Tensor renormalization group study of (1+1)-dimensional U(1) gauge-Higgs model at θ=π with Lüscher's admissibility condition

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Based on
SA-Kuramashi, JHEP09(2024)086, PoS(LATTICE2024)361

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2025.11.19

Formulation of renormalization group

- Block-spin transformation by Kadanoff Kadanoff, Phys. Phys. Fiz. 2(1966)263-272
- Formulation of the RG method via the introduction of scale trans. by Wilson

of cooperative behavior arise. In the renormalization group framework, these qualitative features result from the iterative character of the renormalization group. Namely, there is a transformation τ which converts $\mathcal{H}_0$ to $\mathcal{H}_1$, $\mathcal{H}_1$ to $\mathcal{H}_2$, etc. The transformation is the same whether one is constructing $\mathcal{H}_1$ from $\mathcal{H}_0$ or $\mathcal{H}_2$ from $\mathcal{H}_1$; in each case one is thinning the degrees of freedom by a factor 2. The only difference is in the lengthscale (L_0 versus $2L_0$) which is easily transformed away. So one has a transformation τ which is to be applied repeatedly:

$$\tau(\mathcal{H}_0) = \mathcal{H}_1, \qquad \tau(\mathcal{H}_1) = \mathcal{H}_2, \qquad \tau(\mathcal{H}_2) = \mathcal{H}_3 \qquad \text{etc.} \tag{1.1}$$

This transformation is to be iterated n times where $2^n L_0$ is of order ξ. When ξ is large, the number of iterations is large.

When one has a transformation τ which is iterated many times, the simplest result we can obtain is that the sequence $\mathcal{H}_l$ approaches a fixed point of τ, namely an interaction $\mathcal{H}^*$ satisfying

$$\tau(\mathcal{H}^*) = \mathcal{H}^*. \tag{1.2}$$

This is what will happen in the examples discussed later in this review.

Wilson−Kogut, Phys. Rept. 12(1974)75-199

- As a practical numerical method, the further development has been made based on tensor network description

RG methods as practical tools

- **Numerical RG** method for single impurity Kondo problem Wilson, RMP47(1975)773
  - “0D” problems
- **Density Matrix RG** for 1D quantum lattice models White, PRL69(1992)2863-2866
  - Reduction of the number of states via the density matrix formalism
  - The most accurate numerical method for 1D quantum systems
- **Corner transfer matrix RG** for 2D classical systems Nishino-Okunishi, JPSJ65(1996)891
  - A transfer-matrix RG method for 2D models based on Baxter’s CTM
    Baxter, J. Math. Phys. 9(1968)650-654
- **Tensor Network RG** Levin–Nave, PRL99(2007)120601 Evenbly–Vidal, PRL115(2015)180405
  - Usually formulated within the Lagrangian formalism (**Tensor Network RG approach**)
  - With the advancement of QC, TN methods based on the Hamiltonian formalism have also been increasingly applied to lattice field theory (**Tensor Network State approach**)

TN representations for path integrals

- Most LFTs are ready to be described by network of tensors

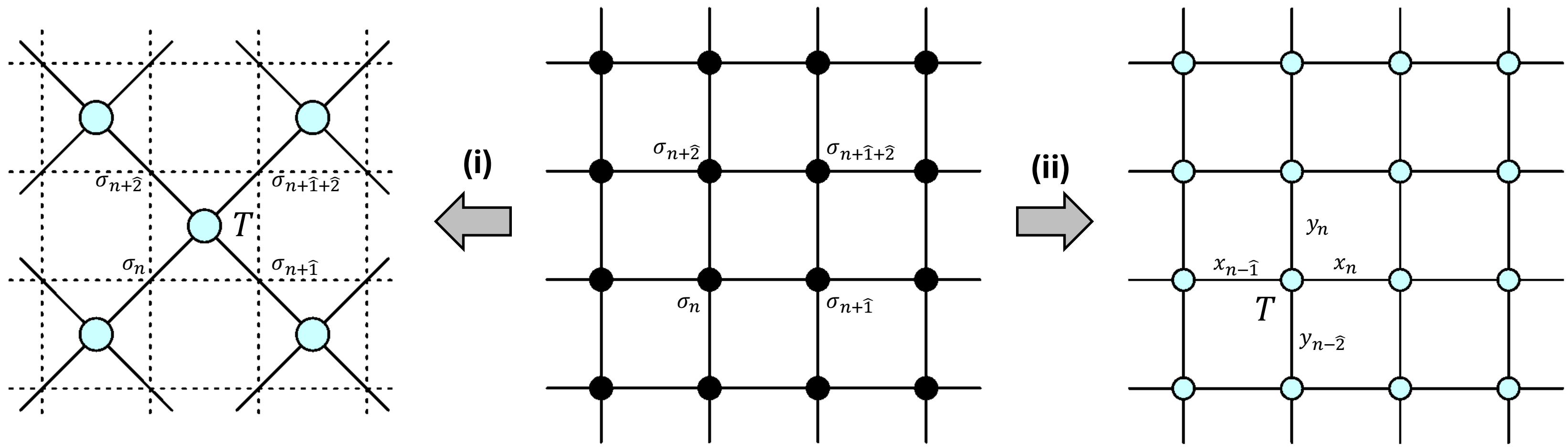
Liu–Meurice–Qin–Unmuth-Yockey–Xiang–Xie–Yu–Zou, PRD88(2013)056005
[Review] Meurice–Sakai–Unmuth-Yockey, RMP94(2022)025005

Boltzmann weight → Tensor elts	Fields → Tensor indices	Interaction → Tensor contraction

- Demonstration in the 2D Ising model

(i) $Z = \Sigma_{\{\sigma=\pm1\}} \prod T_{\sigma_n\sigma_{n+\hat{1}}\sigma_{n+\hat{2}}\sigma_{n+\hat{1}+\hat{2}}}$ w/ $T_{\sigma_n\sigma_{n+\hat{1}}\sigma_{n+\hat{2}}\sigma_{n+\hat{1}+\hat{2}}} := \mathrm{e}^{K(\sigma_n\sigma_{n+\hat{1}}+\sigma_{n+\hat{1}}\sigma_{n+\hat{1}+\hat{2}}+\sigma_{n+\hat{1}+\hat{2}}\sigma_{n+\hat{2}}+\sigma_{n+\hat{2}}\sigma_n)}$

(ii) $Z = \Sigma_{\{x,y=0,1\}} \prod T_{x_n y_n x_{n-\hat{1}} y_{n-\hat{2}}}$ w/ $T_{xyx'y'} := 2\cosh^2 K\left(\sqrt{\tanh K}\right)^{x+y+x'+y'} \delta_{x+y+x'+y' \bmod 2,0}$



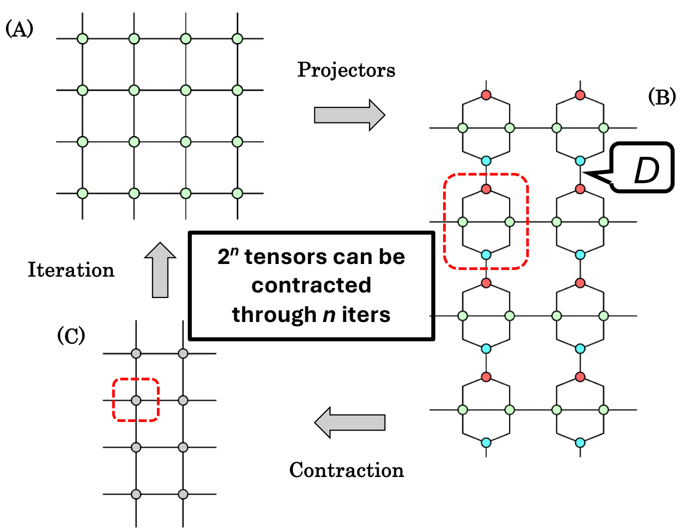
Tensor Network RG

- TNRG recursion formula based on truncated SVD up to a given **bond dimension *D***
  - **TNRG approximately computes path integrals** via their TN rep.

