Towards tensor renormalization group study of lattice QCD

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Extreme Universe

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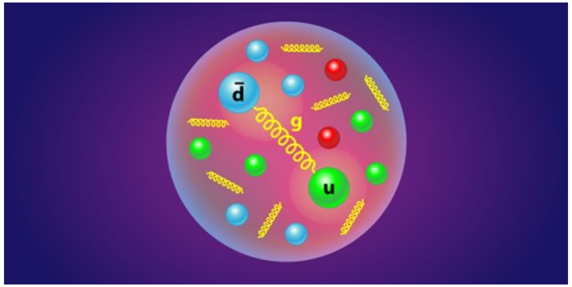
Lattice QCD

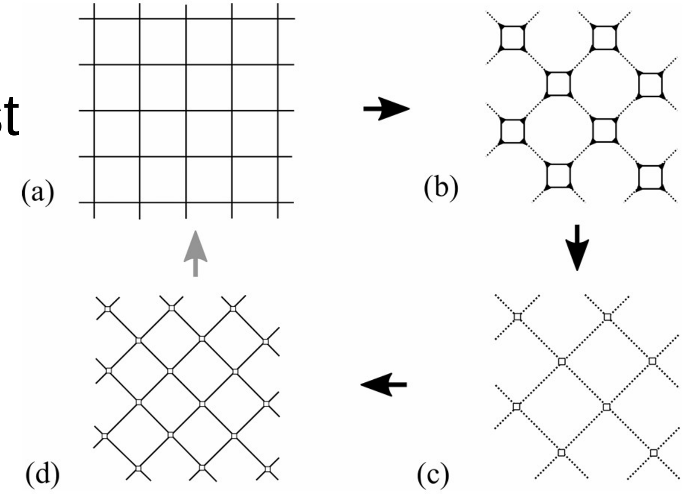
- non-perturbative formulation for quantum chromodynamics
- 4D Euclidean SU(3) gauge theory with Nf=2,3
    - Higher dimensions, Non-abelian, Multiple flavors
- MC computation suffers from the sign problem at finite θ and finite density
- MC computation also suffers from the topology freezing problem toward continuum limit



So far, we do not have a universal method that avoids all the problems...

Tensor renormalization group (TRG)

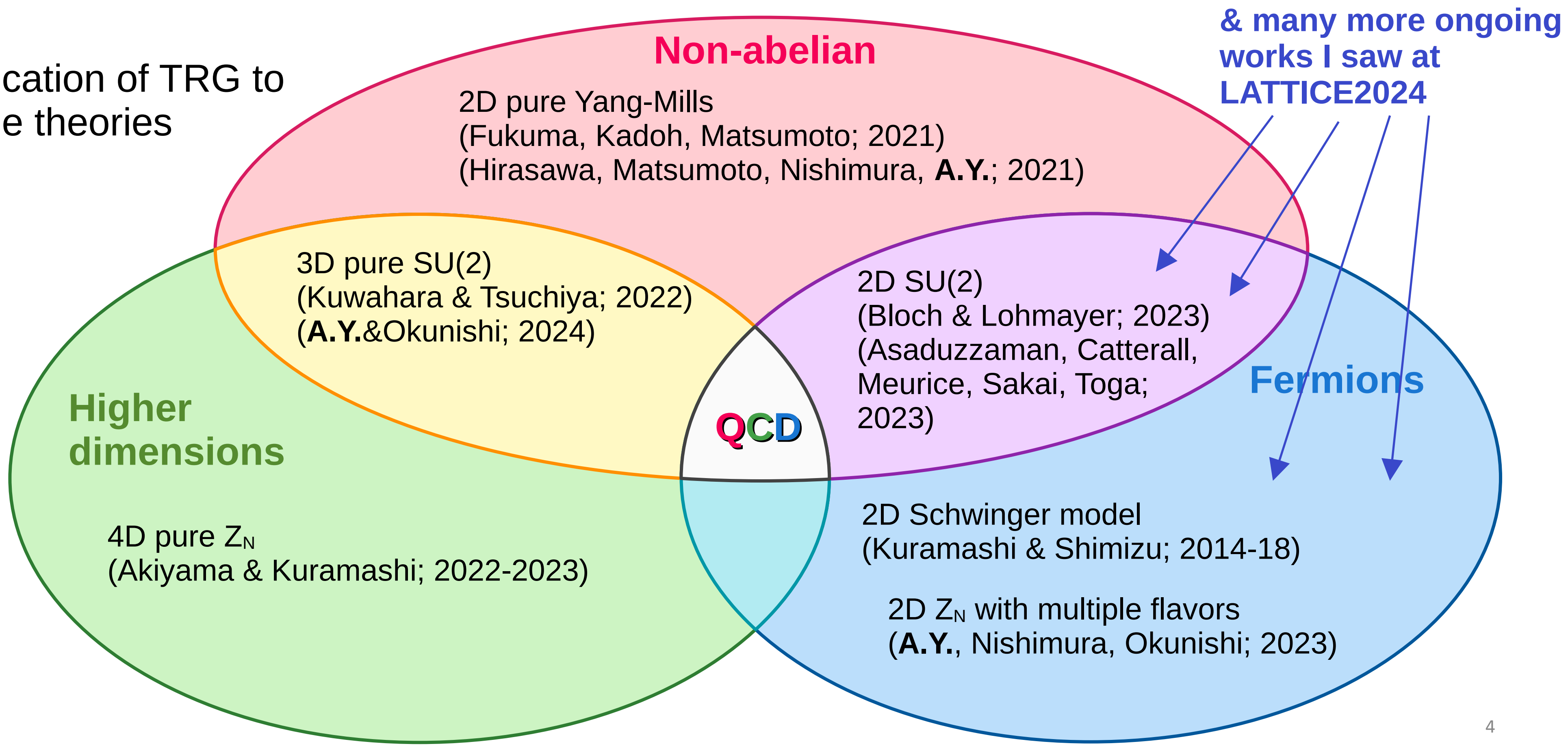
- An alternative to Monte Carlo methods based on coarse graining
- No sampling = No sign problem
- Can access large volumes with log cost
- Can handle fermion/Grassmann numbers directly; Grassmann TRG



[Figures from Okunishi-Nishino-Ueda; 2022]

Progress toward TRG study of Lattice QCD

Application of TRG to gauge theories



Challenges

- TRG can be challenging when the local Hilbert space is large
- By that, I meant QCD
  - Multiple fermion flavors ==> dimension ~ exp(kN_f)
  - Non-abelian gauge symmetry ==> Redundancy in the TN

I will talk about my works on these two directions.

Outline

- Part I: Multi-flavor gauge theory
  - The multi-layer formulation
  - Initial tensor compression
  - Result: *finite density* N_f - flavor 2D Z_N theory
- Part II: Armillary sphere formulation
  - Degeneracy in the tensor network
  - Reduced tensor network formulation
  - Result: 3D SU(2) & SU(3) thoery

Outline

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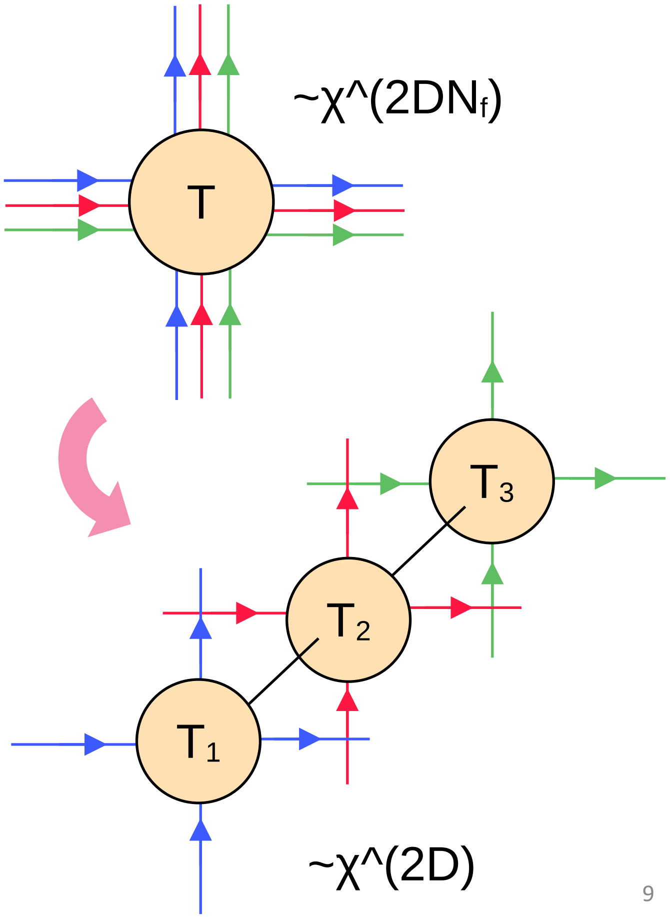
Part I: Multi-layer construction for multi-flavor gauge theory

Based on [JHEP11(2023)187], with **Jun Nishimura** (KEK) and **Kouichi Okunishi** (Niigata U)

Multi-flavor gauge theory

- The number of tensor legs increases with the number of flavors
- This makes it difficult to consider multi-flavor theory in the tensor network
- This can be avoided by separating flavor d.o.f. from each other



