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Universal dynamics of heavy operators in BCFT2

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Based on *JHEP (2022) 156* , arXiv [hep-th 2022.01633] w/ I.Tsiares (Saclay)

This work is based on

Based on *JHEP (2022) 156* , arXiv [hep-th 2022.01633] with

Ioannis Tsiares (IPhT, Scalay)

also see *JHEP (2022) 156* , arXiv [hep-th 2112.10984] by Y.Kusuki (Caltech)

Introduction

Today I will talk about the universal formula for $c > 1$, unitary BCFT$_2$ without extended symmetry

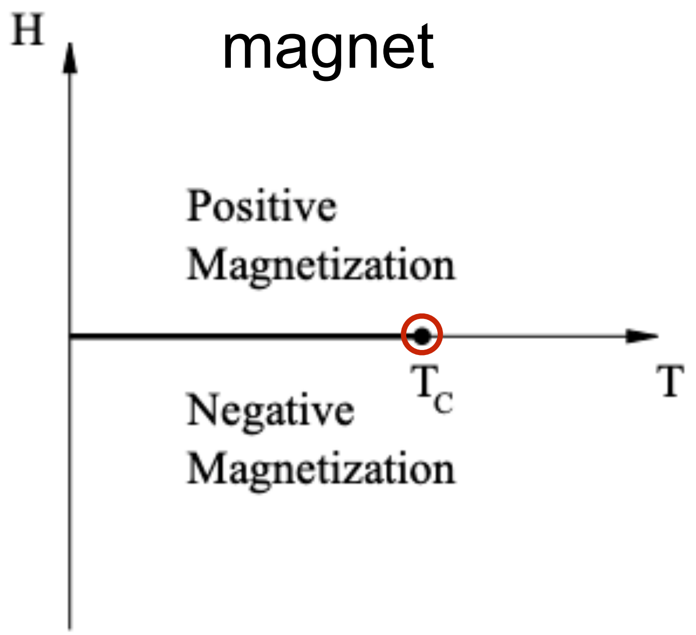
The basic technique is analytic conformal bootstraps those were recently developed.

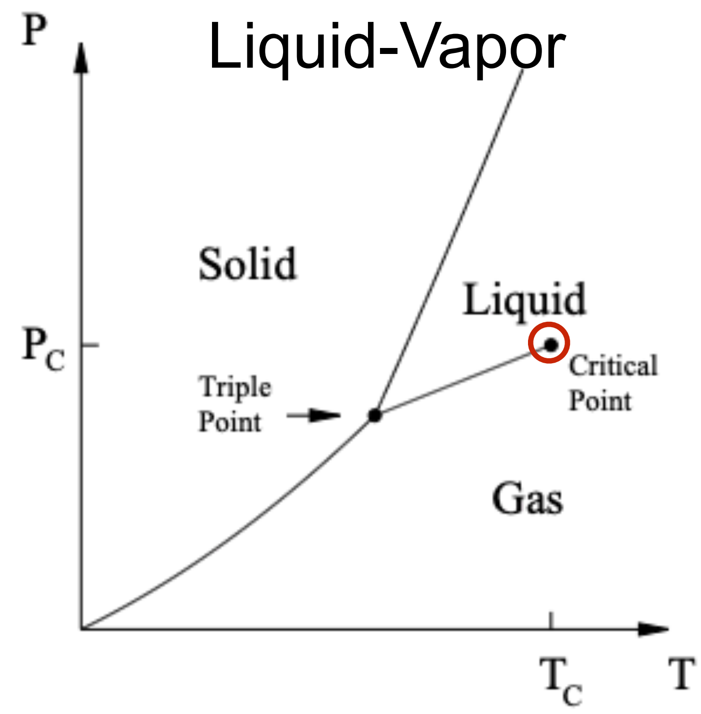
So I start with the motivation to conformal bootstrap approach first.

Cardy formula is a relatively well known example of the consequence of analytic bootstrap, so we review them with a modern point of view with recent techniques.

Phase diagrams

[pictures from Pelisseto-Vicari 02]





FLUID	MAGNET
density: $\rho - \rho_c$ chemical potential: $\mu - \mu_c$	magnetization M magnetic field H
$C_P = -T \left(\frac{\partial^2 \mathcal{F}}{\partial T^2} \right)_P$	$C_H = -T \left(\frac{\partial^2 \mathcal{F}}{\partial T^2} \right)_H$
$C_V = -T \left(\frac{\partial^2 \mathcal{A}}{\partial T^2} \right)_V$	$C_M = -T \left(\frac{\partial^2 \mathcal{A}}{\partial T^2} \right)_M$
$\kappa_T = \frac{1}{\rho} \left(\frac{\partial \rho}{\partial P} \right)_T = -\frac{1}{V} \left(\frac{\partial^2 \mathcal{F}}{\partial P^2} \right)_T$	$\chi = \left(\frac{\partial M}{\partial H} \right)_T = -\left(\frac{\partial^2 \mathcal{F}}{\partial H^2} \right)_T$

Universality: experimental results

Monte Carlo simulation of Ising model

[table from Pelisseto-Vicari 02]

Ref.	info	γ	ν	η	α	β	ω
[513] 1999	FSS ϕ^4	1.2366(15)*	0.6297(5)	0.0362(8)	0.1109(15)*	0.3262(4)*	0.845(10)
[162] 1999	FSS s-$\frac{1}{2}$ $nn,3n$	1.2372(13)*	0.63032(56)	0.0372(10)	0.1090(17)*	0.3269(5)*	0.82(3)
[512] 1999	FSS ϕ^4	1.2367(11)*	0.6296(7)	0.0358(9)	0.1112(21)*	0.3261(5)*	0.845(10)
[98] 1999	FSS s-$\frac{1}{2}$	1.2353(25)*	0.6294(10)	0.0374(12)	0.1118(30)*	0.3265(4)*	0.87(9)
[520] 1999	FSS ϕ^4	1.2366(11)*	0.6298(5)	0.0366(8)	0.1106(15)*	0.3264(4)*	
[296] 1999	FSS s-$\frac{1}{2}$			0.036(2)			
[519] 1998	FSS s-$\frac{1}{2}$		0.6308(10)*		0.1076(30)		
[161] 1995	FSS s-$\frac{1}{2}$ $nn,3n$, s-1	1.237(2)*	0.6301(8)	0.037(3)	0.110(2)*	0.3267(10)*	0.82(6)
[394] 1991	FSS s-$\frac{1}{2}$	1.239(7)*	0.6289(8)	0.030(11)	0.1133(24)*	0.3258(44)*	

Experiments

	Ref.	γ	ν	η	α	β
lv	[1035] 2000	1.14(5)	0.62(3)			
	[530] 1999				$0.1105^{+0.0250}_{-0.0270}$	
	[314] 1998			0.042(6)		
Liquid-Vapor	[680] 1995					0.341(2)
	[1] 1994				0.111(1)	0.324(2)
	[1029] 1993				0.1075(54)	
	[912] 1984	1.233(10)				0.327(2)
ms	[610] 2001	1.14(7)				0.34(2)
	[760] 1995				0.11(3)	
	[761] 1995				0.11(3)	
Magnet	[763] 1994				0.10(2)	
	[770] 1994					0.325(2)
	[974] 1993				0.11(3)	
	[1034] 1993	1.25(2)				0.315(15)
	[122] 1987	1.25(2)	0.64(1)			
	[121] 1983				0.110(5)	0.331(6)

Both agrees well !

Conformal bootstrap

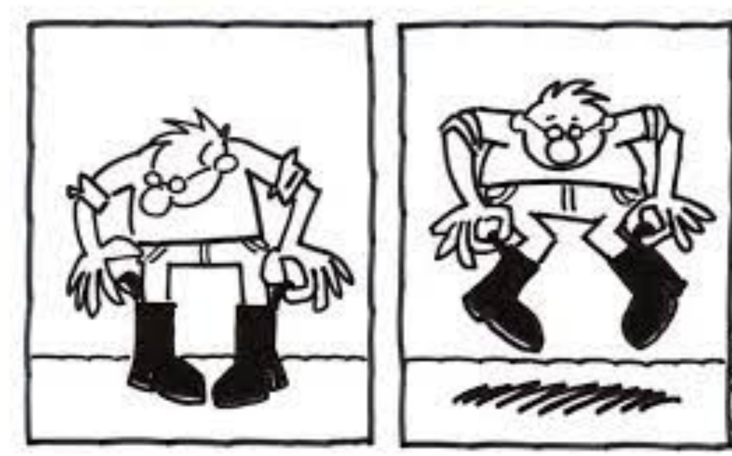
We can study the *lattice Ising model* to study critical exponents of *water*.

but we also wonder whether we can access the exponents themselves *directly* in a model independent way.

→ conformal bootstrap [Ferrara-Gatto-Grillo 73, Polyakov 74]

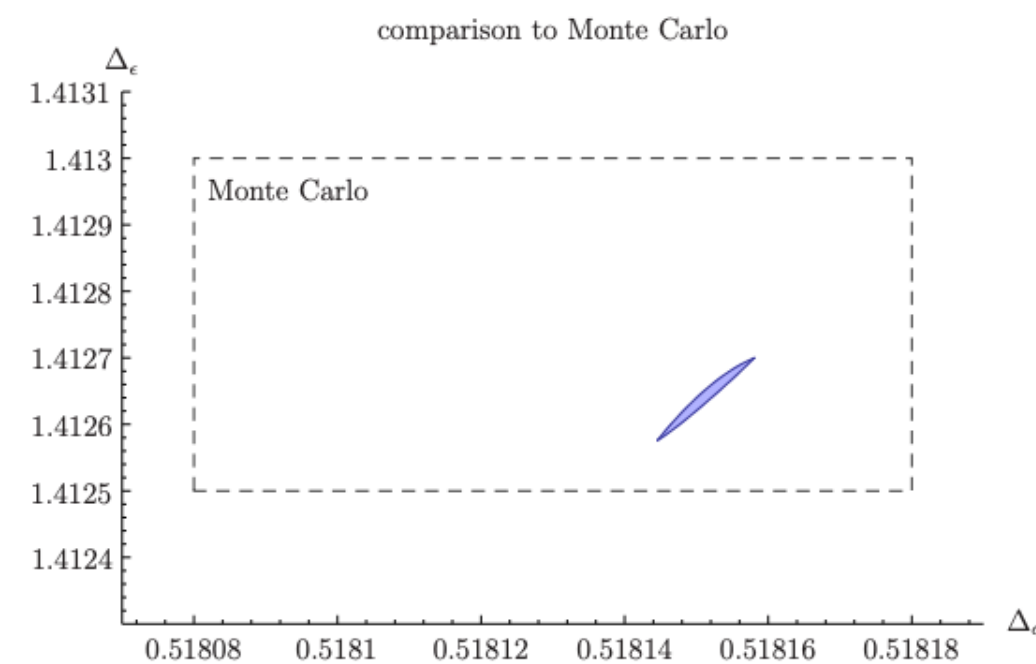
in particular, without relying Lagrangian, duality invariant etc…

even we don't start by writing Hamiltonians!