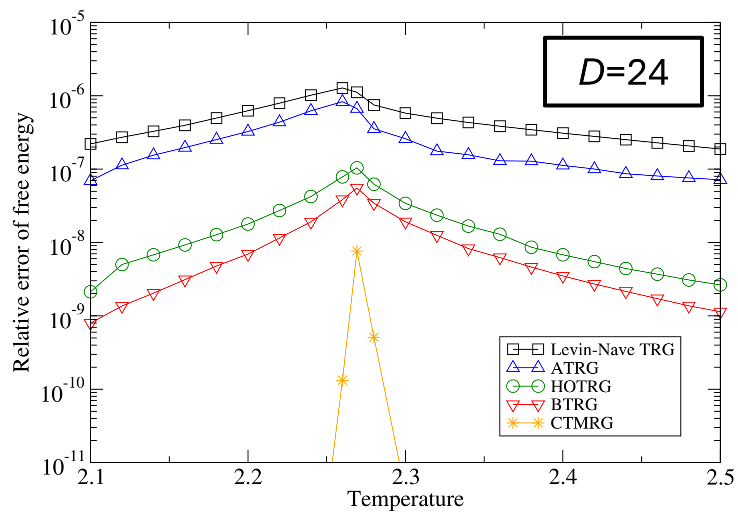
Levin–Nave, PRL99(2007)120601
Xie–Chen–Qin–Zhu–Yang–Xiang, PRB86(2012)045139
Evenbly–Vidal, PRL115(2015)180405
Yang–Gu–Wen, PRL118(2017)110504
Morita–Kawashima, CPC236(2019)65-71

- When $D \to \infty$, the TNRG exactly contracts the given TN rep.
  - **The accuracy can be systematically improved** by increasing D

A typical TNRG recursion formula



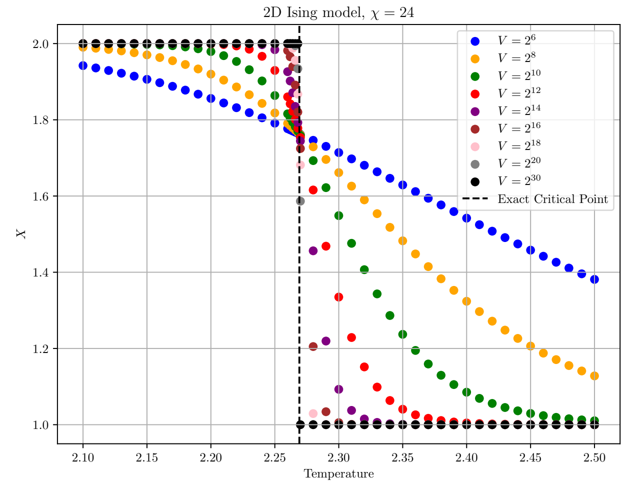
2D Ising model



Hunting Z_n symmetry breaking via partition function

Gu–Wen, PRB80(2009)155131

- Ground-state degeneracy from the renormalized tensors
  - $X \coloneqq Z(N_x, N_y)^2 / Z(2N_x, N_y)$
- Ex. 2D Ising model
  - Symmetric phase: $Z \sim 1 \times e^{\lambda N_x N_y} \Rightarrow X = 1$
  - SSB phase: $Z \sim 2 \times e^{\lambda' N_x N_y} \Rightarrow X = 2$
  - Resulting structure of renormalized tensors are different according to the realized phases ⇨ "Fixed-point tensor"



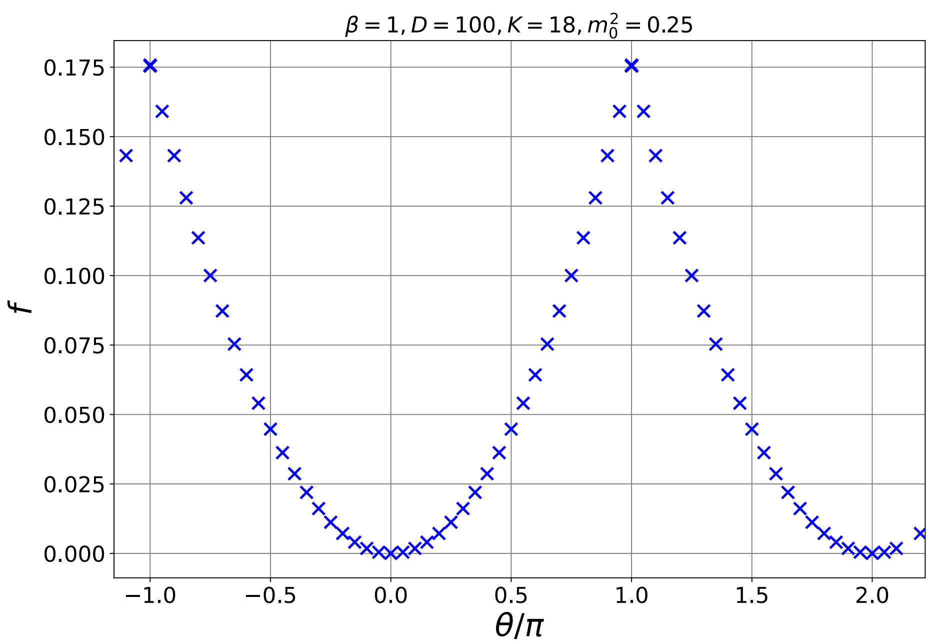
- This method enables us to locate critical points in arbitrary dimensions

