Curved domain-wall fermions

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セミナー　2022 年 11 月 9 日 at 京都大学

arXiv:2203.03782 (published in PTEP)

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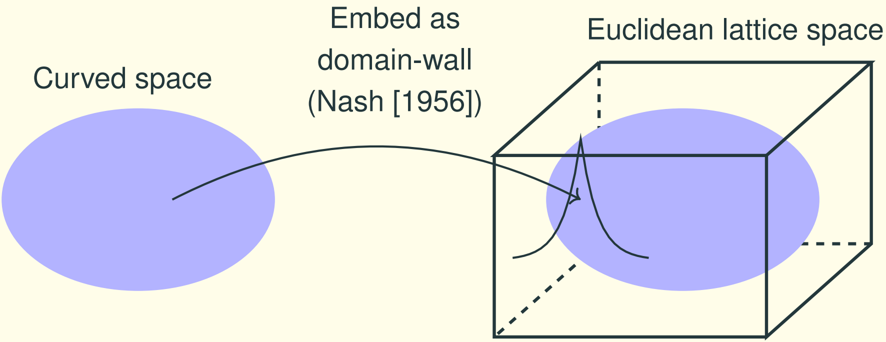
Motivation

Every curved manifold can be isometrically embedded into some higher-dimensional Euclidean spaces.



Localize the edge modes of the curved domain-wall fermion.
= they feel "gravity" by the equivalence principle.

Embedding a curved space

For any n-dim. Riemann space (Y, g), there is an embedding $Y \to \mathbb{R}^m$ $(m \gg n)$ such that Y is identified as

$$x^\mu = x^\mu(y^1, \cdots, y^n) \; (\mu = 1, \cdots, m)$$

$$\left(\begin{array}{ll} x^\mu & : \text{Cartesian coordinates of } \mathbb{R}^m \\ y^i & : \text{coordinates of } Y \end{array} \right.$$

and the metric is written as

$$g_{ij} = \sum_{\mu\nu} \delta_{\mu\nu} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j}.$$

$\longrightarrow$ vielbein and spin connection are also induced!

(Y, g) can be identified as a submanifold of $\mathbb{R}^m$!

Cf. Nash [1956].

Our Work

We consider a Hermitian Dirac operator

$$H = \bar{\gamma}\left(\sum_{i=1}^{n+1} \gamma^i \frac{\partial}{\partial x^i} + m\text{sign}(f)\right)$$

$$\{\gamma^I, \gamma^J\} = 2\delta^{IJ}, \{\bar{\gamma}, \gamma^J\} = 0,\ \bar{\gamma}^2 = 1\ , (I, J = 1, \cdots, n+1)$$

where the smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ determines the domain-wall $Y = \{\, f = 0 \,\}$. The edge modes are

- localized at Y,
- the chiral eigenstate of $\gamma_{\text{normal}} = \boldsymbol{n} \cdot \boldsymbol{\gamma}$,
- and feel gravity throguth the spin connection on Y.

$\longrightarrow$ We confirm the above properties on a square lattice .

$Y = S^1$ or S^2 in this work

Cf. Continuum analysis in condensed matter physics: [Imura et al. [2012], Parente et al. [2011]]

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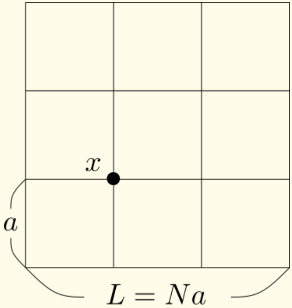
Lattice gauge theory

Lattice approximation of QFT

1. Discretize a spacetime into a lattice with PBC
   $\longrightarrow$ Dofs become finite
2. Compute path integral numerically
3. Continuum limit: $a \to 0,\ L \to \infty$

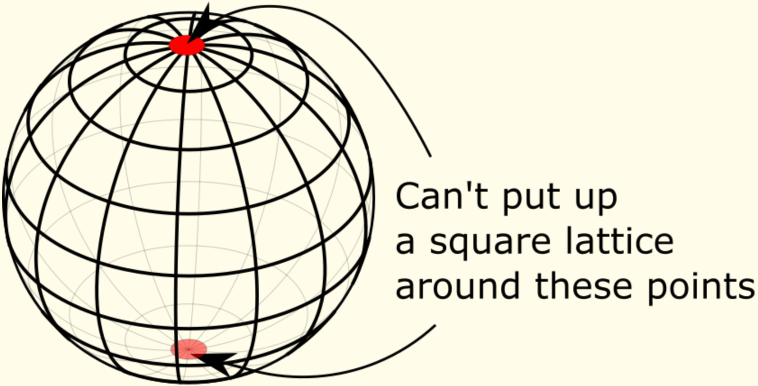


$$\int \mathcal{D}\phi e^{-S[\phi]} = \lim_{a\to 0} \lim_{L\to\infty} \prod_{x\in(a\mathbb{Z}/L\mathbb{Z})^d} \int d\phi_x e^{-S[\phi]}$$

We can deal a QFT NON-perturbatively. Lattice spacing a induces a cut-off of momentum.

Note: We consider only Euclidean space. We can get a physical result by Wick rotation.

Lattice gauge theory on a curved space



We can not approximate d-dim space with d-dim square lattice.

$\longrightarrow$ Can't handle gravity!