For example, by solving bootstrap equations in 2+1d $\mathbb{Z}_2$ symmetric CFT w/ 2 relevant operators (fine tunings), we get critical exponents !

[figure from D.Simons-Duffin 15]



Today, we talk on a universal consequence of *Virasoro symmetry*

Some basics of 1+1d CFT

global conformal symmetry = Poincare + Scale inv + special conformal = $SL(2,\mathbb{C})$

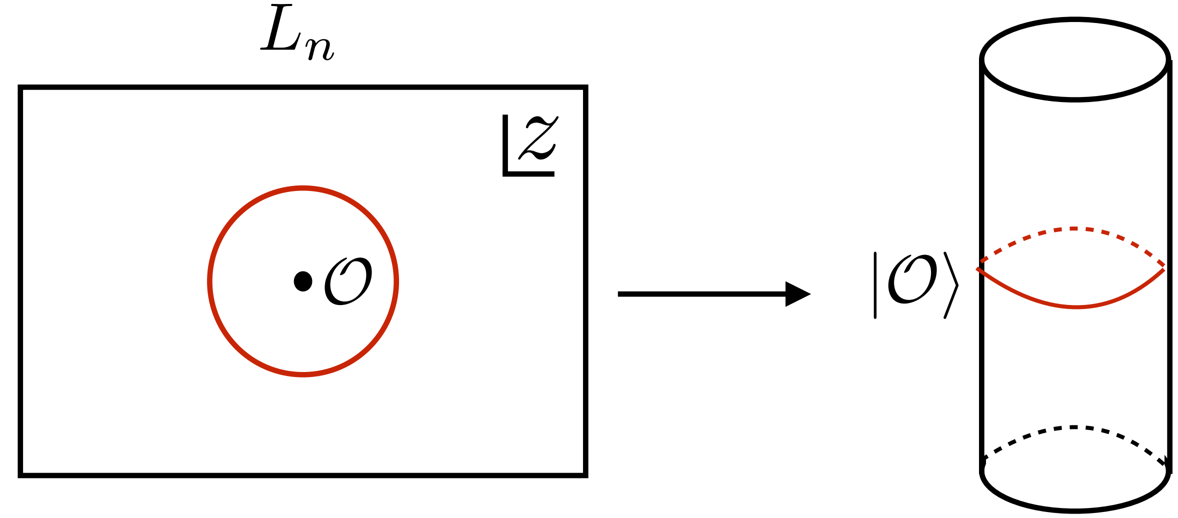
$$z \to z+\epsilon \quad z \to z+\epsilon z \quad z \to z+\epsilon z^2 \quad \epsilon \in \mathbb{C}$$

symmetry generators: L_{-1} L_0 L_1

In 1+1d it is extended to Virasoro symmetry $z \to z+\epsilon z^{n+1}$

L_n

- operator state correspondence



Via Weyl transformation,
we can map the cylinder to plane

States are mapped to local operators

Scaling dimension (+ spin) = eigenvalue of dilatation: $L_0 \left|\mathcal{O}\right\rangle = h_{\mathcal{O}} \left|\mathcal{O}\right\rangle$

- 2 and 3 pt: position dependence are determined by global conformal symmetry

$$\langle \mathcal{O}_i(z_1,\bar{z}_1)\mathcal{O}_j(z_2,\bar{z}_2)\rangle = \frac{\delta_{ij}}{(z_1-z_2)^{h_1+h_2}(\bar{z}_1-\bar{z}_2)^{\bar{h}_1+\bar{h}_2}}$$

$$\langle O_i(z_1,\bar{z}_1)O_j(z_2,\bar{z}_2)O_k(z_3,\bar{z}_3)\rangle = \frac{C_{ijk}}{z_{12}^{h_i+h_j-h_k} z_{23}^{h_j+h_k-h_i} z_{13}^{h_k+h_i-h_j} \bar{z}_{12}^{\bar{h}_i+\bar{h}_j-\bar{h}_k} \bar{z}_{23}^{\bar{h}_j+\bar{h}_k-\bar{h}_i} \bar{z}_{13}^{\bar{h}_k+\bar{h}_i-\bar{h}_j}}$$

A natural parametrization in Virasoro representation

To employ Virasoro symmetry, it is convenient to “Liouville “ parametrization:

$$c = 1 + 6Q^2$$

$$Q = b + b^{-1}$$

$$h = \left(\frac{Q}{2}\right)^2 + P^2 \quad = \alpha(Q - \alpha)$$

cf) parametrization in minimal models

$$c_{(p,q)} = 1 - 6\frac{(p-q)^2}{pq} = 1 + 6(i\sqrt{p/q} + (i\sqrt{p/q}))^2 \equiv 1 + 6Q_{(p,q)}^2$$

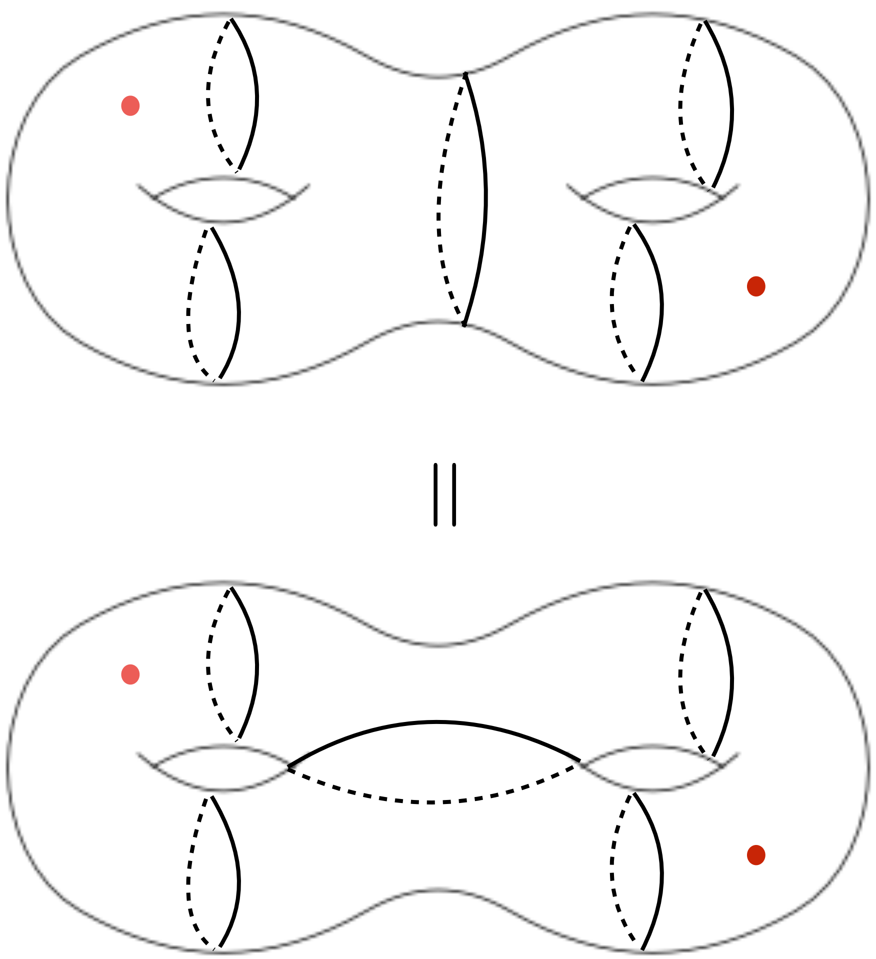
$$Q_{(p,q)} = b_{(p,q)} + b_{(p,q)}^{-1} \qquad b_{(p,q)} = i\sqrt{\frac{p}{q}}$$

$$h_{(m,n)} = \frac{(pm - qn)^2}{pq} + \frac{(p-q)^2}{4pq} = \frac{Q_{(p,q)}^2}{4} + (imb_{(p,q)} + inb_{(p,q)}^{-1})^2 = \frac{Q_{(p,q)}^2}{4} + P_{(m,n)}^2$$

basically the same parametrization

CFT Data

any Riemann surfaces are decomposed into "pants"



= inserting the resolution of identity $1 = \sum |O_i\rangle \langle O_i|$
we can represent the correlation function using spectrum and three point functions

$\rightarrow$ CFT is determined through

- the dimension of primary operators $(h_i, \bar{h}_i)$
- OPE coefficient C_{ijk} that determines the three pt function

$$\langle O_i(z_1, \bar{z}_1) O_j(z_2, \bar{z}_2) O_k(z_3, \bar{z}_3) \rangle$$

$$= \frac{C_{ijk}}{z_{12}^{h_i+h_j-h_k} z_{23}^{h_j+h_k-h_i} z_{13}^{h_k+h_i-h_j} \bar{z}_{12}^{\bar{h}_i+\bar{h}_j-\bar{h}_k} \bar{z}_{23}^{\bar{h}_j+\bar{h}_k-\bar{h}_i} \bar{z}_{13}^{\bar{h}_k+\bar{h}_i-\bar{h}_j}}$$

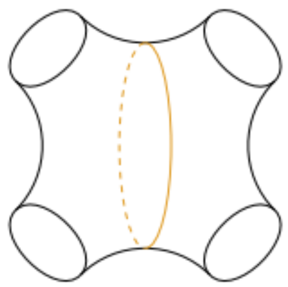
There are many ways to decompose it.
Consistency condition = conformal bootstrap equation !

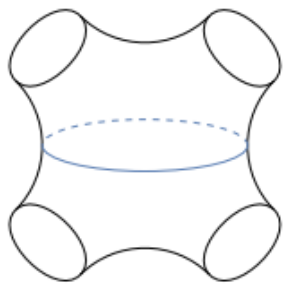
example; bootstrap equation of 4pt functions

$$\langle \mathcal{O}_i(0,0)\mathcal{O}_j(1,1)\mathcal{O}_k(z,\bar{z})\mathcal{O}_l(\infty,\infty)\rangle$$

$$= \sum_p C_{ijp}C_{pkl}\mathcal{F}\begin{bmatrix} i & j \\ k & l \end{bmatrix}(p|z)\mathcal{F}\begin{bmatrix} i & j \\ k & l \end{bmatrix}(p|\bar{z})$$

$$= \sum_q C_{ilq}C_{qjk}\mathcal{F}\begin{bmatrix} i & l \\ j & k \end{bmatrix}(q|1-z)\mathcal{F}\begin{bmatrix} i & l \\ j & k \end{bmatrix}(q|1-\bar{z})$$





→ gives an infinite set of polynomial equations.