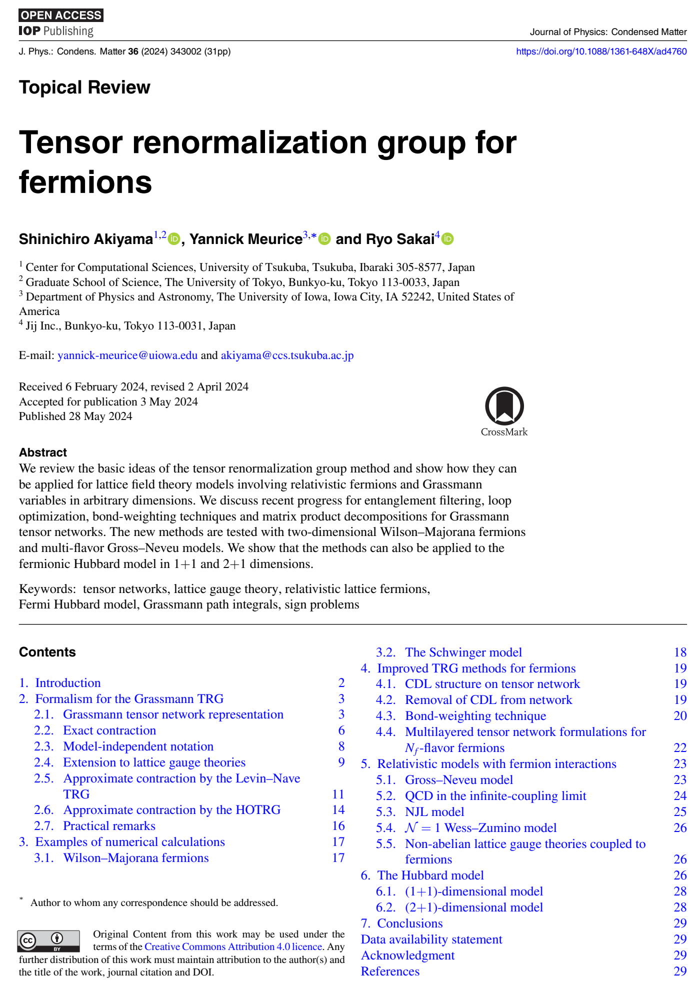
  Wang+, CPL31(2014)070503, SA+, PRD100(2019)054510
  - Finite-size scaling (FSS) for X is recently reported — Morita–Kawashima, PRB111(2025)054433
- Extension to continuous symmetry breaking

  SA–Jha–Unmuth-Yockey, PRD110(2024)034519
  Tanizaki–Maeda, JHEP08(2025)128
  SA–Jha–Maeda–Tanizaki–Unmuth-Yockey, in preparation

Grassmann TNRG approach

- **TNRG can directly deal with the Grassmann path integral w/o pseudo-fermion** Gu–Verstraete–Wen, arXiv.1004.2563, Shimizu–Kuramashi, PRD90(2014)014508, SA–Kadoh, JHEP10(2021)188
  - Introduction to the Grassmann TNRG
    SA–Meurice–Sakai, Journal of Physics: Condensed Matter 36 (2024) 343002
- A study on Nf=2 massive Schwinger model w/ a θ term Kanno–SA–Murakami–Takeda, JHEP11(2025)036

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Topical Review

Tensor renormalization group for fermions

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Abstract
We review the basic ideas of the tensor renormalization group method and show how they can be applied for lattice field theory models involving relativistic fermions and Grassmann variables in arbitrary dimensions. We discuss recent progress for entanglement filtering, loop optimization, bond-weighting techniques and matrix product decompositions for Grassmann tensor networks. The new methods are tested with two-dimensional Wilson–Majorana fermions and multi-flavor Gross–Neveu models. We show that the methods can also be applied to the fermionic Hubbard model in 1+1 and 2+1 dimensions.

Keywords: tensor networks, lattice gauge theory, relativistic lattice fermions, Fermi Hubbard model, Grassmann path integrals, sign problems

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Toward the QCD at finite density

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- MC has played a central role as a practical numerical method for simulating lattice field theories
  - **Information extraction via sampling**
  - However, it suffers from the **sign problem**
    - A solution towards the sign problem

Cf. Tempered Lefschetz thimble method, Fukuma–Umeda, PTEP017(2017)73B01
Worldvolume HMC, Fukuma–Matsumoto, PTEP2021(2021)2023B08
...

- TNRG provides a distinct perspective from MC
  - **Information compression via tensor networks**
  - **No sign problem**
  - However, the accuracy of the compression depends on the **entanglement** in the system
- **Is it possible to explore the QCD at finite density using TNRG?**
  - "From Ising to QCD" [Review] Meurice–Sakai–Unmuth-Yockey, RMP94(2022)025005

II. LATTICE FIELD THEORY

A. The Kogut sequence: From Ising to QCD

In the early 1970s, QCD appeared to be a strong candidate for a theory of strong interactions involving quarks and gluons. However, the perturbative methods that provided satisfactory ways to handle the electroweak interactions of leptons failed to explain confinement, mass gaps, and chiral symmetry breaking. A nonperturbative definition of QCD was needed. In 1974, Wilson proposed (Wilson, 1974) a lattice formulation of QCD where the SU(3) local symmetry is exact. As this four-dimensional model is fairly difficult to handle numerically, a certain number of research groups started considering simpler lattice models in lower dimensions and then increased symmetry and dimensionality. This led to a sequence of models, sometimes called the "Kogut ladder," that appears in the reviews of Kogut (1979, 1983) and was later addressed with small modifications by Polyakov (1987) and Itzykson and Drouffe (1991).

The sequence is approximately the following:
(1) $D = 2$ Ising model
(2) $D = 3$ Ising model and its gauge dual
(3) $D = 2$ O(2) spin and Abelian Higgs models
(4) $D = 2$ fermions and the Schwinger model
(5) $D = 3$ and 4U(1) gauge theory
(6) $D = 3$ and 4 non-Abelian gauge theories
(7) $D = 4$ lattice fermions
(8) $D = 4$ QCD

Lüscher's admissibility condition

Lüscher, NPB549(1999)295-334

- The U(1) gauge action with the admissibility condition:

$$\beta S_g = \begin{cases} \beta \sum_{n,\mu>\nu} \dfrac{1-\mathrm{Re}P_{\mu\nu}(n)}{1-\|1-P_{\mu\nu}(n)\|/\epsilon} & \text{if } \|1-P_{\mu\nu}(n)\| < \epsilon, \\ \infty & \text{otherwise,} \end{cases}$$

$$P_{\mu\nu}(n) := U_\mu(n)U_\nu(n+\hat{\mu})U_\mu^\dagger(n+\hat{\nu})U_\nu^\dagger(n)$$

- The gauge fields are separated into disconnected subspaces, corresponding to topological charge

- In the MC simulation, the topological change is substantially suppressed

Fukaya–Onogi, PRD68(2003)074503

- With a θ term, the naive MC simulation also suffers from **the complex action problem and the topology freezing**

Why don't we take the advantage of TNRG?

- TNRG allows us to directly compute the path integral w/o resorting the probabilistic interpretation on the Boltzmann weight
  - All contributions from every topological sector should be automatically involved in the TNRG computations
- **We demonstrate that the complex action problem and topology freezing issue are simultaneously resolved by the TNRG**
  - Universal information is available from the transfer matrix (CDF data)

(1+1)D U(1) gauge-Higgs model w/ a θ term

- The U(1) gauge fields + complex scalar fields + a θ term

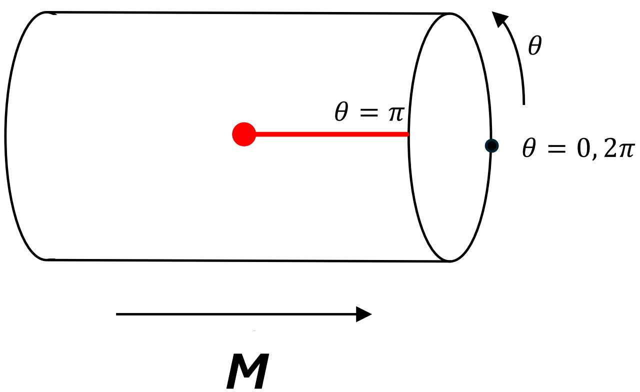
$$S = \beta S_g + S_h + S_\theta$$

$$S_h = -\sum_n \sum_\nu \left[\phi^*(n) U_\nu(n) \phi(n+\hat{\nu}) + \phi^*(n+\hat{\nu}) U_\nu^*(n) \phi(n)\right] + M \sum_n |\phi(n)|^2 + \lambda \sum_n |\phi(n)|^4$$

$$S_\theta = -\frac{\mathrm{i}\theta}{2\pi} \sum_n \ln P_{12}(n)$$

- **The first-order transition at θ = π when *M* > *M*$_c$**
  - 2D Ising universality at ***M* = *M*$_c$**

