格子ゲージ理論における
分数トポロジカル電荷の定式化

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Based on
Abe, Morikawa, Suzuki, arXiv:2210.12967

Recent Studies about 't Hooft Anomaly

- Discussion of the low-energy dynamics of gauge theories based on the mixed 't Hooft anomaly between discrete and higher-form symmetries.
  (Gaiotto, Kapustin, Seiberg, Willett, arXiv:1412.5148
  Gaiotto, Kapustin, Komargodski, Seiberg, arXiv:1703.00501)
- This type of application of the anomaly has been studied vigorously.
  - ✓ Yamaguchi, arXiv:1811.09390
  - ✓ Hidaka, Hirono, Nitta, Tanizaki, Yokokura, arXiv:1903.06389
  - ✓ Honda, Tanizaki, arXiv:2009.10183
  - ✓ etc.
- Keywords: 't Hooft anomaly, higher-form symmetries

't Hooft Anomaly

- 't Hooft anomaly：
  Couple a background gauge field A_μ with the preserved current j_μ related to the symmetry

$$Z[A_\mu] = \left\langle \exp\left(i \int A_\mu j^\mu\right)\right\rangle \qquad Z[A_\mu + \partial_\mu \theta] = Z[A_\mu] \underline{\exp(i\mathcal{A}(\theta, A_\mu))}$$

Phase Gap

- 't Hooft anomaly matching:
  The property of matching the 't Hooft anomaly calculated respectively in both UV and IR theory

➢ Using the prediction of the low-energy physics of gauge theories

$\mathbb{Z}_N$ Zero-form Gauge Symmetry

- Introducing the $U(1)$ gauge field A_μ,

$$S = \int \mathrm{d}^4x\, D_\mu H^\dagger D_\mu H + \cdots\,, \quad D_\mu H = \partial_\mu H - iNA_\mu H$$

- Condense the Higgs H. ϕ is a scalar field.

$$H = he^{i\phi}, \quad \phi \sim \phi + 2\pi$$

$$S = \int \mathrm{d}^4x\, h^2(\partial_\mu\phi - NA_\mu)^2 + \cdots$$

- VEV $h \to \infty$, we get the constraint,

$$\partial_\mu\phi - NA_\mu = 0$$

$\mathbb{Z}_N$ Zero-form Gauge Symmetry

- Constraint: $\partial_\mu \phi = N A_\mu$
- If $N = 1$, A_μ is pure gauge by the constraint, $U(1)$ symmetry is broken completely. On the other hand, if $N > 1$, $\mathbb{Z}_N$ symmetry is remained. Wilson loop is

$$W^N = \left[\exp\left(i \int A_\mu\right)\right]^N = \exp\left(i \int \partial_\mu \phi\right) = 1$$

- By this constraint, a pair, (A_μ, ϕ), $U(1)$ gauge field A_μ and a scalar field ϕ, constructs $\mathbb{Z}_N$ one-form gauge field.
- This pair, (A_μ, ϕ), has the $\mathbb{Z}_N$ zero-form gauge symmetry, and the transformation is

$$\phi \mapsto \phi + N\lambda$$
$$A_\mu \mapsto A_\mu + \partial_\mu \lambda$$

$\mathbb{Z}_N$ One-form Gauge Symmetry

$\mathbb{Z}_N$ zero-form gauge symmetry	$\mathbb{Z}_N$ one-form gauge symmetry

$\mathbb{Z}_N$ zero-form gauge symmetry

- A pair, $U(1)$ gauge field A_μ and a scalar field ϕ, constructs $\mathbb{Z}_N$ one-form gauge field.
- Constraint

$$NA_\mu = \partial_\mu \phi$$

- $\mathbb{Z}_N$ zero-form gauge transformation

$$\phi \mapsto \phi + N\lambda$$
$$A_\mu \mapsto A_\mu + \partial_\mu \lambda$$

$\mathbb{Z}_N$ one-form gauge symmetry

- A pair, $U(1)$ two-form gauge field $B_{\mu\nu}$ and $U(1)$ gauge field C_μ, constructs $\mathbb{Z}_N$ two-form gauge field.
- Constraint

$$NB_{\mu\nu} = \partial_{[\mu} C_{\nu]}$$

In short, we write $NB = \mathrm{d}C$

- $\mathbb{Z}_N$ one-form gauge transformation

$$C_\mu \mapsto C_\mu + N\lambda_\mu$$
$$B_{\mu\nu} \mapsto B_{\mu\nu} + \partial_{[\mu}\lambda_{\nu]}$$

Couple with $SU(N)$ Gauge Theory with θ Term

- Action: $S = -\frac{1}{2g^2}\int \mathrm{tr}\left[(\mathcal{F}-\mathbb{1}B)\wedge\star(\mathcal{F}-\mathbb{1}B)\right] + \frac{\theta}{8\pi^2}\int \mathrm{tr}\left[(\mathcal{F}-\mathbb{1}B)\wedge(\mathcal{F}-\mathbb{1}B)\right] + \frac{1}{2\pi}\int u\wedge(\mathrm{tr}\,\mathcal{F}-NB)$
  $S = -\frac{1}{2g^2}\int \mathrm{tr}(F\wedge\star F) + \frac{\theta}{8\pi^2}\int \mathrm{tr}(F\wedge F)$
- Couple the pair, $(B_{\mu\nu}, C_\mu)$, $\mathbb{Z}_N$ two-form gauge field, with $SU(N)$ gauge theory

➢ Extend the $SU(N)$ gauge theory to the $U(N)$ gauge theory

➢ $\mathcal{A}$: $U(N)$ gauge field, whose traceless part is $SU(N)$ gauge field A.

➢ Eliminate the trace-part by one-form gauge symmetry,

$$\mathcal{A} \mapsto \mathcal{A} + \lambda\mathbb{1}$$

$\mathbb{Z}_N$ One-form Gauge Transformation

$$C \mapsto C + N\lambda$$

$$B \mapsto B + \mathrm{d}\lambda$$

➢ Imposing the constraint,

$$\mathrm{tr}(\mathcal{F}) = NB$$

➢ With the gauge transformation of a pair (B, C), $\mathbb{Z}_N$ two-form gauge field, $F = \mathcal{F} - \mathbb{1}B$ becomes λ gauge invariant.

➢ By this F, we obtain the $SU(N)$ gauge action coupling with the $\mathbb{Z}_N$ two-form gauge field.

Mixed 't Hooft Anomaly in $SU(N)$ Gauge Theory with θ Term

- Action:

$$S = -\frac{1}{2g^2}\int \mathrm{tr}(F\wedge\star F) + \frac{\theta}{8\pi^2}\int \mathrm{tr}(F\wedge F)$$

$\mathcal{T}$-symmetry when $\theta = 0, \pi$

➢ Coupling $\mathbb{Z}_N$ two-form gauge field B as the background gauge field,

$$S = -\frac{1}{2g^2}\int \mathrm{tr}\left[(\mathcal{F}-\mathbb{1}B)\wedge\star(\mathcal{F}-\mathbb{1}B)\right] + \frac{\theta}{8\pi^2}\int \mathrm{tr}\left[(\mathcal{F}-\mathbb{1}B)\wedge(\mathcal{F}-\mathbb{1}B)\right] + \frac{1}{2\pi}\int u\wedge(\mathrm{tr}\,\mathcal{F}-NB)$$

➢ Respecting $\mathbb{Z}_N$ one-form gauge symmetry, when $\theta = \pi$,

$$Z[B] \xrightarrow{\mathcal{T}} Z[B]\exp\left[i\frac{-1+N+2p}{4\pi N}\int NB\wedge NB\right]$$

$2\pi i\times$(Fractional)

➢ This is mixed 't Hooft anomaly between the $\mathbb{Z}_N$ one-form gauge symmetry and the $\mathcal{T}$-symmetry when $\theta = \pi$.

➢ Our motivation is to understand these in a completely regularized framework (lattice field theory).

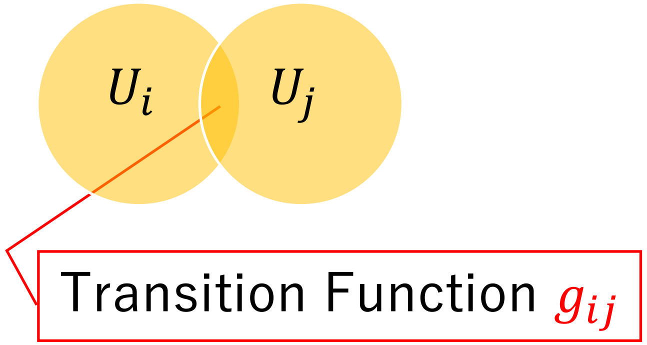
Principal Fiber Bundle

- Covering manifold by patches U_i, each patch has $SU(N)$ gauge field a_i, matter field ϕ_i in an irreducible representation ρ.

- g_{ij} at $U_{ij} = U_i \cap U_j$

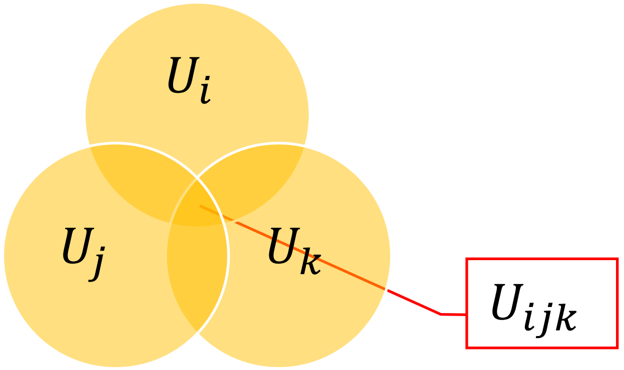
$$a_j = g_{ij}^{-1} a_i g_{ij} - i g_{ij}^{-1} \mathrm{d} g_{ij}$$

$$\phi_j = \rho(g_{ij}^{-1}) \phi_i$$



- Cocycle condition at $U_{ijk} = U_i \cap U_j \cap U_k$

$$g_{ij} g_{jk} g_{ki} = 1$$



$\mathbb{Z}_N$ One-form Gauge Symmetry and Fiber Bundle

- Especially for the adjoint representation, the cocycle condition can become relaxed.

$$g_{ij} g_{jk} g_{ki} = \exp\left(\frac{2\pi i}{N} n_{ijk}\right)$$

$\in \mathbb{Z}_N$

- ➢ $\{n_{ijk}\}$ has the gauge redundancy.
- ➢ The transition function has the transformation:

$$g_{ij} \mapsto \exp\left(\frac{2\pi i}{N} \lambda_{ij}\right) g_{ij}$$

Transition Function in $SU(N)/\mathbb{Z}_N$ Gauge Theory

For the invariance of the cocycle condition,

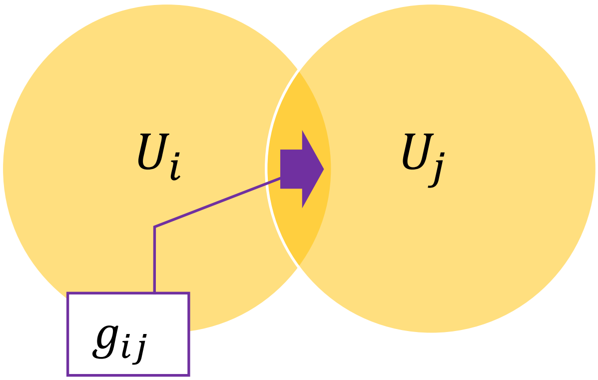
$$n_{ijk} \mapsto n_{ijk} + (\delta\lambda)_{ijk}$$

$$(\delta\lambda)_{ijk} \equiv \lambda_{ij} - \lambda_{ik} + \lambda_{jk}$$

- ➢ This transformation is $\mathbb{Z}_N$ one-form gauge transformation, $\{n_{ijk}\}$ is $\mathbb{Z}_N$ two-form gauge field.