Similar ideas:
- Domain wall fermions (flavor = extra dim)
- MPS-like decomposition (Akiyama; 2023)

separate layers numerically
(still expensive)

Multi-flavor gauge theory

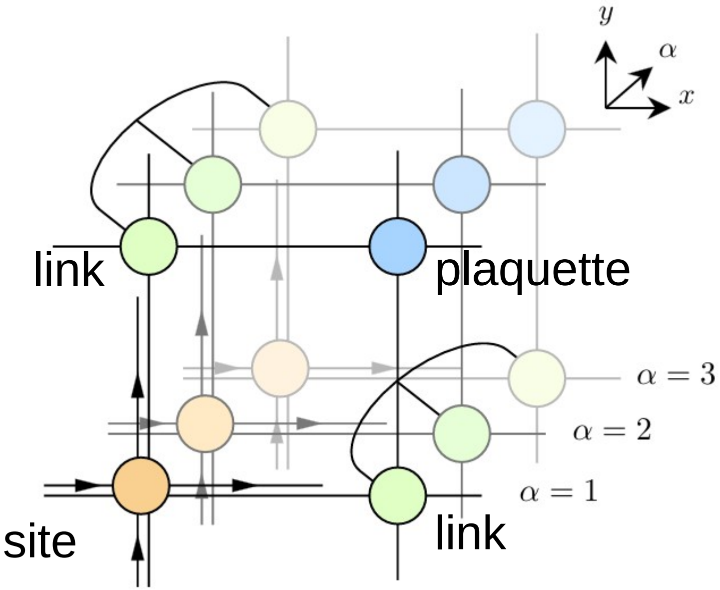
Separate the layers analytically

$$Z = \int D\varphi \prod_{\alpha=1}^{N_{\rm f}} \left(D\varphi^{(\alpha)} D\psi^{(\alpha)} D\bar{\psi}^{(\alpha)}\right) \delta(\varphi^{(\alpha)} - \varphi) e^{-\sum_\alpha S^{(\alpha)}}$$

$$U_{x,\mu} = \exp(iaA_{x,\mu}) \equiv \exp(i\varphi_{x,\mu})$$

local action for each flavor

$$S^{(\alpha)} = \frac{1}{N_{\rm f}} S_{\rm gauge}[\varphi^{(\alpha)}] + \sum_{x\in\Lambda_2} \bar{\psi}^{(\alpha)}_x D\!\!\!/^{(\alpha)} \psi^{(\alpha)}_x$$



- each layer for each flavor
- connected via delta functions
- Grassmann tensors handle fermions directly

The tensor network

To construct the tensor network,

1) Separate the fermionic action into the site terms and the hopping terms

$$\bar\psi_x^{(\alpha)} \not{D}^{(\alpha)} \psi_x^{(\alpha)} = \bar\psi_x^{(\alpha)} W_x^{(\alpha)} \psi_x^{(\alpha)} + \sum_\nu \left(\bar\psi_x^{(\alpha)} H_{x,+\nu}^{(\alpha)} \psi_{x+\hat\nu}^{(\alpha)} + \bar\psi_x^{(\alpha)} H_{x,-\nu}^{(\alpha)} \psi_{x-\hat\nu}^{(\alpha)} \right)$$

site terms

hopping terms

2) For each hopping term, define an auxiliary link fermion

=> The fermionic bond in the tensor network

$$e^{-\bar\psi_x^{(\alpha)} H_{x,\pm\nu}^{(\alpha)} \psi_{x\pm\hat\nu}^{(\alpha)}} = \int d\bar\eta_{x,\pm\nu}^{(\alpha)} d\eta_{x,\pm\nu}^{(\alpha)} e^{-\bar\eta_{x,\pm\nu}^{(\alpha)}\eta_{x,\pm\nu}^{(\alpha)} - \bar\psi_x^{(\alpha)}\eta_{x,\pm\nu}^{(\alpha)} + \bar\eta_{x,\pm\nu}^{(\alpha)} H_{x,\pm\nu}^{(\alpha)} \psi_{x\pm\hat\nu}^{(\alpha)}}$$

hopping term

auxiliary link fermion

The tensor network

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$$\bar{\psi}_x^{(\alpha)} \not{D}^{(\alpha)} \psi_x^{(\alpha)} = \underbrace{\bar{\psi}_x^{(\alpha)} W_x^{(\alpha)} \psi_x^{(\alpha)}}_{\text{site terms}} + \sum_\nu \underbrace{\left(\bar{\psi}_x^{(\alpha)} H_{x,+\nu}^{(\alpha)} \psi_{x+\hat{\nu}}^{(\alpha)} + \bar{\psi}_x^{(\alpha)} H_{x,-\nu}^{(\alpha)} \psi_{x-\hat{\nu}}^{(\alpha)} \right)}_{\text{hopping terms}}$$

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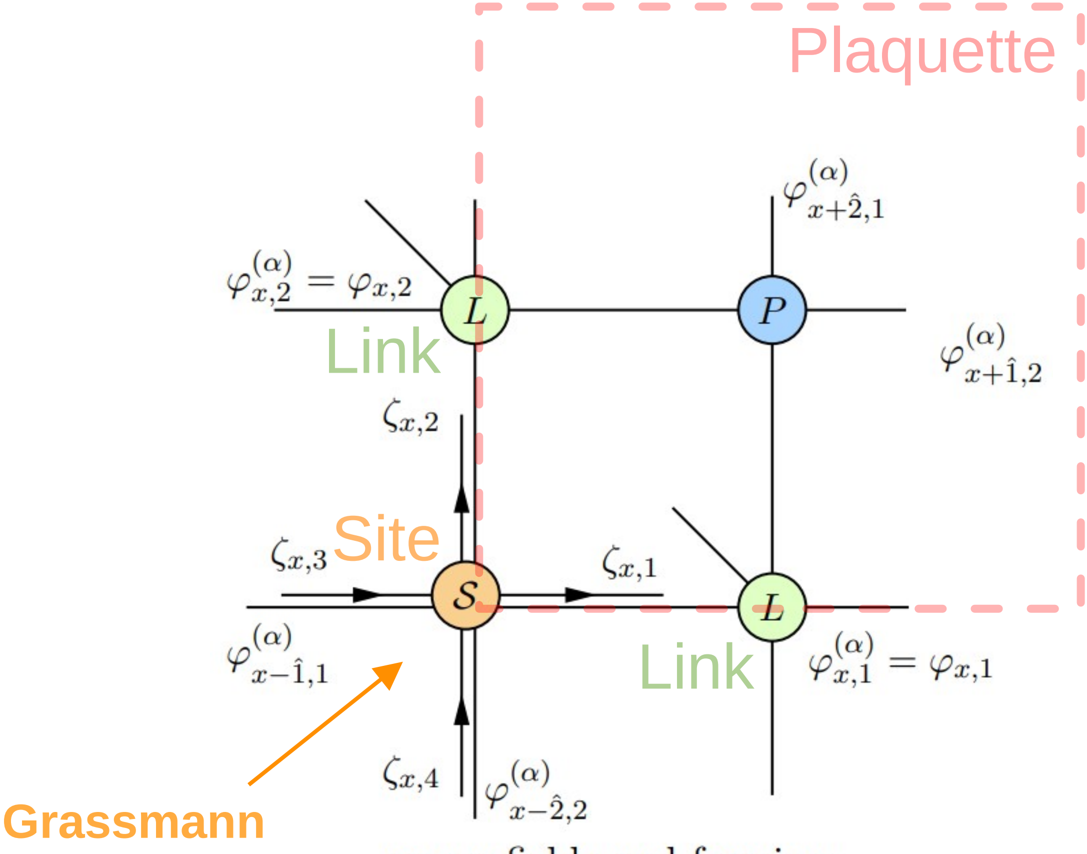
$$\underbrace{e^{-\bar{\psi}_x^{(\alpha)} H_{x,\pm\nu}^{(\alpha)} \psi_{x\pm\hat{\nu}}^{(\alpha)}}}_{\text{hopping term}} = \int \underbrace{d\bar{\eta}_{x,\pm\nu}^{(\alpha)} d\eta_{x,\pm\nu}^{(\alpha)}}_{\text{auxiliary link fermion}} e^{-\bar{\eta}_{x,\pm\nu}^{(\alpha)} \eta_{x,\pm\nu}^{(\alpha)} - \bar{\psi}_x^{(\alpha)} \eta_{x,\pm\nu}^{(\alpha)} + \bar{\eta}_{x,\pm\nu}^{(\alpha)} H_{x,\pm\nu}^{(\alpha)} \psi_{x\pm\hat{\nu}}^{(\alpha)}}$$

3) Integrate out the original site fermions

=> Site fermions are transformed to link fermions

$$\mathcal{S}_x^{(\alpha)} = \int d\psi_x^{(\alpha)} d\bar{\psi}_x^{(\alpha)} \exp\left[-\bar{\psi}_x^{(\alpha)} W_x^{(\alpha)} \psi_x^{(\alpha)} - \sum_{\pm,\nu} \left\{ \bar{\psi}_x^{(\alpha)} \eta_{x,\pm\nu}^{(\alpha)} - \bar{\eta}_{x\mp\hat{\nu},\pm\nu}^{(\alpha)} H_{x\mp\hat{\nu},\pm\nu}^{(\alpha)} \psi_x^{(\alpha)} \right\} \right]$$