Triangular Lattice

Arbitrary space can be discretezed by a triangular lattice. Their length and angle represent the gravity. [Regge [1961]; Ambjørn et al. [2001]; Brower et al. [2017]]

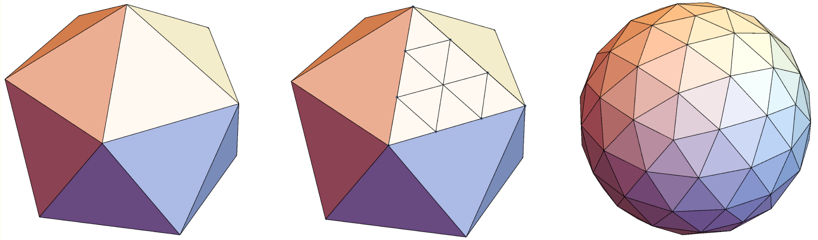


Fig 1: Triangular lattice on 2-dim sphere[Brower et al. [2017]]

However, continuum limit is not unique and symmetry restoration is non-trivial.

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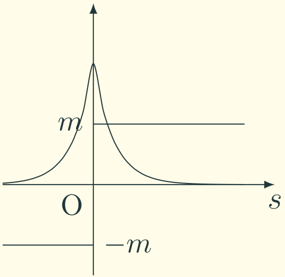
Flat Domain-wall (review)

Domain-wall=A bounday where a sign of mass is flipped.

We put a mass term $m(s) = m\text{sign}(s)$ in 5-dim space.



EoM : $(\sum_{i=1}^{4} \gamma^i \partial_i + \gamma^s \partial_s + m(s))\psi(x, s) = 0$

Sol : $\psi(x, s) = \eta_+(x) e^{-m|s|}, \ \gamma^s \eta_+ = +\eta_+$

A state with $\not{D}_4 = \sum_{i=1}^{4} \gamma^i \partial_i = 0$ and $\gamma^s = +1$ is localized at $s = 0$!

Domain-wall

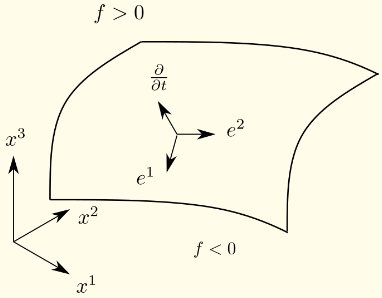
Curved case

Curved domain-wall case:

$$D = \sum_{i=1}^{2m+1} \gamma^i \frac{\partial}{\partial x^i} + m\text{sign}(f)$$

$$\simeq \gamma^{2m+1} \frac{\partial}{\partial t} + F + m\text{sign}(f)$$

$$+ \underbrace{\gamma^a \left(e_a + \frac{1}{4} \sum_{bc} \omega_{bc,a} \gamma^b \gamma^c \right)}_{\mathbb{D}^Y}$$



In the large m limit, $D \to \mathbb{D}^Y_+ = \mathbb{D}^Y \frac{1}{2}(1 + \gamma^{2m+1})$

$\longrightarrow$ A zero mode of $\mathbb{D}^Y$ only appear as the edgemode of D

Hermitian Dirac operator

We consider a Hermitian Dirac operator

$$H = \bar{\gamma}\left(\sum_{i=1}^{n+1} \gamma^i \frac{\partial}{\partial x^i} + m\text{sign}(f)\right) = \bar{\gamma}(\not{D} + m\text{sign}(f))$$

$$\gamma^a = -\sigma_2 \otimes \tilde{\gamma}^a,\ \gamma^{n+1} = \sigma_1 \otimes 1,\ \bar{\gamma} = \sigma_3 \otimes 1$$

$$\{\tilde{\gamma}^a, \tilde{\gamma}^b\} = 2\delta^{a,b},\ (a, b = 1, \cdots, n)$$

where the smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$. The edge modes are

- localized at the domain-wall $Y = \{f = 0\}$,
- the chiral eigenstate of $\gamma_{\text{normal}} = \boldsymbol{n} \cdot \boldsymbol{\gamma}$,
- and feel gravity throguth the spin connection on Y.

Induced spin connection

We take an appropriate coordinate $(y^1, \cdots, y^n, t)$ and vielbein

$$\Big(\underbrace{e^1, \cdots, e^n}_{\text{vielbein on } Y}, \frac{\partial}{\partial t}\Big).$$

We put $\psi = \left(g^{IJ}\frac{\partial f}{\partial x^I}\frac{\partial f}{\partial x^J}\right)^{\frac{1}{4}}\psi'$, H act on ψ' as

$$H' = \begin{pmatrix} \epsilon m & i\not{D}^Y + \frac{\partial}{\partial t} + F \\ i\not{D}^Y - \frac{\partial}{\partial t} - F & -\epsilon m \end{pmatrix},$$

$$i\not{D}^Y = i\sum_{a=1}^{n}\tilde{\gamma}^a\left(e_a + \frac{1}{4}\sum_{bc}\omega_{bc,a}\tilde{\gamma}^b\tilde{\gamma}^c\right)$$

$$F = \frac{1}{4}\frac{\partial}{\partial t}\left(\log\left(g^{IJ}\frac{\partial f}{\partial x^I}\frac{\partial f}{\partial x^J}\right)\right) - \underbrace{\frac{1}{2}\operatorname{tr} h}_{\text{mean curvature}}$$

Spin connection on Y is induced!

Edge mode

In the large m limit, we find an edgemode as

$$\psi = \left(g^{IJ}\frac{\partial f}{\partial x^I}\frac{\partial f}{\partial x^J}\right)^{\frac{1}{4}} e^{-m|t|}\exp\left(-\int_0^t dt' F(y,t')\right)\begin{pmatrix}\chi(y)\\ \chi(y)\end{pmatrix}$$

$$\left(F = \frac{1}{4}\frac{\partial}{\partial t}\left(\log\left(g^{IJ}\frac{\partial f}{\partial x^I}\frac{\partial f}{\partial x^J}\right)\right) - \frac{1}{2}\,\mathrm{tr}\,h\right),$$

where χ is a massless Dirac fermion:

$$i\mathbb{D}^Y|_{t=0}\chi = i\sum_{a=1}^{n}\tilde{\gamma}^a\left(e_a + \frac{1}{4}\sum_{bc}\omega_{bc,a}\tilde{\gamma}^b\tilde{\gamma}^c\right)_{t=0}\chi = \lambda\chi$$

ψ is an eigenstate: $H\psi = \lambda\psi$ and $\gamma^{n+1}\psi = (\sigma_1\otimes 1)\psi = +\psi$

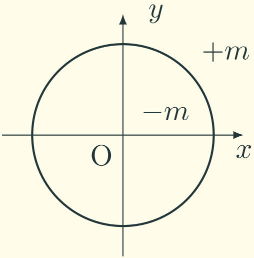
Spin connection on Y is induced and detected by solving the eigenvalue problem of H!