Fiber Bundle in $SU(N)/\mathbb{Z}_N$ Gauge Theory and Fractional Topological Charge

- The topological charge can become fractional by the principal fiber bundle in the $SU(N)/\mathbb{Z}_N$ gauge theory.
  ('t Hooft, Nucl. Phys. B 153 (1979), van Baal, Commun. Math. Phys. 85 (1982))



$$g_{ij} \mapsto \exp\left(\frac{2\pi i}{N}\lambda_{ij}\right) g_{ij}$$

⇩

$$(SU(N)/\mathbb{Z}_N \text{ Transition Function}) \sim \omega_\mu \times (SU(N) \text{ Transition Function})$$

Factor for Fractionality

※Mixed 't Hooft anomaly: $Z[B] \xrightarrow{\mathcal{T}} Z[B] \exp\left[i\frac{-1+N+2p}{4\pi N}\int NB \wedge NB\right]$

$2\pi i \times$(Fractional)

Fractional Topological Charge on the Lattice

- Above discussion about mixed 't Hooft anomaly and the fractional topological charge is all in the continuum.

➢ To understand these in a completely regularized framework (lattice gauge theory).
(cf. Itou, arXiv:1811.05708)

Our Purpose

- Integer topological charge was formulated in the lattice $SU(N)$ gauge theory.
(Lüscher, Commun. Math. Phys. 85 (1982))

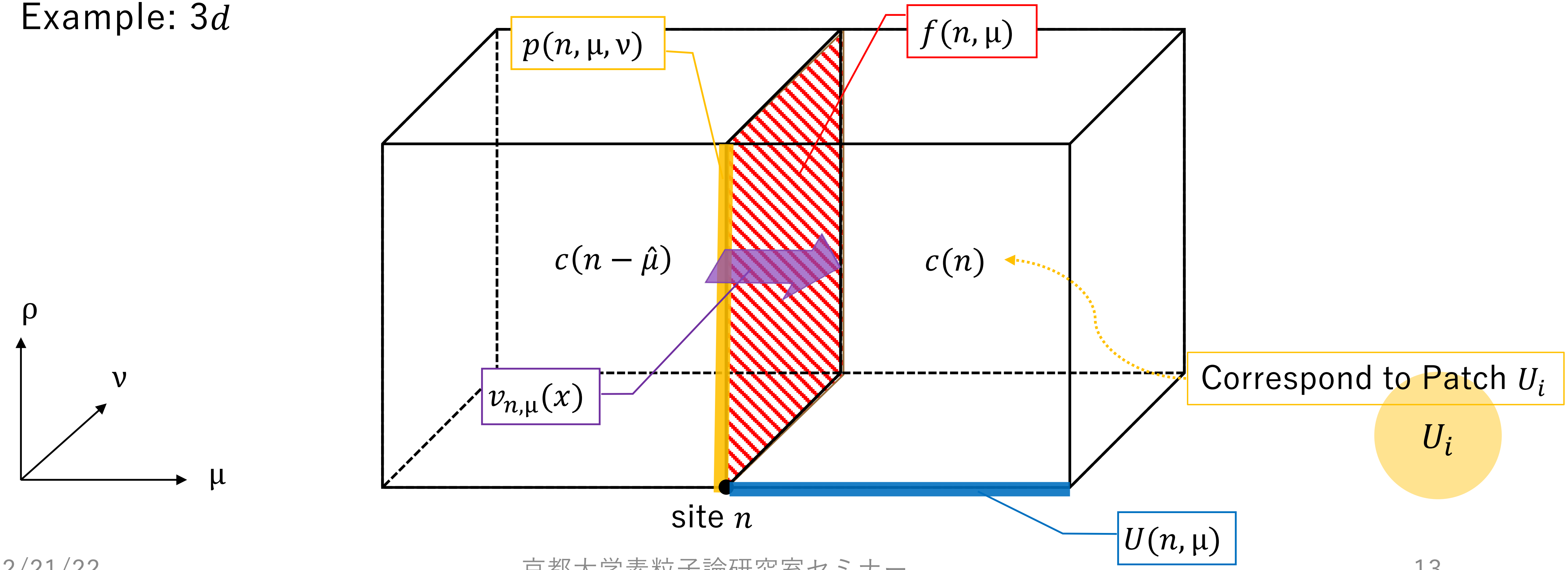
➢ We formulate fractional topological charge in the lattice $U(1)$ gauge theory.

Our Paper

➢ The formulation in the lattice $U(1)$ gauge theory is simpler, so we apply this.
(Fujiwara, Suzuki, Wu, arXiv:0001029)

Fiber Bundle and Lattice Gauge Theory

- Divided the manifold by the hyper cubes $c(n)$
- Example: $3d$

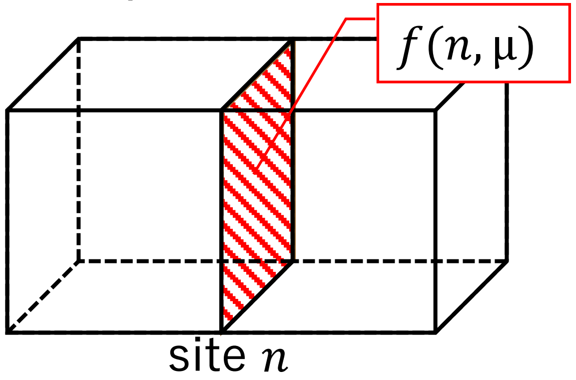


Lüscher's Idea

- Topological charge is defined by the continuum function: transition function $v_{n,\mu}$,

$$Q(v_{n,\mu}) = -\frac{1}{24\pi^2}\sum_{n}\sum_{\mu,\nu,\rho,\sigma}\int_{f(n,\mu)} \mathrm{d}^3x\,\varepsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\Big((v_{n,\mu}^{-1}\partial_\nu v_{n,\mu})(v_{n,\mu}^{-1}\partial_\rho v_{n,\mu})(v_{n,\mu}^{-1}\partial_\sigma v_{n,\mu})\Big)$$
$$+\frac{1}{8\pi^2}\sum_{n}\sum_{\mu,\nu,\rho,\sigma}\int_{p(n,\mu,\nu)} \mathrm{d}^2x\,\varepsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\Big((v_{n,\mu}\partial_\rho v_{n,\mu}^{-1})(v_{n-\hat{\mu},\nu}^{-1}\partial_\sigma v_{n-\hat{\mu},\nu})\Big)$$

- By the interpolate function: “Parallel transporter”, he defined the transition function $v_{n,\mu}$ on the face $f(n,\mu)$.

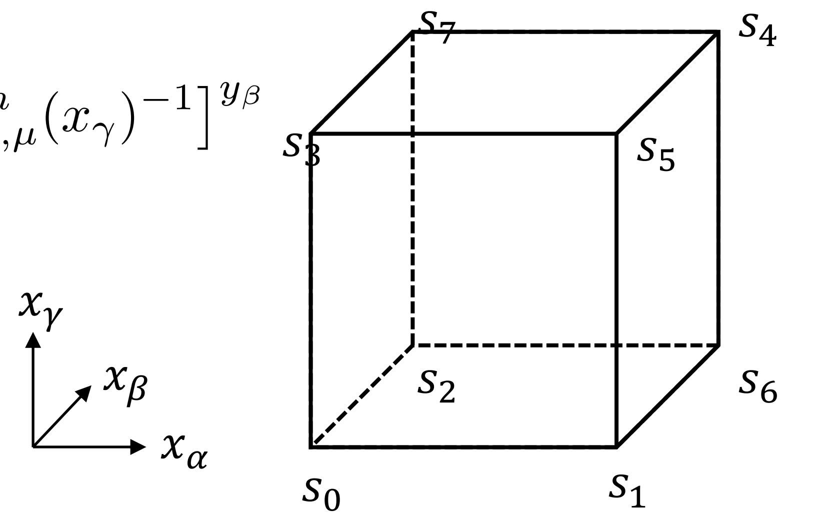


Interpolate Function in $SU(N)$ Gauge Theory

- In $x \in f(n,\mu)$,

$$
\begin{aligned}
f^m_{n,\mu}(x_\gamma) =& (u_{30})^{y_\gamma}(u^m_{03}u^m_{37}u^m_{72}u^m_{20})^{y_\gamma}u^m_{02}(u^m_{27})^{y_\gamma} \\
g^m_{n,\mu}(x_\gamma) =& (u_{51})^{y_\gamma}(u^m_{15}u^m_{54}u^m_{46}u^m_{61})^{y_\gamma}u^m_{16}(u^m_{64})^{y_\gamma} \\
h^m_{n,\mu}(x_\gamma) =& (u_{30})^{y_\gamma}(u^m_{03}u^m_{35}u^m_{51}u^m_{10})^{y_\gamma}u^m_{01}(u^m_{15})^{y_\gamma} \\
k^m_{n,\mu}(x_\gamma) =& (u_{72})^{y_\gamma}(u^m_{27}u^m_{74}u^m_{46}u^m_{62})^{y_\gamma}u^m_{26}(u^m_{64})^{y_\gamma} \\
l^m_{n,\mu}(x_\beta, x_\gamma) =& \left[f^m_{n,\mu}(x_\gamma)^{-1}\right]^{y_\beta}\left[f^m_{n,\mu}(x_\gamma)k^m_{n,\mu}(x_\gamma)g^m_{n,\mu}(x_\gamma)^{-1}h^m_{n,\mu}(x_\gamma)^{-1}\right]^{y_\beta} \\
& \cdot h^m_{n,\mu}(x_\gamma)\left[g^m_{n,\mu}(x_\gamma)\right]^{y_\beta} \\
S^m_{n,\mu}(x_\alpha, x_\beta, x_\gamma) =& (u^m_{03})^{y_\gamma}\left[f^m_{n,\mu}(x_\gamma)\right]^{y_\beta}\left[l^m_{n,\mu}(x_\beta, x_\gamma)\right]^{y_\alpha}
\end{aligned}
$$

Difficult!!