The tensor network



gauge fields and fermions

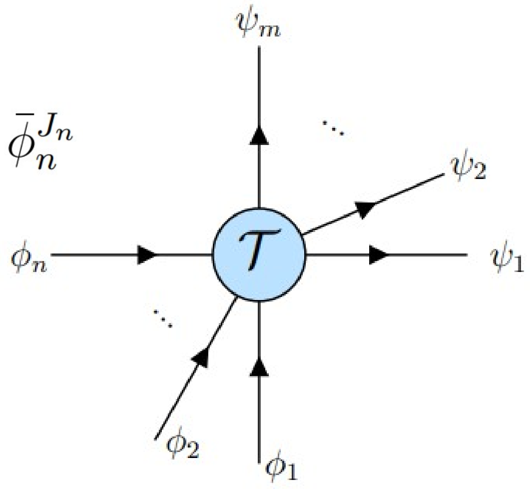
$$\zeta^{K_1}_{x,1} = \eta^{I_1}_{x,+1}\bar{\eta}^{J_1}_{x+\hat{1},-\hat{1}}\,,$$
$$\zeta^{K_2}_{x,2} = \eta^{I_2}_{x,+2}\bar{\eta}^{J_2}_{x+\hat{2},-\hat{2}}\,,$$
$$\bar{\zeta}^{K_3}_{x,3} = (-)^{p(J_3)}\bar{\eta}^{I_3}_{x-\hat{1},+\hat{1}}\eta^{J_3}_{x,-1}\,,$$
$$\bar{\zeta}^{K_4}_{x,4} = (-)^{p(J_4)}\bar{\eta}^{I_4}_{x-\hat{2},+\hat{2}}\eta^{J_4}_{x,-2}\,,$$

Quick intro: Grassmann tensors

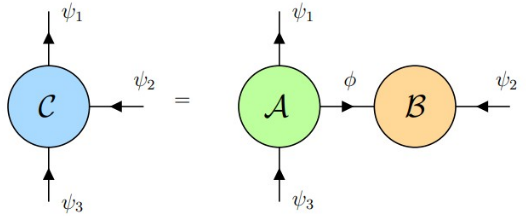
$$\mathcal{T}_{\psi_1\cdots\psi_m\bar{\phi}_1\cdots\bar{\phi}_n} = \sum_{I_1,\cdots,I_m,J_1,\cdots,J_n} T_{I_1\cdots I_m J_1\cdots J_n}\psi_1^{I_1}\cdots\psi_m^{I_m}\bar{\phi}_1^{J_1}\cdots\bar{\phi}_n^{J_n}$$

Coefficient tensor

Multi-component Grassmann numbers



Grassmann tensor contraction



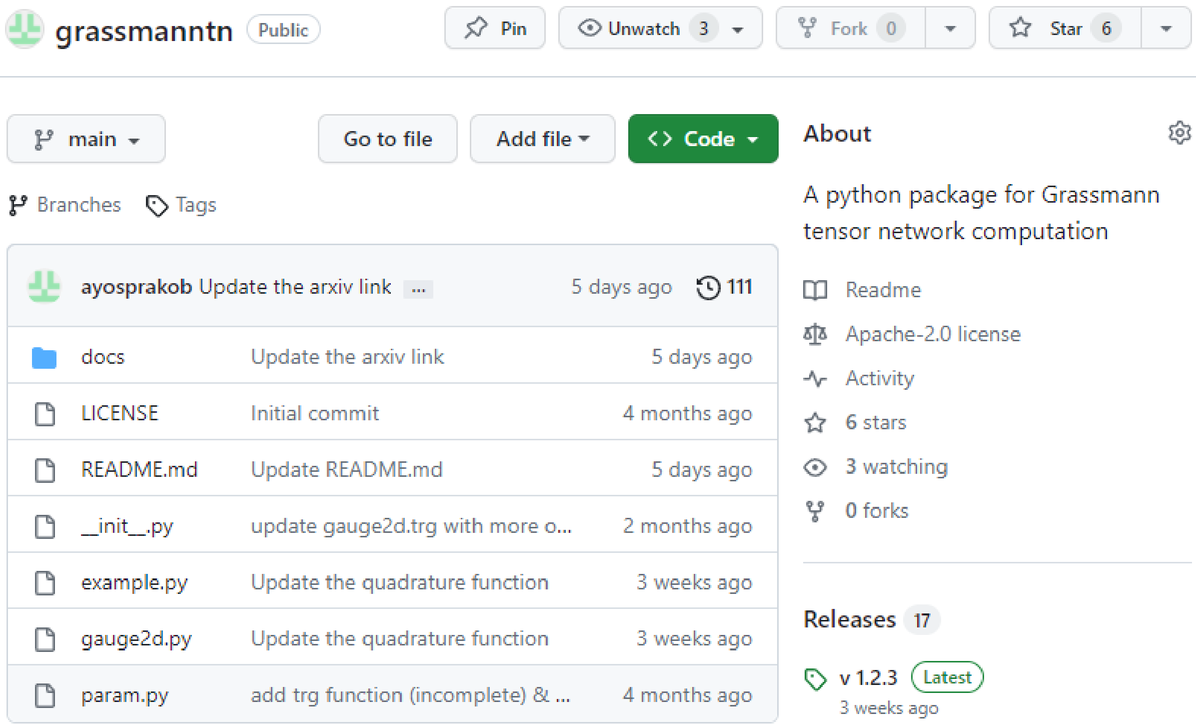
$$C_{IJK} = \sum A_{ILK}B_{JL}s_{JKL}$$

$$s_{JKL} = \sigma_L \times (-)^{p(L)(p(J)+p(K))+p(J)p(K)}$$

No non-local sign factors because signs are computed during the contraction only.

GrassmannTN:

a python package for Grassmann TRG/DMRG



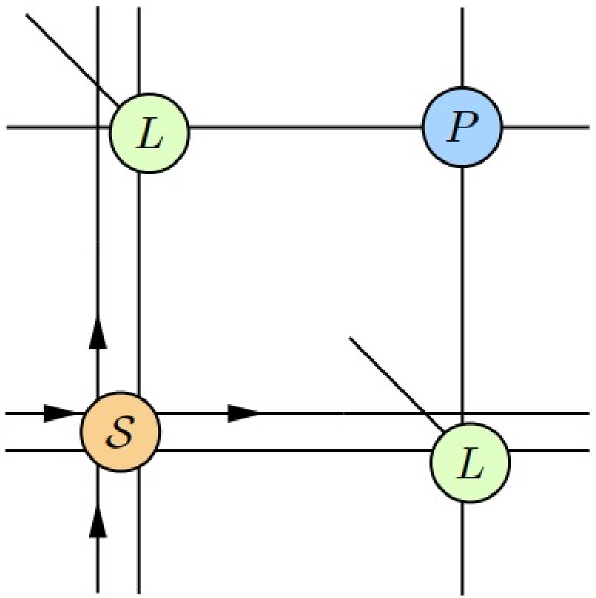
Features:
Grassmann contractions,
Tensor reshapes,
SVD/EigVD,
dense/sparse conversions,
Grassmann arithmatic,
Berezin integrals, etc.

complete tutorial for
1+1D Schwinger model (TRG)

https://github.com/ayosprakob/grassmanntn

Tensor compression

The initial tensor is still too big

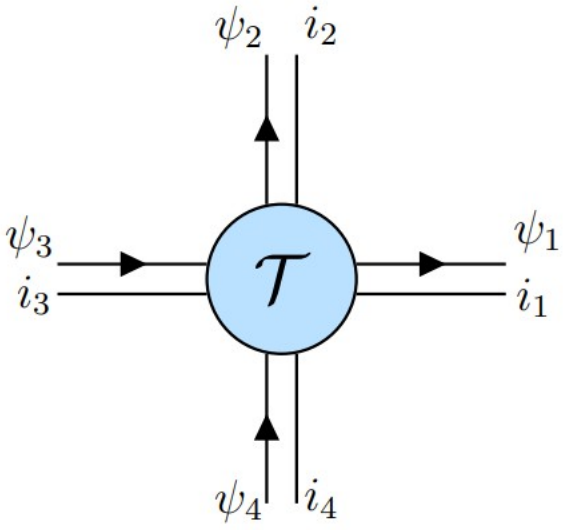


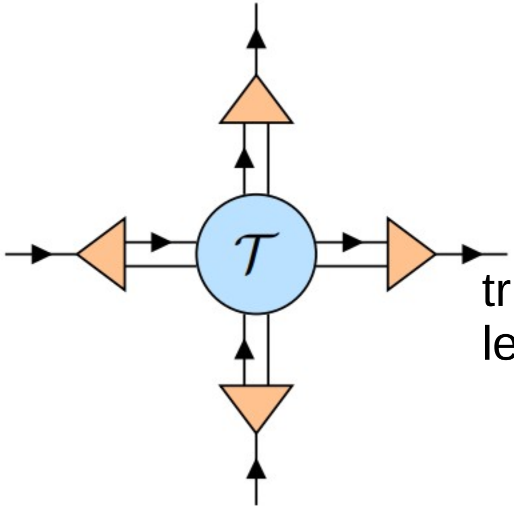
10 bosonic legs & 4 fermionic legs
= K^{10} 16^{4} components

Some compression is needed
to reduce the tensor size first

Tensor compression

The standard method is to use the isometry tensor.

