Krylov complexity of free and interacting scalar QFTs with bounded power spectrum

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[arXiv:2212.14702]

with Hugo A. Camargo, Viktor Jahnke, Keun-Young Kim

Kyoto University, May 24

今日のセミナーで言いたいこと

- Lanczos係数とKrylov complexityは
  量子多体系のoperator growthの指標

- ２点関数から計算できる

- Krylov complexityの指数的増加率λ_Kは
  OTOCのLyapunov指数λ_Lを制限

$$\lambda_L \leq \lambda_K \leq \frac{2\pi}{\beta} \qquad \text{(予想)}$$

- 場の理論のMass gap とUV cutoffが λ_K に影響する

Introduction

Diagnostics for quantum chaos

- Out-of-time-order correlators (OTOC)

$$\frac{\langle W(t,\mathbf{d})V(0,0)W(t,\mathbf{d})V(0,0)\rangle}{\langle W(t,\mathbf{d})W(t,\mathbf{d})\rangle\langle V(0,0)V(0,0)\rangle} \sim 1-\varepsilon e^{\lambda_L(t-\mathbf{d}/v_B)}+\cdots$$

λ_L : Lyapunov exponent $\quad v_B$: butterfly velocity

- Spectral form factor (SFF) $\quad Z(\beta+it)Z(\beta-it)$

Ramp (linear growth) behavior
due to random matrix description

They are well-studied even in hep-th community
due to a connection to black holes.

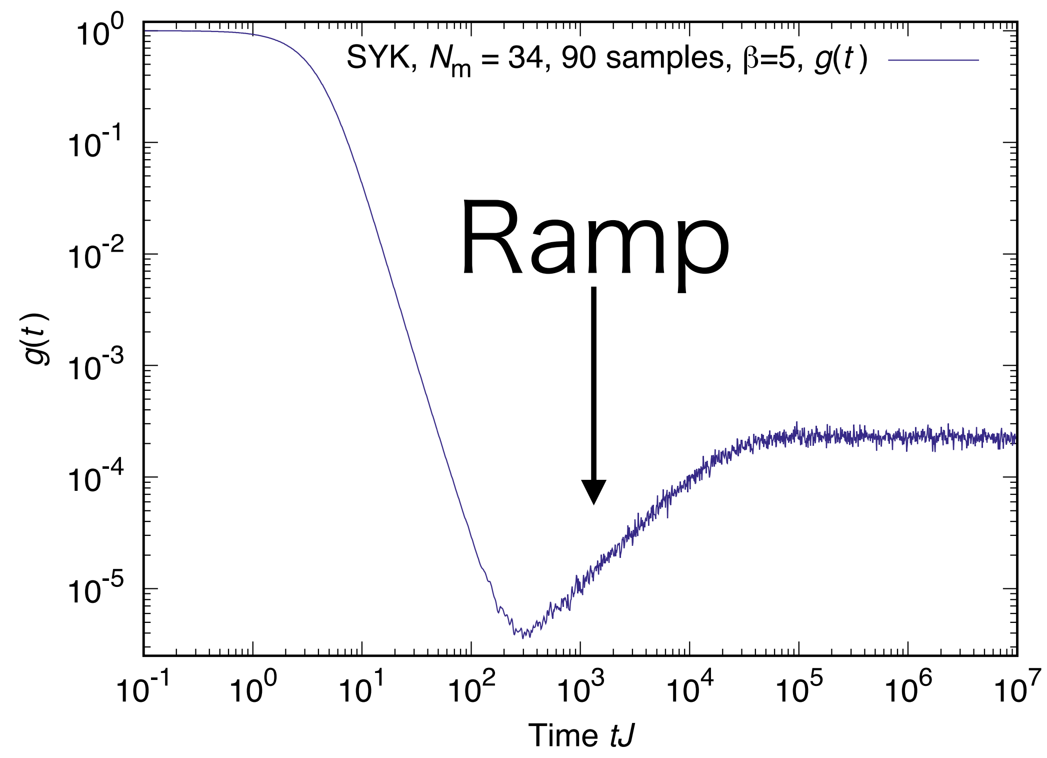
Thermal 2pt function may capture quantum chaos for operators.

Time evolution of SFF

$$Z(\beta+it)Z(\beta-it)=\sum_{i,j}e^{-\beta(E_i+E_j)}\,e^{i\,t(E_i-E_j)}$$

“Black Holes and Random Matrices”

[J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, A. Streicher, M. Tezuka, 2016]



SFF is related to

thermal 2pt function without matrix elements.

$$\langle\mathcal{O}(t-i\beta/2)\mathcal{O}(0)\rangle_\beta=\frac{1}{Z(\beta)}\sum_{i,j}e^{-\frac{\beta}{2}(E_i+E_j)}\,e^{i\,t(E_i-E_j)}|\langle E_j|\mathcal{O}|E_i\rangle|^2$$

Krylov complexity is a measure defined from 2pt functions.

Krylov complexity $K_{\mathcal{O}}(t) := \sum_n n|\varphi_n(t)|^2$

[D. E. Parker, X. Cao, A. Avdoshkin, T. Scaffidi, E. Altman, 2018]

Lanczos algorithm is a mathematical method for Krylov basis $\mathcal{O}_n$ for $\mathcal{O}(t)$.

$$\mathcal{O}(t) = \sum_{n=0} i^n \varphi_n(t) \mathcal{O}_n$$

Lanczos algorithm can determine Krylov basis $\mathcal{O}_n$ Lanczos coefficients b_n , and wave functions $\varphi_n(t)$.

Conjectured properties of Krylov complexity for chaotic systems

[D. E. Parker, X. Cao, A. Avdoshkin, T. Scaffidi, E. Altman, 2018]

- Universal operator growth hypothesis:
  Lanczos coefficient b_n grows linearly at large n.

$$b_n \sim \alpha n + \gamma$$

- Krylov complexity $K_{\mathcal{O}}(t)$ grows exponentially and bounds Lyapunov exponent λ_L .

$$K_{\mathcal{O}}(t) \sim e^{2\alpha t} \qquad \lambda_L \leq 2\alpha$$

These properties have been proved or checked in some systems.

[A. Avdoshkin, A. Dymarsky, 2019], [Y. Gu, A. Kitaev, P. Zhang, 2021], [E. Rabinovici, A. Sánchez-Garrido, R. Shir, J. Sonner, 2020], ⋯

Krylov complexity in CFTs

[A. Dymarsky, M. Smolkin, 2021]

$K_{\mathcal{O}}(t)$ at finite temperature in 2d CFTs, free massless theories, and holographic models were studied.

They found the universal exponential growth behavior $K_{\mathcal{O}}(t) \sim e^{\frac{2\pi}{\beta}t}$ in any theories.

The exponential growth of $K_{\mathcal{O}}(t)$ may not be chaotic behavior in CFTs.

My motivation

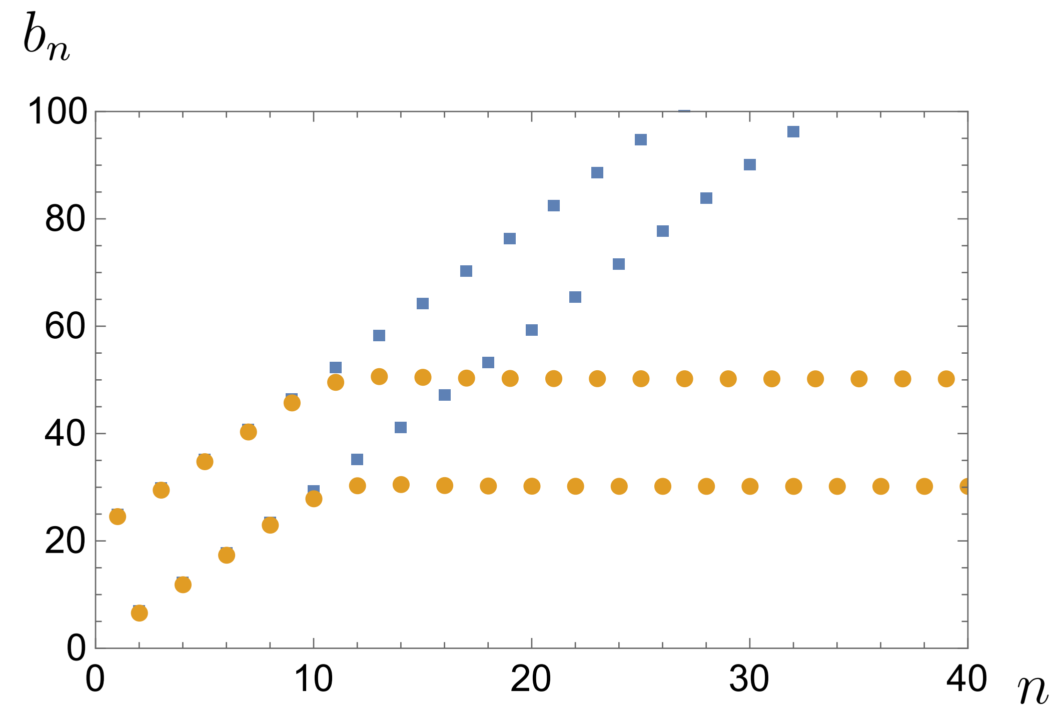
- Understand the meaning of CFT results in [A. Dymarsky, M. Smolkin, 2021]