Parallel Transporter in the Lattice $U(1)$ Gauge Theory

- By the parallel transporter $w^m(x)$, we obtain the transition function $v_{n,\mu}$ in the continuum point x: $v_{n,\mu}(x) = w^{n-\hat{\mu}}(x) w^n(x)^{-1}$

lattice

$$w^n(\bar{x}) = U(n,1)^{\sigma_1} U(n+\sigma_1\hat{1},2)^{\sigma_2} \cdots U(n+\sigma_1\hat{1}+\sigma_2\hat{2}+\cdots+\sigma_{D-1}\widehat{D-1},D)^{\sigma_D}$$

$$\bar{x} = n + \sum_{\mu=1}^{D} \sigma_\mu \hat{\mu} \quad (\sigma_\mu = \{0,1\})$$

Interpolate

Continuum

$$w^m(x) = \prod_{\{\sigma_k=0,1\}_{k=1,\cdots,D-1}} w^m\left(n + \sum_{k=1}^{D-1} \sigma_k \hat{\mu}_k\right)^{\prod_{k=1}^{D-1}(\sigma_k y_k + (1-\sigma_k)(1-y_k))}$$

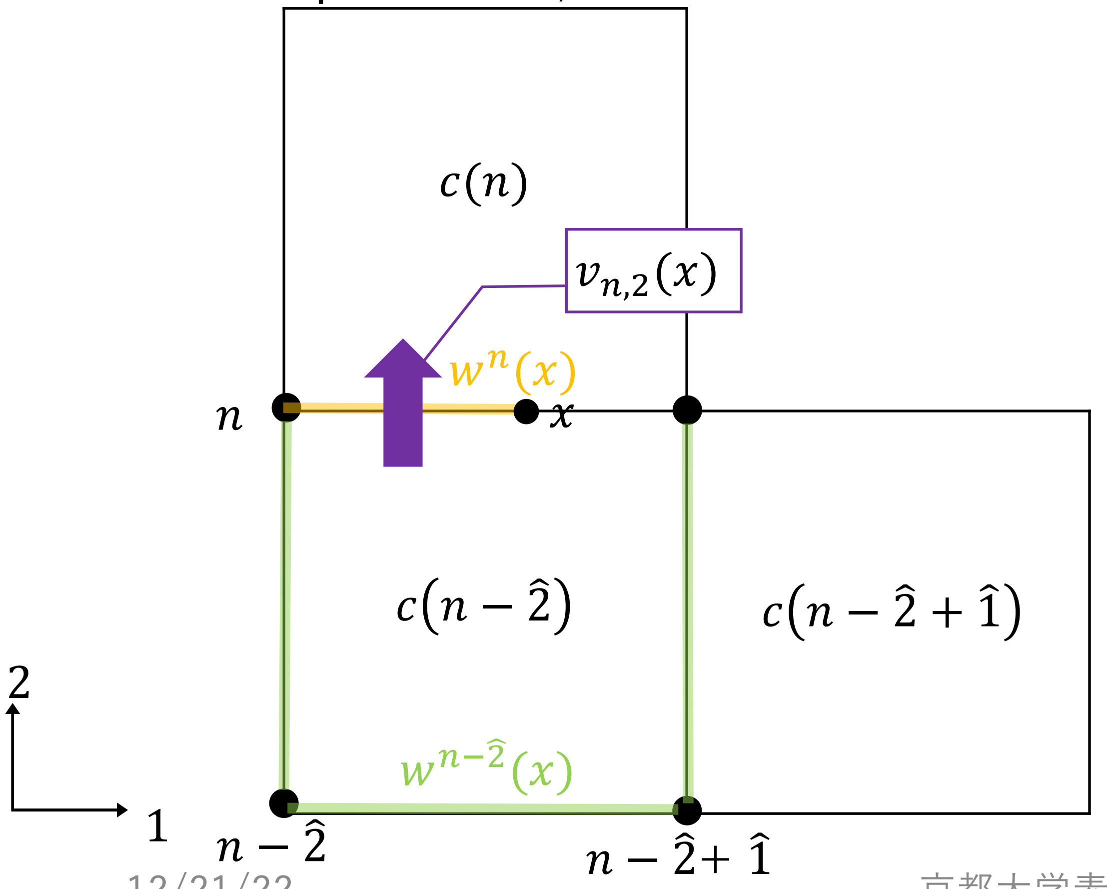
m: $n, n-\hat{\mu}$

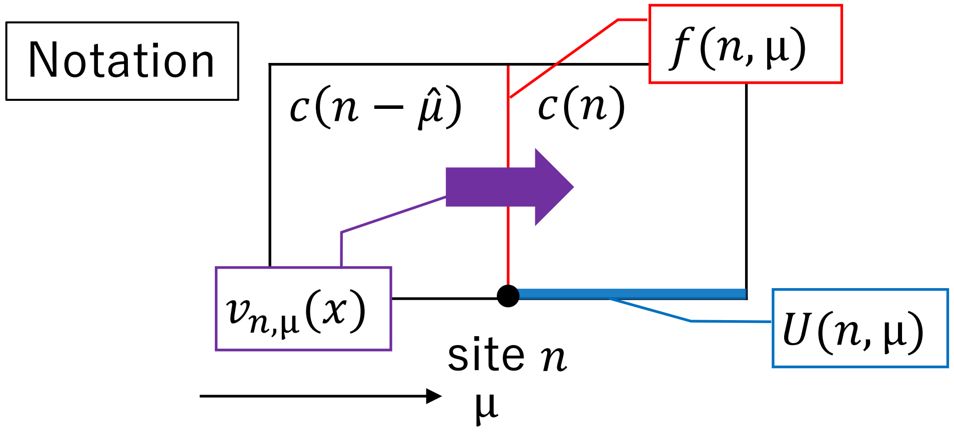
Interpolate Parameter y_k

$$x = n + \sum_{k=1}^{D-1} y_k \hat{\mu}_k, \quad 0 \le y_k \le 1$$

Image of Parallel Transporter

➢ Example: in $2d$,





Parallel Transporter

$0 \leq y_1 \leq 1$

$$w^n(x) = U(n,1)^{y_1}$$

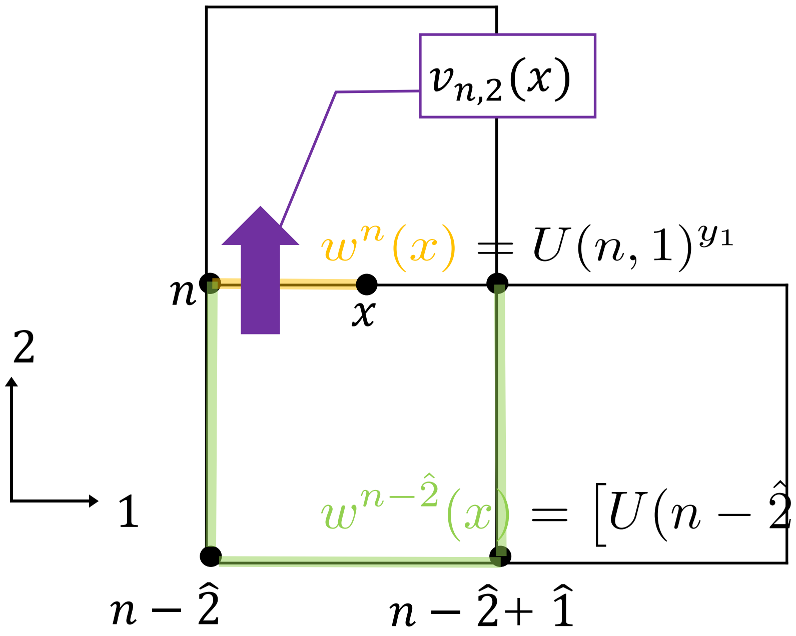
$$w^{n-\hat{2}}(x) = \left[U(n-\hat{2},1)U(n-\hat{2}+\hat{1},2)\right]^{y_1} U(n-\hat{2},2)^{1-y_1}$$

Transition Function on the Lattice in 2d

- By using the parallel transport function, the transition function is,

$$v_{n,\mu}(x) = w^{n-\hat{\mu}}(x) w^{n}(x)^{-1}$$

➢ Example: in 2d,



$$w^{n}(x) = U(n,1)^{y_1}$$

$$w^{n-\hat{2}}(x) = \left[U(n-\hat{2},1)U(n-\hat{2}+\hat{1},2)\right]^{y_1} U(n-\hat{2},2)^{1-y_1}$$

$$\begin{aligned} v_{n,2}(x) &= w^{n-\hat{2}}(x) w^{n}(x)^{-1} \\ &= U(n-\hat{2},2)\left[U(n-\hat{2},1)U(n-\hat{2}+\hat{1},2)U(n,1)^{-1}U(n-\hat{2},2)^{-1}\right]^{y_1} \\ &= U(n-\hat{2},2)\exp\left[i y_1 F_{12}(n-\hat{2})\right] \end{aligned}$$

$$F_{\mu\nu}(n) = \frac{1}{i}\ln\left[U(n,\mu)U(n+\hat{\mu},\nu)U(n+\hat{\nu},\mu)^{-1}U(n,\nu)^{-1}\right]$$

Transition Function on the Lattice in $4d$

In $2d$

$$v_{n,2}(x) = U(n-\hat{2},2)\exp\left[iy_1 F_{12}(n-\hat{2})\right]$$

- In $4d$,

$$\begin{aligned}
v_{n,1}(x) &= U(n-\hat{1},1) \\
&\quad\times \exp\Big[iy_4F_{14}(n-\hat{1}) + iy_3y_4F_{13}(n-\hat{1}+\hat{4}) + iy_3(1-y_4)F_{13}(n-\hat{1}) \\
&\qquad + iy_2y_3y_4F_{12}(n-\hat{1}+\hat{3}+\hat{4}) + iy_2y_3(1-y_4)F_{12}(n-\hat{1}+\hat{3}) \\
&\qquad + iy_2(1-y_3)y_4F_{12}(n-\hat{1}+\hat{4}) + iy_2(1-y_3)(1-y_4)F_{12}(n-\hat{1})\Big], \\
v_{n,2}(x) &= U(n-\hat{2},2)\exp\left[iy_4F_{24}(n-\hat{2}) + iy_3y_4F_{23}(n-\hat{2}+\hat{4}) + iy_3(1-y_4)F_{23}(n-\hat{2})\right], \\
v_{n,3}(x) &= U(n-\hat{3},3)\exp\left[iy_4F_{34}(n-\hat{3})\right], \\
v_{n,4}(x) &= U(n-\hat{4},4)
\end{aligned}$$

- Field strength is

$$F_{\mu\nu}(n) = \frac{1}{i}\ln\left[U(n,\mu)U(n+\hat{\mu},\nu)U(n+\hat{\nu},\mu)^{-1}U(n,\nu)^{-1}\right]$$

New Transition Function on the Lattice

Fractionality in the Continuum $SU(N)/\mathbb{Z}_N$ Gauge Theory

$$(SU(N)/\mathbb{Z}_N \text{ Transition Function}) \sim \omega_\mu \times (SU(N) \text{ Transition Function})$$

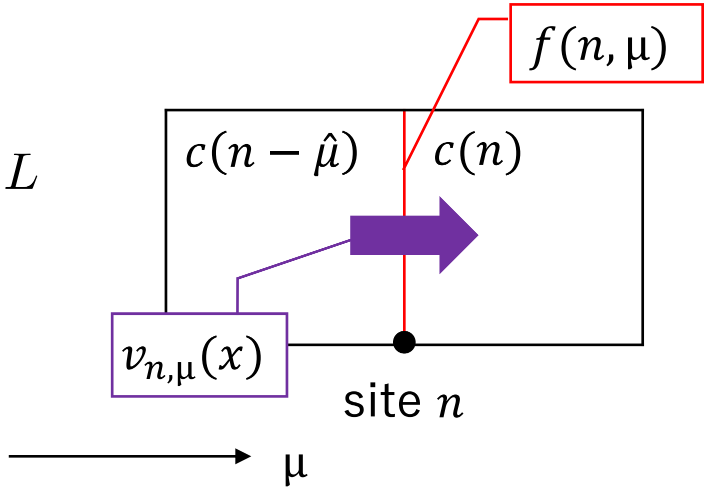
- At $x \in f(n,\mu)$, constructing transition function $v_{n,\mu}$ on the lattice $U(1)/\mathbb{Z}_q$ gauge theory,

$$v_{n,\mu}(x) = \omega_\mu(x)\check{v}_{n,\mu}(x)$$

➢ ω_μ is the factor for relaxing the cocycle condition,

$$\omega_\mu(x) \equiv \begin{cases} \exp\left(\frac{\pi i}{q}\sum_{\nu\neq\mu}\frac{z_{\mu\nu}x_\nu}{L}\right) & \text{for } x_\mu = 0 \bmod L \\ 1 & \text{otherwise} \end{cases}$$

➢ $z_{\mu\nu} \in \mathbb{Z}$ and $z_{\mu\nu} = -z_{\nu\mu}$



Cocycle Condition on the Lattice

New Transition Function

$$v_{n,\mu}(x) = \omega_\mu(x)\check{v}_{n,\mu}(x) \quad \text{at } x \in f(n,\mu)$$