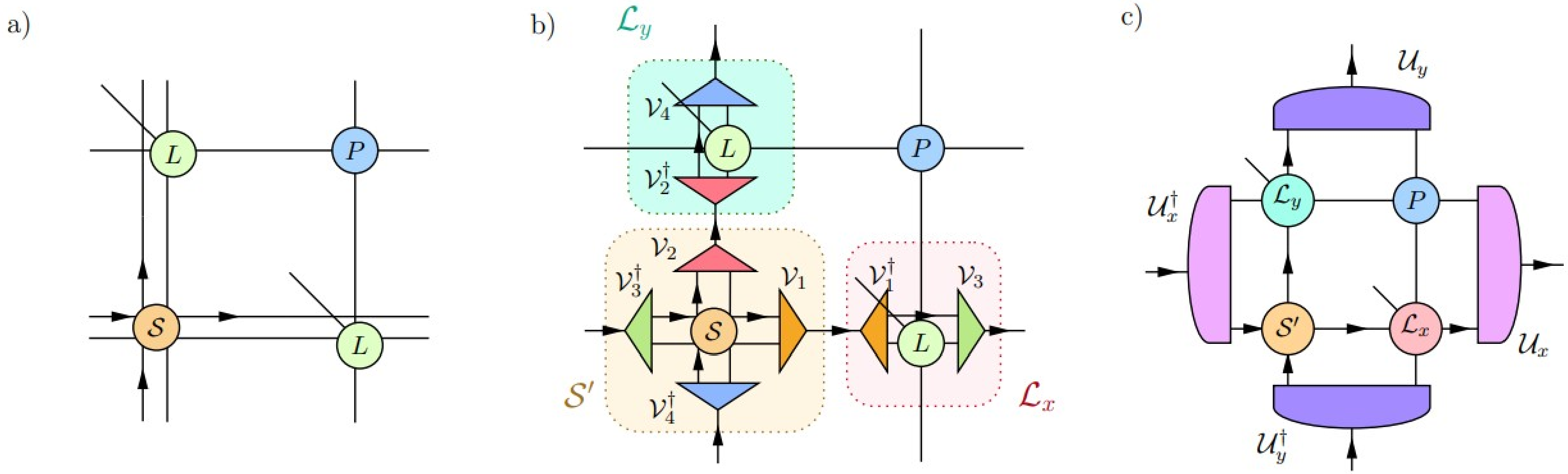




The truncation is done based on the HOSVD.

Tensor compression

Proposed compression scheme:



Isometries are first applied around the Grassmann tensor *S*: a → b
Then another set is applied around the whole tensor: b → c

Compression performance

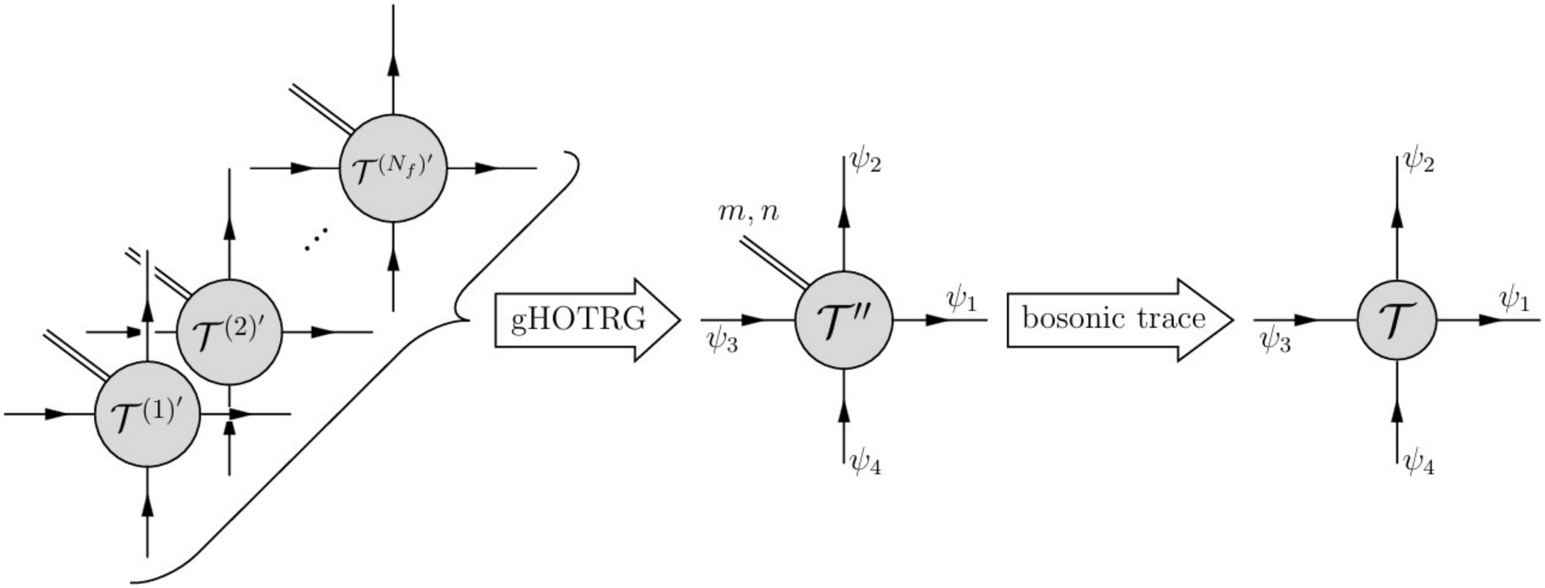
Physical parameters

β	$\tilde{\mu}$	N_f	K	original size	compressed size	compression ratio	D_x	D_y
0.0	0.0	1	2	67108864	1024	1.53×10^{-5}	4	4
0.0	0.0	1	3	3869835264	2304	5.95×10^{-7}	4	4
0.0	0.0	1	4	68719476736	4096	5.96×10^{-8}	4	4
0.0	0.0	1	5	640000000000	6400	1.00×10^{-9}	4	4
2.0	0.0	1	2	67108864	16384	2.44×10^{-4}	8	8
2.0	0.0	2	2	67108864	16384	2.44×10^{-4}	8	8
2.0	3.0	1	2	67108864	16384	2.44×10^{-4}	8	8
2.0	3.0	2	2	67108864	16384	2.44×10^{-4}	8	8

All of these compressions are done without any spectrum truncation!
Such high performance is due to the sparse nature of fermions.