Contents

S^1 domain-wall

Domain wall:

$$\epsilon(r) = \text{sign}(r - r_0) = \begin{cases} -1 & (r < r_0) \\ 1 & (r \geq r_0) \end{cases},$$



Hermitian Dirac operator:

$$H = \sigma_3 \left(\sum_{i=1,2} \left(\sigma_i \frac{\partial}{\partial x^i} \right) + m\epsilon \right)$$
$$= \begin{pmatrix} m\epsilon & e^{-i\theta}(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \theta}) \\ -e^{i\theta}(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}) & -m\epsilon \end{pmatrix}.$$

Effective Dirac operator and Edge modes

We consider a normalized edge state as

$$\psi_{\text{edge}} = \rho(r)\begin{pmatrix}\chi(\theta)\\ e^{i\theta}\chi(\theta)\end{pmatrix},\ \chi(\theta+2\pi)=\chi(\theta)$$

$$\int_0^\infty dr 2r\rho^2 = 1,\ \int_0^{2\pi} d\theta \chi^\dagger \chi = 1$$

and let $2r\rho^2 \to \delta(r-r_0)$ $(m\to\infty)$. Then we obtain

$$\int dxdy\psi^\dagger_{\text{edge}} H\psi_{\text{edge}} \to \int_0^{2\pi} d\theta\chi^\dagger \underbrace{\frac{1}{r_0}\left(-i\frac{\partial}{\partial\theta}+\frac{1}{2}\right)}\chi$$

Effective Dirac op $i\mathbb{D}^{S^1}_{eff}$!!

The factor $\frac{1}{2}$ means induced spin connection.

Spectrum of Edge modes

Effective Dirac operator:

$$i\mathbb{D}^{S^1}_{eff} = \frac{1}{r_0}\left(-i\frac{\partial}{\partial\theta} + \underset{\sim}{\frac{1}{2}}\right)$$

Spin^c connection

$\longrightarrow$The edge modes is effectively anit-periodic spinor.

(trivial element of the spin bordism group)

Eigenvalue:

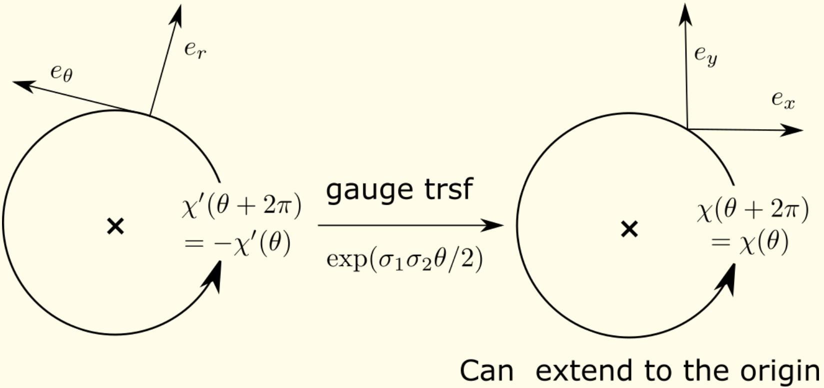
$$E = \pm\frac{n+\frac{1}{2}}{r_0}\ (n = 0, 1, \cdots).$$

$\longrightarrow$ Gravity appears as the gap of the spectrum

Periodicity of Edge modes

S^1 admits two spin structures:

$\longrightarrow$ periodic spinor and anti-periodic spinor.



Only anti-periodic spinors appear at the boundary.

Lattice domain-wall fermion

Let $(\mathbb{Z}/N\mathbb{Z})^2$ be a two-dim. lattice.

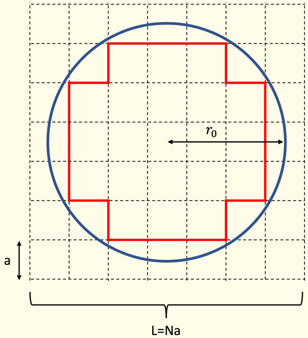
The domain-wall is given by

$$\epsilon(x) = \begin{cases} -1 & (r < r_0) \\ 1 & (r \geq r_0) \end{cases},$$

and the (Wilson) Dirac op is

$$H = \sigma_3 \left(\sum_{i=1,2} \left[\sigma_i \frac{\nabla_i - \nabla_i^\dagger}{2} + \frac{1}{2} \nabla_i \nabla_i^\dagger \right] + \epsilon m a \right),$$

$$(\nabla_i \psi)_x = \psi_{x+\hat{i}} - \psi_x, \; (\nabla_i^\dagger \psi)_x = \psi_{x-\hat{i}} - \psi_x$$



$+$ PBC for all direction.