- For the ordinary transition function $\check{v}_{n,\mu}$, the cocycle condition is

$$\check{v}_{n-\hat{\mu},\nu}(x)\check{v}_{n,\mu}(x)\check{v}_{n,\nu}(x)^{-1}\check{v}_{n-\hat{\nu},\mu}(x)^{-1} = 1$$

Lattice Version of $g_{ij}g_{jk}g_{ki} = 1$

- For the new transition function $v_{n,\mu}$, owing to ω_μ,

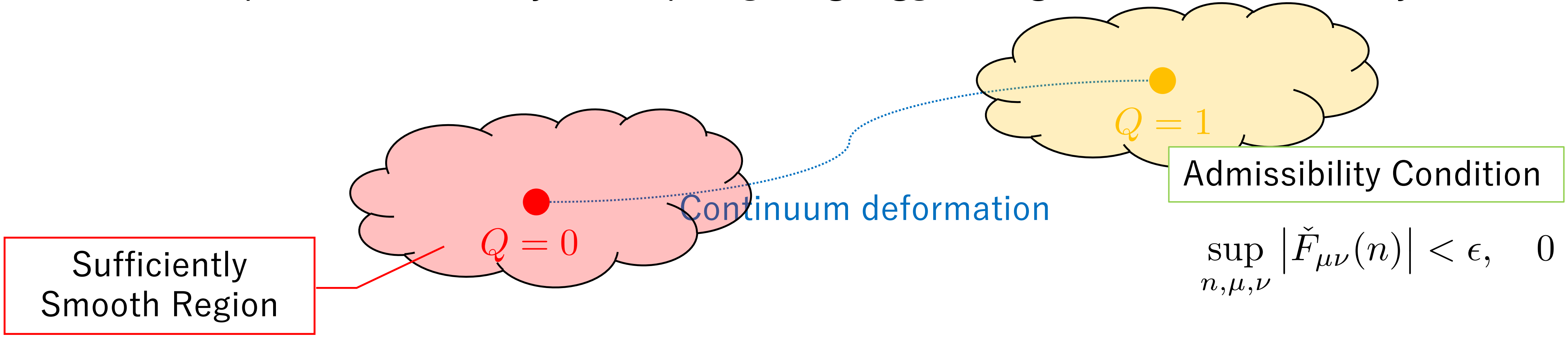
$$v_{n-\hat{\mu},\nu}(x)v_{n,\mu}(x)v_{n,\nu}(x)^{-1}v_{n-\hat{\nu},\mu}(x)^{-1}$$
$$= \begin{cases} \exp\left(\frac{2\pi i}{q} z_{\mu\nu}\right) & \text{for } x_\mu = x_\nu = 0 \bmod L \\ 1 & \text{otherwise} \end{cases}$$

$\in \mathbb{Z}_q$

$\mathbb{Z}_q$ “Relax”

Admissibility Condition

➢ It is impossible to define the topological charge which is integer on the lattice.
➢ Under the “Admissibility condition”, the gauge configuration is sufficiently smooth.



$$\sup_{n,\mu,\nu} \left|\check{F}_{\mu\nu}(n)\right| < \epsilon, \quad 0 < \epsilon < \frac{\pi}{3q}$$

- Field strength is

$$\check{F}_{\mu\nu}(n) = \frac{1}{iq} \ln \left[U(n,\mu)U(n+\hat{\mu},\nu)U(n+\hat{\nu},\mu)^{-1}U(n,\nu)^{-1}\right]^q$$

※ q is needed for the invariance under the $\mathbb{Z}_q$ one-form transformation.

$\mathbb{Z}_q$ One-form Global Symmetry on the Lattice

- The factor of fractionality ω_μ is related to the $\mathbb{Z}_q$ one-form transform.

➤ Link variable

$$U(n,\mu) \to \exp\left(\frac{2\pi i}{q} z_\mu\right) U(n,\mu) \qquad n_\mu = 0$$

$\in \mathbb{Z}_q$

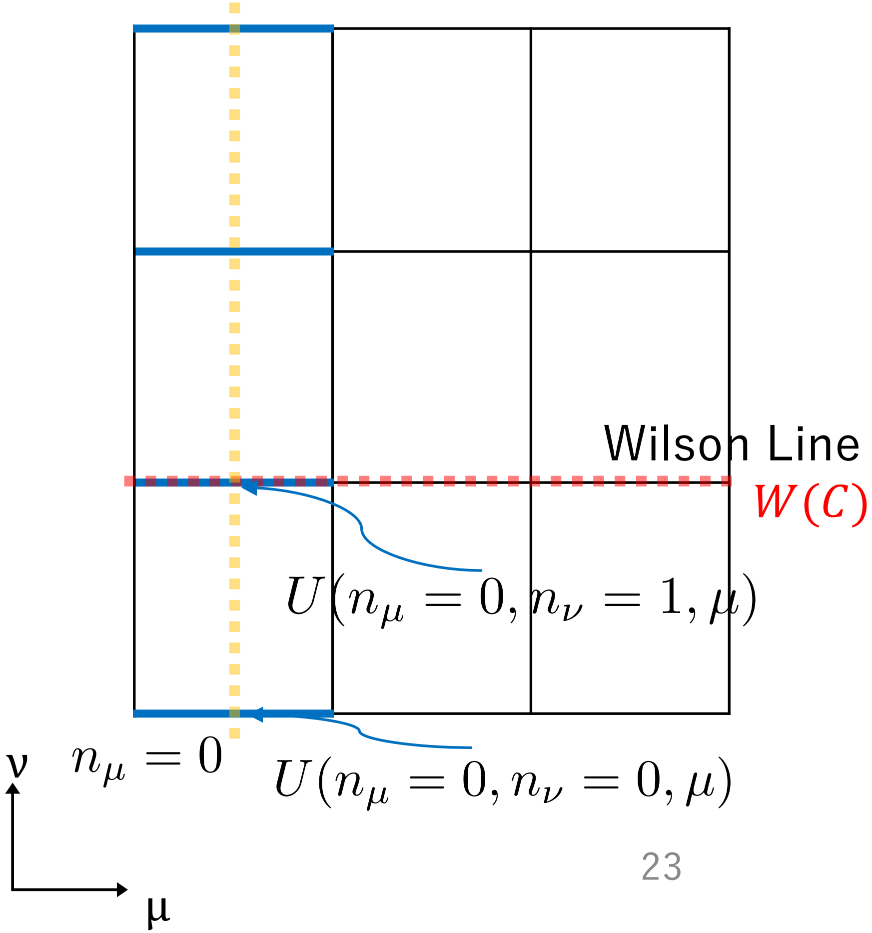
➤ Transition function

$$\check{v}_{n,\mu}(x) \to \begin{cases} \exp\left(\frac{2\pi i}{q} z_\mu\right) \check{v}_{n,\mu}(x) & \text{for } x_\mu = 1 \\ \check{v}_{n,\mu}(x) & \text{otherwise} \end{cases}$$

➤ Cocycle condition

Not $\mathbb{Z}_q$ “Relax”

$$\check{v}_{n-\hat{\nu},\mu}(x)\check{v}_{n,\nu}(x)\check{v}^{-1}_{n,\mu}(x)\check{v}^{-1}_{n-\hat{\mu},\nu}(x) = 1$$



$\mathbb{Z}_q$ One-form Gauge Symmetry on the Lattice

- The factor of fractionality ω_μ is related to the $\mathbb{Z}_q$ one-form transform.

➤ Link variable

$$U(n,\mu) \to \exp\left[\frac{2\pi i}{q} z_\mu(n)\right] U(n,\mu)$$

$\in \mathbb{Z}_q$

➤ Transition function

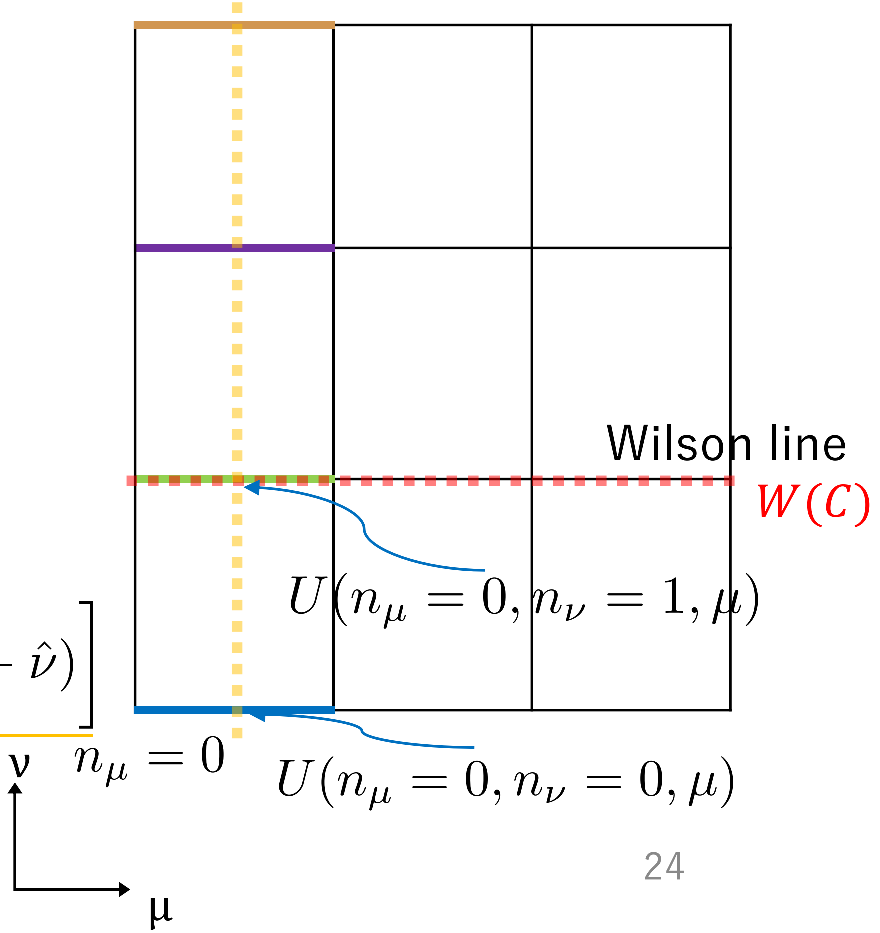
$$v_{n,\mu}(x) \to \exp\left[\frac{2\pi i}{q} z_\mu(n-\hat{\mu})\right] v_{n,\mu}(x) \qquad x \in f(n,\mu)$$

➤ Cocycle condition

$\mathbb{Z}_q$ "relax"

$$v_{n-\hat{\nu},\mu}(x) v_{n,\nu}(x) v_{n,\mu}(x)^{-1} v_{n-\hat{\mu},\nu}(x)^{-1} \equiv \exp\left[\frac{2\pi i}{q} z_{\mu\nu}(n-\hat{\mu}-\hat{\nu})\right]$$

$\in \mathbb{Z}_q$



Fractional Topological Charge on the Lattice

New Transition Function

$$v_{n,\mu}(x) = \omega_\mu(x)\check{v}_{n,\mu}(x)$$

- In the continuum,

$$Q = \frac{1}{32\pi^2}\int_{T^4} \mathrm{d}^4x\, \varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x)$$

- Topological charge is calculated by the transition function,

$$Q = -\frac{1}{8\pi^2}\sum_{n}\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma}\int_{p(n,\mu,\nu)} \mathrm{d}^2x\,[v_{n,\mu}(x)\partial_\rho v_{n,\mu}(x)^{-1}][v_{n-\hat{\mu},\nu}(x)^{-1}\partial_\sigma v_{n-\hat{\mu},\nu}(x)]$$

Factor of Fractionality

$$\omega_\mu(x) \sim \exp\left(\frac{\pi i}{q}\sum_{\nu\neq\mu}\frac{z_{\mu\nu}x_\nu}{L}\right)$$

- For the new transition function $v_{n,\mu}$,

$$Q = \frac{1}{8q^2}\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma} + \frac{1}{8\pi q}\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu}\sum_{n_\mu=0}\check{F}_{\rho\sigma}(n)$$

$$+ \frac{1}{32\pi^2}\sum_{n}\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma}\check{F}_{\mu\nu}(n)\check{F}_{\rho\sigma}(n+\hat{\mu}+\hat{\nu})$$

Fractional!!