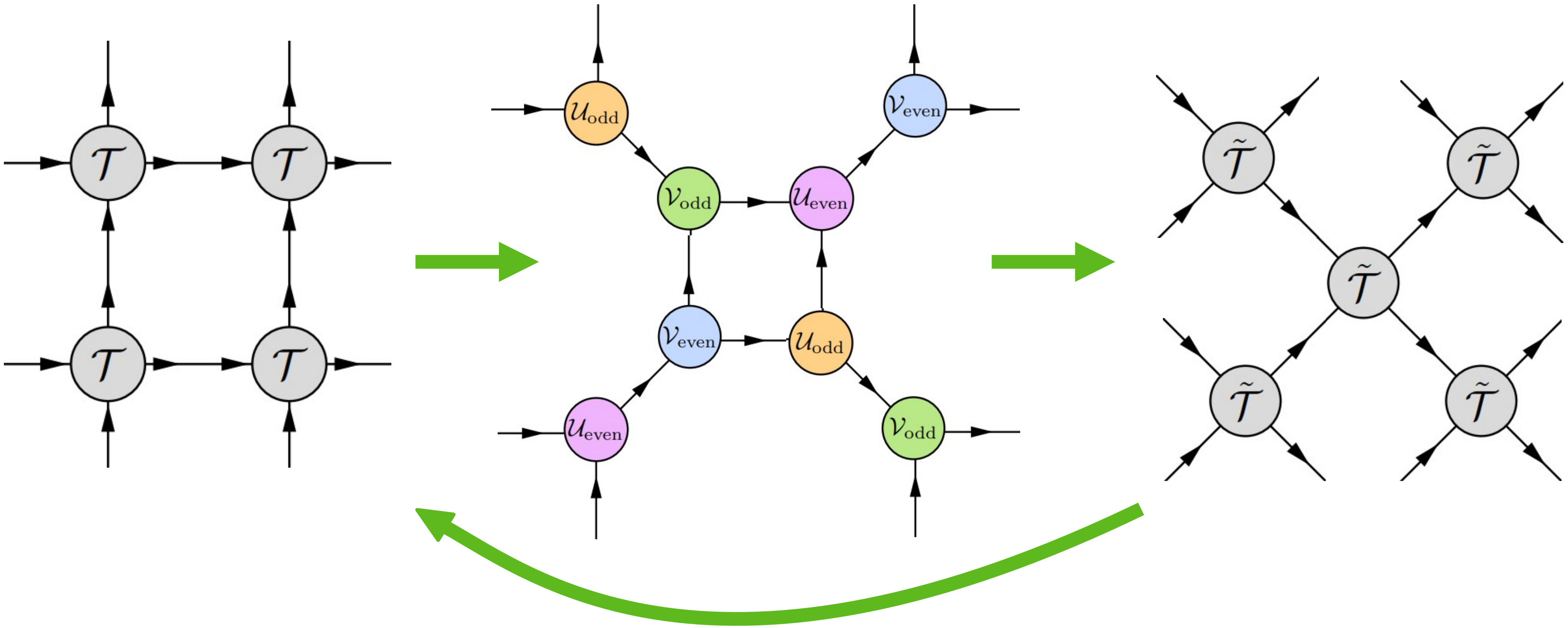
Coarse-graining schemes

Multiple flavors are then combined via a TRG algorithm like an extra dimension



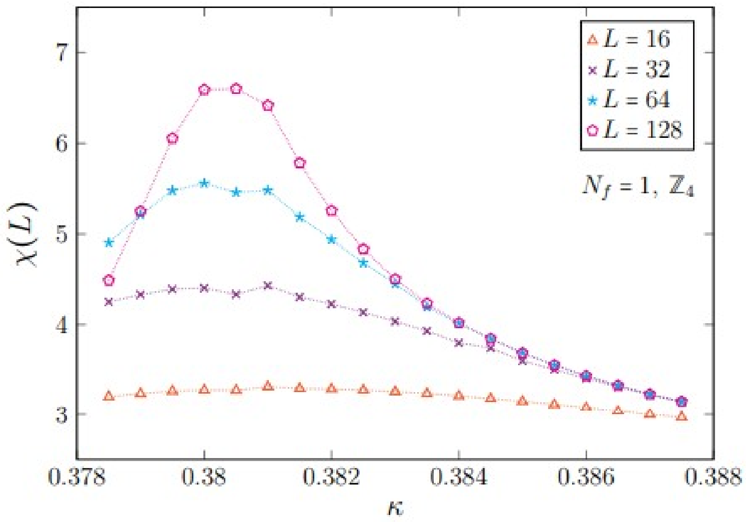
Coarse-graining schemes

We use the Levin-Nave TRG for 2D coarse-graining

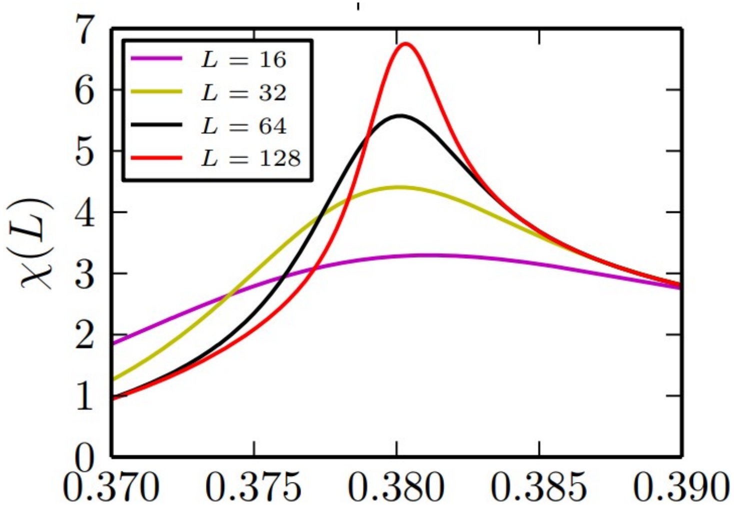


Results

- Wilson fermions break chiral sym explicitly
- The chiral sym is restored at a critical hopping parameter

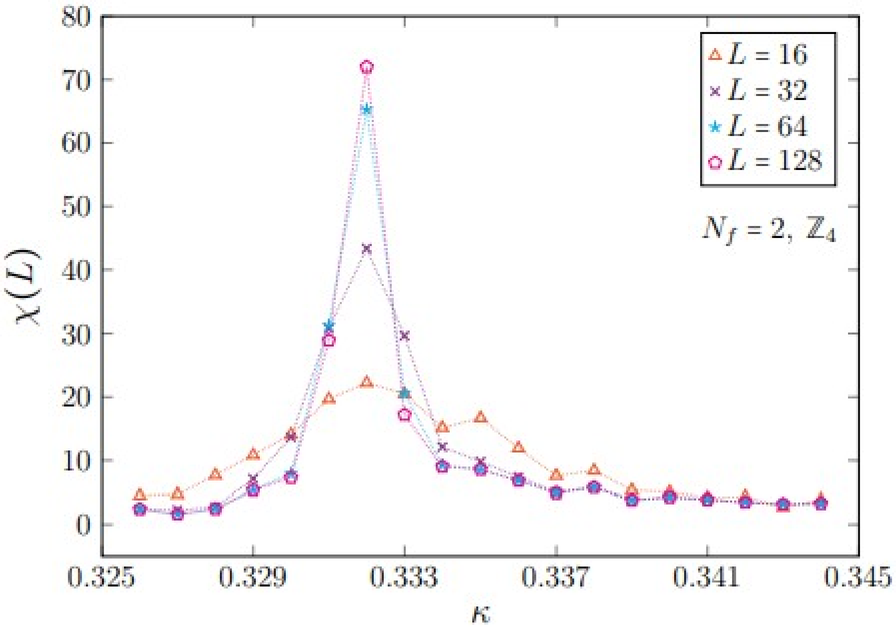


One flavor result (Z4 theory)
κ ~ 0.38



Kuramashi-Shimizu '14
(U(1) theory)
κ ~ 0.3807

Results



$N_\mathrm{f} = 2$ at $\beta = 0$

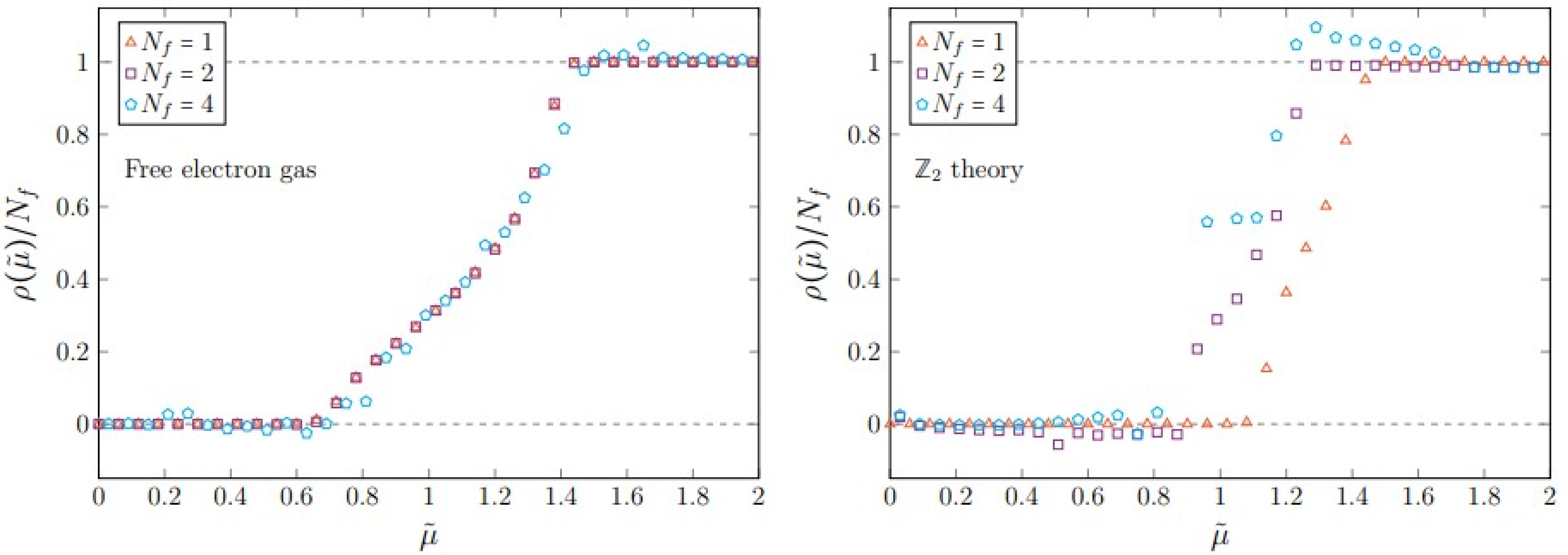
Gauge group	Algorithm	κ_c
$\mathbb{Z}_2$	gTRG	0.335(1)
$\mathbb{Z}_4$	gTRG	0.332(1)
U(1)	gTRG	0.332(1)
U(1)	Monte Carlo [48]	0.3296983759

Hip et al. ‘98

Critical behavior can also be observed in 2 flavors

Results

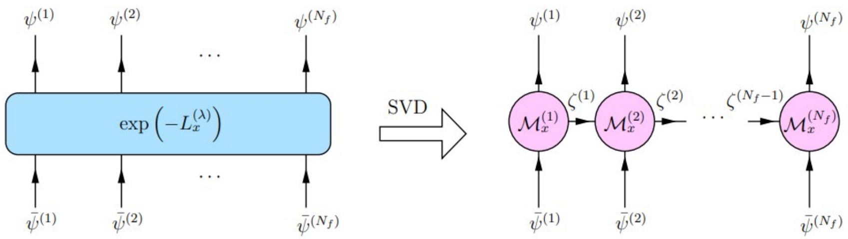
- Finite density and Silver Blaze phenomena