Cf. Kaplan [1992] studied a flat domain-wall in $\mathbb{R}^{2m+1}$

Spectrum

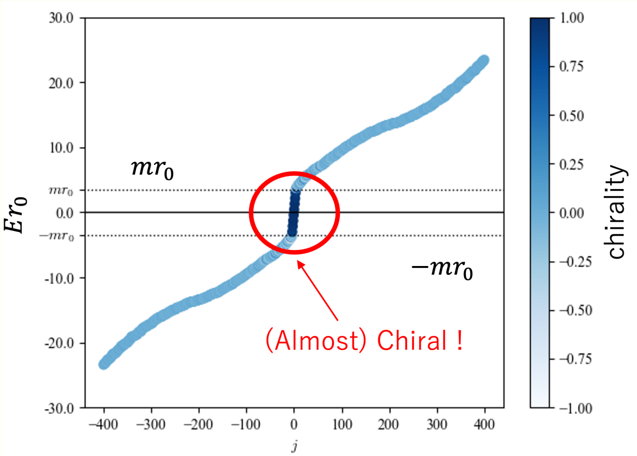
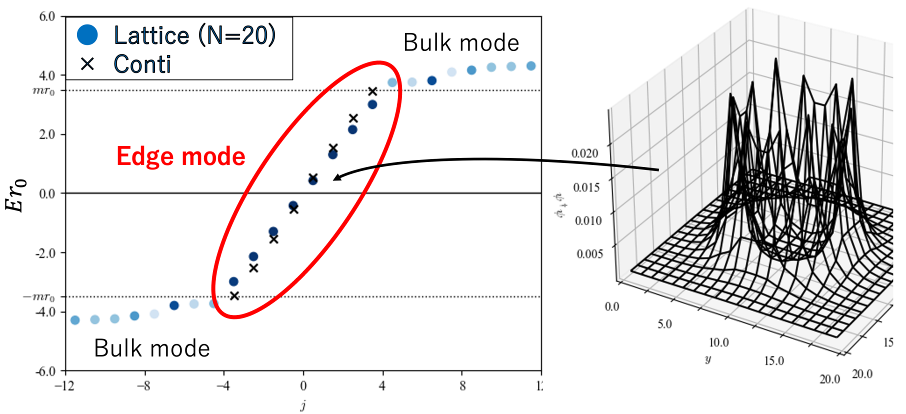


Fig 2: The Dirac eigenvalue spectrum: $ma = 0.7, r_0 = L/4, N = 20$

The color = chirality: $\gamma_{\text{normal}} = \frac{x}{r}\sigma_1 + \frac{y}{r}\sigma_2$

Edge modes



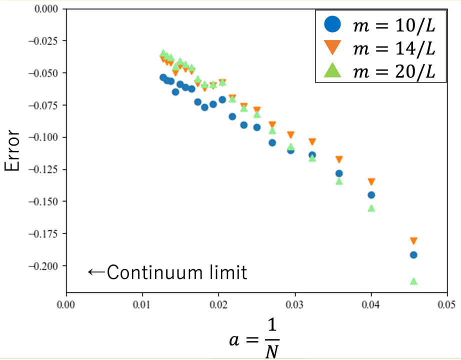
The edge modes

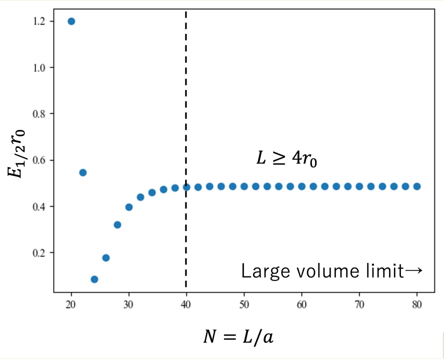
- are chiral: $\gamma_{\text{normal}} = \frac{x}{r}\sigma_1 + \frac{y}{r}\sigma_2$
- have a gap from zero (as a gravitational effect)
- agree well with the continuum prediction

Continuum limit and Finite-volume effect

Continuum limit $a = 1/N \to 0$

Large volume limit $L = Na \to \infty$





Fixed parameter:

$L = Na, r_0 = Na/4, m = 14/L$

Agree well with

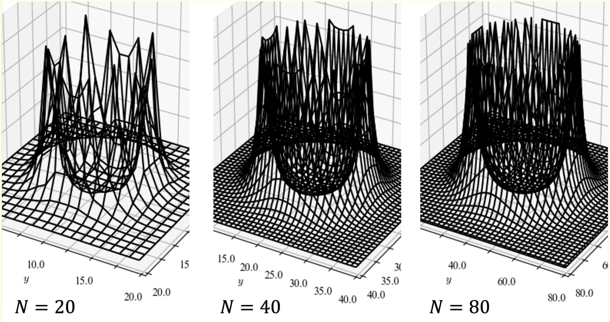
the conti. prediction!

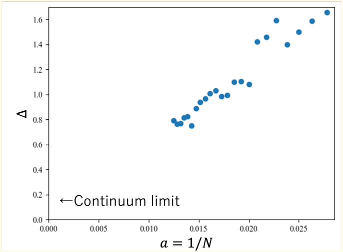
Fixed parameter:

$r_0 = 10a$

Saturates when $L \geq 4r_0$!

Recovery of Rotational symmetry in the continuum limit





$$\Delta = (\max(\text{peak}) - \min(\text{peak}))/a^2$$

The rotational symmetry automatically recovers in the continuum limit!

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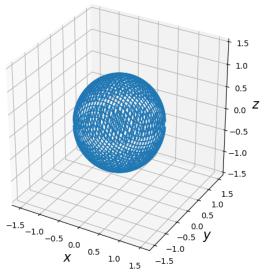
S^2 domain-wall

Domain-wall:



$$\epsilon(r) = \text{sign}(r - r_0) = \begin{cases} -1 & (r < r_0) \\ 1 & (r \geq r_0) \end{cases},$$

Hermitian Dirac operator

$$H = \bar{\gamma}\left(\gamma^j \frac{\partial}{\partial x^j} + m\epsilon\right) = \begin{pmatrix} m\epsilon & \sigma^j \partial_j \\ -\sigma^j \partial_j & -m\epsilon \end{pmatrix}$$

$$(\bar{\gamma} = \sigma_3 \otimes 1,\ \gamma^j = \sigma_1 \otimes \sigma^j)$$

acts on two-flavors of two-component spinors.