- Understand how much difference of Krylov complexity in lattice and continuum theories

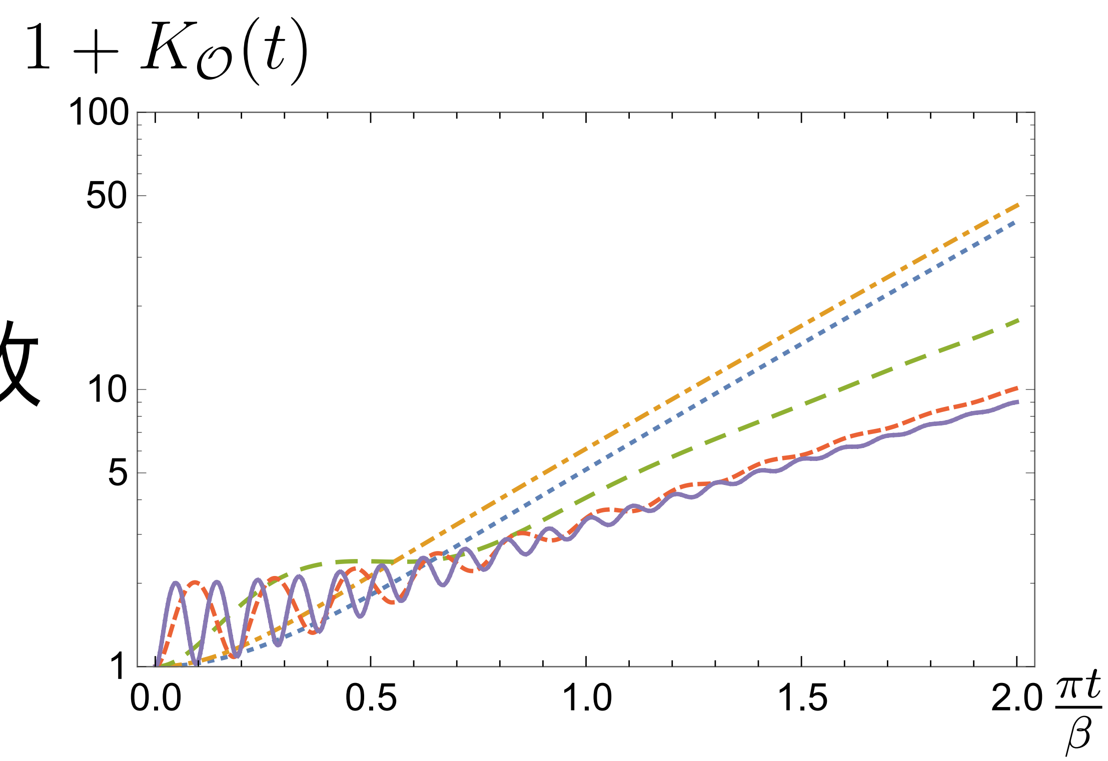
- Compute Krylov complexity of familiar and simple theories in QFT's textbooks

我々がやったこと

- 自由massive scalar場の理論のLanczos係数とKrylov complexityを調べた



- Mass gap とUV cutoffの効果を調べた



- 4d ϕ^3 と4d ϕ^4 理論のLanczos係数を摂動的に調べた

Outline

1. Lanczos coefficients and Krylov complexity

2. Conjectures for quantum chaos

3. Lanczos coefficients and Krylov complexity in scalar QFTs

4. Summary

Lanczos coefficients and Krylov complexity

- Lanczos係数とKrylov complexityは
  量子多体系のoperator growthの指標

- ２点関数から計算できる

- Krylov complexity $K_{\mathcal{O}}(t) := \sum_n n|\varphi_n(t)|^2$ の増加は
  φ_0 から φ_n への伝搬を意味する

$$\mathcal{O}(t) = \sum_{n=0} i^n \varphi_n(t) \mathcal{O}_n$$

Expansion of $\mathcal{O}(t)$ and inner product

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}e^{-iHt} = \sum_{n=0} \frac{(it)^n}{n!}\mathcal{L}^n\mathcal{O} \qquad \mathcal{L}\mathcal{O} := [H, \mathcal{O}]$$

We want to construct an orthonormal basis for $\{\mathcal{L}^n\mathcal{O}\}$ choosing an inner product.

Choices of inner products

$$(A|B) := \mathrm{Tr}[A^\dagger B]/\mathrm{Tr}[1]$$ Infinte temperature

$$(A|B)^S_\beta := \frac{1}{2Z}\mathrm{Tr}[e^{-\beta H}(A^\dagger B + BA^\dagger)]$$ Standard inner product

$$(A|B)^W_\beta := \frac{1}{Z}\mathrm{Tr}[e^{-\beta H/2}A^\dagger e^{-\beta H/2}B]$$ Wightman inner product

Construction of orthonormal basis

$$|\mathcal{O}_0) = |\mathcal{O}), \quad (\mathcal{O}_0|\mathcal{L}|\mathcal{O}_0) := (\mathcal{O}_0|\mathcal{L}\mathcal{O}_0)$$

$$a_0 =(\mathcal{O}_0|\mathcal{L}|\mathcal{O}_0), \quad |A_1) := \mathcal{L}|\mathcal{O}_0) - a_0|\mathcal{O}_0),$$

$$b_1 =\sqrt{(A_1|A_1)}, \quad |\mathcal{O}_1) = b_1^{-1}|A_1).$$

$$a_1 =(\mathcal{O}_1|\mathcal{L}|\mathcal{O}_1), \quad |A_2) := \mathcal{L}|\mathcal{O}_1) - a_1|\mathcal{O}_1) - b_1|\mathcal{O}_0),$$

$$b_2 =\sqrt{(A_2|A_2)}, \quad |\mathcal{O}_2) = b_2^{-1}|A_2).$$

We can construct $|\mathcal{O}_n)$ such that $(\mathcal{O}_m|\mathcal{O}_n) = \delta_{mn}$

Lanczos algorithm

An algorithm for tridiagonalization of a Hermitian matrix

If $(\mathcal{L}^m\mathcal{O}|\mathcal{L}|\mathcal{L}^n\mathcal{O}) := (\mathcal{L}^m\mathcal{O}|\mathcal{L}^{n+1}\mathcal{O})$ is Hermitian,
one can construct an orthonormal basis $|\mathcal{O}_n)$

$$|\mathcal{O}_{-1}) := 0,\ |\mathcal{O}_0) := |\mathcal{O}),\ \mathcal{L}|\mathcal{O}_n) = a_n|\mathcal{O}_n) + b_n|\mathcal{O}_{n-1}) + b_{n+1}|\mathcal{O}_{n+1})$$

Krylov subspace

$\mathrm{Span}\{\mathcal{L}^n\mathcal{O}\}$

Krylov basis

$|\mathcal{O}_n)$

$$(\mathcal{O}_m|\mathcal{L}|\mathcal{O}_n) = \begin{pmatrix} a_0 & b_1 & 0 & 0 & \cdots \\ b_1 & a_1 & b_2 & 0 & \cdots \\ 0 & b_2 & a_2 & b_3 & \cdots \\ 0 & 0 & b_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

a_n, b_n : Lanczos coefficients

Lanczos coefficients can be determined from a 2pt function.