$\mathcal{F}\begin{bmatrix} i & j \\ k & l \end{bmatrix}(p|z)$: conformal block

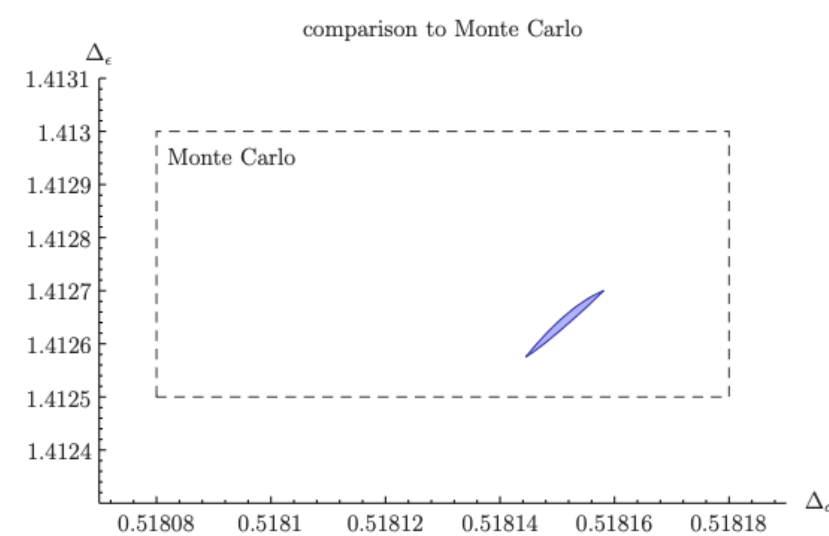
in 1+1d for minimal models this equation is solved (= RCFT is solved!)

(equations are for finite number of OPE coefficients) [cf: Belavin-Polyakov-Zamolodchikov 84]

in 2+1d $\mathbb{Z}_2$ symmetric CFT with 2 relevant ops, this is numerically solved

[El Showk-Paulos-Poland-Rychkov-Simons Duffin-Vichi 12]

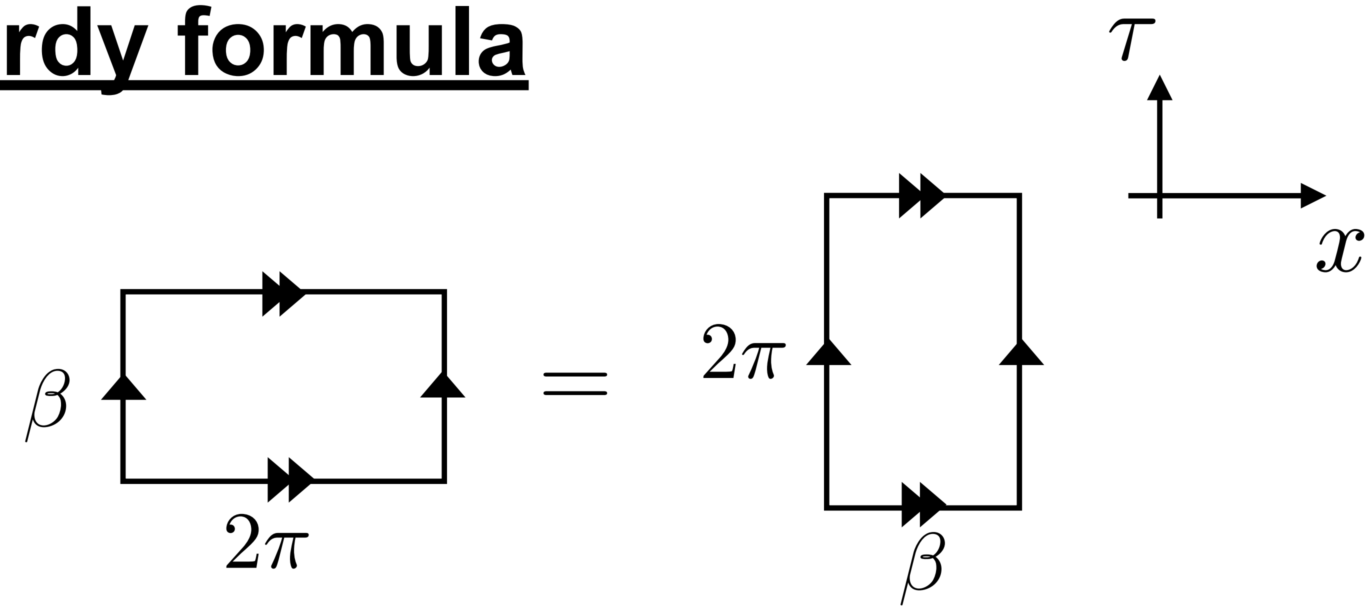
[Kos-Poland-Simons Duffin 14] …



Modular bootstrap and Cardy formula

$$Z(\beta) = Z(4\pi^2/\beta)$$

$$Z(\beta) = \int N(E) e^{-\beta E} dE$$



The density of state is derived by the inverse Laplace transformation

$$N(E) = \int d\beta Z(\beta) e^{\beta E}$$

Using the modular invariance, the high energy is dominated in the low temperature regime in the dual channel:

$$N(E) = \int d\beta Z(4\pi^2/\beta) e^{\beta E} = \int d\beta \, \underline{e^{\frac{\pi^2 c}{3\beta}}} (1 + \text{excited states})) e^{\beta E}$$

Casimir energy: fixed by conformal anomaly !

$$\sim \boxed{e^{2\pi\sqrt{\frac{cE}{3}}}} \quad E \to \infty$$

saddle pt. Cardy formula

Virasoro Characters

Virasoro characters for generic representation

$$\chi_P(\tau) = \frac{q^{P^2}}{\eta(\tau)}$$

$$\chi_{\mathbf{1}}(\tau) = \chi_{\frac{i}{2}(b+b^{-1})}(\tau) - \chi_{-\frac{i}{2}(b-b^{-1})}(\tau)$$: vacuum character

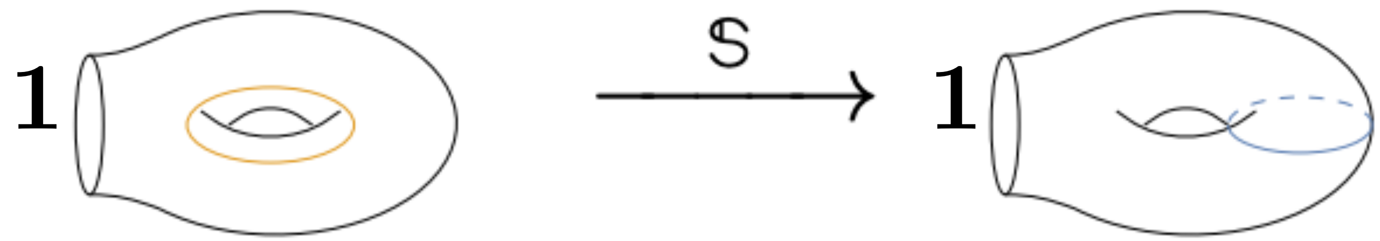
By simple Fourier trasformations, these transform as

$$\chi_P(-1/\tau) = \int \frac{dP}{2} \chi_{P'}(\tau) \mathbb{S}_{P'P}[\mathbf{1}]$$

$$\mathbb{S}_{PP'}[\mathbf{1}] = 2\sqrt{2}\cos(4\pi PP')$$

$$\mathbb{S}_{P\mathbf{1}}[\mathbf{1}] = \mathbb{S}_{P,\frac{i}{2}(b+b^{-1})} - \mathbb{S}_{P,\frac{i}{2}(-b+b^{-1})} = 4\sqrt{2}\sinh(2\pi bP)\sinh(2\pi b^{-1}P)$$

Cardy formula as crossing Kernel



$$Z(\tau,\bar{\tau}) = \int \frac{dP}{2}\frac{d\bar{P}}{2}\rho(P,\bar{P})\chi_P(\tau)\chi_{\bar{P}}(\bar{\tau}) \qquad \rho(P,\bar{P}) = \sum_i \delta(P-P_i)\delta(\bar{P}-\bar{P}_i)$$

$$Z(-1/\tau,-1/\bar{\tau}) = \int \frac{dP}{2}\frac{d\bar{P}}{2}\frac{dP'}{2}\frac{d\bar{P}'}{2}\mathbb{S}_{P'P}[\mathbf{1}]\mathbb{S}_{\bar{P}'\bar{P}}[\mathbf{1}]\rho(P,\bar{P})\chi_{\bar{P}'}(\tau)\bar{\chi}_{\bar{P}'}(\bar{\tau})$$

Modular invariance:

$$\rho(P,\bar{P}) = \int \frac{dP'}{2}\frac{d\bar{P}'}{2}\mathbb{S}_{PP'}[\mathbf{1}]\mathbb{S}_{\bar{P}\bar{P}'}[\mathbf{1}]\rho(P',\bar{P}')$$

$$\rho(P,\bar{P}) \sim \mathbb{S}_{P\mathbf{1}}[\mathbf{1}]\mathbb{S}_{\bar{P}\mathbf{1}}[\mathbf{1}] \qquad P\to\infty \quad \bar{P}\to\infty$$

$$\sim 2e^{2\pi QP}e^{2\pi Q\bar{P}}$$

Cardy formula (for primaries) as crossing Kernel !

The universal OPE density

We derived an asymptotic formula for the density of states.

So next we should consider is the statistics of OPE coefficients!

→Any asymptotics of OPE densities are uniformly expressed using the function

$$C_0(P_1, P_2, P_3) = \frac{1}{\sqrt{2}} \frac{\Gamma_b(2Q)}{\Gamma_b(Q)^3} \frac{\prod_{\pm} \Gamma_b(\frac{Q}{2} \pm iP_1 \pm iP_2 \pm iP_3)}{\prod_{i=1}^{3} \Gamma_b(Q + 2iP_k)\Gamma_b(Q - 2iP_k)}$$

where

$$\log \Gamma_b(x) = \int_0^\infty \frac{dt}{t} \left[\frac{e^{-xt} - e^{-\frac{Qt}{2}}}{(1 - e^{-bt})(1 - e^{b^{-1}t})} - \frac{(\frac{Q}{2} - x)^2}{2} e^{-t} - \frac{\frac{Q}{2} - x}{t} \right]$$

$$= \log \frac{\Gamma_2(x|b, b^{-1})}{\Gamma_2(Q/2|b, b^{-1})}$$

$$\Gamma_2(x|\omega_1, \omega_2) = \text{``} \prod_{n,m=0}^{\infty} (x + n\omega_1 + m\omega_2)^{-1} \text{``}$$: double gamma function

The asymptotic formula for OPE density

For any of H-L-L, H-H-L, H-H-H cases,

the average of OPE density is universally summarized as

$$\overline{|C_{ijk}|^2} \sim C_0(P_i, P_j, P_k) C_0(\bar{P}_i, \bar{P}_j, \bar{P}_k)$$