Gattringer+, NPB935 (2018) 344-364
Komargodski+, SciPost Phys. 6 (2019) 003



Path integral & its regularization

- The path integral on a lattice
  - Link variable is parametrized by $U_\nu(n) = \mathrm{e}^{\mathrm{i}\vartheta_\nu(n)}$
  - Complex scalar by $\phi(n) = r(n)\mathrm{e}^{\mathrm{i}\varphi(n)}$

$$Z = \prod_{n,\nu} \int_{-\pi}^{\pi} \frac{\mathrm{d}\vartheta_\nu(n)}{2\pi} \prod_{n} \int_{\mathbb{C}} \frac{\mathrm{d}\phi(n)}{2\pi} \exp(-S)$$

  - Unitary gauge eliminates the angular field $\varphi(n)$ from the path integral
- The path integral can be discretized by the Gauss quadrature

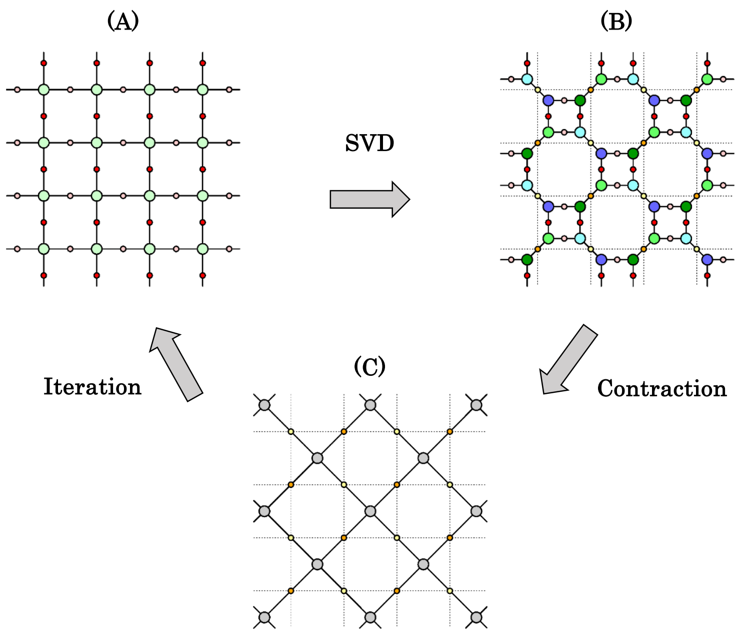
Kuramashi–Yoshimura, JHEP04(2020)089, Kadoh+, JHEP02(2020)161

  - $\int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_\nu(n)}{2\pi} \cdots \simeq \sum_{i=1}^{\boldsymbol{K_g}} \frac{w_i}{2} \cdots$ by the Gauss-Legendre quadrature
  - $\int_0^\infty r(n)\mathrm{d}r(n) \cdots \simeq \sum_{i=1}^{\boldsymbol{K_h}} \omega_i \cdots$ by the Gauss-Laguerre quadrature

TN formulation

- The path integral is approximately represented as a 2D tensor network
  - $Z(K_g, K_h) = \mathrm{tTr}\left[\prod_n T_n\right]$
  - The local tensor involves three parts: $(T_n)_{xyx'y'} = T^{(g)}_{x_g y_g x'_g y'_g} T^{(\theta)}_{x_g y_g x'_g y'_g} T^{(h)}_{x_h y_h x'_g x'_h y'_g y'_h}$
- TN rep. is approximately contracted by the bond-weighted TRG algorithm

Adachi–Okubo–Todo, PRB105 (2022) L060402



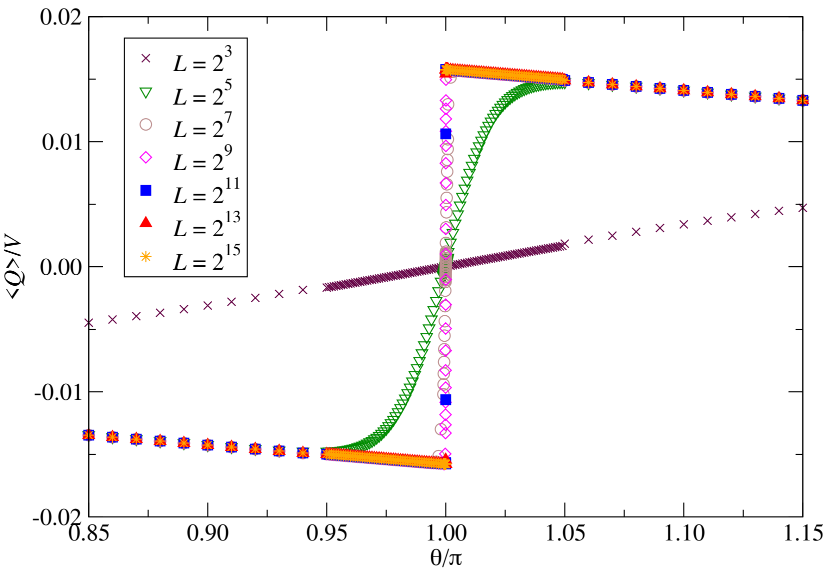
Pure gauge theory 1/2

w/ $\beta = 3, \epsilon = 1, D = K_g = 30$

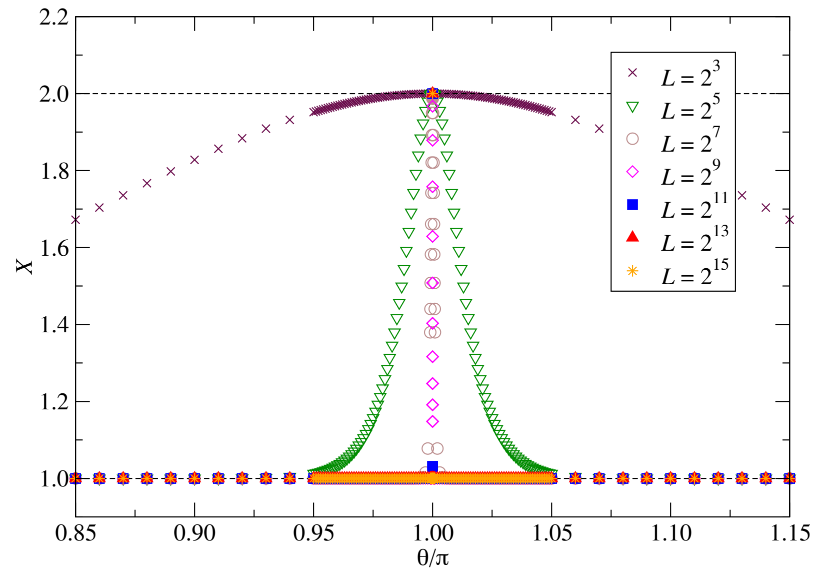
- A Clear signal of the first-order transition in the topological charge
- The two-fold ground state degeneracy at θ = π is also observed