Cross Term

Integer

Mixed 't Hooft Anomaly

- A lattice action is

$$S \equiv \frac{1}{4g_0^2} \sum_n \sum_{\mu,\nu} \check{F}_{\mu\nu}(n) \check{F}_{\mu\nu}(n) + S_{\mathrm{matter}} - i q \theta Q$$

By the Witten Effect
Honda, Tanizaki, arXiv:2009.10183

➢ Topological charge is

$$qQ = \frac{1}{8q} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma} + \mathbb{Z}$$

✓ invariant under the $\mathbb{Z}_q$ one-form gauge transformation

✓ odd under the lattice $\mathcal{T}$-transformation: $qQ \xrightarrow{\mathcal{T}} -qQ$

➢ We discussed the mixed 't Hooft anomaly between the $\mathbb{Z}_q$ one-form gauge symmetry and the $\mathcal{T}$-symmetry when $\theta = \pi$.

Mixed 't Hooft Anomaly

- e^{iS} is ,under the $\mathcal{T}$-transformation, when $\theta = \pi$,

$$e^{i\pi qQ} \overset{\mathcal{T}}{\to} e^{-i\pi qQ} = e^{-2\pi iqQ} \cdot e^{i\pi qQ}$$

$$= \exp\left(-\frac{2\pi i}{8q} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma} \right) e^{i\pi qQ}$$

➢ Introducing a local counter term which is invariant under the $\mathbb{Z}_q$ one-form gauge transformation,

$$e^{-S_{\mathrm{counter}}} \equiv \exp\left(\frac{2\pi ik}{4q} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma} \right)$$

Mixed 't Hooft Anomaly

- Including this local counter term, e^{iS} is ,under the $\mathcal{T}$-transformation, when $\theta = \pi$,

$$e^{i\pi qQ}e^{-S_{\mathrm{counter}}} \xrightarrow{\mathcal{T}} e^{-i\pi qQ}e^{+S_{\mathrm{counter}}} = e^{-2i\pi qQ}e^{2S_{\mathrm{counter}}}e^{i\pi qQ}e^{-S_{\mathrm{counter}}}$$

$$= \exp\left[-\frac{2\pi i(4k+1)}{8q}\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma}z_{\mu\nu}z_{\rho\sigma}\right]e^{i\pi qQ}e^{-S_{\mathrm{counter}}}$$

$0, \pm 8, \pm 16, \cdots$

- The anomaly is canceled for $4k+1 = 0 \mod q$.
- This is
  - impossible for $q \in 2\mathbb{Z}$.
  - possible for $q \in 2\mathbb{Z}+1$.
- This implies the mixed 't Hooft anomaly between the $\mathbb{Z}_q$ one-form gauge symmetry and the $\mathcal{T}$-symmetry for $q \in 2\mathbb{Z}$ when $\theta = \pi$.

Conclusion and Future Work

- Conclusion

➢ We formulated the fractional topological charge in the lattice $U(1)$ gauge theory.

➢ Our construction provides the mixed 't Hooft anomaly between the $\mathbb{Z}_q$ one-form gauge symmetry and the $\mathcal{T}$-symmetry for $q \in 2\mathbb{Z}$ when $\theta = \pi$.

- Future works

➢ The formulation of the fractional topological charge in the lattice $SU(N)$ gauge theory

➢ The formulation of the Witten effect on the lattice

Back Up

’t Hooft Anomaly Matching Condition

➢ Application example

✓ Restricting the low-energy effective theory of QCD, this condition requires lagrangian to have the Wess-Zumino-Witten term.

✓ Since a part of the background gauge field exists as the gauge field in Electro-Weak gauge theory, ’t Hooft anomaly can be observed in the collapse of neutral π meson. To match the experiment with this theory, the strong field theory is detected to $SU(3)$ gauge theory.

Gaiotto, Kapustin, Seiberg, Willett, arXiv:1412.5148

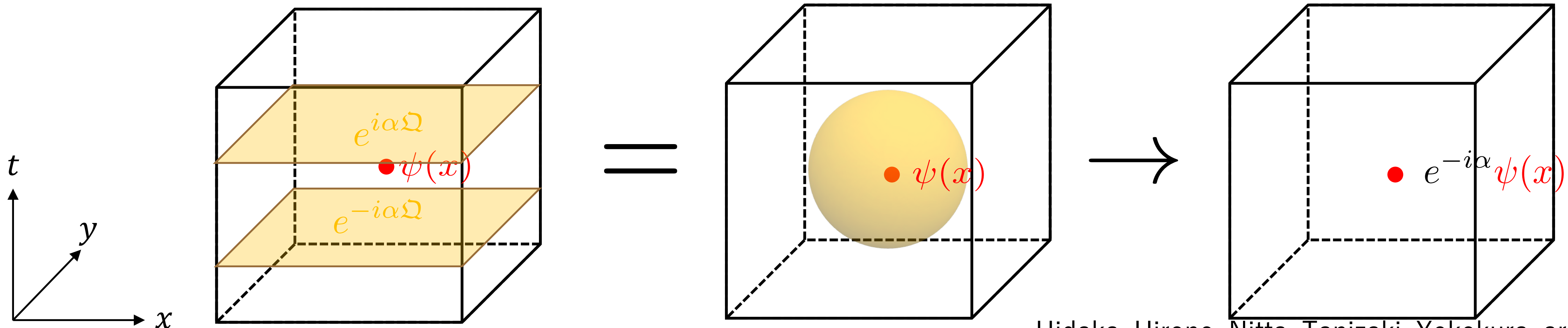
Higher Form Symmetry

- Reconsider an ordinary symmetry (zero-form symmetry) by the expansion of the space.

➢ In (2+1) dimension
By the transformation of field $\psi(x)$, the charge $\mathfrak{Q}$ has the 2-dimmension expansion.

graphical

$$e^{i\alpha\mathfrak{Q}}\psi(x)e^{-i\alpha\mathfrak{Q}} = e^{-i\alpha}\psi(x), \quad \mathfrak{Q} = \int_{M^2} \mathrm{d}^2x\, j^0(x), \quad j^\mu(x) = i\bar{\psi}(x)\gamma^\mu\psi(x)$$



Hidaka, Hirono, Nitta, Tanizaki, Yokokura, arXiv:1903.06389

Higher Form Symmetry

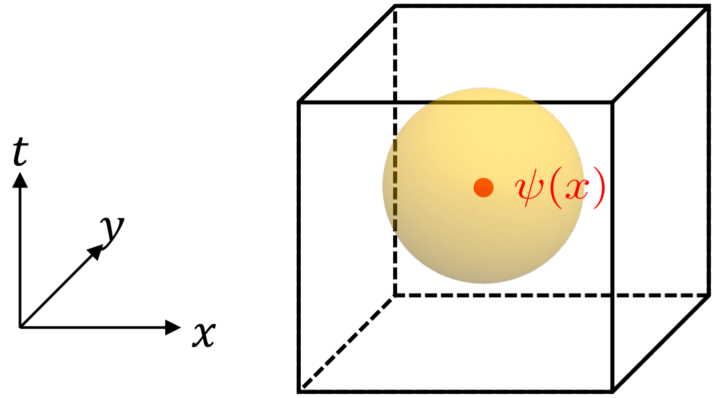
- Extend the object to it with the higher expansion

➢In (2+1) dimension

- zero-form symmetry

✓Point $(0d)$ $\psi(x)$ is changed.

✓Surrounded by the face $(2d)$

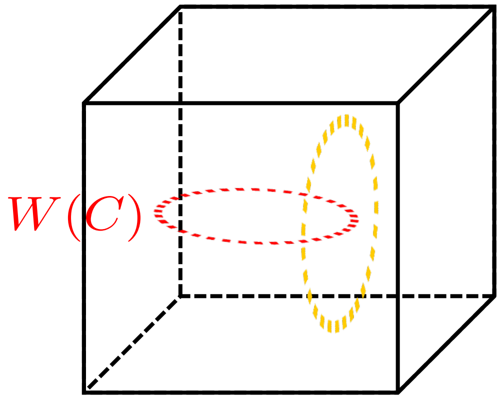


Extend

- one-form symmetry

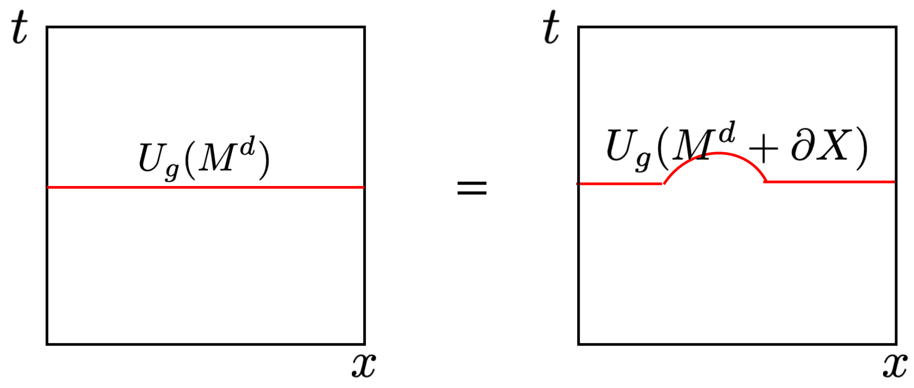
✓loop $(1d)$ $W(C)$ is changed.

✓ Linked by the line $(1d)$



Symmetry Operator's Topological Invariance

- Infinitisimal transformation of M^d,



$$\delta Q = \int_{M^d+\delta M^d} j - \int_{M^d} j = \int_{\partial X^d} j = \int_{X^d} \mathrm{d}j = 0$$

$\mathbb{Z}_N$ One-form Gauge Symmetry

- An example of higher form symmetries, $\mathbb{Z}_N$ one-form gauge symmetry, is not familiar.

