_j$

$$\mathbf{T}_j(\lambda) = \mathrm{tr}_{\pi_j}(\mathbf{M}_j(\lambda))$$

Yang-Baxter equation leads to

$$[\mathbf{T}_j(\lambda), \mathbf{T}_{j'}(\lambda')] = 0, \quad [\mathbf{T}_j(\lambda), \mathbf{I}_{2k-1}] = 0$$

Eigenvalue of T-operator

$$\mathbf{T}_j \left| p \right\rangle = T_j \left| p \right\rangle$$

$$T_j(q^{\frac{1}{2}}\lambda) T_j(q^{-\frac{1}{2}}\lambda) = 1 + T_{j+\frac{1}{2}}(\lambda) T_{j-\frac{1}{2}}(\lambda)$$

Y-function

$$Y_j(\theta) = T_{j+\frac{1}{2}}(\lambda) T_{j-\frac{1}{2}}(\lambda), \quad \lambda = \exp(\frac{\theta}{1+\xi})$$

$$Y_j(\theta + \frac{\pi i \xi}{2}) Y_j(\theta - \frac{\pi i \xi}{2}) = \left(1 + Y_{j+\frac{1}{2}}(\theta)\right)\left(1 + Y_{j-\frac{1}{2}}(\theta)\right)$$

Finite T-/Y-system

1. $\mathbf{T}_j$ can be regarded as generating function for the IM.
2. As we consider minimal model $M_{2,2N+3}$, the functional relations are truncated. This leads to an A_{2N}-type T-system.

$$\mathbf{T}_{N+\frac{1}{2}} = \mathbf{1}, \quad \mathbf{T}_{N+1} = 0, \quad \text{only } (j = \frac{1}{2}, 1, \cdots, N) \text{ are nontrivial}$$

3. One can also construct the Q and Y-system [BLZ '94 '96 '98].

More about ODE/CFT correspondence

[Dorey-Dunning-Suzuki-Tateo '07] generalize the ODE/CFT to ABCD-type ODE:

$$D(g) = (\frac{d}{z} - \frac{g}{z}), \quad P_K = (x^{hM/K} - E)^K, \quad h: \text{Coxeter numbers}$$

$$D_n(\mathbf{g}) = D(g_{n-1} - (n-1))D(g_{n-2} - (n-2)) \cdots D(g_1 - 1)$$

$A_{n-1}(su(n))$:

$$\left[(-1)^n D_n(\mathbf{g}) - P_K(x, E)\right] \psi(x) = 0$$

$D_n(so(2n))$: $\cdots$