2pt function $C(t) := (\mathcal{O}|\mathcal{O}(-t)) = \sum_{n=0} M_n \frac{(-it)^n}{n!}$

Moments $M_n := \frac{1}{(-i)^n} \frac{d^n C(t)}{dt^n}\Big|_{t=0} = (\mathcal{O}_0|\mathcal{L}^n|\mathcal{O}_0)$

Moments determine Lanczos coefficients.

$$M_1 = (\mathcal{O}_0|\mathcal{L}|\mathcal{O}_0) = a_0,$$

$$M_2 = (\mathcal{O}_0|\mathcal{L}^2|\mathcal{O}_0) = a_0^2 + b_1^2,$$

$$M_3 = (\mathcal{O}_0|\mathcal{L}^3|\mathcal{O}_0) = a_0^3 + 2a_0 b_1^2 + a_1 b_1^2\,,$$

$$M_4 = (\mathcal{O}_0|\mathcal{L}^4|\mathcal{O}_0) = (a_0 + a_1)^2 b_1^2 + (a_0^2 + b_1^2)^2 + b_1^2 b_2^2.$$

Time evolution of $\varphi_n(t)$

$$|\mathcal{O}(t)) = \sum_{n=0} i^n \varphi_n(t)|\mathcal{O}_n), \quad \varphi_n(t) := i^{-n}(\mathcal{O}_n|\mathcal{O}(t))$$

$$|\mathcal{O}_{-1}) := 0,\ |\mathcal{O}_0) := |\mathcal{O}),\ \mathcal{L}|\mathcal{O}_n) = a_n|\mathcal{O}_n) + b_n|\mathcal{O}_{n-1}) + b_{n+1}|\mathcal{O}_{n+1})$$

$$\varphi_{-1}(t) := 0, \varphi_0(t) = C(-t), \frac{d\varphi_n(t)}{dt} = ia_n\varphi_n(t) + b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$$

From $C(t)$, we can determine a_n, b_n
and solve $\varphi_n(t)$ recursively.

Krylov complexity $K_{\mathcal{O}}(t) := \sum_n n|\varphi_n(t)|^2$

[D. E. Parker, X. Cao, A. Avdoshkin, T. Scaffidi, E. Altman, 2018]

- $K_{\mathcal{O}}(0) = 0$ due to $\varphi_0(0) = 1, \sum_n |\varphi_n(t)|^2 = 1$

$$|\mathcal{O}(t)) = \sum_{n=0} i^n \varphi_n(t) |\mathcal{O}_n), \quad \varphi_n(t) := i^{-n} (\mathcal{O}_n | \mathcal{O}(t))$$

- For large $K_{\mathcal{O}}(t)$, $\varphi_n(t)$ with large n should have nonzero values.

- Increase of $K_{\mathcal{O}}(t)$ under time evolution means spreading from $\varphi_0(t)$ to $\varphi_n(t)$.

Lanczos coefficients and Krylov complexity

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$$\mathcal{O}(t) = \sum_{n=0} i^n \varphi_n(t) \mathcal{O}_n$$

Conjectures for quantum chaos

- 量子カオス $\Longrightarrow$ $b_n \sim \alpha n + \gamma$ (予想)

- Krylov complexityの指数的増加率λ_Kは OTOCのLyapunov指数λ_Lを制限

$$\lambda_L \leq \lambda_K \leq \frac{2\pi}{\beta}$$ (予想)

- $b_n \sim \alpha n + \gamma$ $\not\Longrightarrow$ 量子カオス

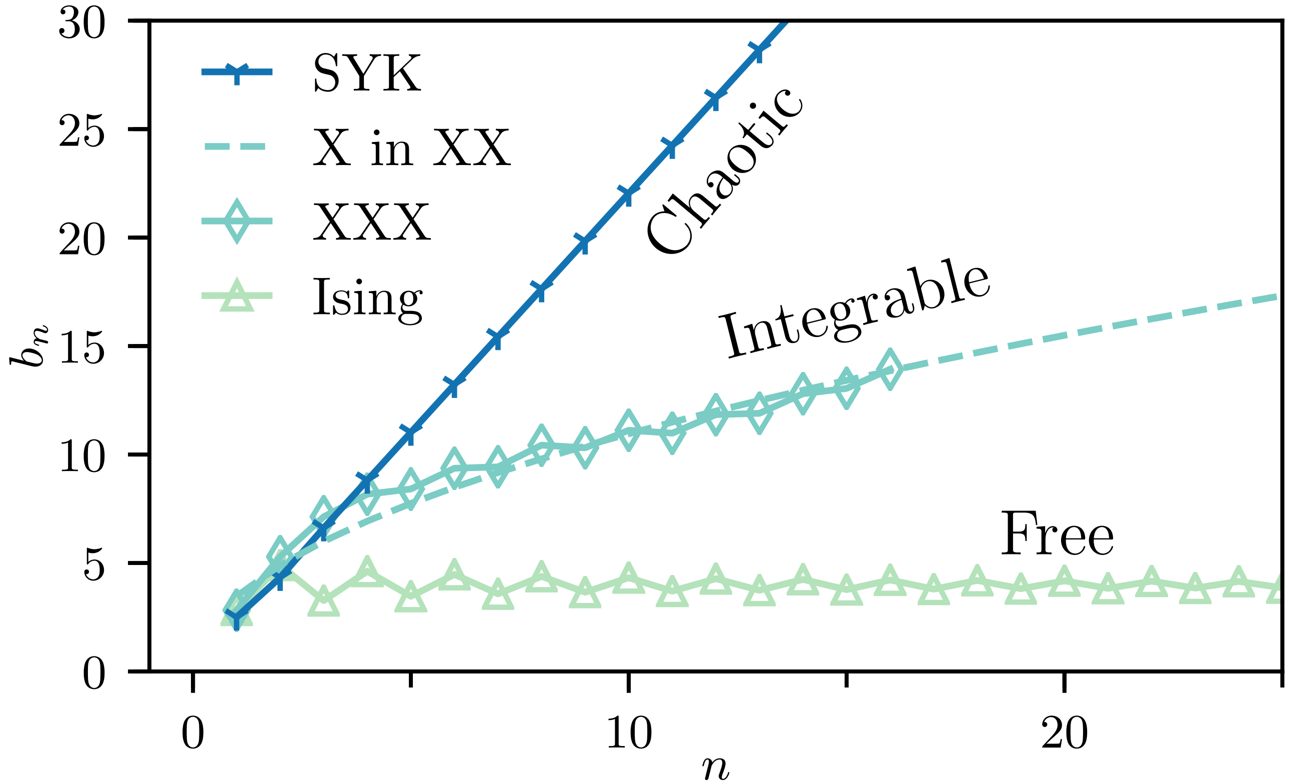
Universal operator growth hypothesis

b_n of chaotic quantum many-body systems with local interactions grows linearly.

$$b_n \sim \alpha n + \gamma \quad \text{at large} \quad n$$

[D. E. Parker, X. Cao, A. Avdoshkin, T. Scaffidi, E. Altman, 2018]

$$|\mathcal{O}_{-1}) := 0, \ |\mathcal{O}_0) := |\mathcal{O}), \ \mathcal{L}|\mathcal{O}_n) = a_n|\mathcal{O}_n) + b_n|\mathcal{O}_{n-1}) + b_{n+1}|\mathcal{O}_{n+1})$$



Large-N, large-q limit of SYK

$$b_n^W = \begin{cases} v\pi T\sqrt{2/q} + O(1/q) & n = 1 \\ v\pi T\sqrt{n(n-1)} + O(1/q) & n > 1 \end{cases}$$

$$\frac{T}{\mathcal{J}} = \frac{\cos\frac{\pi v}{2}}{\pi v}$$

Lyapunov exponent

$$\lambda_L = 2v\pi T = 2\alpha$$

λ_K bounds λ_L

Smooth linear behavior $b_n \sim \alpha n + \gamma$ implies the exponential growth behavior $K_{\mathcal{O}}(t) \sim e^{2\alpha t} = e^{\lambda_K t}$