Gu–Wen, PRB80(2009)155131

Topological charge density



Ground-state degeneracy

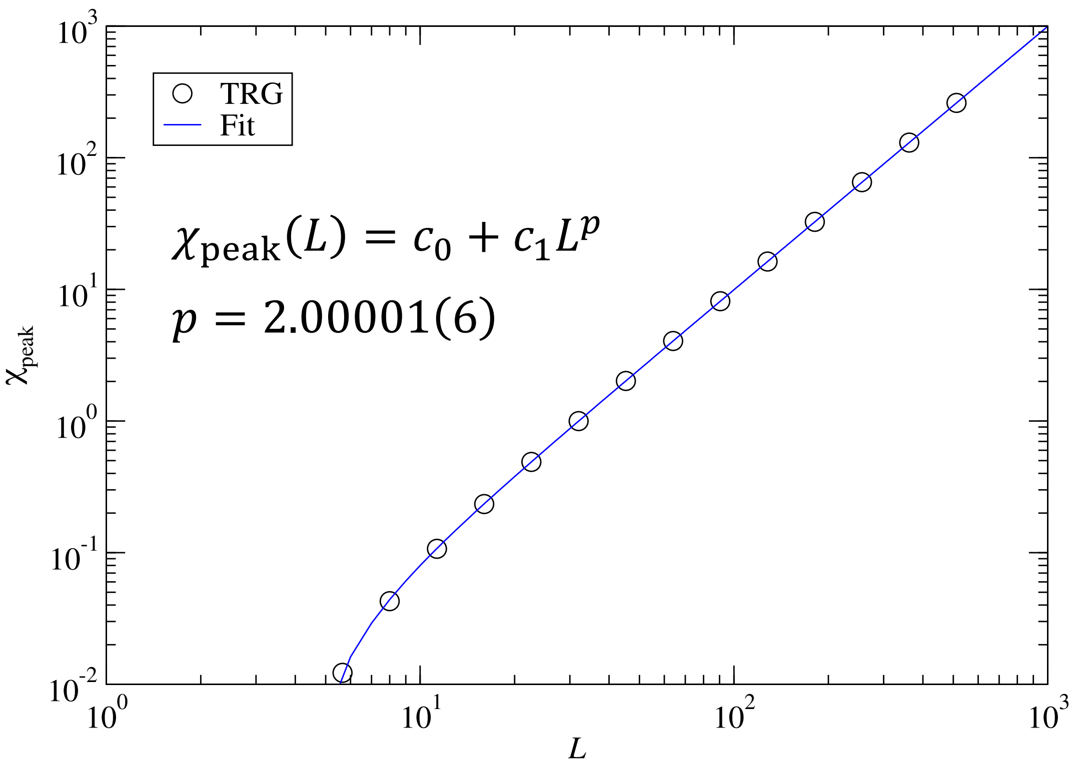


Pure gauge theory 2/2

w/ $\beta = 3, \epsilon = 1, D = K_g = 30$

- The peak height of the topological susceptibility is proportional to the volume
- **TRG is successfully dealing with the Lüscher gauge action**

Peak height of topological susceptibility

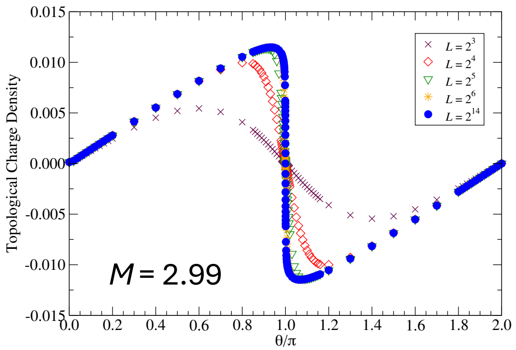


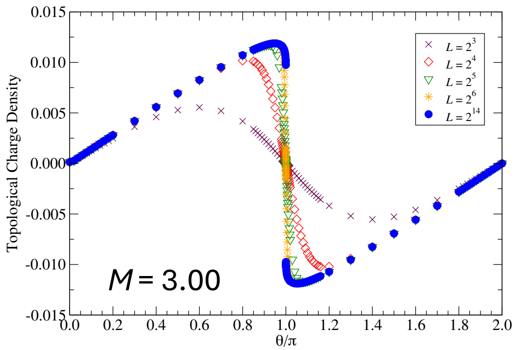
The gauge-Higgs model

w/ $\beta = 3, \lambda = 0.5, \epsilon = 1, K_g = K_h = 20, D = 160$

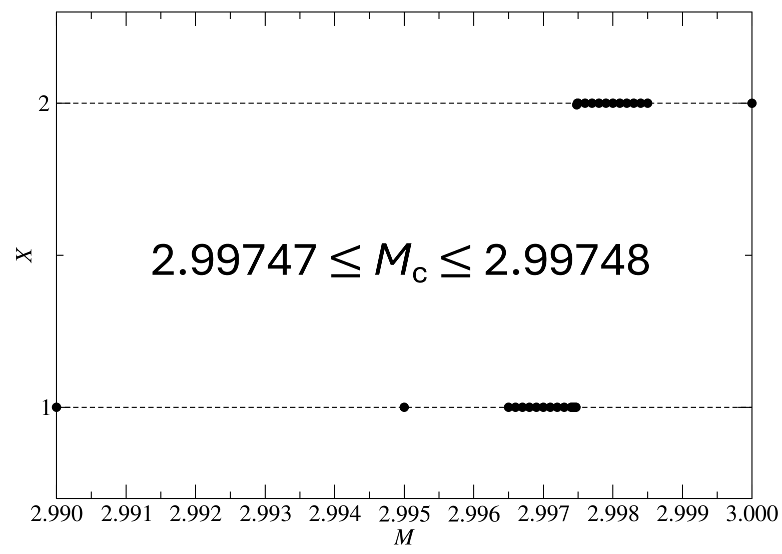
- Discontinuity in the topological charge is vanishing by decreasing the mass M
- Computing the ground-state degeneracy, we can bound the critical mass M_c

Topological charge density





Ground-state degeneracy



Identification of the universality class

w/ $\beta = 3, \lambda = 0.5, \epsilon = 1, K_g = K_h = 20, D = 160$

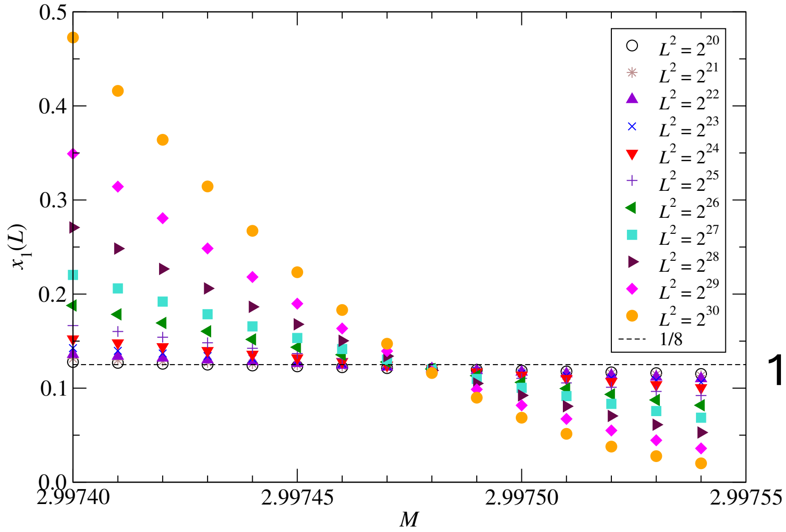
- Transfer matrix *T* is easily obtained from the TN representation