➢ Rough method of making $\mathbb{Z}_N$ one-form gauge symmetry

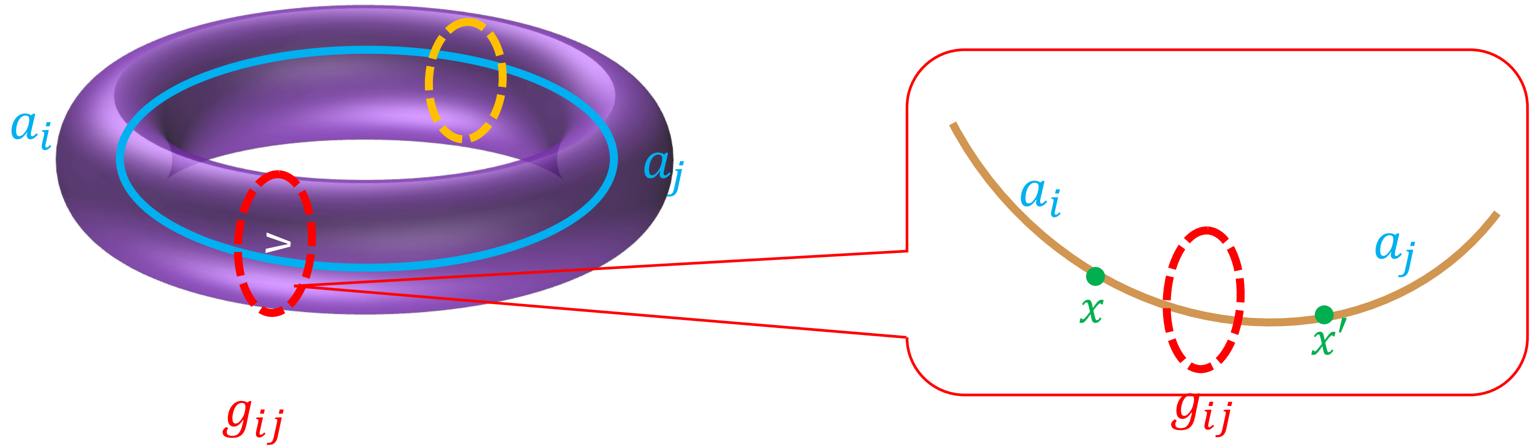
✓ Consider $\mathbb{Z}_N$ zero-form gauge symmetry

✓ Raise the rank of the derivative

✓ Consider $\mathbb{Z}_N$ one-form gauge symmetry

Wilson Loop and Transition Function

- Divided the torus into two part, $g_{ji} = 1$



$$W(C) = e^{i\int_{y'}^{x'} a_j} e^{i\int_{y'}^{y} a_j} e^{i\int_{y}^{x} a_i} e^{i\int_{x}^{x'} a_j}$$

$$\xrightarrow[x \to x',\, y \to y']{} g_{ji} e^{i\int_{x}^{y} a_i} g_{ij} e^{i\int_{y}^{x} a_i}$$

$$= g_{ij} e^{i\int_{C} a_i}$$

Integration of $B^{(2)}$ in $2d$ Manifold

- Surface operator: $U_k(\Sigma) = \exp\left(ik\int_\Sigma B_{\mu\nu}\right)$

➢ To be invariant under the gauge transformation: $B_{\mu\nu} \mapsto B_{\mu\nu} + \partial_{[\mu}\lambda_{\nu]}$,k should be quantized.

➢ By the constraint: $NB_{\mu\nu} = \partial_{[\mu}C_{\nu]}$,

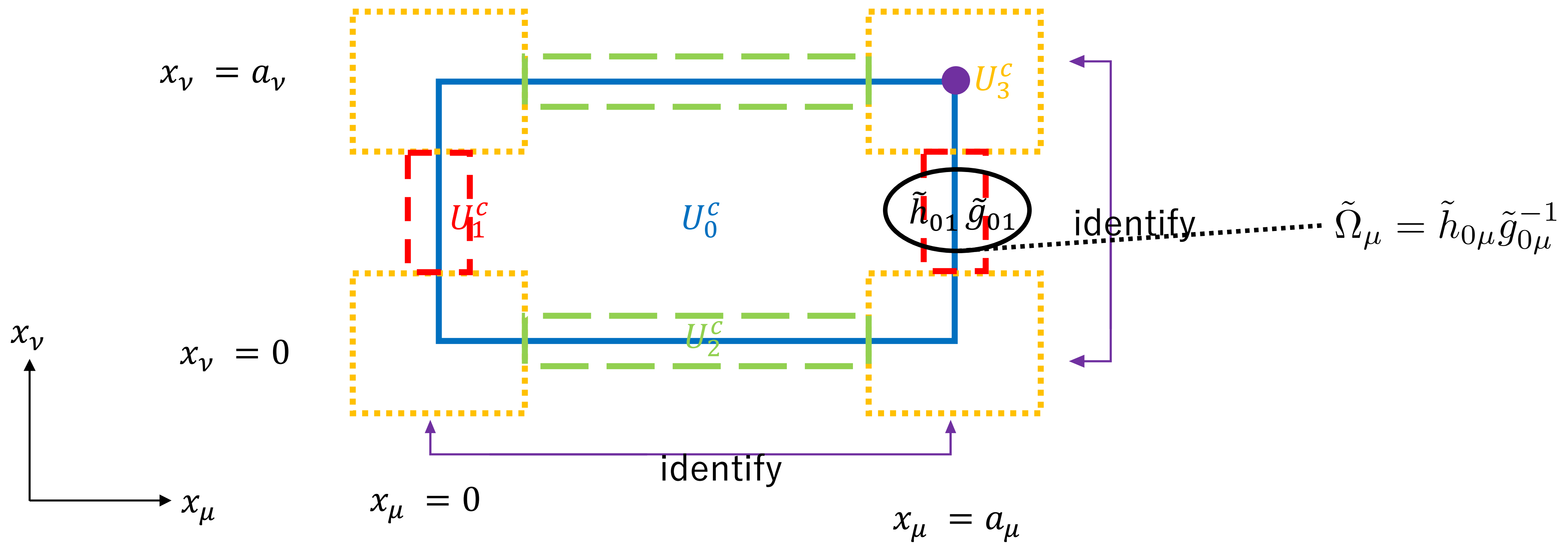
$$\exp\left(ik\int_\Sigma B^{(2)}\right) = \exp\left(\frac{ik}{N}\int_\Sigma \mathrm{d}B^{(1)}\right) = \exp\left(\frac{ik}{N}\cdot 2\pi\mathbb{Z}\right)$$

➢ The integration of $B^{(2)}$ in $2d$ manifold is

$$\int_\Sigma B^{(2)} \in \frac{2\pi}{N}\mathbb{Z}$$

Transition Function in $SU(N)$ Gauge Theory

- Transition function is defined in nontrivial patches.
- In $2d$, the manifold T^2 is divided by four patches



Transition Function in $SU(N)$ Gauge Theory

- By the transition function $\tilde{\Omega}_\mu$, the cocycle condition is

$$\tilde{\Omega}_\mu(x_\nu = a_\nu)\tilde{\Omega}_\nu(x_\mu = 0)\tilde{\Omega}_\mu^{-1}(x_\nu = 0)\tilde{\Omega}_\nu^{-1}(x_\mu = a_\mu) = 1$$

- To consider the fractional topological charge, we redefine the transition function Ω_μ. (Making $SU(N)/\mathbb{Z}_N$ bundle)

$$\Omega_\mu = \tilde{h}_{0\mu}\omega_\mu \tilde{g}_{0\mu}^{-1}$$

factor of fractionality

$$\omega_\mu = \exp\left(\frac{\pi i}{N}\sum_\nu \frac{n_{\mu\nu}x_\nu}{a_\nu}T_1\right)$$

$SU(N)$'s generator

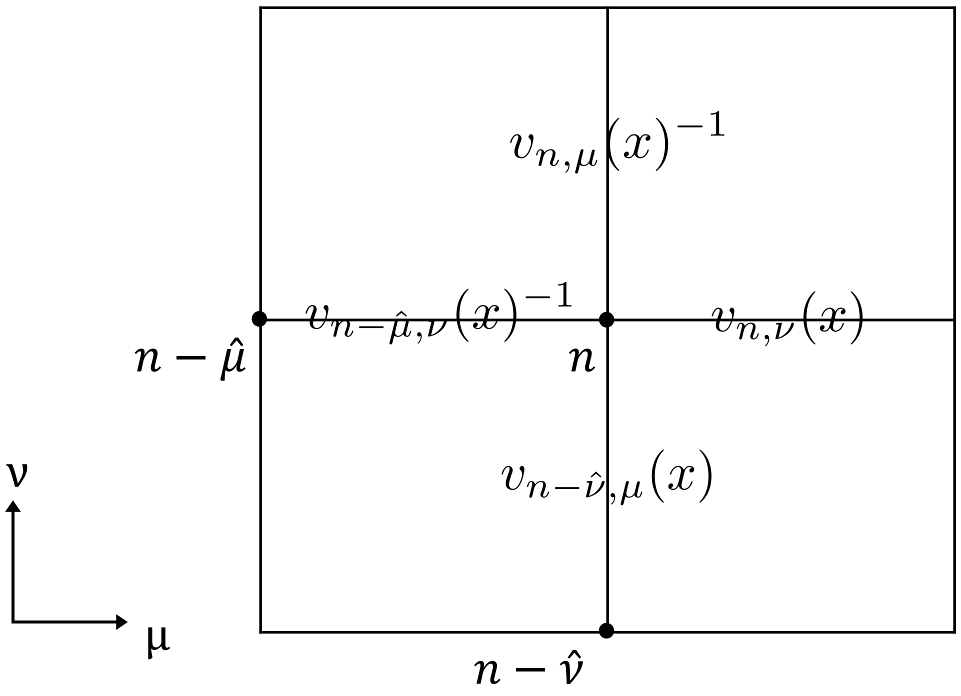
➢ The cocycle condition is relaxed,

$$\Omega_\mu(x_\nu = a_\nu)\Omega_\nu(x_\mu = 0)\Omega_\mu^{-1}(x_\nu = 0)\Omega_\nu^{-1}(x_\mu = a_\mu) = \exp\left(\frac{2\pi i}{N}n_{\mu\nu}\right)$$

Cocycle Condition on the Lattice

$$\text{(new transition function)} \sim \omega_\mu \times \text{(normal transition function)}$$

- By the new transition function, the cocycle condition is



ordinary

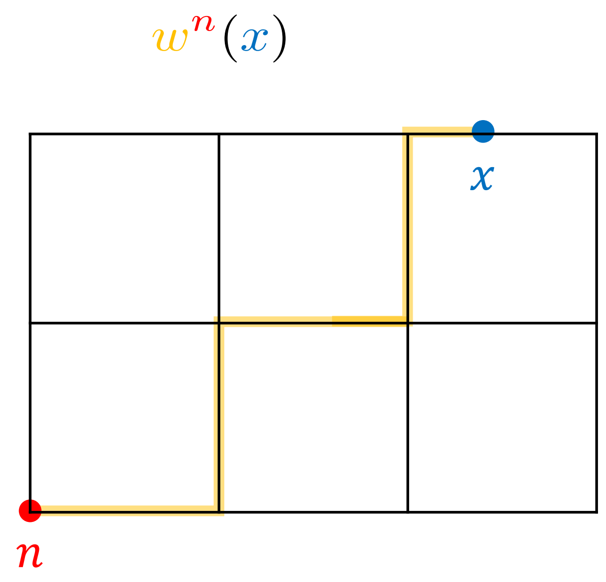
$$\check{v}_{n-\hat{\mu},\nu}(x)\check{v}_{n,\mu}(x)\check{v}_{n,\nu}(x)^{-1}\check{v}_{n-\hat{\nu},\mu}(x)^{-1} = 1$$

new

$$v_{n-\hat{\mu},\nu}(x)v_{n,\mu}(x)v_{n,\nu}(x)^{-1}v_{n-\hat{\nu},\mu}(x)^{-1} = \exp\left(\frac{2\pi i}{N} z_{\mu\nu}\right)$$

Parallel Transport Function

- Parallel transport function's image is "by the interpolate parameter y, the transition function is defined as the function on an arbitrarily point x on the link".



Link Variables

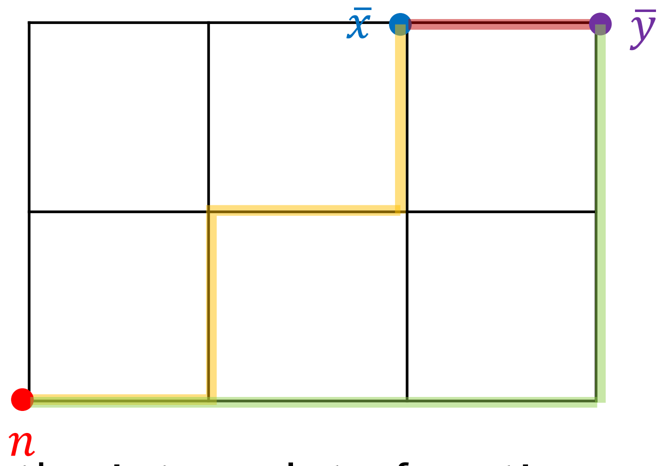
- In $SU(N)$ gauge field, this process is very complicated.