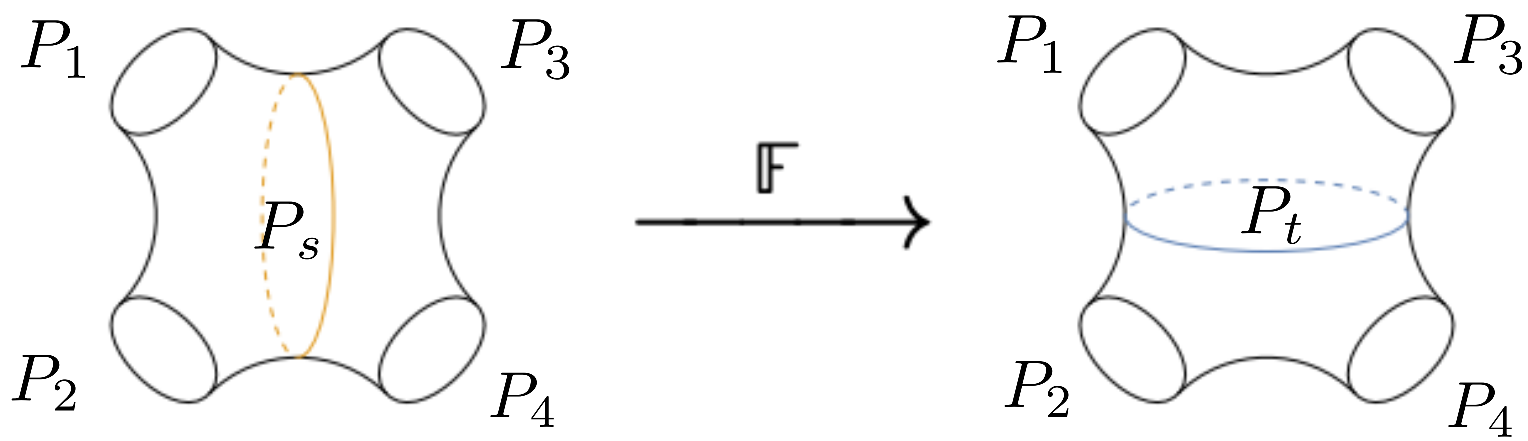
[Collier-Maloney-Maxfield-Tsiares 19]

(at least one of P_i , P_j , P_k is large)

Elementary Kernels

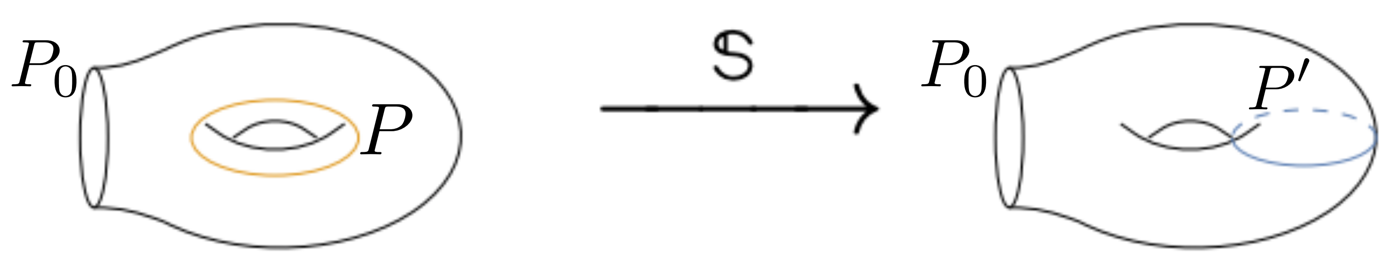
[Ponsot-Tschner 99,01]

Fusion transformation: transformation of sphere 4pt conformal blocks



$$\underbrace{\mathcal{F}\begin{bmatrix} P_2 & P_1 \\ P_3 & P_4 \end{bmatrix}(P_s|z)}_{\text{conformal block}} = \int \frac{dP_t}{2} \underbrace{\mathbb{F}_{P_s P_t}\begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}}_{\text{kernel}} \underbrace{\mathcal{F}\begin{bmatrix} P_2 & P_3 \\ P_1 & P_4 \end{bmatrix}(P_t|1-z)}_{\text{conformal block}}$$

Modular S transformation: transformation of torus 1pt conformal blocks



$$\tau^{h_0}\underline{\mathcal{F}[P_0](P|\tau)} = \int \frac{dP'}{2} \underline{\mathbb{S}_{P'P}[P_0]}\,\underline{\mathcal{F}[P_0](P|-1/\tau)}$$

These are *purely kinematical* (fixed by Virasoro *symmetry*)

Analogy with hypergeometric function

$$\underbrace{{}_2F_1(\alpha,\beta,\gamma;z)}_{\text{“block”}} = \underbrace{\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}}_{\text{kernel}}(-z)^{\alpha}\underbrace{{}_2F_1(\alpha,\alpha+1-\gamma,\alpha+1-\beta;1/z)}_{\text{“block”}}$$
$$+\underbrace{\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}}_{\text{kernel}}(-z)^{-\beta}\underbrace{{}_2F_1(\beta,\beta+1-\gamma,\beta+1-\alpha;1/z)}_{\text{“block”}}$$

where

$$\log\Gamma(z) = \int_0^\infty \frac{dt}{t}\left[\frac{e^{-zt}-e^{-t}}{1-e^{-t}}+(z-1)e^{-t}\right]$$: ”single” gamma function

$$\left(\log\Gamma_b(x) = \int_0^\infty \frac{dt}{t}\left[\frac{e^{-xt}-e^{-\frac{Qt}{2}}}{(1-e^{-bt})(1-e^{b^{-1}t})} - \frac{(\frac{Q}{2}-x)^2}{2}e^{-t} - \frac{\frac{Q}{2}-x}{t}\right] \right)$$

Actually if one of 4 operators is in a degenerate representation, conformal blocks is represented by hypergeometric functions.

[minimal models; Belavin-Polyakov-Zamolodchikov 84]

[in Liouville theory; Tschner 95]

difference: We do not know blocks explicitly, but still can write down kernels!

Analogy with SU(2) 6j symbol

6j-symbol: a transformation of basis of 3 angular momentums:

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \prod_{i=1}^{4} \sqrt{\Delta(\hat{\alpha}_i)} \sum_k (-1)^k \frac{(k+1)!}{\prod_{j=1}^{4}(k-\alpha_j)! \prod_{l=1}^{3}(\beta_l - k)!}$$

Fusion matrix is basically 6j symbol for Virasoro group

(generically for representations of any group G we have fusion matrices)

(Actually 6j symbol is a data of fusion category.
The representation of G , Rep_G , is a fusion category.

Modular tensor category, which is the underlying structure in 2d (R)CFT,
is also automatically a fusion category.)

Elementary Kernels; explicit form

[Ponsot-Tschner 99,01]

[cf; Kusuki 18, Collier-Gobeil-Maxfield-Perlmutter 18]

Fusion kernel:

$$\mathbb{F}_{P_s P_t}\begin{bmatrix} P_2 & P_1 \\ P_3 & P_4 \end{bmatrix} = P_b(P_i; P_s, P_t) P_b(P_i; -P_s, -P_t) \int \frac{ds}{i} \prod_{i=1}^{4} \frac{S_b(s+U_k)}{S_b(s+V_k)}$$

Modular S kernel:

$$\mathbb{S}_{PP'}[P_0] = \frac{\rho_0(P)}{S_b(\frac{Q}{2}+iP_0)} \frac{\Gamma_b(Q+2iP')\Gamma_b(Q-2iP')\Gamma_b(\frac{Q}{2}+i(2P-P_0)\Gamma_b(\frac{Q}{2}-i(2P+P_0))}{\Gamma_b(Q+2iP)\Gamma_b(Q-2iP))\Gamma_b(\frac{Q}{2}+i(2P'-P_0))\Gamma_b(\frac{Q}{2}-i(2P'+P_0))}$$

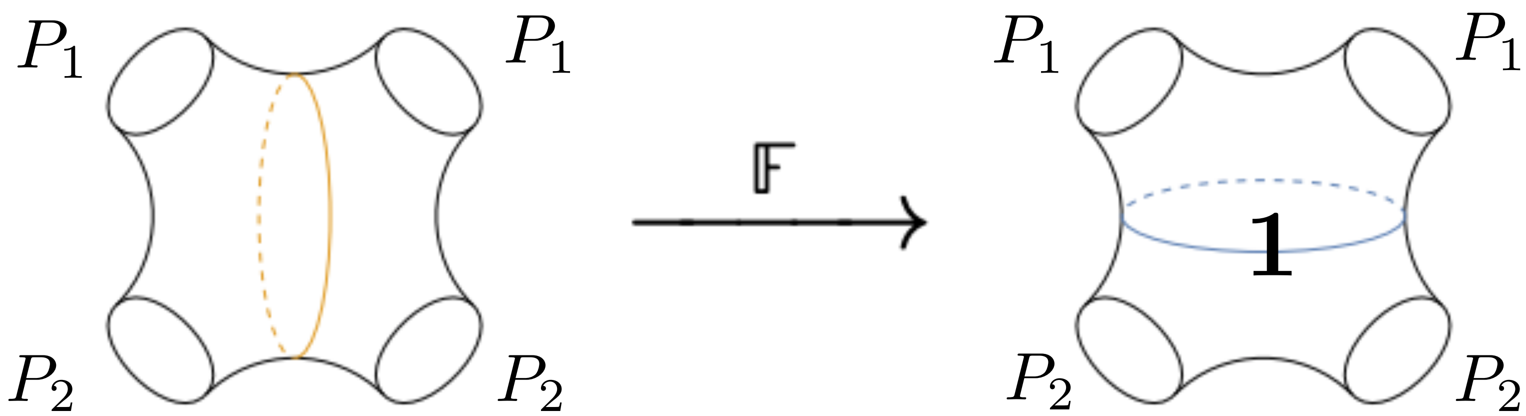
$$\times \int \frac{d\xi}{i} e^{-4\pi P'\xi} \frac{S_b(\xi+\frac{Q}{4}+i(P+\frac{1}{2}P_0))S_b(\xi+\frac{Q}{4}-i(P-\frac{1}{2}P_0))}{S_b(\xi+\frac{3Q}{4}+i(P-\frac{1}{2}P_0))S_b(\xi+\frac{3Q}{4}-i(P+\frac{1}{2}P_0))}$$

What is important is the following limits:

$$\mathbb{F}_{P_s \mathbf{1}}\begin{bmatrix} P_2 & P_1 \\ P_2 & P_1 \end{bmatrix} = \rho_0(P_s) C_0(P_1, P_2, P_s)$$ (OPE density)

$$\mathbb{S}_{P_s \mathbf{1}}[\mathbf{1}] = 4\sqrt{2}\sinh(2\pi b P)\sinh(2\pi b^{-1} P) = \rho_0(P)$$ (Cardy density)

Heavy-Light-Light case



$$\sum_{\mathcal{O}_s} C_{12s}C_{12s}\mathcal{F}\begin{bmatrix} P_2 & P_1 \\ P_2 & P_1 \end{bmatrix}(P_s|z)\mathcal{F}\begin{bmatrix} \bar{P}_2 & \bar{P}_1 \\ \bar{P}_2 & \bar{P}_1 \end{bmatrix}(\bar{P}_s|z) = \int \frac{dP_s}{2}\frac{d\bar{P}_s}{2}\underline{\rho_s(P_s,\bar{P}_s)}\mathcal{F}\begin{bmatrix} P_2 & P_1 \\ P_2 & P_1 \end{bmatrix}(P_s|z)\mathcal{F}\begin{bmatrix} \bar{P}_2 & \bar{P}_1 \\ \bar{P}_2 & \bar{P}_1 \end{bmatrix}(\bar{P}_s|z)$$