Effective Dirac op for S^2 domain-wall

We consider a normalized edge state as

$$\psi_{\text{edge}} = \rho(r) \begin{pmatrix} \chi(\theta, \phi) \\ \frac{\boldsymbol{x} \cdot \boldsymbol{\sigma}}{r} \chi(\theta, \phi) \end{pmatrix}$$

$$\int_0^\infty dr r^2 2\rho^2 = 1, \ \int_{S^2} \chi^\dagger \chi = 1,$$

and we assume $2r^2\rho^2 \to \delta(r - r_0)$ $(m \to \infty)$. Thus

$$\int dx^3 \psi^\dagger_{\text{edge}} H \psi_{\text{edge}} = \int_0^\infty dr 2r^2 \rho^2 \int_{S^2} \chi^\dagger \frac{1}{r} (\boldsymbol{\sigma} \cdot \boldsymbol{L} + 1) \chi$$

$$\to \int_{S^2} \chi^\dagger \underbrace{\frac{1}{r_0} (\boldsymbol{\sigma} \cdot \boldsymbol{L} + 1)}_{} \chi \ (m \to \infty),$$

Effective Dirac op H_{S^2} !!

where $\boldsymbol{L}$ is an orbital angular momentum.

Effective Dirac op and Dirac op. of S^2

The gauge transformation using

$$s = \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) & -e^{-i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right) \\ e^{i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right) & e^{i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) \end{pmatrix} e^{i\frac{\phi}{2}}$$

changes $\chi \to s^{-1}\chi$ and

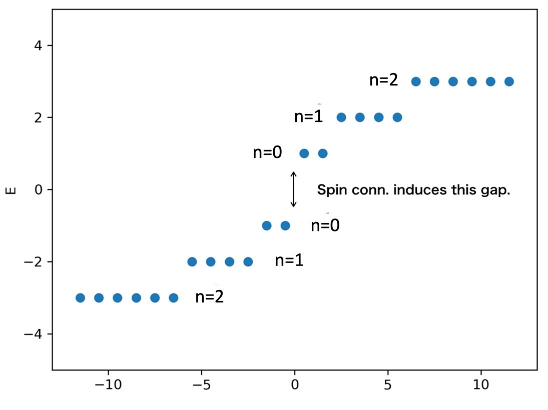
$$\begin{aligned} H_{S^2} \to & s^{-1} H_{S^2} s \\ &= -\frac{\sigma_3}{r_0}\left(\sigma_1 \frac{\partial}{\partial\theta} + \sigma_2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \underbrace{+ \frac{i}{2\sin\theta} - \frac{\cos\theta}{2\sin\theta}\sigma_1\sigma_2}_{\text{Spin conn. of } S^2}\right)\right) \\ &= -\frac{\sigma_3}{r_0} \not{D}_{S^2}. \end{aligned}$$

Edge states are affected by the spin connection of the spherical domain-wall [Takane and Imura [2013]].

Spectrum of Edgemodes

$$i\mathbb{D}^{S^2}_{eff} = -\frac{\sigma_3}{r_0}\left(\sigma_1\frac{\partial}{\partial\theta} + \sigma_2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} + \frac{i}{2\sin\theta} - i\frac{\cos\theta}{2\sin\theta}\sigma_3\right)\right)$$

Spin^c connection



Eigenvalue:

$$E = \pm\frac{n+1}{r_0}$$

$$(n = 0, 1, \cdots)$$

Degeneracy:

$$2n+2$$

Lattice Domain-wall Fermion

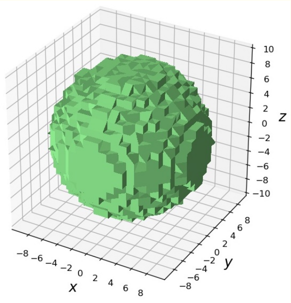
Let $(\mathbb{Z}/N\mathbb{Z})^3$ be a three-dim. lattice.
The domain-wall is given by

$$\epsilon(x) = \begin{cases} -1 & (r < r_0) \\ 1 & (r \geq r_0) \end{cases},$$

and the (Wilson) Dirac op is

$$H = \bar{\gamma}\left(\sum_{i=1,2,3}\left[\gamma^i \frac{\nabla_i - \nabla_i^\dagger}{2} + \frac{1}{2}\nabla_i \nabla_i^\dagger\right] + \epsilon m a\right).$$

$$(\nabla_i \psi)_x = \psi_{x+\hat{i}} - \psi_x,\ (\nabla_i^\dagger \psi)_x = \psi_{x-\hat{i}} - \psi_x$$