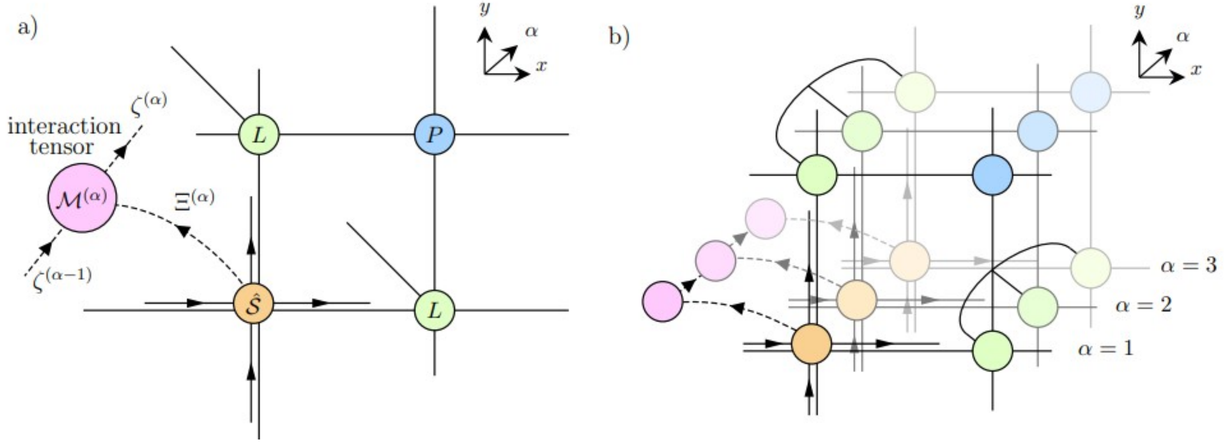


This cannot be observed in traditional Monte Carlo simulation because of the sign problem!

Interaction terms

- Possible concern: Fermionic terms cannot be separated into layers with interaction terms!
- Answer: We first have to perform an MPO-like decomposition on the interaction terms.





Part II:
Armillary sphere
Non-abelian gauge theory in higher dimensions

Based on [PTEP 2024 (2024) 7, 073B05] (Formulation)
and [arXiv:2406.16763] (Numerical) with **Kouichi Okunishi** (Niigata U)

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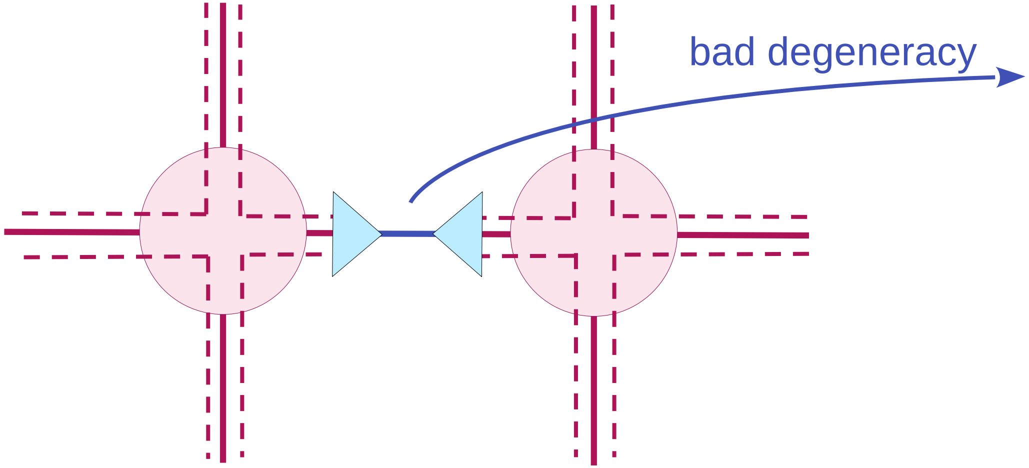
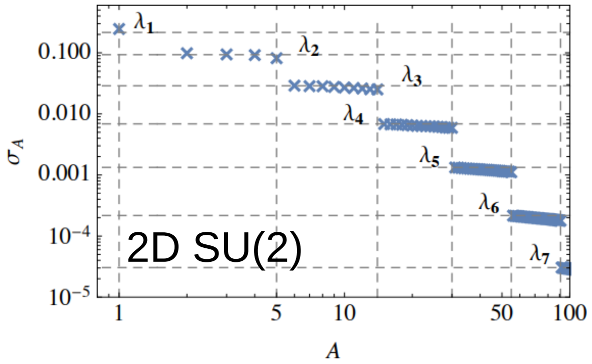
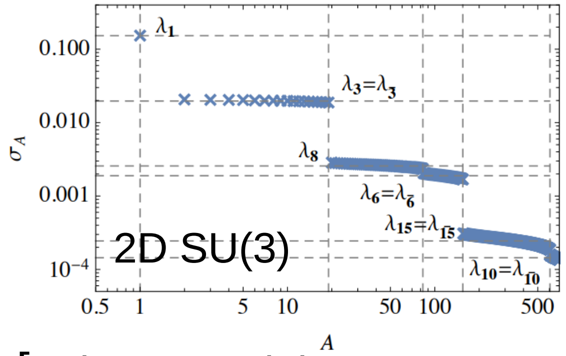
Why is non-abelian tensor network difficult?

Internal symmetry in SU(N) is a redundancy in the tensor network that cannot be truncate by an SVD







The entanglement structure is nonlocal...

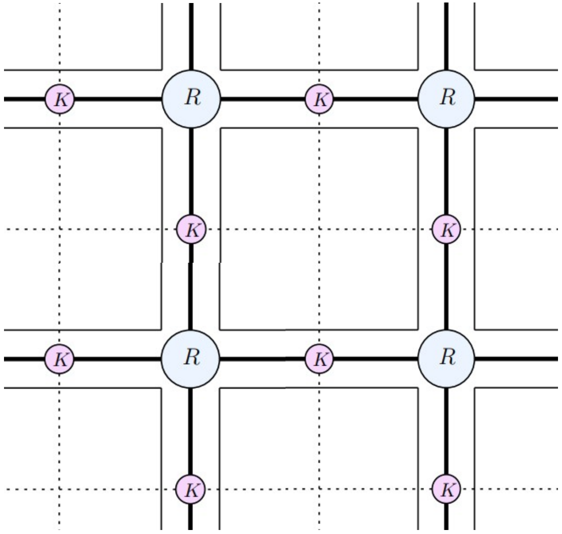
Figures from [Fukuma-Kadoh-Matsumoto; 2021]

Why is non-abelian tensor network difficult?

- Lesson from 1+1D: the (matrix) index loops can be traced out if we use character expansion

[Hirasawa, Matsumoto, Nishimura, **A.Y.**; 2021]

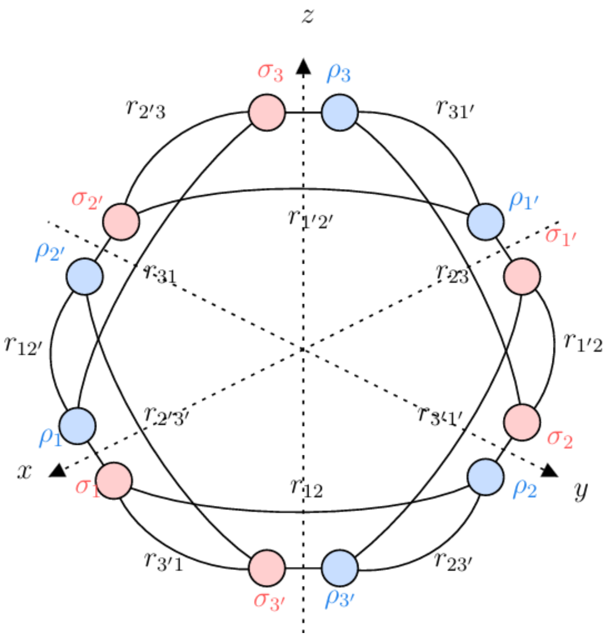
- Degeneracy is completely eliminated

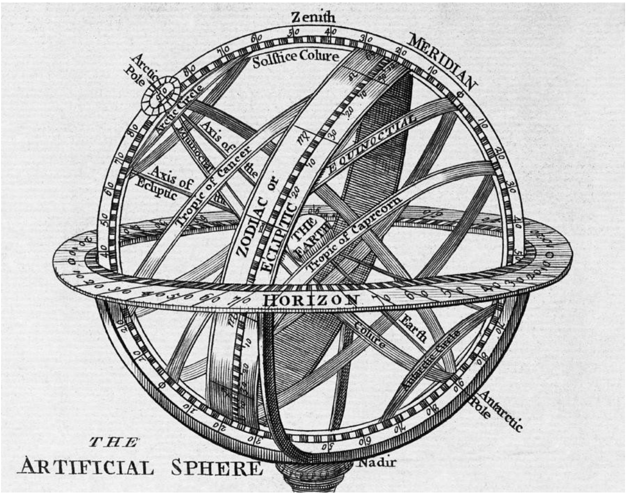


Question: Can we do the same thing for any dimension?

The armillary sphere

Yes! There is a similar closed network in any dimension
Which we call the armillary sphere





armillary sphere
= intersecting circles

This was first noticed by [Oeckl & Pfeiffer;2001] in the context of the spin foam model.