OPE density

$$\sum_{\mathcal{O}_t} C_{11t}C_{22t}\mathcal{F}\begin{bmatrix} P_2 & P_2 \\ P_1 & P_1 \end{bmatrix}(P_t|1-z)\mathcal{F}\begin{bmatrix} \bar{P}_2 & \bar{P}_2 \\ \bar{P}_1 & \bar{P}_1 \end{bmatrix}(\bar{P}_t|1-\bar{z})$$

$$= \int \frac{dP_t}{2}\frac{d\bar{P}_t}{2}\rho_t(P_t,\bar{P}_t)\mathcal{F}\begin{bmatrix} P_2 & P_2 \\ P_1 & P_1 \end{bmatrix}(P_t|1-z)\mathcal{F}\begin{bmatrix} \bar{P}_2 & \bar{P}_2 \\ \bar{P}_1 & \bar{P}_1 \end{bmatrix}(\bar{P}_s|1-\bar{z})$$

Kernel: $\mathcal{F}\begin{bmatrix} P_2 & P_1 \\ P_3 & P_4 \end{bmatrix}(P_s|z) = \int \frac{dP_t}{2}\mathbb{F}_{P_sP_t}\begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}\mathcal{F}\begin{bmatrix} P_2 & P_3 \\ P_1 & P_4 \end{bmatrix}(P_t|1-z)$

Bootstrap eq: $\rho_s(P_s,\bar{P}_s) = \int \frac{dP_t}{2}\frac{d\bar{P}_t}{2}\mathbb{F}_{P_sP_t}\mathbb{F}_{\bar{P}_s\bar{P}_t}\rho_t(P_t,\bar{P}_t)$

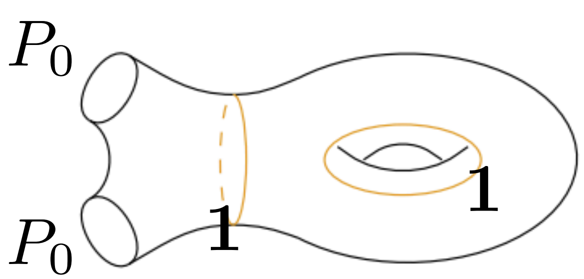
$$\rho_0(P_s)\rho_0(\bar{P}_s)\overline{|C_{12s}|^2} \sim \mathbb{F}_{P_s\mathbf{1}}\begin{bmatrix} P_2 & P_1 \\ P_2 & P_1 \end{bmatrix}\mathbb{F}_{P_s\mathbf{1}}\begin{bmatrix} P_2 & P_1 \\ P_2 & P_1 \end{bmatrix} \qquad P_s,\bar{P}_s \to \infty$$

$$= \rho_0(P_s)\rho_0(\bar{P}_s)\underline{C_0(P_1,P_2,P_s)C_0(\bar{P}_1,\bar{P}_2,\bar{P}_s)}$$

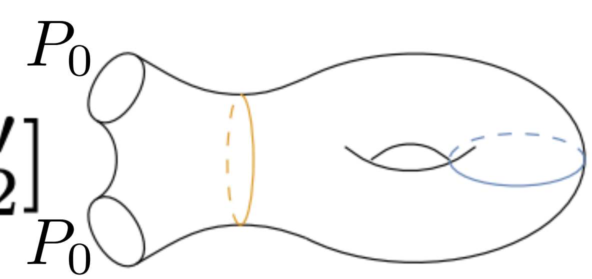
Heavy-Heavy-Light case

The game is basically the same:

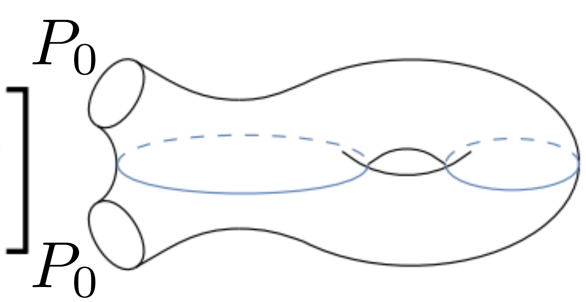
Start with OPE square, join holes for heavy operators, then transform to the dual channel where identity dominates



$$= \int \frac{dP_1}{2} \mathbb{S}_{P_1 P_1'}[P_2']$$



$$= \int \frac{dP_1}{2} \frac{dP_2}{2} \mathbb{S}_{P_1 P_1'}[P_2'] \mathbb{F}_{P_2 P_2'} \begin{bmatrix} P_0 & P_0 \\ P_1 & P_1 \end{bmatrix}$$

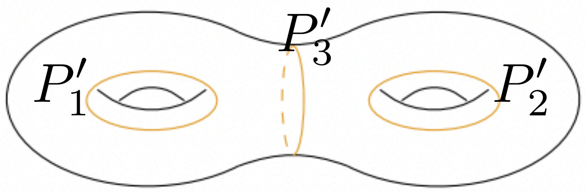


$$\rho_0(P_1)\rho_0(\bar{P}_1)\rho_0(P_2)\rho_0(\bar{P}_2)\overline{|C_{012}|^2} \sim \mathbb{S}_{P_1 \mathbf{1}}[\mathbf{1}]\mathbb{F}_{P_2 \mathbf{1}} \begin{bmatrix} P_0 & P_1 \\ P_0 & P_1 \end{bmatrix} \mathbb{S}_{\bar{P}_1 \mathbf{1}}[\mathbf{1}]\mathbb{F}_{\bar{P}_2 \mathbf{1}} \begin{bmatrix} \bar{P}_0 & \bar{P}_1 \\ \bar{P}_0 & \bar{P}_1 \end{bmatrix}$$

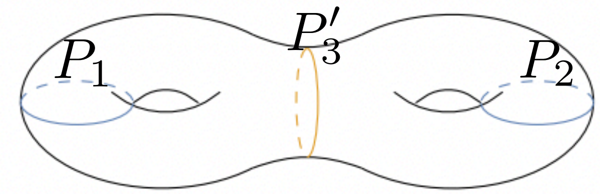
$$= \rho_0(P_1)\rho_0(\bar{P}_1)\rho_0(P_2)\rho_0(\bar{P}_2) C_0(P_0, P_1, P_2) C_0(\bar{P}_0, \bar{P}_1, \bar{P}_2)$$

$$P_1, \bar{P}_1, P_2, \bar{P}_2 \to \infty$$

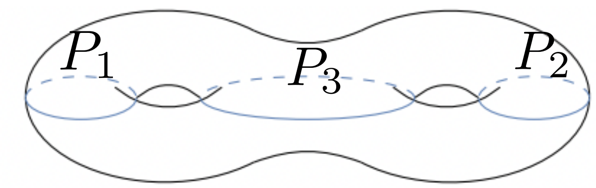
Heavy-Heavy-Heavy case



$$= \int \frac{dP_1}{2}\frac{dP_2}{2}\mathbb{S}_{P_1 P_1'}[P_3']\mathbb{S}_{P_2 P_2'}[P_3']$$



$$= \int \frac{dP_1}{2}\frac{dP_2}{2}\frac{dP_3}{2}\mathbb{S}_{P_1 P_1'}[P_3']\mathbb{S}_{P_2 P_2'}[P_3']\mathbb{F}_{P_3 P_3'}\begin{bmatrix} P_1 & P_1 \\ P_2 & P_2 \end{bmatrix}$$



$$\rho_0(P_1)\rho_0(\bar{P}_1)\rho_0(P_2)\rho_0(\bar{P}_2)\rho_0(P_3)\rho_0(\bar{P}_3)\overline{|C_{123}|^2}$$

$$\sim \mathbb{S}_{P_1\mathbf{1}}[\mathbf{1}]\mathbb{S}_{P_3\mathbf{1}}[\mathbf{1}]\mathbb{F}_{P_2\mathbf{1}}\begin{bmatrix} P_1 & P_3 \\ P_1 & P_3 \end{bmatrix}\mathbb{S}_{\bar{P}_1\mathbf{1}}[\mathbf{1}]\mathbb{S}_{\bar{P}_3\mathbf{1}}[\mathbf{1}]\mathbb{F}_{\bar{P}_2\mathbf{1}}\begin{bmatrix} \bar{P}_1 & \bar{P}_3 \\ \bar{P}_1 & \bar{P}_3 \end{bmatrix}$$

$$= \rho_0(P_1)\rho_0(\bar{P}_1)\rho_0(P_2)\rho_0(\bar{P}_2)C_0(P_1,P_2,P_3)C_0(\bar{P}_1,\bar{P}_2,\bar{P}_3)$$

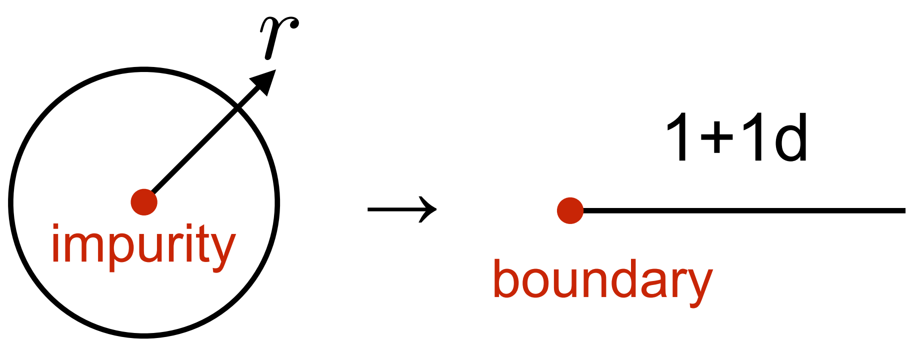
$$P_1, P_2, P_3, \bar{P}_1, \bar{P}_2, \bar{P}_3 \to \infty$$

Boundary conformal field theory

Appears in many physics contexts like:

- boundary critical phenomena
- Kondo effect
- monopole-fermion scattering
- open strings and D-branes
- black holes coupled to a bath

etc…



I think 1+1d BCFT is *ubiquitous* since the radial direction has a boundary and only s-wave can reach the core of impurities.

According to Pauli, “God made solids, but surfaces were the work of the devil”. So boundaries are complicated.