$+$PBC for all direction

Spectrum

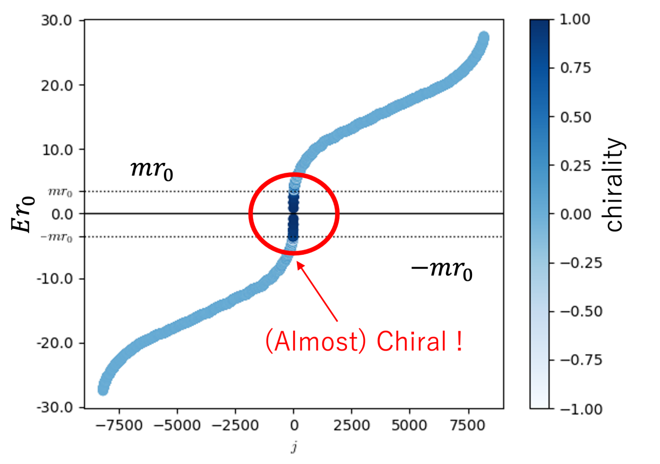
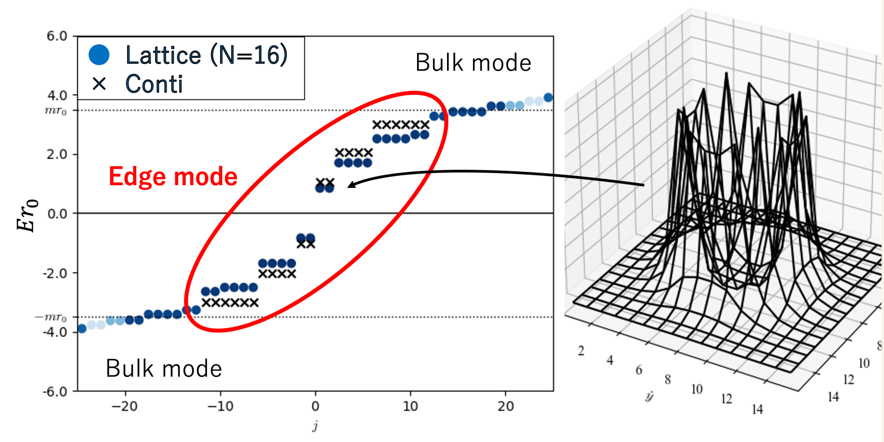


Fig 3: The Dirac eigenvalue spectrum: $ma = 0.7, r_0 = L/4, N = 16$

The color = chirality: $\gamma_{\text{normal}} = \frac{x}{r}\gamma^1 + \frac{y}{r}\gamma^2 + \frac{z}{r}\gamma^3$

Edge modes

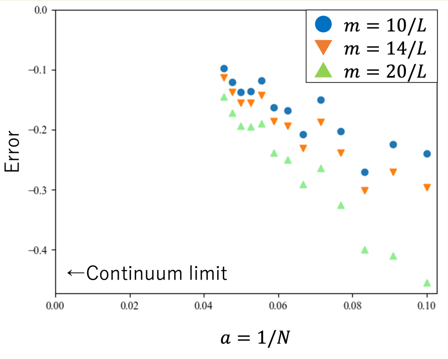


The edge modes

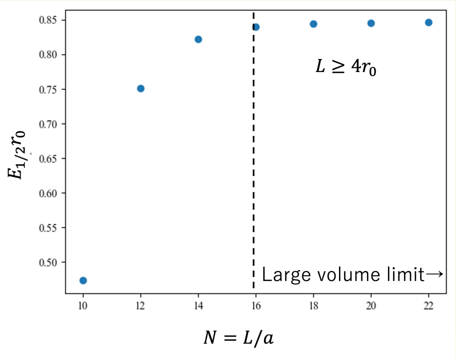
- are chiral: $\gamma_{\text{normal}} = \frac{x}{r}\gamma^1 + \frac{y}{r}\gamma^2 + \frac{z}{r}\gamma^3$
- have a gap from zero (as a gravitational effect)
- agree well with the continuum prediction

Continuum limit and Finite volume effect

Continuum limit $a = 1/N \to 0$



Large volume limit $L = Na \to \infty$



Fixed parameter:

$$L = Na, r_0 = Na/4, m = 14/L$$

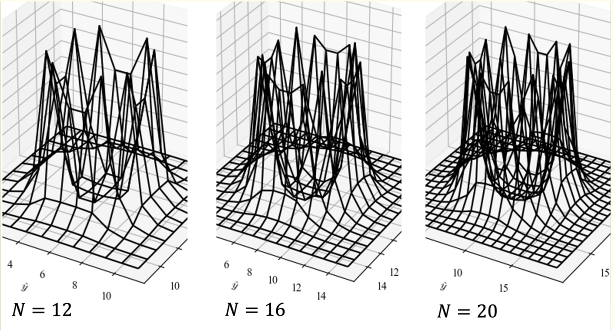
Fixed parameter:

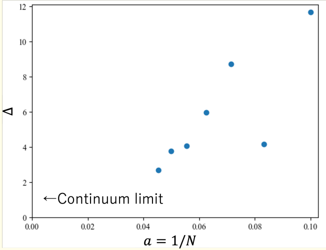
$$r_0 = 4a$$

Agree well with the conti. prediction!

Saturates when $L \geq 4r_0$!

Recovery of Rotational symmetry in the continuum limit





(slice at $z = N/2$)

$$\Delta = (\max(\text{peak}) - \min(\text{peak}))/a^3$$

The rotational symmetry automatically recovers in the continuum limit!

Contents

Summary and Outlook

[Summary]

In cases S^1 and S^2, we embodied Nash's thm in domain-wall.

- Masless chiral edge sates appear on the domain-wall.
- Edge states feel gravity through the induced spin connection.
- The rotational symmetry recovers in the conti limit.

[Outlook]

- Gravitational anomaly inflow
- Index theorem with a nontrivial curvature
- Formulate real projective space

Reference i

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Reference ii

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Contents

General domain-wall

$\psi = \left(g^{IJ}\frac{\partial f}{\partial x^I}\frac{\partial f}{\partial x^J}\right)^{\frac{1}{4}}\psi'$ とすれば、ψ' に対して

$$
\begin{aligned}
H' =& \bar{\gamma}\gamma^{n+1}\left(\frac{\partial}{\partial t} + \frac{1}{4}\frac{\partial}{\partial t}\left(\log\left(g^{IJ}\frac{\partial f}{\partial x^I}\frac{\partial f}{\partial x^J}\right)\right) - \underbrace{\frac{1}{2}\operatorname{tr} h}_{\text{平均曲率}} + \gamma^{n+1} m\epsilon\right) \\
&+ \underbrace{\bar{\gamma}\gamma^a\left(e_a + \frac{1}{4}\sum_{bc}\omega_{bc,a}\gamma^b\gamma^c\right)}_{Y \text{ の Dirac 演算子 } i\not{D}^Y}
\end{aligned}
$$