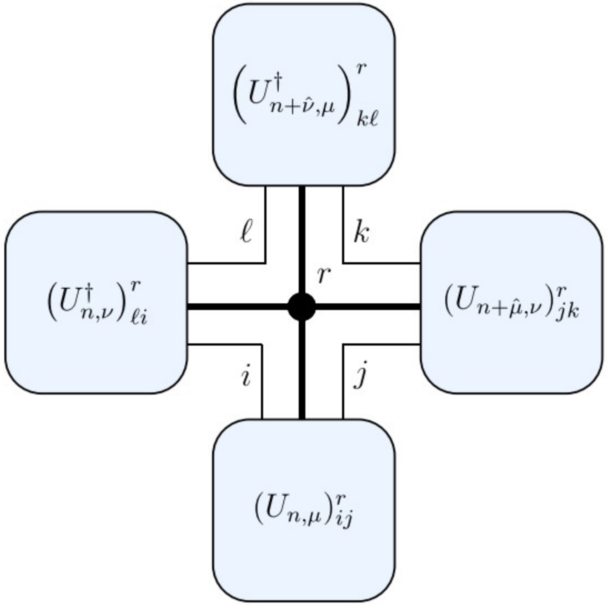
The armillary sphere

Step 1: perform character expansion on the Boltzmann weight

$$e^{\beta \Re\, \mathrm{tr}\, P_{n,\mu\nu}} = \sum_r f_r \, \mathrm{tr}_r (U_{n,\mu} U_{n+\hat{\mu},\nu} U^\dagger_{n+\hat{\nu},\mu} U^\dagger_{n,\nu})$$

$$= \sum_r f_r \sum_{i,j,k,l} (U_{n,\mu})^r_{ij} (U_{n+\hat{\mu},\nu})^r_{jk} (U^\dagger_{n+\hat{\nu},\mu})^r_{kl} (U^\dagger_{n,\nu})^r_{li}$$

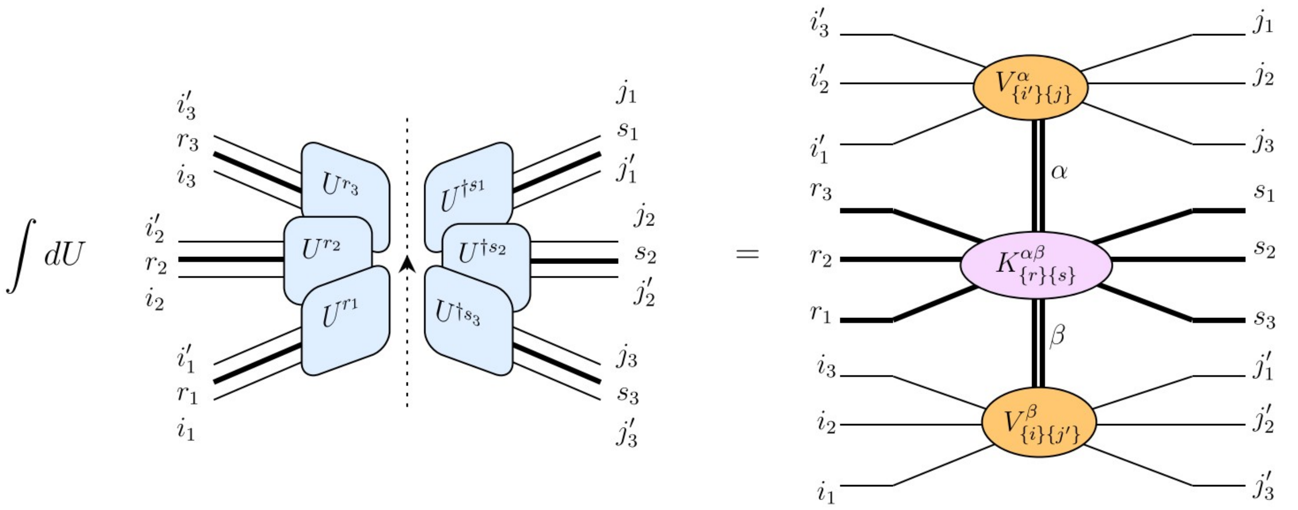
$\mathrm{tr}_r P_{n,\mu\nu} =$



The armillary sphere

Step 2: perform group integral on each link variable

$$\int dU_{n,\mu}(U_{n,\mu})^{r_1}_{i_1 i'_1}\cdots(U_{n,\mu})^{r_{d-1}}_{i_{d-1}i'_{d-1}}(U^\dagger_{n,\mu})^{s_1}_{j_1 j'_1}\cdots(U^\dagger_{n,\mu})^{s_{d-1}}_{j_{d-1}j'_{d-1}} = \sum_{\hat r\in D_{\otimes r}}\sum_{\hat s\in D_{\otimes s}}\sum_{\hat\imath,\hat\jmath}\frac{1}{\dim(\hat r)}C^{\hat r\hat\imath}_{\{r\}\{i\}}C^{\hat r\hat\jmath}_{\{r\}\{i'\}}C^{\hat s\hat\jmath}_{\{s\}\{j\}}C^{\hat s\hat\imath}_{\{s\}\{j'\}}\delta(\hat r\simeq\hat s)$$



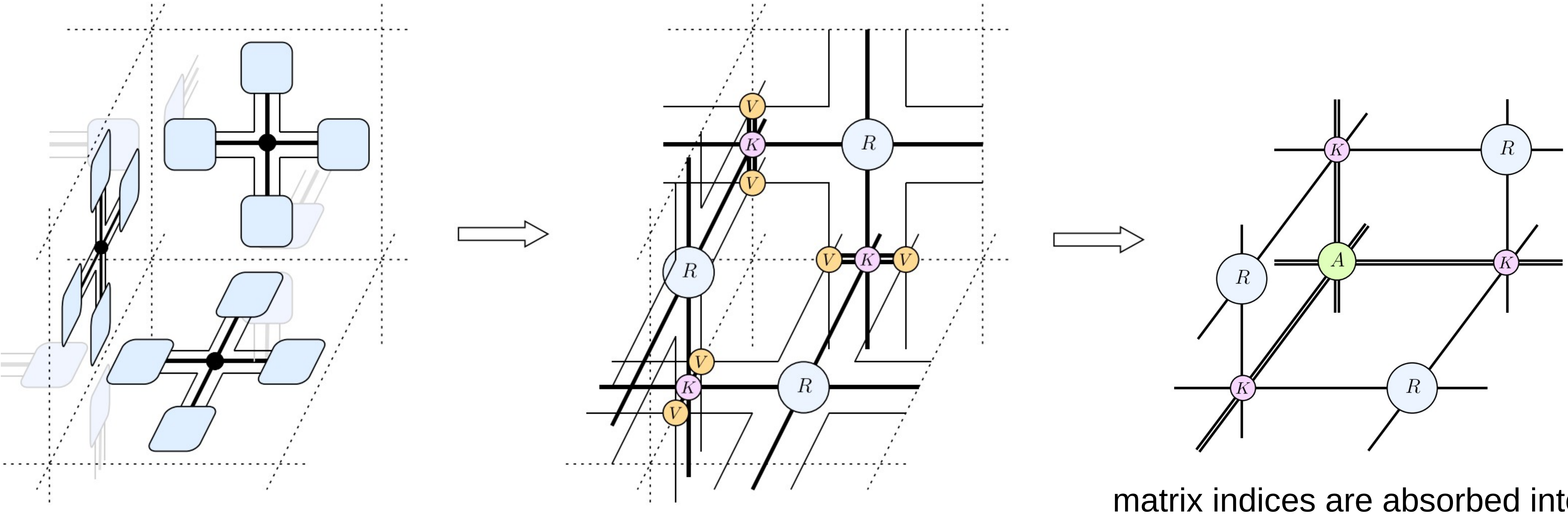
$$V^\alpha_{\{i\}\{j'\}} = \delta(\hat r\simeq\hat s)\sum_{\hat\imath} C^{\hat r\hat\imath}_{\{r\}\{i\}}C^{\hat s\hat\imath}_{\{s\}\{j'\}},$$

$$K^{\alpha\beta}_{\{r\}\{s\}} = \frac{1}{\dim(\hat r)}\delta_{\hat r\hat r'}\delta_{\hat s\hat s'}\delta(\hat r\in D_{\otimes r})\delta(\hat s\in D_{\otimes s});$$

Note: matrix indices (thin lines) are neatly separated into two layers

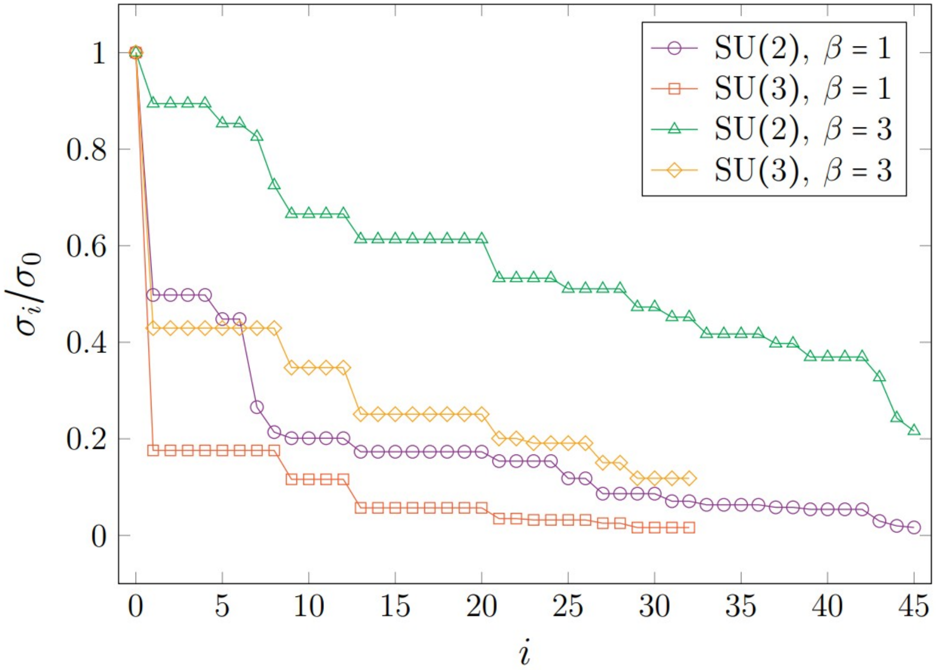
The armillary sphere

Step 3: Contract the matrix indices



matrix indices are absorbed into
the armillary tensor ‘A’