[J.L.F. Barbon, E. Rabinovici, R. Shir, R. Sinha, 2019]

An exact example

$$C(t) = \frac{1}{(\cosh(\alpha t))^\eta}, \quad b_n = \alpha\sqrt{n(n-1+\eta)}, \quad K_{\mathcal{O}}(t) = \eta \sinh^2(\alpha t)$$

Generalized chaos bound

$\lambda_L \leq \lambda_K = 2\alpha$

proved $(T = \infty)$

[D. E. Parker, X. Cao, A. Avdoshkin, T. Scaffidi, E. Altman, 2018]

$\lambda_L \leq \lambda_K \leq 2\pi T$

conjecture $(\text{finite } T)$

[A. Avdoshkin, A. Dymarsky, 2019], [Y. Gu, A. Kitaev, P. Zhang, 2021]

$b_n \sim \alpha n + \gamma \;\; \not\Rightarrow$ Chaos

- Linear growth from saddle-dominated scrambling

[B. Bhattacharjee, X. Cao, P. Nandy, T. Pathak, 2022]

- Linear growth in free CFTs

[A. Dymarsky, M. Smolkin, 2021]

In momentum space, a free scalar QFT includes continuous infinite harmonic oscillators with $\omega^2 = m^2 + k^2$

This subtlety in QFTs is solved by UV cutoff.

Conjectures for quantum chaos

- 量子カオス $\Longrightarrow$ $b_n \sim \alpha n + \gamma$ (予想)

- Krylov complexityの指数的増加率λ_Kは OTOCのLyapunov指数λ_Lを制限

$$\lambda_L \leq \lambda_K \leq \frac{2\pi}{\beta}$$ (予想)

- $b_n \sim \alpha n + \gamma$ $\not\Longrightarrow$ 量子カオス

Lanczos coefficients and Krylov complexity in scalar QFTs

- 自由 massive scalar場の理論のLanczos係数とKrylov complexityを調べた

- Mass gapの効果: $b_{\rm odd}$ と $b_{\rm even}$ のずれ
  $K_{\mathcal{O}}(t) \sim e^{\lambda_K t}$ の指数の減少 $\lambda_K < 2\pi/\beta$

- UV cutoffの効果: b_n のsaturation
  $K_{\mathcal{O}}(t)$ の線形増加

- 4d ϕ^3 と4d ϕ^4 理論のLanczos係数を摂動的に調べた

How to compute b_n and $K_{\mathcal{O}}(t) := \sum_n n|\varphi_n(t)|^2$

2pt function

$$C(t) = \langle \phi(t - i\beta/2, \mathbf{0})\phi(0, \mathbf{0})\rangle_\beta$$

Wightman power spectrum

$$f^W(\omega) := \int \mathrm{d}t\, C(t) e^{i\omega t}$$

$$= \frac{1}{\sinh[\beta\omega/2]} \int \frac{\mathrm{d}^{d-1}\mathbf{k}}{(2\pi)^{d-1}} \rho(\omega, \mathbf{k})$$

Moments

$$M_{2n} := \frac{1}{(-i)^{2n}} \frac{d^{2n}C(t)}{dt^{2n}}\Big|_{t=0}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, \omega^{2n} f^W(\omega)$$

From a given spectral function $\rho(\omega, \mathbf{k})$,

we can compute $C(t), M_{2n}, b_n, K_{\mathcal{O}}(t)$

d-dim free scalar with odd d , $\beta m \gg 1$

$$\rho(\omega, \mathbf{k}) = \frac{N}{\epsilon_k}[\delta(\omega - \epsilon_k) - \delta(\omega + \epsilon_k)] \qquad \epsilon_k = \sqrt{|\mathbf{k}|^2 + m^2}$$

$$f^W(\omega) \sim N(m, \beta, d)\, e^{-\beta|\omega|/2} \left(\omega^2 - m^2\right)^{(d-3)/2} \underline{\Theta(|\omega| - m)}$$

Mass gap

$$C^{(d)}(t) = c_1^{(d)}(t) \left(c_2^{(d)}(t) \sin(m\,t) + c_3^{(d)}(t) \cos(m\,t) \right)$$

Oscillation due to mass

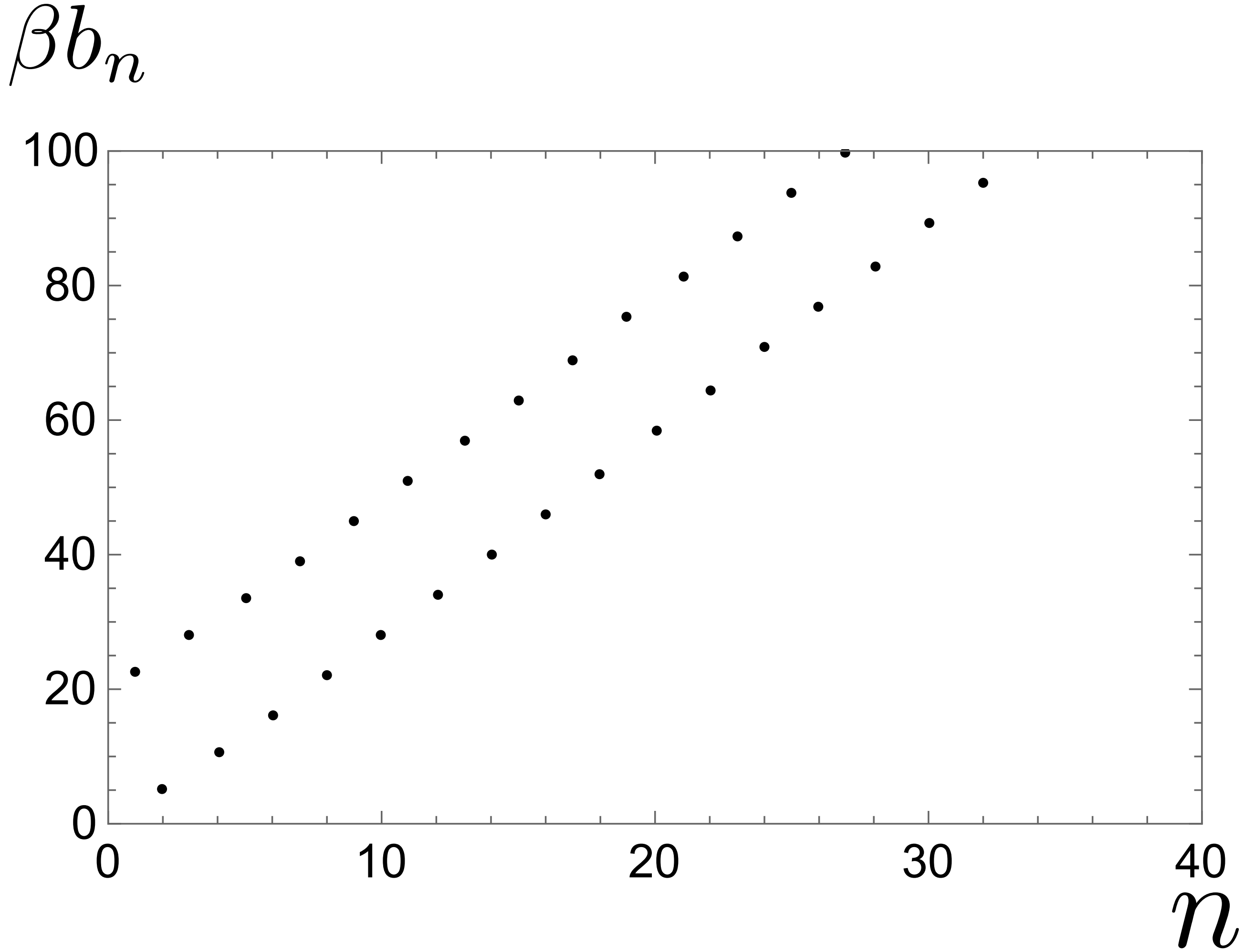
$$c_1^{(5)}(t) = \frac{\beta^3}{(\beta m + 2)\left(\beta^2 + 4t^2\right)^3}$$