が作用する。$m \to \infty$ の極限でエッジ状態は

$$
\psi = \left(g^{IJ}\frac{\partial f}{\partial x^I}\frac{\partial f}{\partial x^J}\right)^{\frac{1}{4}} e^{-m|t|}\exp\left(-\int_0^t dt' F(y,t')\right)\chi(y)
$$

$$
(i\not{D}^Y\chi = \lambda\chi,\ \gamma^{n+1}\chi = \chi,\ F = -\frac{1}{2}\operatorname{tr} h + \frac{1}{4}\frac{\partial}{\partial t}\left(\log\left(g^{IJ}\frac{\partial f}{\partial x^I}\frac{\partial f}{\partial x^J}\right)\right))
$$

となり、H の固有値は λ となる。

Edge states

M が十分大きい時に，エッジ状態は

$$\psi_{\text{edge}}^{E,j} \simeq \sqrt{\frac{M}{4\pi r}} e^{-M|r-r_0|} \begin{pmatrix} e^{i(j-\frac{1}{2})\theta} \\ e^{i(j+\frac{1}{2})\theta} \end{pmatrix}.$$

しかも

$$\begin{aligned}\gamma_{\text{normal}} &:= \sigma_1 \cos\theta + \sigma_2 \sin\theta \\ &= \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix},\end{aligned}$$

の固有値 $+1$ を持つ．

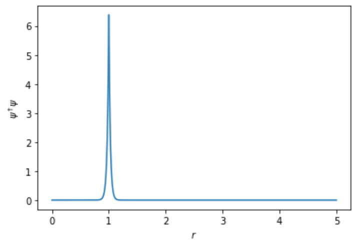


Fig 4: Edge state when $M = 5, r_0 = 1$

Edge states and Their spectrum

M が十分大きい時，エッジ状態は

$$\psi_{\text{edge}}^{\pm E,j,j_3} \simeq \sqrt{\frac{M}{2}} \frac{e^{-M|r-r_0|}}{r} \begin{pmatrix} \chi_{j,j_3}^{(\pm)} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{r}\chi_{j,j_3}^{(\pm)} \end{pmatrix}.$$

$$E \simeq \frac{j+\frac{1}{2}}{r_0} \left(j = \frac{1}{2}, \frac{3}{2}, \cdots \right)$$

さらに"chiral"なエッジ状態で

$$\gamma_{\text{normal}} := \sum_{i=1}^{3} \frac{x^i}{r}\gamma^i = \begin{pmatrix} 0 & \frac{\boldsymbol{x}\cdot\boldsymbol{\sigma}}{r} \\ \frac{\boldsymbol{x}\cdot\boldsymbol{\sigma}}{r} & 0 \end{pmatrix}$$

の固有値 $+1$ を持つ．

有効 Dirac 演算子は

$$H_{S^2} = \frac{1}{r_0}(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1),$$

で与えられ，2 成分 spinor χ に作用する．

Effective Dirac op and Dirac op. of S^2

有効 Dirac 演算子は

$$H_{S^2} = \frac{1}{r_0}(\boldsymbol{\sigma} \cdot \boldsymbol{L} + 1)$$

$$\Big\downarrow \; s = \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) & -e^{-i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right) \\ e^{i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right) & e^{i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$ で gauge 変換すると

$$s^{-1} H_{S^2} s = -\frac{\sigma_3}{r_0}\left(\sigma_1 \frac{\partial}{\partial\theta} + \sigma_2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \underbrace{- \frac{\cos\theta}{2\sin\theta}\sigma_1\sigma_2}_{\text{Spin conn. of } S^2}\right)\right)$$

$$= -\frac{\sigma_3}{r_0} \not{D}_{S^2}.$$

エッジ状態は S^2 domain-wall の重力を感じる

Cf. [Takane and Imura [2013]].

Euler number of S^2

スピン接続

$$\omega_\Delta = -\frac{\cos\theta}{2\sin\theta}\sigma_1\sigma_2\sin\theta d\phi = -\frac{1}{2}i\sigma_3\cos\theta d\phi,$$

から Levi-Civita 接続 ω，Riemann 曲率 R が得られる．

$$\omega = \begin{pmatrix} 0 & -\cos\theta d\phi \\ \cos\theta d\phi & 0 \end{pmatrix}$$

$$\frac{R}{2\pi} = \frac{d\omega + \omega^2}{2\pi} = \begin{pmatrix} 0 & \frac{\sin\theta}{2\pi}d\theta d\phi \\ -\frac{\sin\theta}{2\pi}d\theta d\phi & 0 \end{pmatrix}$$

Euler class of S^2

S^2 の Euler 数は

$$\chi(S^2) = \int_{S^2}\frac{\sin\theta}{2\pi}d\theta d\phi = 2.$$

で与えられる．

Edge state

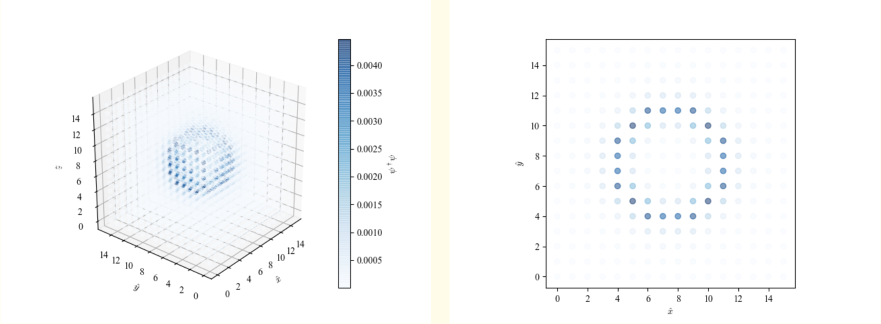


Fig 5: The Edge mode when $ma = 0.7, r_0 = L/4, N = 16$.