Gu–Wen, PRB80(2009)155131

- Ratio of the largest eigenvalue of *T* to smaller one: $x_n(L) = \frac{1}{2\pi} \ln \frac{\lambda_0(L)}{\lambda_n(L)}$

- These are nothing but the **scaling dimensions** when the system is sufficiently large and at criticality

- The volume independence in $x_1(L)$ is observed w/ $x_1(L) = 1/8$, which agrees with the 2D Ising universality class



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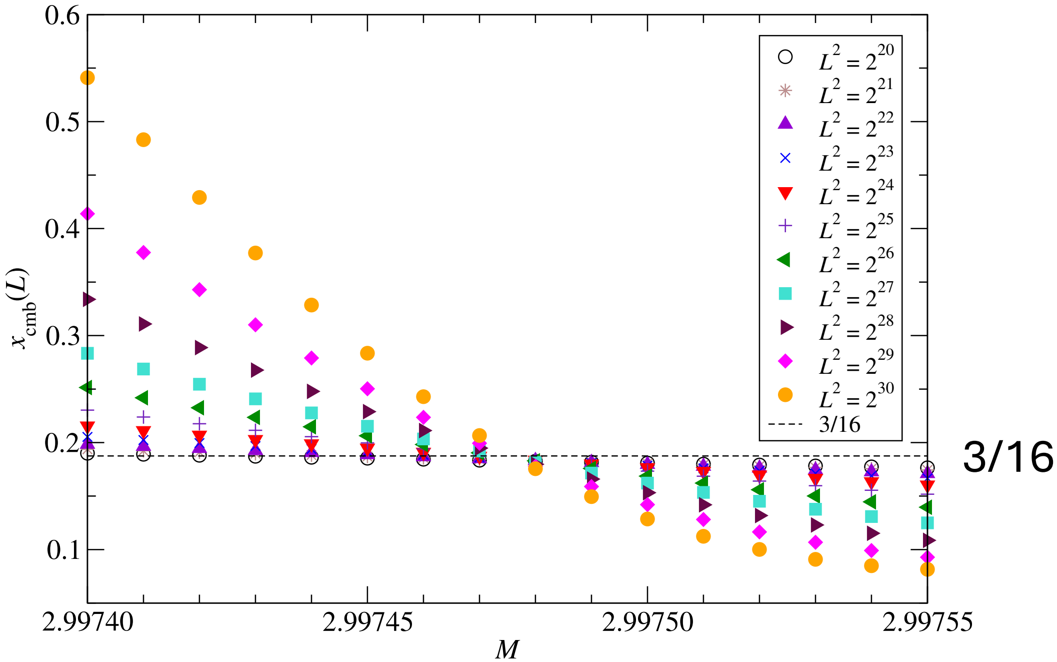
Tensor-network-based level spectroscopy

- Assuming the 2D Ising universality class, we employ the level spectroscopy to determine the critical mass M_c from scaling dimensions intersections

Ueda–Oshikawa, PRB108(2023)024413

- We particularly use the intersections of $x_{\mathrm{cmb}} = x_1 + x_2/16$ to remove the effect of the leading irrelevant perturbation



Critical point and central charge

w/ $\beta = 3, \lambda = 0.5, \epsilon = 1, K_g = K_h = 20, D = 160$

- The resulting critical mass is M_c = 2.997480(2)
  - Consistent not only with the bound from the ground-state degeneracy, but also comparable with the previous MC result based on dual representation employing the Villain-type gauge action: M_c = 2.989(2) Gattringer+, NPB935(2018)344-364
- Investigating the finite-size correction for the free energy, the central charge is obtained as c = 0.50(7), in agreement with the 2D Ising universality class
- The algorithmic-parameter dependence of M_c seems well suppressed

K_g	K_h	χ	D	M_c
24	20	8	192	2.9982886(1)
22	20	8	176	2.9998263(13)
20	20	8	160	2.9974765(14)
24	10	6	144	2.9929635(1)
22	10	6	132	2.9945222(9)
20	10	7	140	2.9921698(6)

χ is another algorithmic parameter to compress the initial bond dimension from $K_g K_h$ to $K_g \chi$

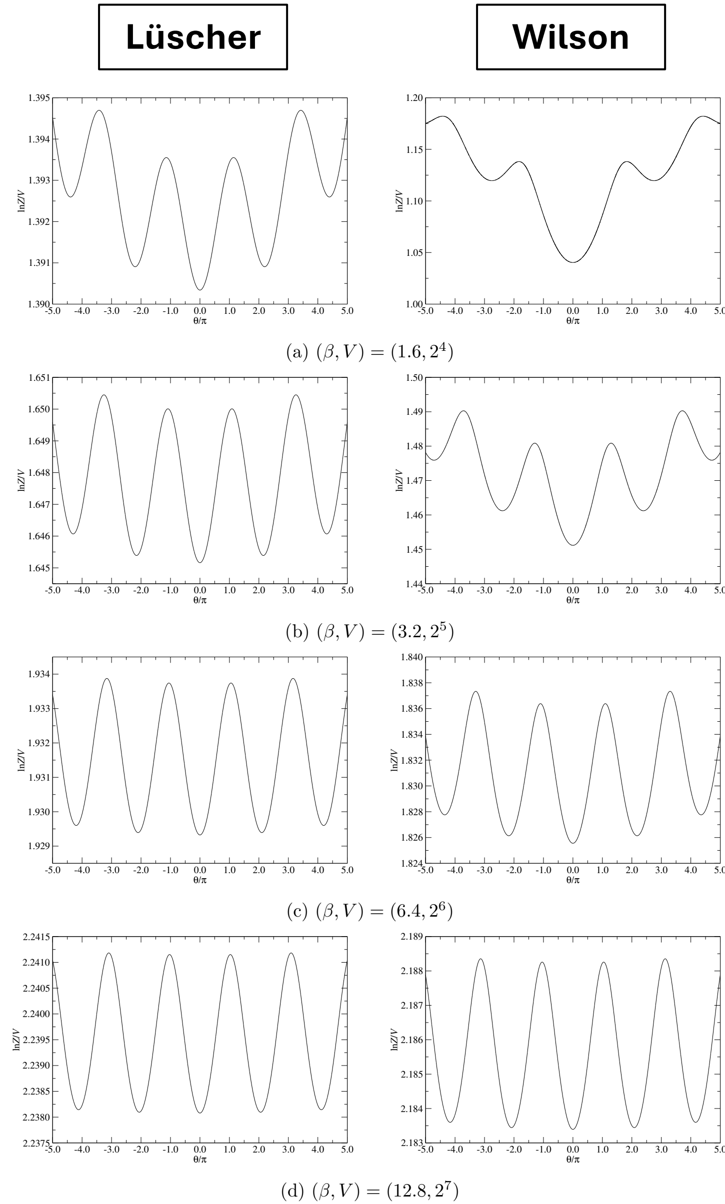
Bonus: Lüscher vs Wilson

- The field-theoretical def. of a θ term:

$$S_\Theta = -\frac{\mathrm{i}\theta}{2\pi}\sum_n \mathrm{Im}\, P_{12}(n)$$

- The 2π periodicity is restored only in the continuum limit

- The Lüscher action should show the faster convergence toward the continuum limit than the Wilson action