So Boundary CFT may be rich enough to lack universal formula…

Some basics of 1+1d BCFT

global conformal symmetry = Translation + Scale inv + special conformal $SL(2,\mathbb{R})$

$$z \to z+\epsilon \quad z \to z+\epsilon z \quad z \to z+\epsilon z^2 \qquad \epsilon \in \mathbb{R}$$

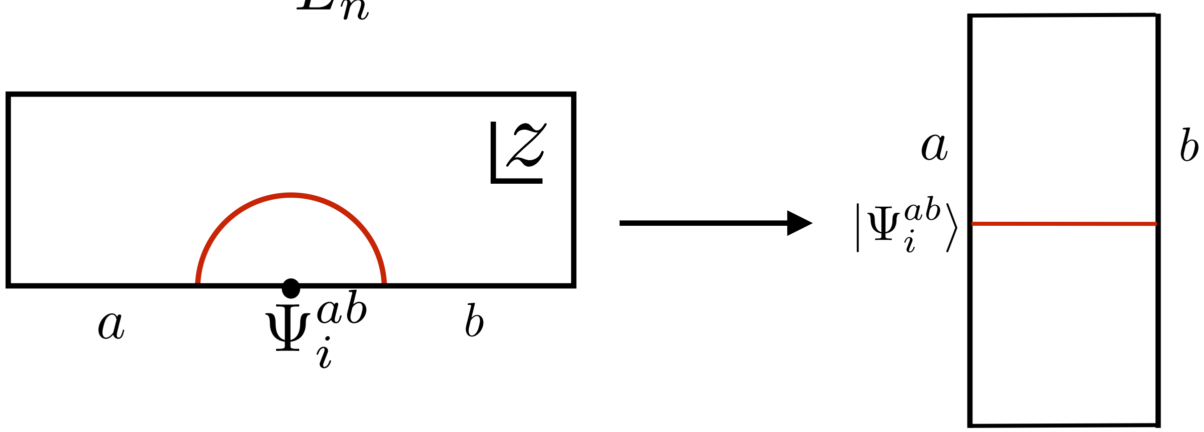
symmetry generators: L_{-1} L_0 L_1

In 1+1d it is extended to Virasoro symmetry $z \to z+\epsilon z^{n+1}$

L_n

- operator state correspondence

Via Weyl transformation,
we can map the strip to UHP



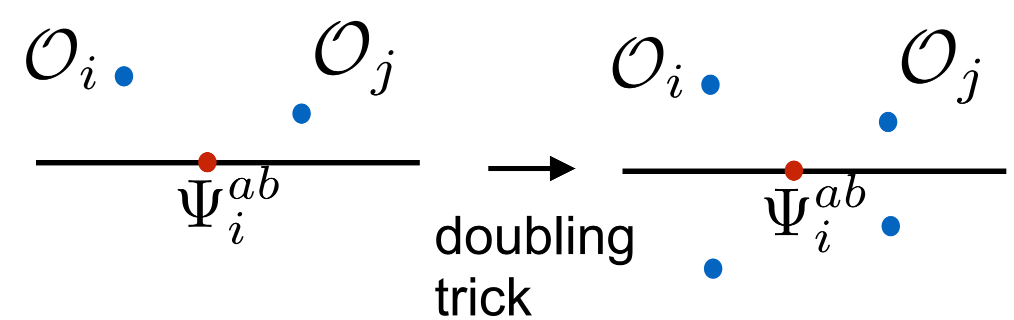
States are mapped to boundary operators

Scaling dimension = eigenvalue of dilatation: $L_0\,|\Psi_i^{(ab)}\rangle = \Delta_i\,|\Psi_i^{(ab)}\rangle$

- 2 and 3 pt: position dependence are determined by global conformal symmetry

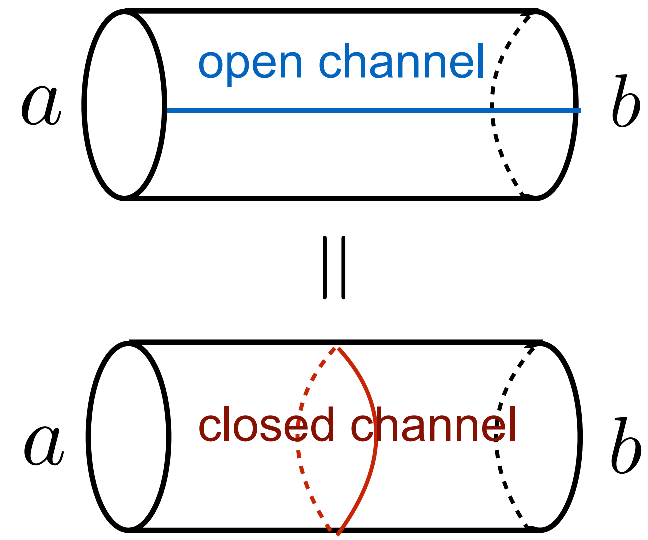
$$\langle \Psi_i^{(ab)}(x_1)\Psi_j^{(ba)}(x_2)\rangle = \frac{\mathfrak{g}_{ij}^{(ab)}\delta_{ij}}{|x_1-x_2|^{2h_i}} \qquad \langle \mathcal{O}_\alpha(z,\bar{z})\Psi_i^{aa}(x)\rangle = \frac{C_{\alpha i}^{(a)}}{|z-\bar{z}|^{2h_\alpha-\Delta_i}|z-x|^{2\Delta_i}}$$

$$\langle \Psi_i^{ab}(x_1)\Psi_j^{bc}(x_2)\Psi_k^{ca}(x_3)\rangle = \frac{C_{ijk}^{(abc)}}{|x_{12}|^{\Delta_{12}}|x_{23}|^{\Delta_{23}}|x_{13}|^{\Delta_{13}}}$$



Boundary entropy

Cardy condition ("modular invariance" for cylinder) relates open channel and closed channel:



$$\sum_{\Psi_i \in \mathcal{H}_{ab}} n^i_{ab} \chi_i(\tau) = \sum_{\mathcal{O}_i \in \mathcal{H}^{\mathrm{scalar}}_{\mathrm{closed}}} \mathcal{B}^i_a \mathcal{B}^i_b \chi_i(-1/\tau)$$

where $n^i_{ab} \in \mathbb{Z}$ is a number of representation i

Expansion of Boundary state $|B_a\rangle = \sum_{\mathcal{O}_i \in \mathcal{H}^{\mathrm{scalar}}_{\mathrm{closed}}} \mathcal{B}^i_a |P_i\rangle\rangle$ by Ishibashi state

Ishibashi state

[Ishibashi 89, Ishibashi-Onogi 89]

(r.h.s and l.h.s are different quantities. This differs from modular invariance)

Similar to the torus case, in $\tau \to 0$ we obtain the Cardy formula for open spectrum

$$\rho^{\mathrm{open}}_{ab}(P) \sim g_a g_b \rho_0(P) \qquad P \to \infty$$

$g_a \equiv \mathcal{B}^0_a$ is essentially a disc partition function

$\log g_a$: boundary entropy [Affleck-Ludwig 91]

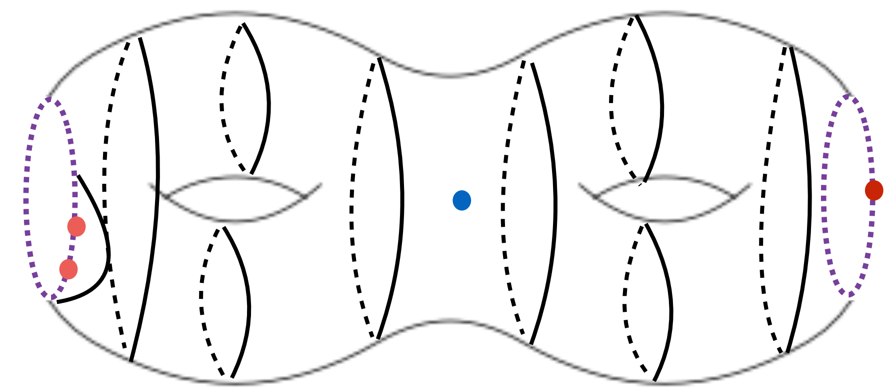
Boundary entropy is the only new ingredient for universal formula in BCFT!

BCFT Data

[Cardy 89] [Lewellen 92] [Runkel's PhD thesis 00] …

Similarly to bulk CFT,

inserting the resolution of identity $1 = \sum |O_i\rangle \langle O_i|$
we can represent the correlation function
using boundary operator spectrum,
bulk-boundary functions and boundary 3pt function

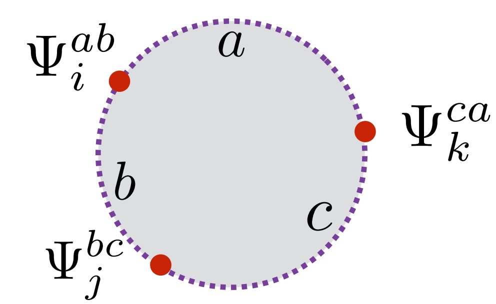


On top of bulk data, BCFT is characterized by

- the dimension Δ_i of boundary primary operators $\Psi_i^{(ab)}$

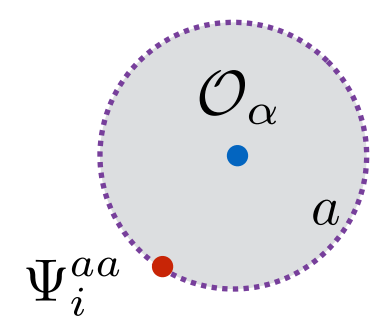
- OPE coefficient $C_{ijk}^{(abc)}$ that determines the 3pt function

$$\langle \Psi_i^{ab}(x_1)\Psi_j^{bc}(x_2)\Psi_k^{ca}(x_3)\rangle = \frac{C_{ijk}^{(abc)}}{|x_{12}|^{\Delta_{12}}|x_{23}|^{\Delta_{23}}|x_{13}|^{\Delta_{13}}}$$



- OPE coefficient $C_{\alpha i}^{(a)}$ that determines the three pt function

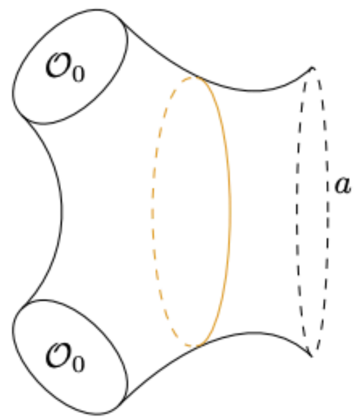
$$\langle \mathcal{O}_\alpha(z,\bar{z})\Psi_i^{aa}(x)\rangle = \frac{C_{\alpha i}^{(a)}}{|z-\bar{z}|^{2h_\alpha-\Delta_i}|z-x|^{2\Delta_i}}$$



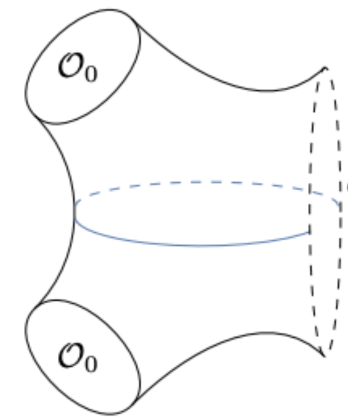
Boundary Kernels

fixed by doubling trick

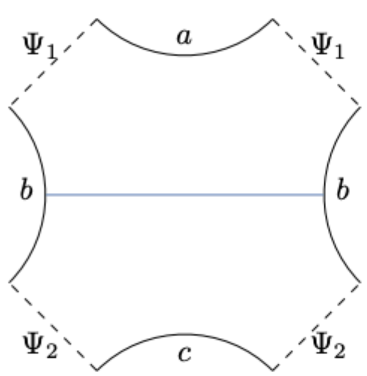
transformation of bulk 2pt conformal blocks



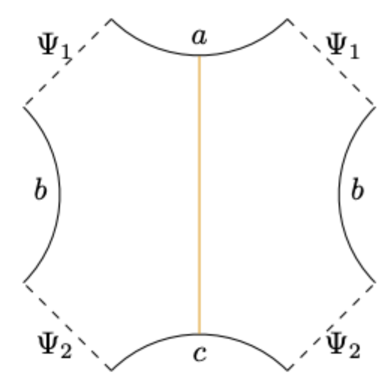
$$= \int \frac{dP}{2} \mathbb{F}_{PP'} \begin{bmatrix} P_0 & \bar{P}_0 \\ P_0 & \bar{P}_0 \end{bmatrix}$$