➢ By the parallel transport function, we defined the new link variable.

$$u_{xy}^n = w^n(\bar{x})U(\bar{x},\mu)w^n(\bar{y})^{-1} \qquad (\bar{y} = n + \hat{\mu})$$

$$u_{xy}^n = (u_{xy}^n)^{-1} \qquad (\bar{y} = n - \hat{\mu})$$

Image of u_{xy}^n



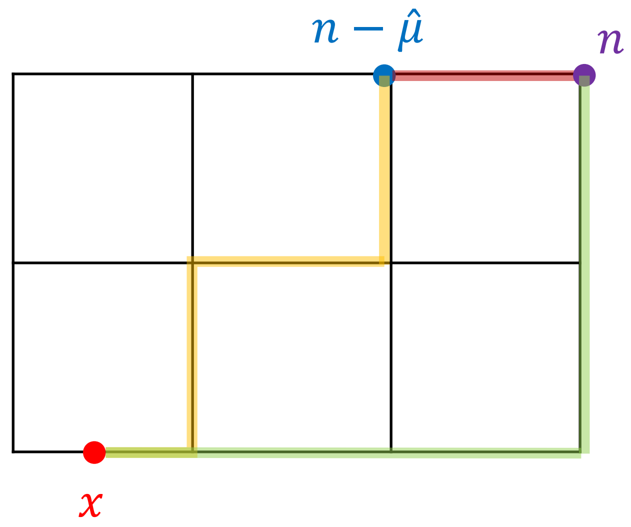
➢ By this link variable, we define the interpolate function.

Transition Function

- By the interpolate function made from the new link variable, we define the transition function as continuum function on the lattice .

$$\tilde{v}_{n,\mu}(x) = S_{n,\mu}^{n-\hat{\mu}}(x)^{-1}\tilde{v}_{n,\mu}(n)S_{n,\mu}^{n}(x)$$

Image of $\tilde{v}_{n,\mu}$



Cocycle Condition

- Check the cocycle condition by this new transition function

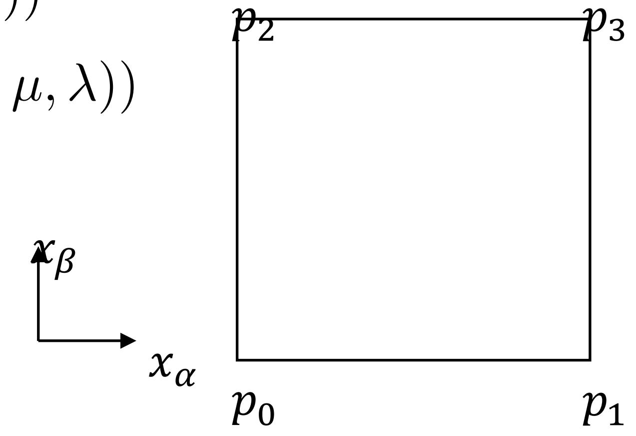
➢ In $x \in p(n, \mu, \nu)$, we define new function,

$$P^m_{n,\mu\nu}(x_\alpha, x_\beta) = (u^m_{p_0p_2})^{y_\beta}\left[(u^m_{p_2p_0})^{y_\beta}\left(u^m_{p_0p_2}u^m_{p_2p_3}u^m_{p_3p_1}u^m_{p_1p_0}\right)^{y_\beta}u^m_{p_0p_1}(u^m_{p_1p_3})^{y_\beta}\right]^{y_\alpha}$$

➢ The relation with $S^m_{n,\mu}(x)$ is

$$S^m_{n,\mu}(x) = P^m_{n,\mu\lambda}(x) \qquad (x \in p(n,\mu,\lambda))$$

$$S^m_{n,\mu}(x) = R^m_{n,\mu;\lambda}P^m_{n+\hat{\lambda},\mu\lambda}(x) \qquad (x \in p(n+\hat{\lambda},\mu,\lambda))$$



Cocycle Condition

➢ R^m is

$$
\begin{aligned}
R^m_{n,\mu;\alpha}(x_\beta, x_\gamma) =& \left[\left(u^m_{03}u^m_{37}u^m_{72}u^m_{20}\right)^{y_\gamma} u^m_{02} \right. \\
& \cdot \left(u^m_{27}u^m_{74}u^m_{46}u^m_{62}\right)^{y_\gamma} u^m_{26}u^m_{61} \left(u^m_{16}u^m_{64}u^m_{45}u^m_{51}\right)^{y_\gamma} \\
& \left. \cdot\ u^m_{10} \left(u^m_{01}u^m_{15}u^m_{53}u^m_{30}\right)^{y_\gamma} \right]^{y_\beta} \left(u^m_{03}u^m_{35}u^m_{51}u^m_{10}\right)^{y_\gamma} u^m_{01} \\
R^m_{n,\mu;\beta}(x_\alpha, x_\gamma) =& \left(u^m_{03}u^m_{37}u^m_{72}u^m_{20}\right)^{y_\gamma} u^m_{02} \\
R^m_{n,\mu;\gamma}(x_\alpha, x_\beta) =& u^m_{03}
\end{aligned}
$$

Cocycle Condition

➢ By the new interpolate function, in $x \in p(n, \mu, \nu)$, the cocycle condition is

$$\begin{aligned}\tilde{v}_{n-\hat{\mu},\nu}(x)\tilde{v}_{n,\mu}(x) &= \left(P^{n-\hat{\nu}-\hat{\mu}}_{n,\mu\nu}(x)^{-1} v_{n-\hat{\mu},\nu}(n) P^{n-\hat{\mu}}_{n,\mu\nu}(x)\right)\left(P^{n-\hat{\mu}}_{n,\mu\nu}(x)^{-1} v_{n,\nu}(n) P^{n}_{n,\mu\nu}(x)\right)\\ &= P^{n-\hat{\nu}-\hat{\mu}}_{n,\mu\nu}(x)^{-1} v_{n-\hat{\mu},\nu}(n) v_{n,\nu}(n) P^{n}_{n,\mu\nu}(x)\end{aligned}$$

$$\tilde{v}_{n-\hat{\mu},\nu}(x)\tilde{v}_{n,\mu}(x) = P^{n-\hat{\nu}-\hat{\mu}}_{n,\mu\nu}(x)^{-1} v_{n-\hat{\nu},\mu}(n) v_{n,\mu}(n) P^{n}_{n,\mu\nu}(x)$$

➢ When (cocycle condition)=1 is satisfied at each site,

$$\tilde{v}_{n-\hat{\mu},\nu}(x)\tilde{v}_{n,\mu}(x)\tilde{v}_{n,\nu}(x)^{-1}\tilde{v}_{n-\hat{\mu},\nu}(x)^{-1} = 1$$

Topological Charge

- By the new transition function, the topological charge is

$$
\begin{aligned}
P(\tilde{v}_{n,\mu}) =& \frac{1}{24\pi^2} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \Bigg\{ 3 \int_{p(n+\hat{\mu}+\hat{\nu},\mu,\nu)} \mathrm{d}^2 x \, \mathrm{Tr}\left[P^n_{n+\hat{\mu}+\hat{\nu},\mu\nu} \partial_\rho (P^n_{n+\hat{\mu}+\hat{\nu},\mu\nu})^{-1} (R^n_{n+\hat{\mu},\mu;\nu})^{-1} \partial_\sigma R^n_{n+\hat{\mu},\mu;\nu} \right] \\
& - 3 \int_{p(n+\hat{\nu},\mu,\nu)} \mathrm{d}^2 x \, \mathrm{Tr}\left[P^n_{n+\hat{\nu},\mu\nu} \partial_\rho (P^n_{n+\hat{\nu},\mu\nu})^{-1} (R^n_{n,\mu;\nu})^{-1} \partial_\sigma R^n_{n,\mu;\nu} \right] \\
& - \int_{f(n+\hat{\mu},\mu)} \mathrm{d}^3 x \, \mathrm{Tr}\left[S^n_{n+\hat{\mu},\mu} \partial_\nu (S^n_{n+\hat{\mu},\mu})^{-1} S^n_{n+\hat{\mu},\mu} \partial_\rho (S^n_{n+\hat{\mu},\mu})^{-1} S^n_{n+\hat{\mu},\mu} \partial_\sigma (S^n_{n+\hat{\mu},\mu})^{-1} \right] \\
& + \int_{f(n,\mu)} \mathrm{d}^3 x \, \mathrm{Tr}\left[S^n_{n,\mu} \partial_\nu (S^n_{n,\mu})^{-1} S^n_{n,\mu} \partial_\rho (S^n_{n,\mu})^{-1} S^n_{n,\mu} \partial_\sigma (S^n_{n,\mu})^{-1} \right] \Bigg\}
\end{aligned}
$$

Topological Charge in the $SU(N)$ Gauge Theory

- By the new transition function, we calculate topological charge $Q(v_{n,\mu})$.
- In $4d$ continuum theory, (van Baal, Commun. Math. Phys. 85 (1982))

$$Q(v_{n,\mu}) = \frac{1}{24\pi^2} \sum_{\mu} \int \mathrm{d}_3\sigma_{\mu} \varepsilon_{\mu\nu\alpha\beta} \,\mathrm{Tr}\Big((v_{n,\mu}\partial_{\nu} v_{n,\mu}^{-1})(v_{n,\mu}\partial_{\alpha} v_{n,\mu}^{-1})(v_{n,\mu}\partial_{\beta} v_{n,\mu}^{-1}) \Big)$$

$$+ \frac{1}{8\pi^2} \sum_{\mu,\nu} \int \mathrm{d}_2 S_{\mu\nu} \varepsilon_{\mu\nu\alpha\beta} \,\mathrm{Tr}\Big((v_{n,\nu}^{-1}\partial_{\alpha} v_{n,\nu})_{x_\mu = a_\mu} (v_{n,\mu}\partial_{\beta} v_{n,\mu}^{-1})_{x_\nu = 0} \Big)$$

$$= \mathbb{Z} + \frac{N-1}{N} \cdot \frac{1}{8} \varepsilon_{\mu\nu\alpha\beta} z_{\mu\nu} z_{\alpha\beta}$$

integer

fractional

Differential Calculus on the Lattice

- k-form function: $f(n) \equiv \frac{1}{k!} \sum_{\mu_1,\cdots,\mu_k} f_{\mu_1\cdots\mu_k}(n)\, \mathrm{d}x_{\mu_1} \cdots \mathrm{d}x_{\mu_k}$
- The definition of extra derivative: $\mathrm{d}x_\mu\, f_{\mu_1\cdots\mu_k}(n) = f_{\mu_1\cdots\mu_k}(n+\hat{\mu})\, \mathrm{d}x_\mu$

➢ By this extra derivative on the lattice, the Leibniz rule on the lattice is

$$\mathrm{d}[f(n)g(n)] = \mathrm{d}f(n) \cdot g(n) + (-1)^k f(n) \cdot \mathrm{d}g(n)$$

➢ Example:

$$f(n) = \frac{1}{2} \sum_{\mu,\nu} f_{\mu\nu}(n) dx_\mu dx_\nu \quad \Rightarrow \quad \begin{aligned} f(n)f(n) &= \frac{1}{4} \sum_{\mu,\nu,\rho,\sigma} f_{\mu\nu}(n) f_{\rho\sigma}(n+\hat{\mu}+\hat{\nu}) dx_\mu dx_\nu dx_\rho dx_\sigma \\ &= \frac{1}{4} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} f_{\mu\nu}(n) f_{\rho\sigma}(n+\hat{\mu}+\hat{\nu}) dx_1 dx_2 dx_3 dx_4 \end{aligned}$$

$\mathbb{Z}_q$ One-form Global Symmetry and Gauge Symmetry

- $\mathbb{Z}_q$ one-form symmetry is corresponding to multiplying the $\mathbb{Z}_q$ element by the transition function from the point of fiber bundle.