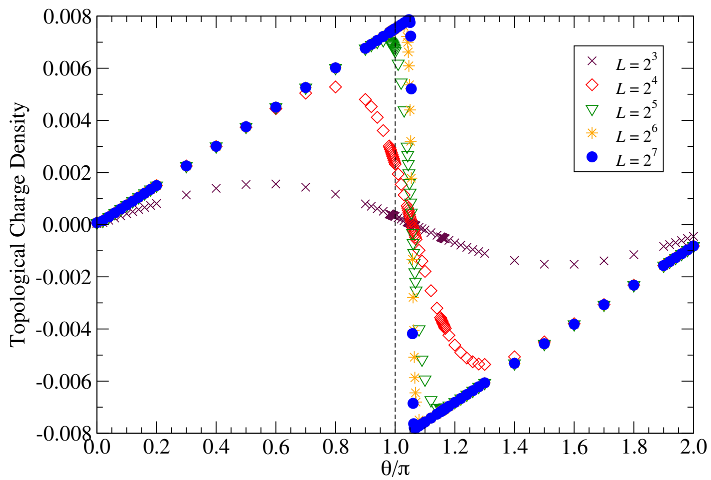


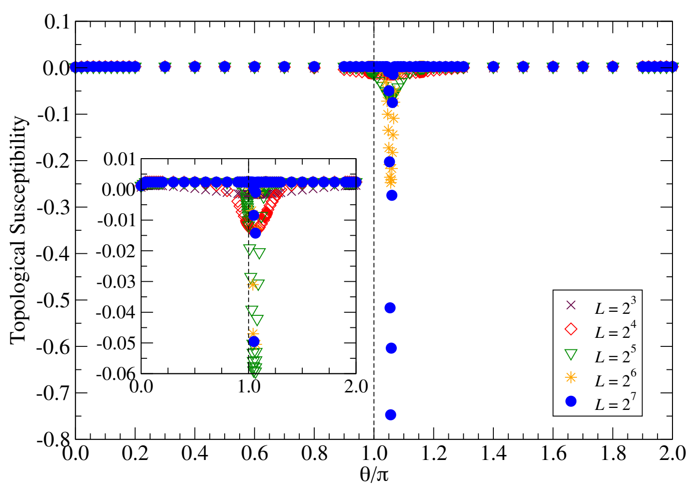
TRG vs MC

w/ $\boldsymbol{\beta} = \mathbf{10}, \lambda = 0.5, M = 4, K_g = K_h = 20, D = 160$