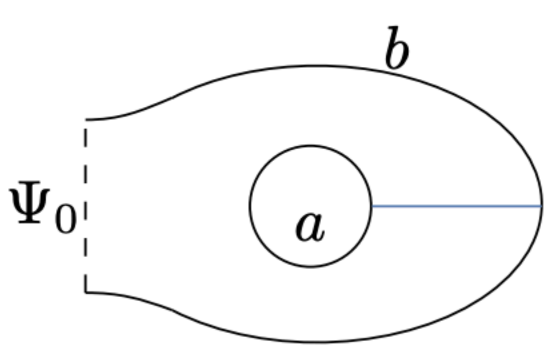
transformation of boundary 4pt conformal blocks



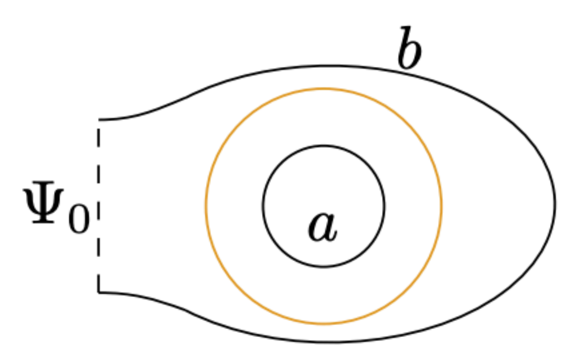
$$= \int \frac{dP}{2} \mathbb{F}_{PP'} \begin{bmatrix} P_2 & P_1 \\ P_2 & P_1 \end{bmatrix}$$



open closed duality

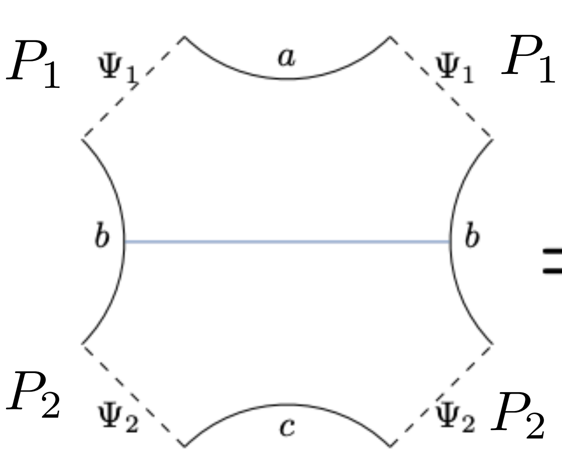


$$= \int \frac{dP'}{2} \mathbb{S}_{P'P}[P_0]$$

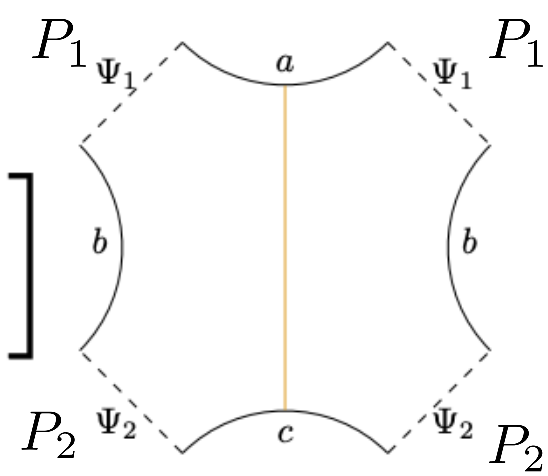


Boundary 3pt: Heavy-Light-Light case

[TN-Tsiares 22]



$$= \int \frac{dP}{2} \mathbb{F}_{PP'} \begin{bmatrix} P_2 & P_1 \\ P_2 & P_1 \end{bmatrix}$$



$$\sum_{\Psi_k \in \mathcal{H}^{a,c}_{\text{open}}} C^{(abc)k}_{12} C^{(bca)}_{21k} \mathcal{F} \begin{bmatrix} P_2 & P_1 \\ P_2 & P_1 \end{bmatrix} (P_k|\eta) \quad = \sum_{\Psi_i \in \mathcal{H}^{b,b}_{\text{open}}} C^{(bab)i}_{11} C^{(bcb)}_{22i} \mathcal{F} \begin{bmatrix} P_2 & P_2 \\ P_1 & P_1 \end{bmatrix} (P_i|\eta)$$

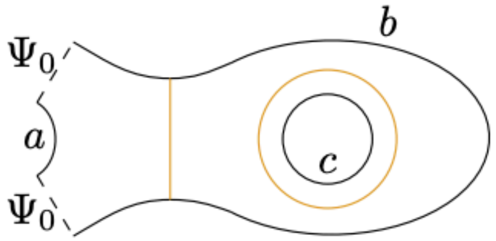
using $\quad C^{(bab)\mathbf{1}}_{11} C^{(bcb)}_{22\mathbf{1}} = g_b^{-1} \mathfrak{g}^{(ab)}_{11} \mathfrak{g}^{(bc)}_{22}$

$\mathfrak{g}^{(ab)}_{ij}$: two point function (boundary Zamolodchikov metric)

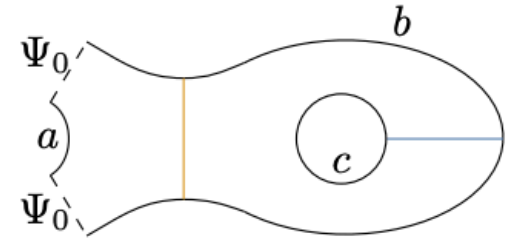
$$\overline{|C^{(abc)P}_{12}|^2} \sim g_a^{-1} g_b^{-1} g_c^{-1} C_0(P_1, P_2, P)$$

where $\quad |C^{(abc)P}_{12}|^2 \equiv C^{(abc)P}_{12} C^{(cba)21}_{P}$

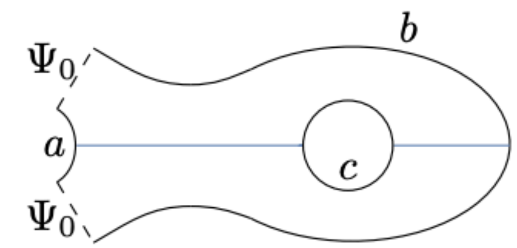
Boundary 3pt: Heavy-Heavy-Light case



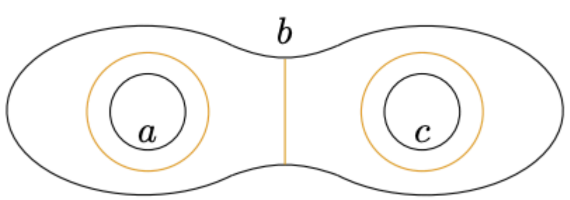
$$= \int \frac{dP_1}{2} \mathbb{S}_{P_1 P_1'}[P_2']$$



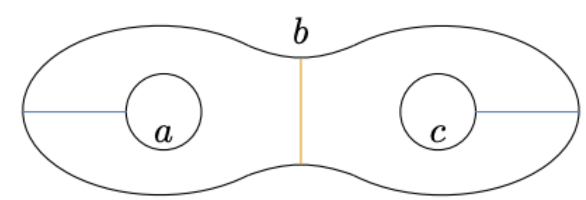
$$= \int \frac{dP_1}{2} \frac{dP_2}{2} \mathbb{S}_{P_1 P_1'}[P_2'] \mathbb{F}_{P_2 P_2'} \begin{bmatrix} P_1 & P_0 \\ P_1 & P_0 \end{bmatrix}$$



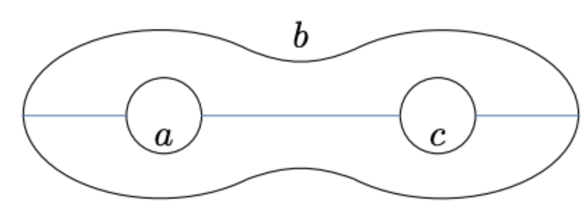
Boundary 3pt: Heavy-Heavy-Heavy case



$$= \int \frac{dP_1}{2} \frac{dP_2}{2} \mathbb{S}_{P_1 P_1'}[P_3'] \mathbb{S}_{P_2 P_2'}[P_3']$$

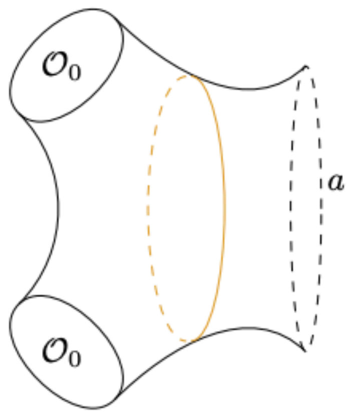


$$= \int \frac{dP_1}{2} \frac{dP_2}{2} \frac{dP_3}{2} \mathbb{S}_{P_1 P_1'}[P_3'] \mathbb{S}_{P_2 P_2'}[P_3'] \mathbb{F}_{P_3 P_3'} \begin{bmatrix} P_1 & P_2 \\ P_1 & P_2 \end{bmatrix}$$

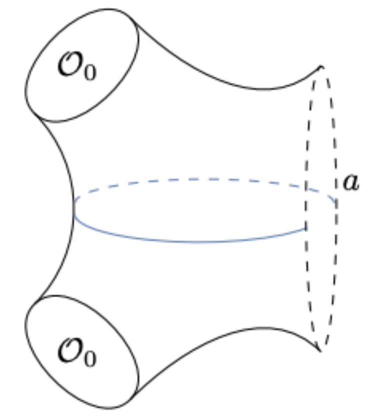


essentially we are using the doubling trick

Bulk-boundary: Heavy-Light case



$$= \int \frac{dP}{2} \mathbb{F}_{PP'} \begin{bmatrix} P_0 & \bar{P}_0 \\ P_0 & \bar{P}_0 \end{bmatrix}$$



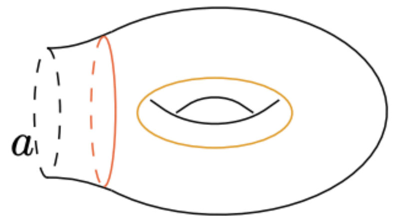
$$\sum_{\Psi_i \in \mathcal{H}^{a,a}_{\text{open}}} C_1^{(a)i} C_{2i}^{(a)} \mathcal{F} \begin{bmatrix} P_2 & \bar{P}_1 \\ P_1 & \bar{P}_2 \end{bmatrix} (P_i|\eta) = \sum_{\Psi_i \in \mathcal{H}^{\text{scalar}}_{\text{closed}}} C_{12k} C^{(a)}_{k\mathbf{1}} \mathcal{F} \begin{bmatrix} P_2 & P_1 \\ \bar{P}_1 & \bar{P}_2 \end{bmatrix} (P_k|1-\eta)$$

$$g_a^2 \rho_0(P_i) \overline{|C^{(a)}_{\alpha i}|^2} = \underbrace{C^{(a)}_{\mathbf{11}}}_{g_a} \underbrace{\mathbb{F}_{P_i \mathbf{1}} \begin{bmatrix} P_\alpha & \bar{P}_\alpha \\ P_\alpha & \bar{P}_\alpha \end{bmatrix}}_{\rho_0(P_i) C_0(P_\alpha, \bar{P}_\alpha, P_i)}$$