Power-law decay

$$c_2^{(5)}(t) = -\,4t\left(\beta^2(\beta m + 3) + 4t^2(\beta m - 1)\right)$$

$$c_3^{(5)}(t) = 2\beta^3 + m\left(\beta^4 - 16t^4\right) - 24\beta t^2$$

b_n of free massive scalar theory



$$b_n \sim \frac{\pi}{\beta} n + \gamma_{\rm odd} \quad (\text{odd } n)$$

$$b_n \sim \frac{\pi}{\beta} n + \gamma_{\rm even} \quad (\text{even } n)$$

$$\gamma_{\rm odd} - \gamma_{\rm even} \sim m$$

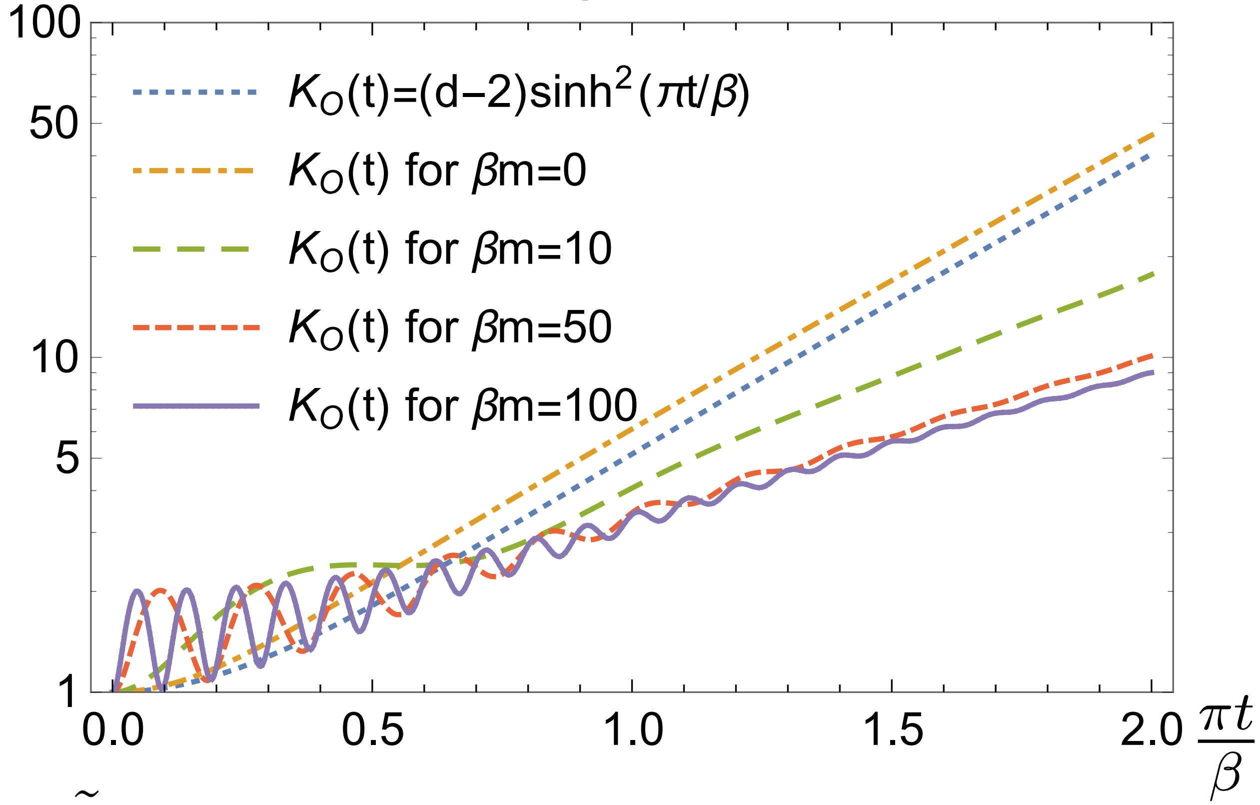
$$d = 5, \beta = 1, m = 20$$

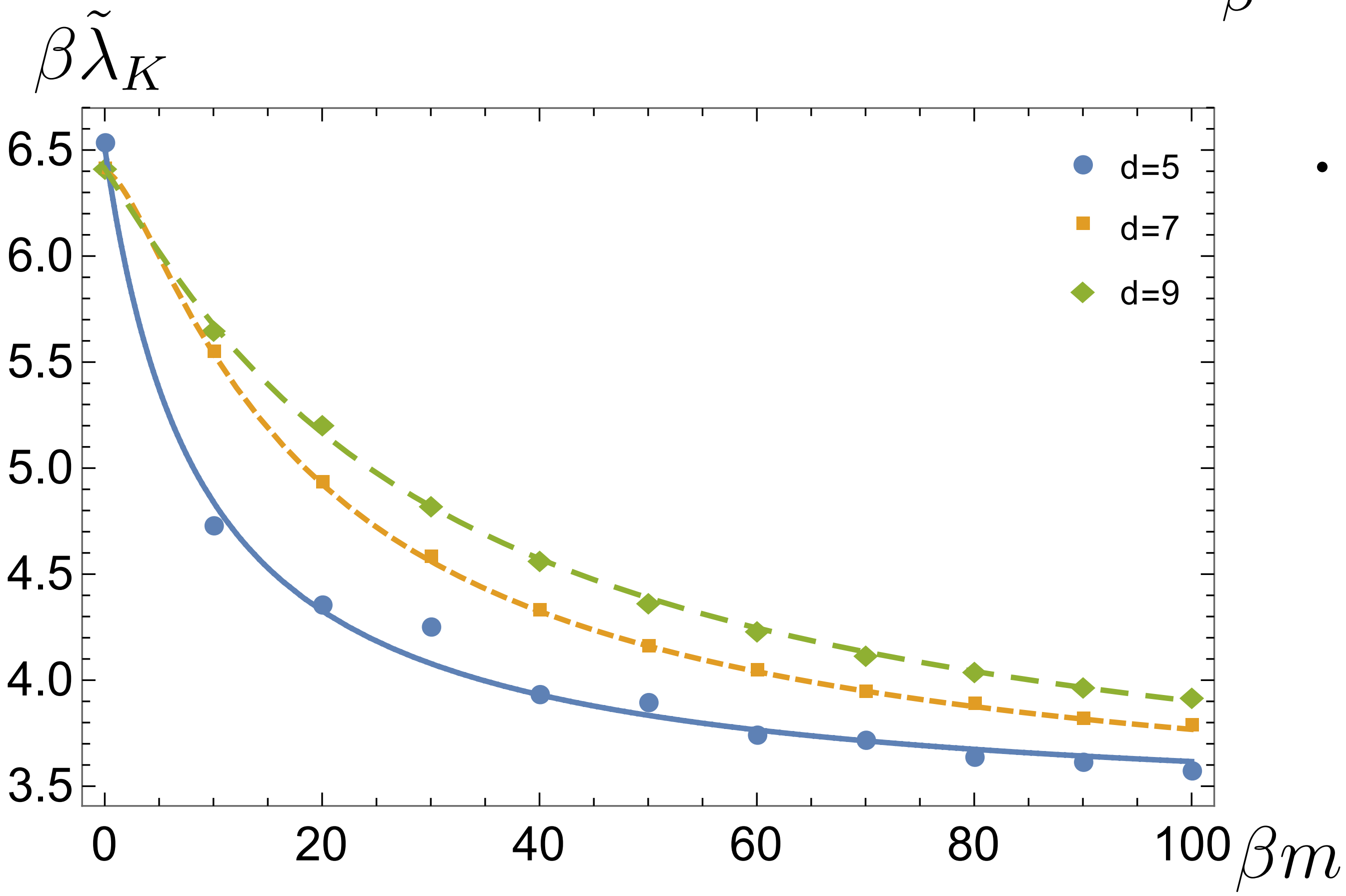
Mass m causes the difference between $b_{\rm odd}$ and $b_{\rm even}$.

b_n is not smooth with respect to n due to mass.