- **Quantitatively agreement btw the TRG and MC (dual lattice simulation)**

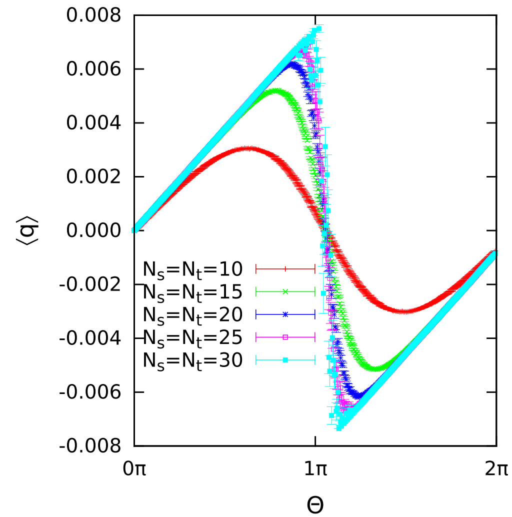
TRG

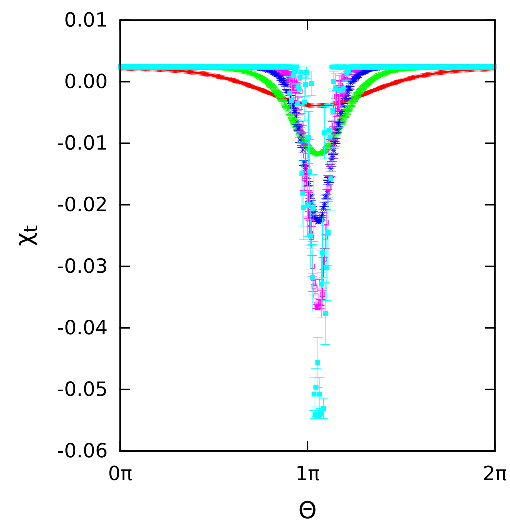




Dual simulation

Gattringer+, PRD92(2015)114508



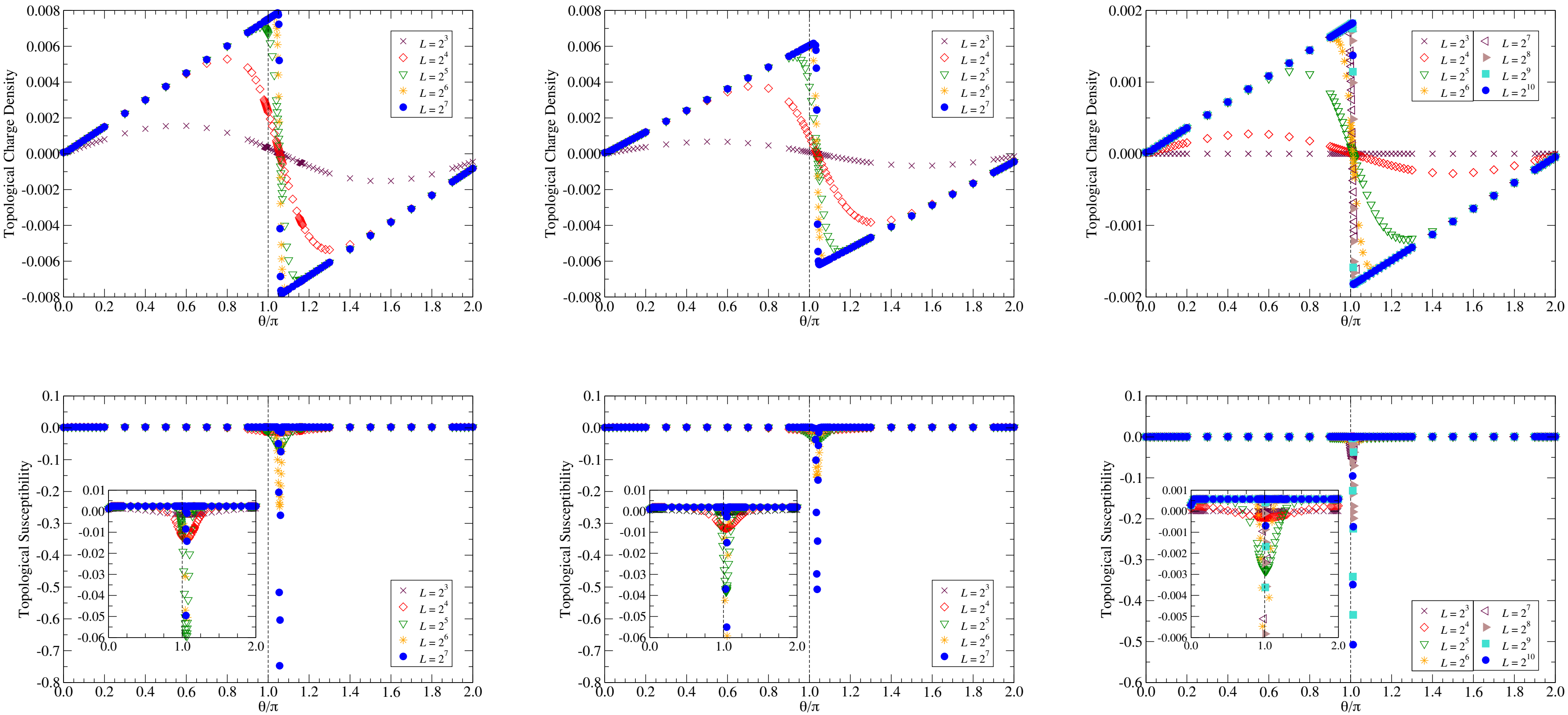


Bonus: Lüscher vs Wilson

w/ $\boldsymbol{\beta} = \mathbf{10}, \lambda = 0.5, M = 4, K_g = K_h = 20, D = 160$

- Transition point at fixed β is pushed toward θ = π, by decreasing ε
  - Although the finite-size effect is enhanced, it doesn't pose any issue for TRG

Wilson | Lüscher w/ $\epsilon = 1$ | Lüscher w/ $\epsilon = 0.1$



Summary and outlook

- Tensor network offers a novel formulation of real-space RG, whose accuracy can be systematically improved
  - No sign problem, path integral is available, CFT data from transfer matrices
- The critical behavior in the (1+1)D gauge-Higgs model with a θ term has been investigated by the TRG, **employing the Lüscher gauge action**
  - The critical mass is precisely determined
  - The **2D Ising universality class** is confirmed at $\theta = \pi$ as expected
  - Faster convergence toward the continuum limit than the Wilson action
- All numerical results show that the TNRG is a promising approach to study the lattice gauge theories with Lüscher's admissibility condition

Appendix

Tensor-network based level spectroscopy

- Assuming the 2D Ising universality class, we employ a level spectroscopy to determine the critical point T_c

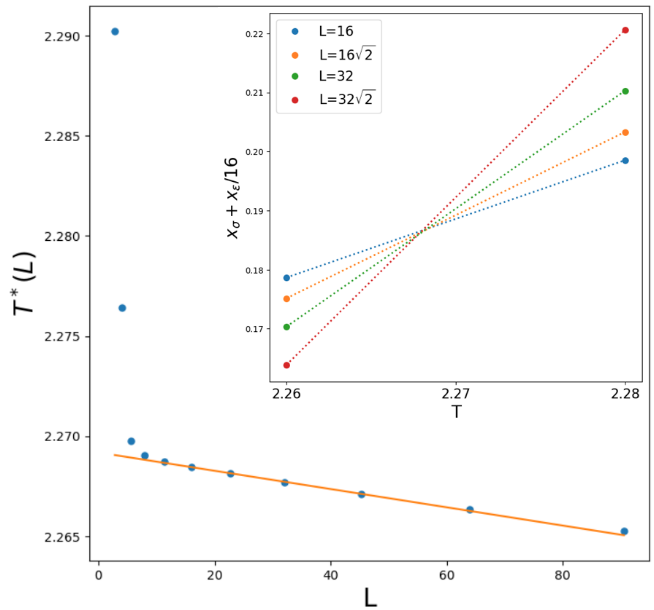
Ueda–Oshikawa, PRB108(2023)024413

(i) Choose two mass parameter $T^{(\pm)}$ such that $T^{(-)} \leq T_c \leq T^{(+)}$

(ii) At these two points, compute $x_{\text{cmb}}(L) = x_1(L) + x_2(L)/16$. This combination removes the effect from the leading irrelevant perturbation associated with the scaling dimension 4

(iii) Perform liner interpolations of $x_{\text{cmb}}(L) - 3/16$ btw $T^{(-)}$ and $T^{(+)}$ at each system size and find a crossing point $T^*(L)$ of two lines with the system sizes L and $\sqrt{2}L$

(iv) Fit $T^*(L)$ by $T^*(L) = T_c + aL$, and we finally obtain the critical point T_c



TN representation 1/2

- The discretized path integral is described by a four-leg local tensor T:
  - $$Z(K_g, K_h) = \mathrm{tTr}\left[\prod_n T_n\right]$$
  - $$(T_n)_{xyx'y'} = T^{(g)}_{x_g y_g x'_g y'_g} T^{(\theta)}_{x_g y_g x'_g y'_g} T^{(h)}_{x_h y_h x'_g x'_h y'_g y'_h}$$
  - $$T^{(g)}_{x_g y_g x'_g y'_g} = \begin{cases} \dfrac{\sqrt{w_{x_g} w_{y_g} w_{x'_g} w_{y'_g}}}{2^2} \exp\left[-\beta \dfrac{1-\cos\pi\left(y'_g + x_g - y_g - x'_g\right)}{1-\left[1-\cos\pi\left(y'_g + x_g - y_g - x'_g\right)\right]/\epsilon}\right] & \text{if admissible} \\ 0 & \text{otherwise} \end{cases},$$
  - $$T^{(\theta)}_{x_g y_g x'_g y'_g} = \exp\left(\frac{\mathrm{i}\theta}{2\pi} \ln\left[\mathrm{e}^{\mathrm{i}\pi\left(y'_g + x_g - y_g - x'_g\right)}\right]\right)$$

TN representation 2/2

- Compression for the hopping term:
  - $$H_{\tilde{\ell}(n)\tilde{\theta}_\nu(n)\tilde{\ell}(n+\hat{\nu})} = \frac{\sqrt[4]{w_{\tilde{\ell}(n)} w_{\tilde{\ell}(n+\hat{\nu})}} \mathrm{e}^{(\tilde{\ell}(n)+\tilde{\ell}(n+\hat{\nu}))/4}}{\sqrt{2}} \times \exp\left[2\sqrt{\tilde{\ell}(n)\tilde{\ell}(n+\hat{\nu})}\cos\pi\tilde{\theta}_\nu(n) - \frac{M}{4}\left(\tilde{\ell}(n)+\tilde{\ell}(n+\hat{\nu})\right) - \frac{\lambda}{4}\left(\tilde{\ell}(n)^2+\tilde{\ell}(n+\hat{\nu})^2\right)\right]$$
  - $$H_{\tilde{\ell}(n)\tilde{\vartheta}_\nu(n)\tilde{\ell}(n+\hat{\nu})} \simeq \sum_{\alpha=1}^{\chi} A_{\tilde{\ell}(n)\tilde{\vartheta}_\nu(n)\alpha} B_{\tilde{\ell}(n+\hat{\nu})\alpha}$$
  - $$T^{(h)}_{x_h y_h x'_g x'_h y'_g y'_h} = \sum_{\tilde{\ell}(n)} A_{\tilde{\ell}(n) y'_g x_h} A_{\tilde{\ell}(n) x'_g y_h} B_{\tilde{\ell}(n) x'_h} B_{\tilde{\ell}(n) y'_h}$$