➢ Consider the transformation of the transition function on the lattice

➢ Firstly, consider the $\mathbb{Z}_q$ one-form global symmetry

Admissibility Condition

- Field strength is

$$\check{F}_{\mu\nu}(n) = \frac{1}{iq} \ln \left[U(n,\mu) U(n+\hat{\mu},\nu) U(n+\hat{\nu},\mu)^{-1} U(n,\nu)^{-1} \right]^q \quad |F_{\mu\nu}(n)| < \pi$$

➢ Invariant under the $\mathbb{Z}_q$ one-form gauge transformation

➢ We require the admissibility condition to the field strength,

$$\sup_{n,\mu,\nu} \left| \check{F}_{\mu\nu}(n) \right| < \epsilon, \quad 0 < \epsilon < \frac{\pi}{3q}$$

➢ Under this condition, the Bianchi identity is satisfied.

$$\sum_{\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \Delta_\nu \check{F}_{\rho\sigma}(n) = 0$$

Proof of Admissibility Condition

- Field strength is

$$\begin{aligned} F_{\mu\nu}(n) =& \frac{1}{iq} \ln \left[e^{i(a_\mu(n)+a_\nu(n)+\hat{\mu}-a_\mu(n+\hat{\nu})-a_\nu(n))} \right]^q \\ =& \frac{1}{iq} \left[i \left(a_\mu(n) + a_\nu(n) + \hat{\mu} - a_\mu(n+\hat{\nu}) - a_\nu(n) \right) \cdot q + 2\pi i N_{\mu\nu}(n) \right] \\ =& \Delta_\nu a_\mu(n) - \Delta_\mu a_\nu(n) + \frac{2\pi}{q} N_{\mu\nu}(n) \end{aligned}$$

➢ $N_{\mu\nu}$ is the function for taking $F_{\mu\nu}$ back to the range $[-\pi, \pi]$.

Proof of Admissibility Condition

- By the admissibility condition,

$$\sum_{\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \Delta_\nu F_{\mu\nu}(n) < 6\epsilon$$

➢ By definition,

$$\sum_{\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \Delta_\nu \left(\Delta_\rho a_\sigma(n) - \Delta_\rho a_\sigma(n) + \frac{2\pi}{q} N_{\rho\sigma}(n) \right) = \frac{2\pi}{q} \sum_{\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \Delta_\nu N_{\rho\sigma}(n)$$

➢ By $\varepsilon_{\mu\nu\rho\sigma} \Delta_\nu N_{\rho\sigma}(n) < 1$

$$0 < 6\epsilon < \frac{2\pi}{q} \quad \Rightarrow \quad 0 < \epsilon < \frac{\pi}{3q}$$

$\mathbb{Z}_q$ Two-form Gauge Field

- $\mathbb{Z}_q$ two-form gauge field is defined by

$$z_{\mu\nu}(n) = z_{\mu\nu}\delta_{n_\mu,L-1}\delta_{n_\nu,L-1} + \Delta_\mu z_\nu(n) - \Delta_\nu z_\mu(n) + qN_{\mu\nu}(n) \quad \in \mathbb{Z}$$

➢ To protect the antisymmetric value,

$$\begin{cases} 0 \leq z_{\mu\nu}(n) < q & \text{for } \mu < \nu, \\ z_{\mu\nu}(n) \equiv -z_{\nu\mu}(n) & \text{for } \mu > \nu \end{cases}$$

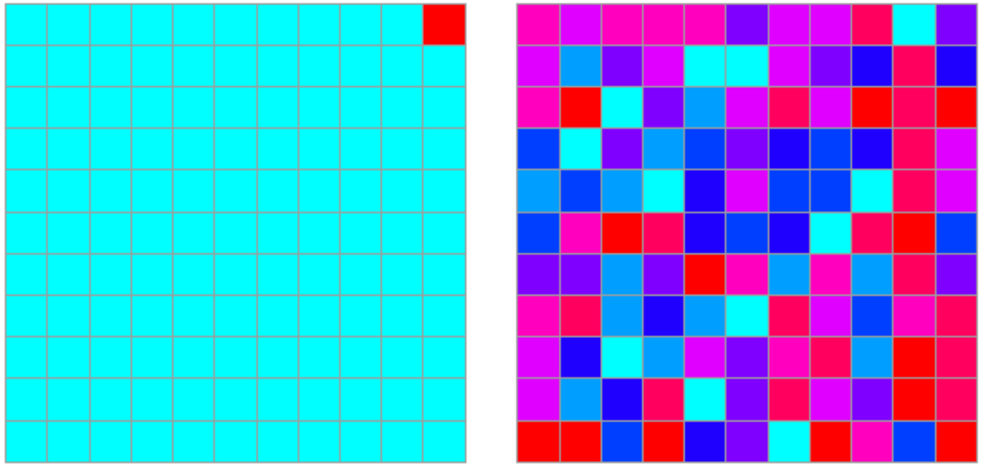
➢ Under the $\mathbb{Z}_q$ one-form gauge transformation, $\mathbb{Z}_q$ two-form field is

$$z_{\mu\nu}(n) \to z_{\mu\nu}(n) + \Delta_\mu z_\nu(n) - \Delta_\nu z_\mu(n) + qN_{\mu\nu}(n)$$

$\mathbb{Z}_q$ Two-form Gauge Field

- This $\mathbb{Z}_q$ two-form gauge field is connected to an arbitrary gauge configuration by the $\mathbb{Z}_q$ one-form gauge transformation.

$$z_{\mu\nu}(n) = z_{\mu\nu}\delta_{n_\mu, L-1}\delta_{n_\nu, L-1} + \Delta_\mu z_\nu(n) - \Delta_\nu z_\mu(n) + qN_{\mu\nu}(n) \quad \in \mathbb{Z}$$



Fractional Topological Charge by $\mathbb{Z}_q$ Two-form Gauge Field

$$Q = \frac{1}{32\pi^2} \sum_{n\in\Lambda} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \left[F_{\mu\nu}(n) + \frac{2\pi}{q} z_{\mu\nu}(n) \right] \left[F_{\rho\sigma}(n+\hat{\mu}+\hat{\nu}) + \frac{2\pi}{q} z_{\rho\sigma}(n+\hat{\mu}+\hat{\nu}) \right]$$

$$z_{\mu\nu}(n) = z_{\mu\nu}\delta_{n_\mu,L-1}\delta_{n_\nu,L-1} + \Delta_\mu z_\nu(n) - \Delta_\nu z_\mu(n) + qN_{\mu\nu}(n) \quad \in \mathbb{Z}$$

$$Q = \frac{1}{8q^2} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma} + \frac{1}{8\pi q} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} \sum_{n_\mu=0} \check{F}_{\rho\sigma}(n)$$
$$+ \frac{1}{32\pi^2} \sum_{n} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \check{F}_{\mu\nu}(n) \check{F}_{\rho\sigma}(n+\hat{\mu}+\hat{\nu})$$

Mixed 't Hooft Anomaly

- e^{iS} is ,under the $\mathcal{T}$-transformation,

$$e^{i\pi qQ} \xrightarrow{\mathcal{T}} e^{-i\pi qQ} = e^{-2\pi iqQ} \cdot e^{i\pi qQ}$$
$$= \exp\left(-\frac{2\pi i}{8q} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}\right) e^{i\pi qQ}$$

➢ Introducing a local counter term which is invariant under the $\mathbb{Z}_q$ one-form gauge transformation,

$$e^{-S_{\text{counter}}} \equiv \exp\left[\frac{2\pi ik}{4q} \sum_n \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu}(n) z_{\rho\sigma}(n+\hat{\mu}+\hat{\nu})\right]$$
$$= \exp\left(\frac{2\pi ik}{4q} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}\right)$$

Time Reversal Symmetry

$$U(n,\mu) \xrightarrow{\mathcal{T}} \begin{cases} U(\bar{n},\mu) & \text{for } \mu \neq 4, \\ U(\bar{n}-\hat{4},4)^{-1} & \text{for } \mu = 4, \end{cases}$$

$$\check{F}_{\mu\nu}(n) \xrightarrow{\mathcal{T}} \begin{cases} \check{F}_{\mu\nu}(\bar{n}) & \text{for } \mu \neq 4,\ \nu \neq 4, \\ -\check{F}_{4\nu}(\bar{n}-\hat{4}) & \text{for } \mu = 4, \\ -\check{F}_{\mu 4}(\bar{n}-\hat{4}) & \text{for } \nu = 4. \end{cases}$$

$$z_{\mu\nu}(n) \xrightarrow{\mathcal{T}} \begin{cases} z_{\mu\nu}(\bar{n}) & \text{for } \mu \neq 4,\ \nu \neq 4, \\ -z_{4\nu}(\bar{n}+\hat{4}) & \text{for } \mu = 4, \\ -z_{\mu 4}(\bar{n}+\hat{4}) & \text{for } \nu = 4, \end{cases}$$

$$z_{\mu\nu} \xrightarrow{\mathcal{T}} \begin{cases} z_{\mu\nu} & \text{for } \mu \neq 4,\ \nu \neq 4, \\ -z_{4\nu} & \text{for } \mu = 4, \\ -z_{\mu 4} & \text{for } \nu = 4. \end{cases}$$

Witten Effect

- Setting magnetic monopole with magnetic charge g, electric charge q is induced by θ term.

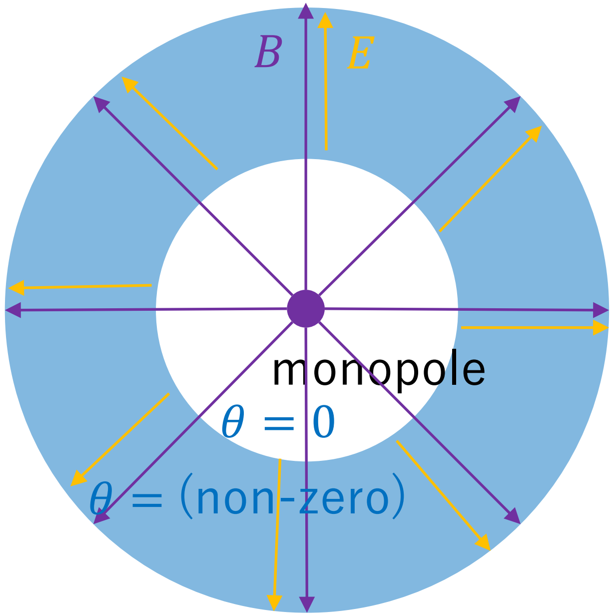
$$S = -\frac{1}{2g^2}\int \mathrm{tr}(f \wedge \star f) + \frac{\theta}{8\pi^2}\int \mathrm{tr}(f \wedge f)$$

➢ In the abelian gauge theory, EOM is

$$\partial_\mu F^{\mu\nu} = \frac{g^2}{4\pi^2}\varepsilon_{\mu\nu\rho\sigma}\partial_\mu\left(\theta\partial_\rho A_\sigma\right)$$

$$\nabla \cdot \mathbf{E} = -\frac{g^2}{4\pi^2\epsilon_0}\nabla\theta \cdot \mathbf{B}$$

ρ/ϵ_0



➢ Dirac quaternization is condition: $gq = \theta$