$K_{\mathcal{O}}(t)$ of free massive scalar theory

$1+K_{\mathcal{O}}(t)$ Log plot $d=5$





$$K_{\mathcal{O}}(t) \sim e^{\tilde{\lambda}_K t} \quad \left(1.5 \leq \frac{\pi t}{\beta} \leq 2.0\right)$$

- For $\beta m = 0$, $K_{\mathcal{O}}(t) \sim e^{\frac{2\pi}{\beta}t}$

  [A. Dymarsky, M. Smolkin, 2021]

- For $\beta m \neq 0$, $\tilde{\lambda}_K$ decreases due to mass.

- Mass violates the smoothness of b_n for $K_{\mathcal{O}}(t) \sim e^{2\alpha t}$ from $b_n \sim \alpha n + \gamma$

Universal behavior of b_n in QFTs

High frequency behavior of $f(\omega)$ is related to b_n at large n

[D. Lubinsky, H. Mhaskar, E. Saff, 1988]

$$f^W(\omega) \sim N(m, \beta, d)\, \underline{e^{-\beta|\omega|/2}}\, \left(\omega^2 - m^2\right)^{(d-3)/2} \Theta(|\omega| - m)$$

$$b_n \sim \frac{\pi}{\beta} n$$

The leading term of b_n in QFTs are governed by UV CFTs, but the sub-leading terms depend on IR like mass.

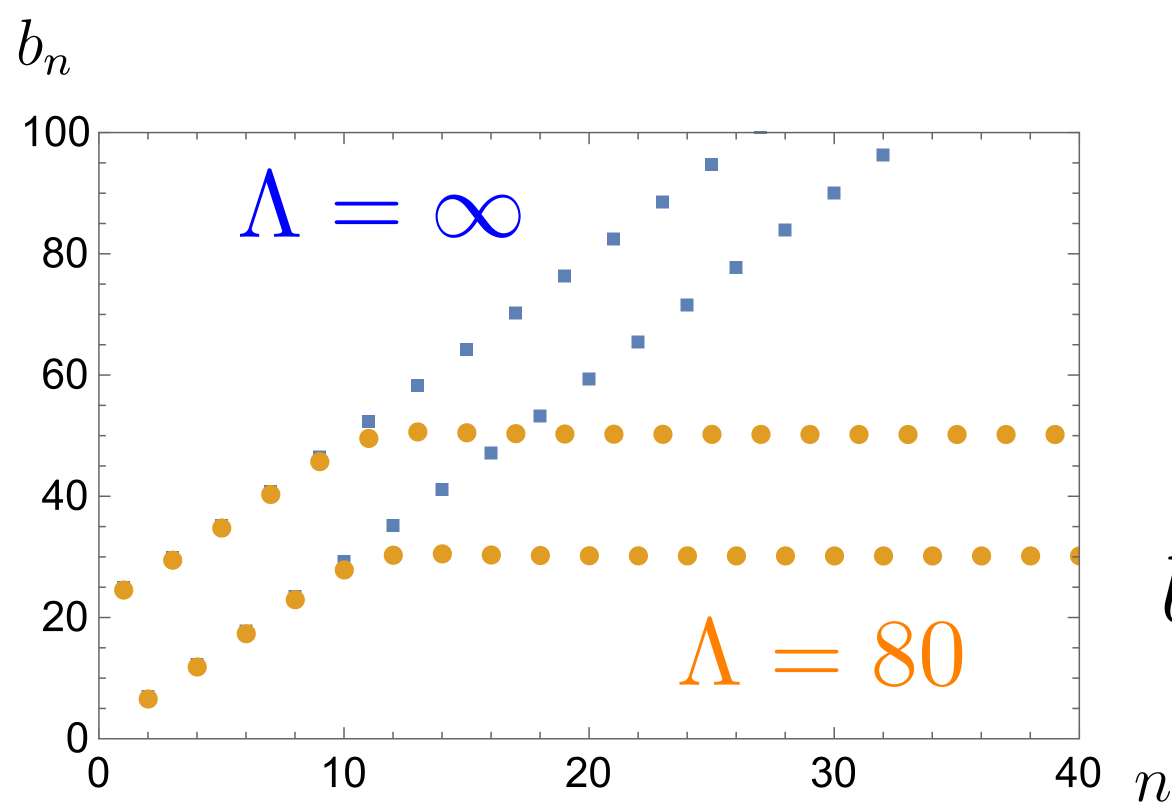
Introducing UV lattice cutoff changes b_n .

[A. Dymarsky's talk, 2022] [A. Avdoshkin, A. Dymarsky, M. Smolkin, 2022]

We introduce hard momentum cutoff.

b_n With finite UV cutoff Λ $(d=5)$

$$f^W(\omega) \sim N(m,\beta,\Lambda)\,(\omega^2-m^2)\,e^{-\frac{\beta|\omega|}{2}}\,\Theta\,(|\omega|-m\,,\,\Lambda-|\omega|)$$



$b_{\mathrm{odd}} \sim (\Lambda+m)/2$

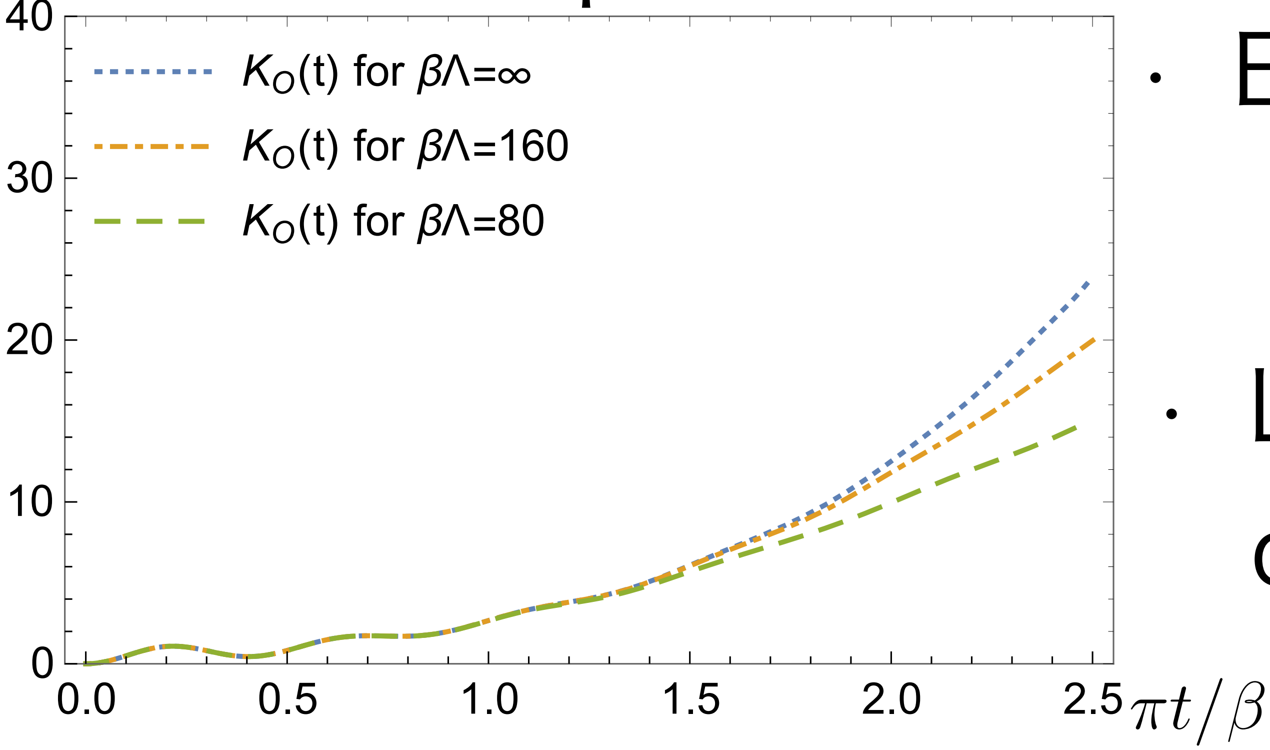
$b_{\mathrm{even}} \sim (\Lambda-m)/2$

$$(d=5, m=20, \beta=1)$$

UV cutoff Λ causes the saturation of b_n .