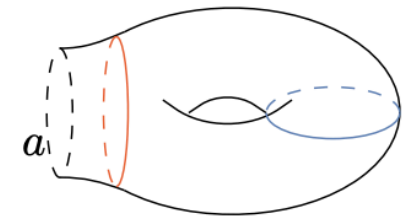
$$\boxed{\overline{|C^{(a)}_{\alpha i}|^2} \sim g_a^{-1} C_0(P_\alpha, \bar{P}_\alpha, P_i)}$$

where $|C^{(a)}_{\alpha i}|^2 \equiv C^{(a)}_{\alpha i} C^{(a)i}_{\alpha}$

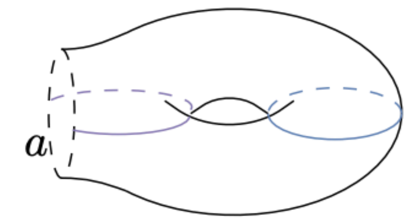
Bulk-boundary: Heavy-Heavy case



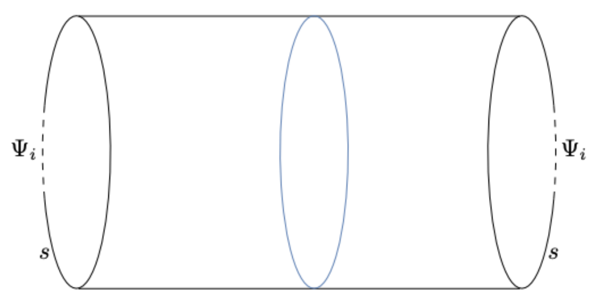
$$= \int \frac{dP_1}{2}\frac{d\bar{P}_1}{2}\mathbb{S}_{P_1 P_1'}[P_2']\mathbb{S}_{\bar{P}_1 \bar{P}_1'}[P_2']$$



$$= \int \frac{dP_1}{2}\frac{d\bar{P}_1}{2}\frac{dP_2}{2}\mathbb{S}_{P_1 P_1'}[P_2']\mathbb{S}_{\bar{P}_1 \bar{P}_1'}[P_2']\mathbb{F}_{P_2 P_2'}\begin{bmatrix} P_1 & \bar{P}_1 \\ P_1 & \bar{P}_1 \end{bmatrix}$$



Bulk-boundary: Light-Heavy case



$$= \mathbb{K}^{(\text{cyl-2pt})}_{P\bar{P};P_1P_2}[P_i]$$

$$\mathbb{K}^{(\text{cyl-2pt})}_{P\bar{P};P_1P_2}[P_i] = \int \frac{dP'}{2}\mathbb{F}^{-1}_{P'P_2}\begin{bmatrix} P_1 & P_i \\ P_1 & P_i \end{bmatrix}\mathbb{S}_{PP_1}[P']\mathbb{F}_{\bar{P}P'}\begin{bmatrix} P & P_i \\ P & P_i \end{bmatrix}$$

We could not derive this kernel only using the basic fusion transformations.
we rather employ the doubling trick directly.

Some differences with bulk formula

- We have boundary entropy factors

$$\overline{|C_{12}^{(abc)P}|^2} \sim \underline{g_a^{-1} g_b^{-1} g_c^{-1}} C_0(P_1, P_2, P)$$

$$\overline{|C_{\alpha i}^{(a)}|^2} \sim \underline{g_a^{-1}} C_0(P_\alpha, \bar{P}_\alpha, P_i)$$

- We couldn't set the boundary two point function to be 1

(boundary Zamolodchikov metric can be diagonal but can not be set to identity matrix)

and no canonical nomalization, so we care the upper and lower indices

$$|C_{12}^{(abc)P}|^2 \equiv C_{12}^{(abc)\underline{P}} C_{\underline{P}}^{(cba)21}$$

- Like Cardy formula for open spectrum, bootstrap eq relates

different OPE coefficients.
It is still powerful enough to derive the universal formula.

(cf: (Selberg)-zeta functions and the asymptotics of the length of primary geodesics)

large c limit:

[Collier-Maloney-Maxfield-Tsiares 19]

Since our formula is universal, and in particular we can take the large c limit

- in BTZ limit, we recover the spectral density of particle around BTZ BH

$$P_1 = b^{-1}p + b\delta \qquad P_2 = b^{-1}p - b\delta \qquad P_3 = i(Q/2 - bh) \qquad b \to 0$$

$$\langle BH_1|\mathcal{O}|BH_2\rangle = \rho_0(b^{-1}p)C_0(P_1,P_2,P_3) \sim \frac{(2p)^{2h}}{2\pi b}\frac{\Gamma(h+2i\delta)\Gamma(h-2i\delta)}{\Gamma(2h)}$$

(heavy operators = BH microstates)

- in Shwarzian limit, we recover the near extremal BTZ BH dynamics:

[Ghosh-Triaci-Maxfield 19]

$$P_1 = bk_1 \qquad P_2 = bk_2 \qquad P_3 = i(Q/2 - bh)$$

$$\rho_0(bk) \sim 8\sqrt{2}b^2 k \sinh(2\pi k)$$

$$C_0(bk_1, bk_2, i(Q/2 - bh)) \sim \frac{b^{4h}}{\sqrt{2}(2\pi b)^3}\frac{\prod_{\pm\pm}\Gamma(h \pm ik_1 \pm ik_2)}{\Gamma(2h)}$$

large c limit: BCFT case

[TN-Tsiares 22]

Since our formula is universal, and in particular we can take the large c limit

- in BTZ limit, we recover the spectral density of particle around BTZ BH with End of the World branes = 2BH coupled to a bath (bulk CFT)

[cf: Gen-Lust-Mishra-Wakeham, 21]

$$P_1 = b^{-1}p + b\delta \qquad P_2 = b^{-1}p - b\delta \qquad P_3 = i(Q/2 - bh) \qquad b \to 0$$

$$_{ab}\langle BH_1|\Psi^{bc}|BH_2\rangle_{ac} = \rho_0(b^{-1}p)C_0(P_1,P_2,P_3) \sim \frac{(2p)^{2h}}{2\pi b}\frac{\Gamma(h+2i\delta)\Gamma(h-2i\delta)}{\Gamma(2h)}$$

- in Shwarzian limit, we recover the near extremal BTZ BH dynamics:

$$P_1 = bk_1 \qquad P_2 = bk_2 \qquad P_3 = i(Q/2 - bh)$$

$$\rho^{(ab)}_{\text{open}}(bk) \sim 8\sqrt{2}g_a g_b b^2 k \sinh(2\pi k)$$

$$C_0(bk_1, bk_2, i(Q/2-bh)) \sim \frac{b^{4h}}{\sqrt{2}(2\pi b)^3}\frac{\prod_{\pm\pm}\Gamma(h \pm ik_1 \pm ik_2)}{\Gamma(2h)}$$

Some comments:

In the last discussion, we do not specify how we can extend the validity of formulae.

In holographic CFT, we expect that the Cardy formula is valid at relatively low energy

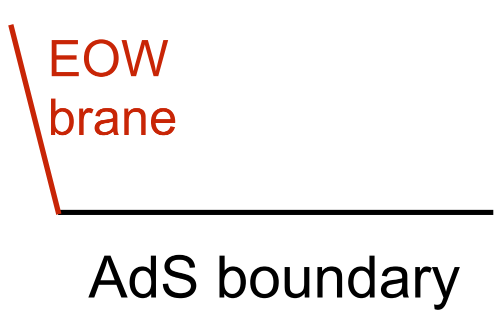
$$E \sim c/6 \qquad c \to \infty$$

whereas the universal derivation of Cardy formula we take $E \to \infty$ first.

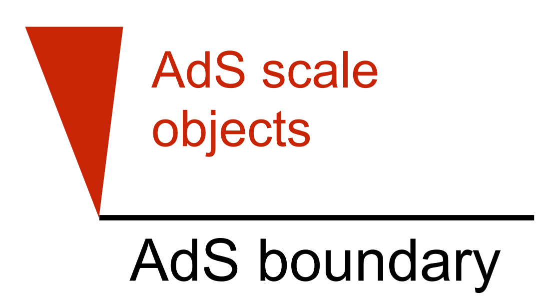
This extension requires the sparse spectrum for low lying operators [Hartman-Keller-Stoica 15]

for OPE we need stronger condition [B.Michel 19] .

We do not know in the case of BCFT we do not know similar argument
so we are not deriving the AdS/(B)CFT description [Takayanagi 11] [Randall-Sundrum 98]
still give some formula which AdS/BCFT should also reproduce



or



Some Related works:

- As we discussed, our formula is a straightforward generalization of former work

  [Collier-Maloney-Maxfield-Tsiares 19]

- For Cardy formula, people also include the width of smearing (“Tauberian theorem”)

  [Mukhametzhanov-Zhiboedov 19]

- There are many papers on universal formula with (generalized) symmetries

  [Pal-Sun 20] [Ooguri-Harlow 21] [Magan 21] [Kang-Lee-Ooguri 22] [Lin-Okuda-Seifnashri-Tachikawa 22] …

  Their formulas are formula with *topological* defects, whereas our formulas are a sort of those with *non-topological* defect

  (any defects are formulated as boundary problem by folding [Affleck-Oshikawa 96])

Summary

- We discussed a motivation to Conformal Bootstrap approach.
- Review of Cardy formula with a Kernel point of veiw
- Review of former work by Collier-Maxfield-Maloney-Tsiares:

$$\overline{|C_{ijk}|^2} \sim C_0(P_i, P_j, P_k) C_0(\bar{P}_i, \bar{P}_j, \bar{P}_k)$$

  asympototics of average of OPE coefficients are given by C_0

  which is a combination of double gamma functions

- Universal formula for BCFT; essentially the same with the bulk but with a factor of boundary entropy

$$\overline{|C_{12}^{(abc)P}|^2} \sim g_a^{-1} g_b^{-1} g_c^{-1} C_0(P_1, P_2, P)$$

$$\overline{|C_{\alpha i}^{(a)}|^2} \sim g_a^{-1} C_0(P_\alpha, \bar{P}_\alpha, P_i)$$

- Evil vs Universality: at least asymptotics of 1+1d BCFT data, they are universal.

Future works

- Include symmetries
- Tauberian
- condition for extended validity for large c BCFT
- Extension to non-unitary BCFT
- Test in explicit examples of c>1 unitary (B)CFT
- We also find that the normalization of boundary Zamolodchikov metric $\mathfrak{g}_{ij}^{ab} = \sqrt{g_a g_b}\delta_{ij}$ is a natural one. Make use of it.
- We focus on the analytic bootstrap. It is interesting to study the numerical one.

  [Collier-Mazac-Wang, 21] find that numerical bootstrap of Cardy condition constrains boundary entropy. it is interesting to study other bootstrap equations.