Result: singular value spectrum



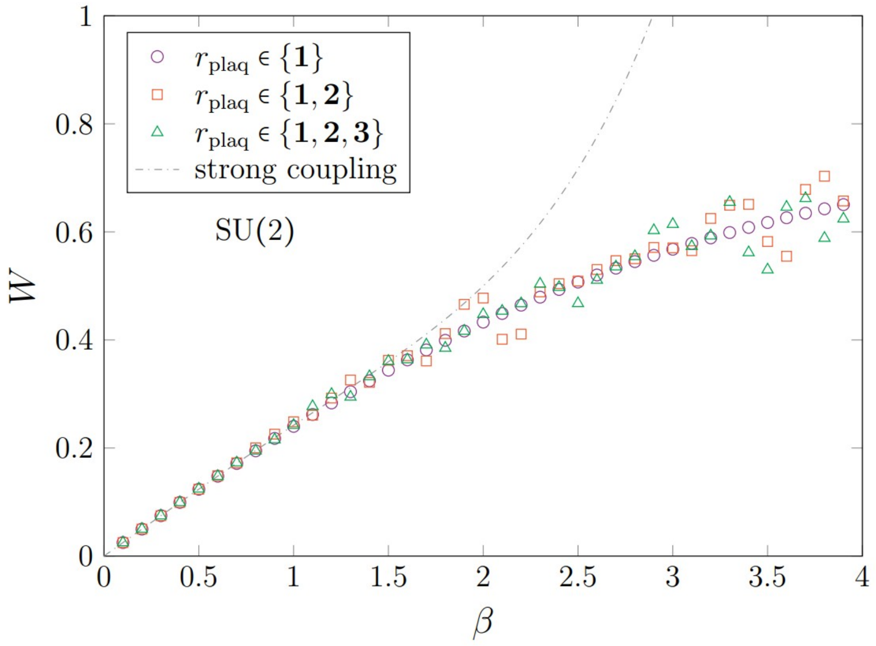
Singular value spectrum of the initial tensor do not have large degeneracy

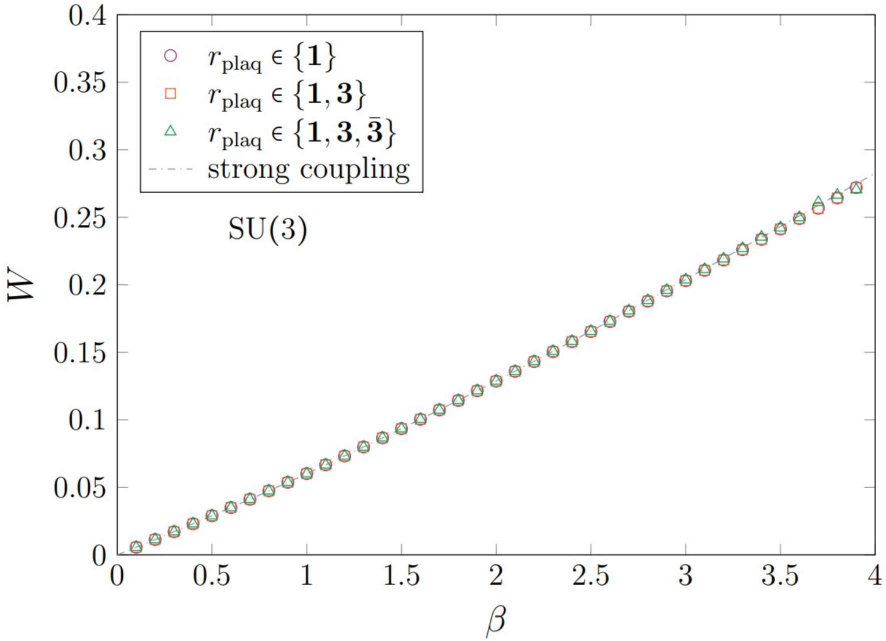
Result: average plaquette @ zero temperature

pure 2+1D SU(2) and SU(3) gauge theory

ATRG; $V = 16^3$; $D_{cut} = 16$

Average plaquette – consistent with strong coupling expansion

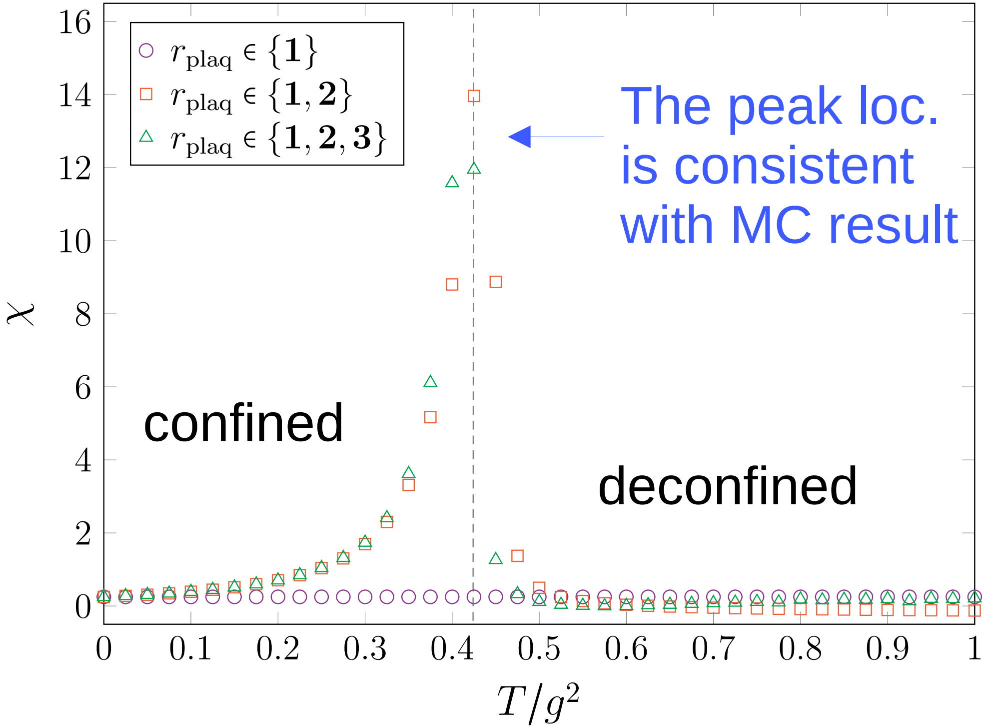


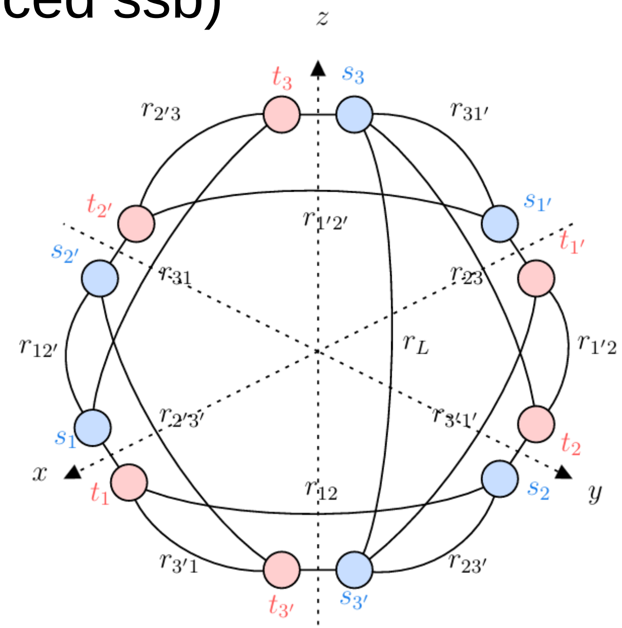


Result: deconfinement @ finite temperature

TRG; V = 1x1024^2; D_{cut} = 64

Polyakov loop susceptibility (with induced ssb)





Armillary sphere with polyakov loop

Summary

- With the multi-layer construction, it is possible to handle multiple-flavor with small local tensors

- The armillary sphere formalism helps eliminate the redundancy in the tensor network

- Both of these developments are essential for TRG analysis of lattice QCD

Future prospect

- Can we reduce the tensor network without character expansion? (Some variation of Gilt-TNR?) [Hauru,Delcamp,Mizera; 2017]
- Reduced TN with matter fields
- More in-depth analysis (any physical meaning? gauge fixing?)
- 4D gauge theory + theta term
- Etc.