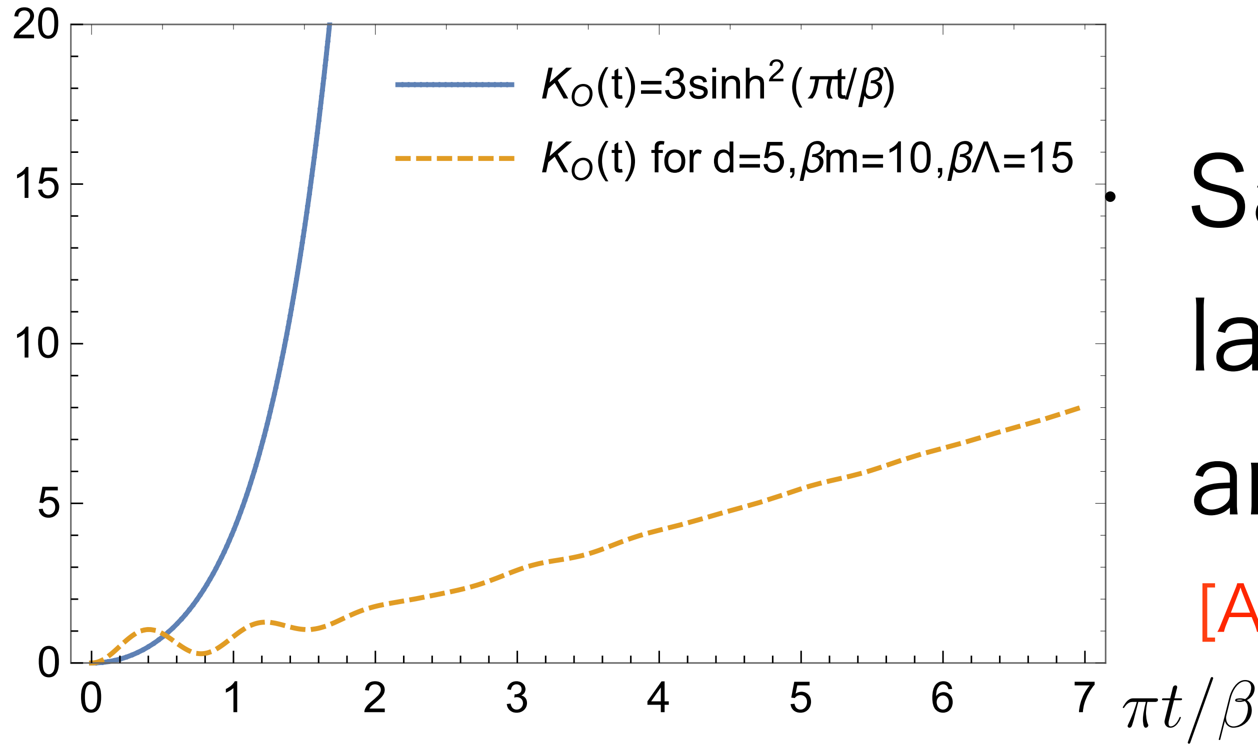
$K_{\mathcal{O}}(t)$ With finite UV cutoff Λ

$K_{\mathcal{O}}(t)$ Linear plot



- Early-time exponential growth, independent of Λ
- Late-time linear growth due to saturation of b_n

$K_{\mathcal{O}}(t)$ Linear plot



- Saturation of b_n and late-time linear growth of $K_{\mathcal{O}}(t)$ are consistent with free lattice.

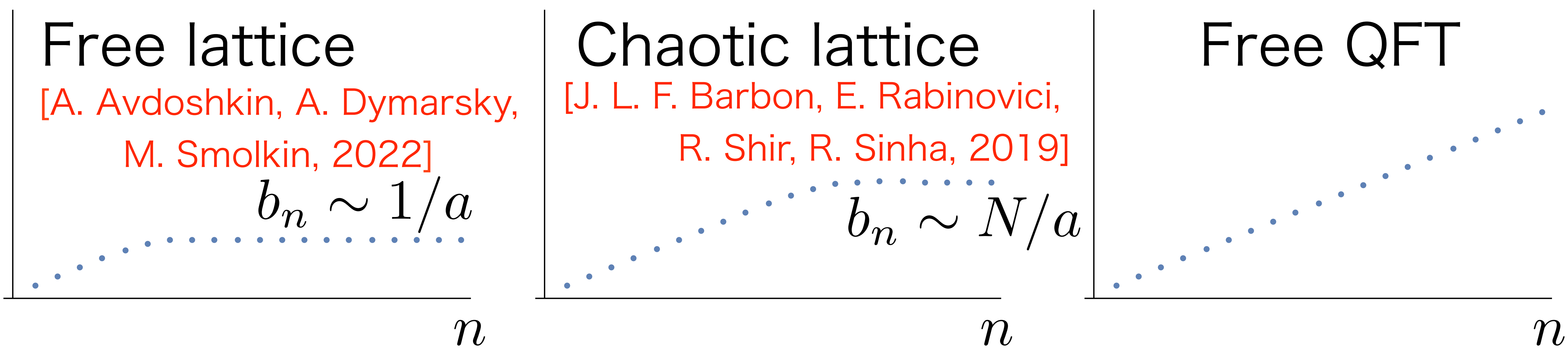
[A. Avdoshkin, A. Dymarsky, M. Smolkin, 2022]

b_n in lattice and continuum theories

Dispersion relation of free massless scalar

periodic lattice $\omega = \frac{2}{a}\sin[ka/2]$ continuum limit $\omega = k$

Schematic plots of b_n (N lattice points, lattice spacing a)



In the continuum limit $a \to 0$, we cannot distinguish

$1/a \sim \infty$ and $N/a \sim \infty$.

b_n in interacting scalar QFTs

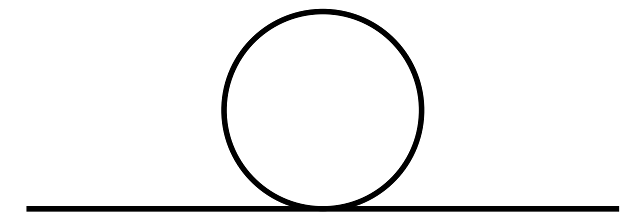
From a given spectral function $\rho(\omega, \mathbf{k})$,
we can compute $C(t), M_{2n}, b_n, K_{\mathcal{O}}(t)$

We consider 4d perturbative theory
and one-loop effect.

1. $L_{int} = g\phi^4/4!$

2. $L_{int} = g\phi^3/3!$

b_n in 4d $g\phi^4/4!$ theory

One-loop self energy $\Pi_E =$ 

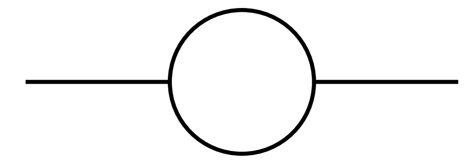
Thermal mass $m_{\rm eff}^2 = m^2 + m_{\rm th}^2 = m^2 + \frac{g}{24\beta^2}$

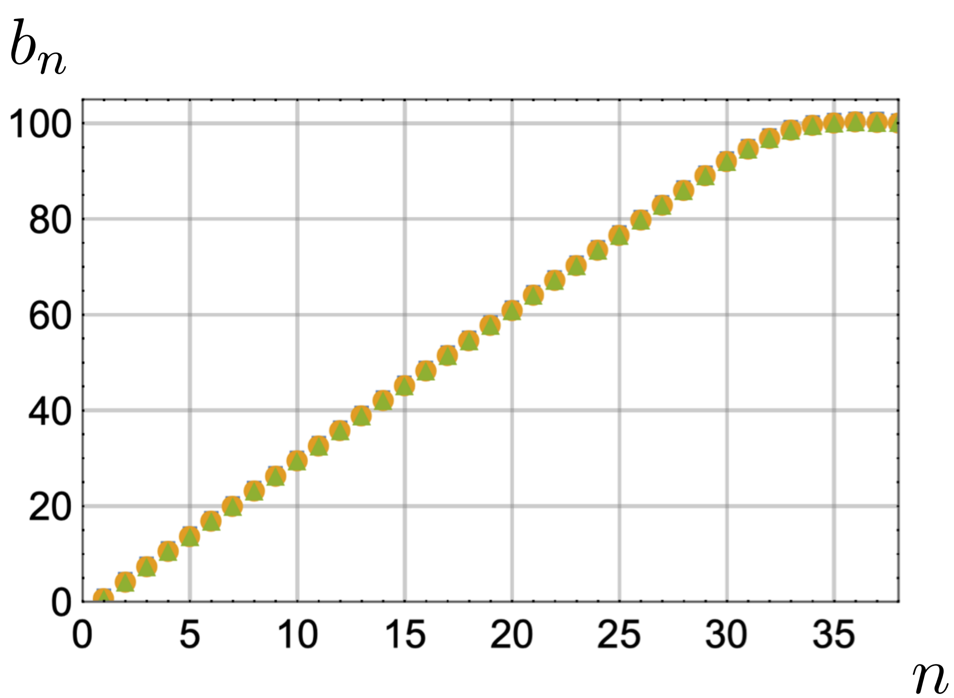
The effect of $g\phi^4/4!$ is
similar to massive free scalar.