Weyl fermion

For $f : \mathbb{R}^{2n+1} \to \mathbb{R}$, massive Dirac operator

$$D = \sum_{i=1}^{2n+1} \gamma^i \frac{\partial}{\partial x^i} + m\,\mathrm{sign}(f)$$

$$\{\gamma^I, \gamma^J\} = 2\delta^{IJ},\ (I, J = 1, \cdots, n+1)$$

is not hermitian. In the large m limit, a Weyl fermion described by

$$\not{D}_Y^+ = \not{D}_Y \frac{1}{2}(1 + \gamma_{\mathrm{normal}})$$

is localized at $Y = \{f = 0\}$.

topological insulator との対応

Domain-wall がないとき、運動量空間で

$$H(k) = \bar{\gamma}\left(\sum_{j=1}^{n+1}\left(\gamma^j i \sin k_j + 1 - \cos k_j\right) + am\right)$$

となる。この演算子は下の表の赤文字に属す。

$n+1$	0	1	2	3	4	5	6	7	8
AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Chirality of Edge mode

Edge mode ψ is an eigenstate $\gamma^{n+1} = +1$.
$\longrightarrow$ ψ looks chiral for observers on $\mathbb{R}^{n+1}$.

However, ψ is regarded as Dirac fermion for observers on Y .

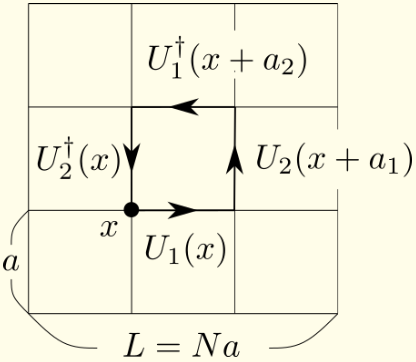
A chiral fermion on $Y \subset \mathbb{R}^{2m+1}$ is described by

$$D = \sum_{i=1}^{2m+1} \gamma^i \frac{\partial}{\partial x^i} + m\text{sign}(f)$$

$$\simeq \underbrace{\gamma^a \left(e_a + \frac{1}{4} \sum_{bc} \omega_{bc,a} \gamma^b \gamma^c \right)}_{\not{D}^Y} + \gamma^{2m+1} \frac{\partial}{\partial t} + F + m\text{sign}(f)$$

In the large m limit, $D \to \not{D}^Y_+ = \not{D}^Y \frac{1}{2}(1 + \gamma^{2m+1})$ and localized at $Y = \{f = 0\}$.

Gauge field on lattice



Gauge field A:

$$U_\mu(x) = P\exp\left(\int_x^{x+a_\mu} A\right) \simeq \exp(aA_\mu).$$

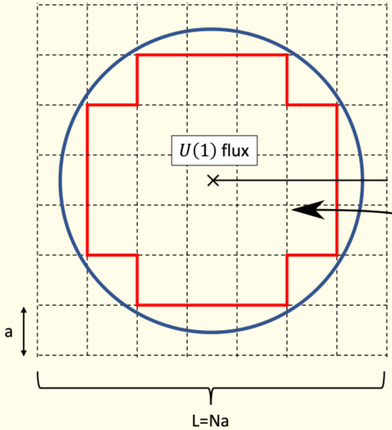
Covariant derivative:

$$(\nabla_\mu\psi)_x = U_\mu(x+a_\mu)\psi_{x+a_\mu} - \psi_x$$

The curvature F is determined by

$$\mathrm{tr}\Big(U_\nu^\dagger(x)U_\mu^\dagger(x+a_\nu)U_\nu(x+a_1)U_\mu(x)\Big) \simeq \exp\big(a^2F_{\mu\nu}\big).$$

S^1 domain-wall with $U(1)$ flux

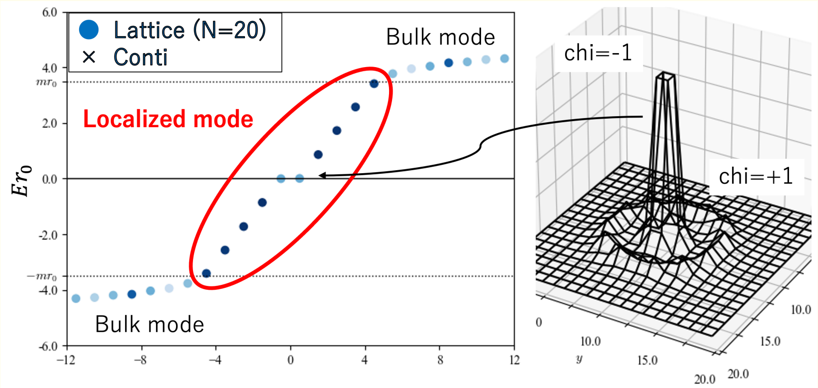


When a state crosses this line, the state get -1

$$\nabla_2\psi_{(x,y)} = -\psi_{(x,y+a)} - \psi_{(x,y)}$$

$\longrightarrow$Edge modes are periodic effectively.

Spectrum and zero modes



- Two zero modes appear at the domain-wall and the flux!!
- The flux mode tunnels with an edge mode.
- The flux has chirality -1 (Edge modes has chirality $+1$).

Equivalence principle

Equivalence principle: A person in free fall does not feel gravity.

An action for particle bound to a sphere is

$$S = \int dt \Big(\frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \lambda(x^2 + y^2 + z^2 - r_0^2) \Big)$$

$$\longrightarrow \underbrace{S_{eff} = \int dt \frac{m}{2} r_0^2 (\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2).}_{\text{Action on a Sphere}}$$

Particles bound to a space feel the gravity of that space.