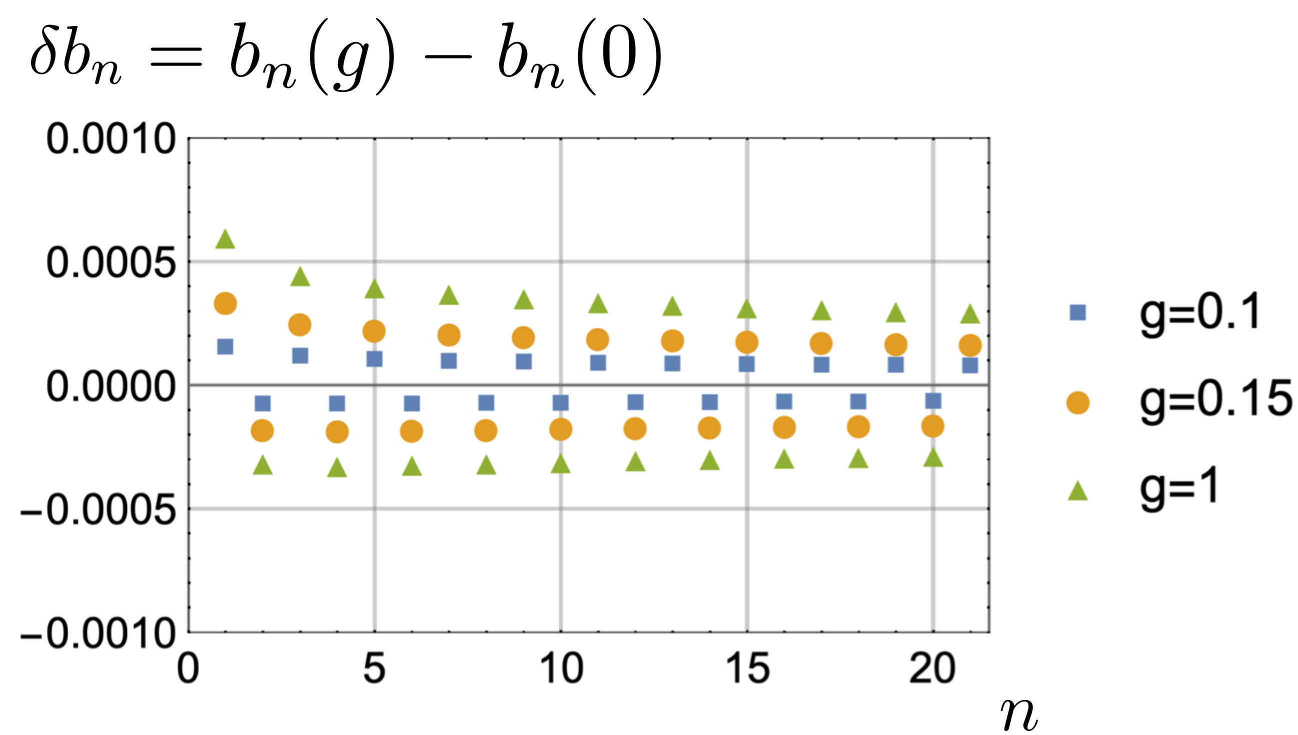
$g\phi^4/4!$ decreases the exponential growth rate

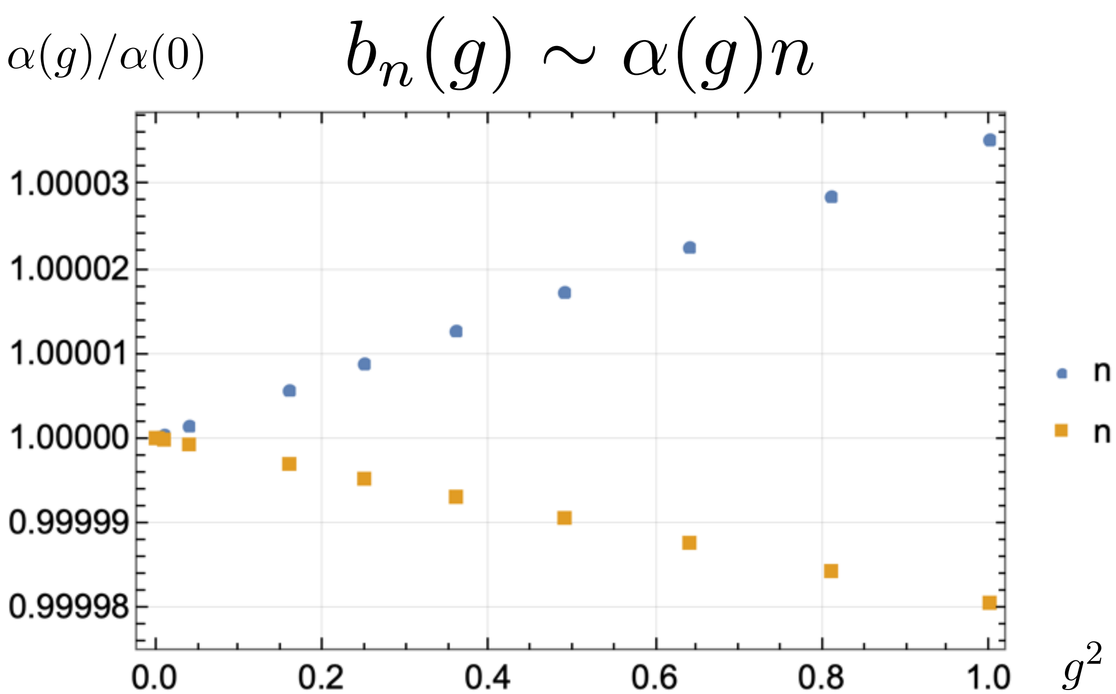
$K_{\mathcal{O}}(t) \sim e^{\tilde{\lambda}_K t}$ $\tilde{\lambda}_K(g) \leq \tilde{\lambda}_K(0) \leq 2\pi/\beta$

b_n in 4d $g\phi^3/3!$ theory $(m=0, \beta=1, \Lambda=200)$

One-loop self energy $\Pi_E =$ 







$g\phi^3/3!$ causes the difference between $b_{\rm odd}$ and $b_{\rm even}$.

But, the difference is small because of the perturbation.

まとめ

- Lanczos係数 b_n とKrylov complexity $K_{\mathcal{O}}(t)$ は
  量子多体系のoperator growthの指標

- 自由 massive scalar場の理論のLanczos係数と
  Krylov complexityを調べた

- 場の理論のMass gap とUV cutoffが b_n $K_{\mathcal{O}}(t)$ に影響

- 4d ϕ^3 と4d ϕ^4 理論のLanczos係数を摂動的に調べた
  ただし、摂動なので効果はすごく小さい

展望

- Mass gapがある場合のλ_Lの計算および λ_K との比較

$$\lambda_L \leq \lambda_K \leq \frac{2\pi}{\beta}$$

- 他の理論での解析

ϕ^4 matrix theory, TTbar deformed QFT

- Krylov complexityの重力双対