Resurgence structure in large-N sigma model

-how renormalon ambiguity is canceled-

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based on the collaboration with T.Fujimori, H.Nishimura, M.Nitta, N.Sakai
JHEP 06(2022) 151 [arXiv:2112.13999]

Review

Path integral and Saddle points

$$Z = \int \mathcal{D}\phi \, \exp(-S[\phi]) = \sum_{\sigma \in \text{saddles}} Z_\sigma$$

: stationary points

Trivial (perturbative) saddle

$$Z_0 = \sum_{q=0}^{\infty} a_q g^{2q}$$

Perturbative series

Nontrivial saddles $\frac{\delta S}{\delta \phi} = 0$

$$Z_\sigma \propto e^{-S_{\text{sol}}} \sim e^{-\frac{A}{g^2}}$$

Non-perturbative contribution

<u>Relation between Pert. and Non-pert.</u>

$$\sum_{q=0} a_q g^{2q} \quad \not\longleftrightarrow \quad \exp\left[-\frac{A}{g^2}\right]$$

Perturbative series **Non-perturbative contribution**

"They are not connected ?
We just have independent contributions ?"

No, it is not correct !

Perturbation and Borel resummation

$$\left[H_0 + g^2 H_{\mathrm{pert}}\right]\psi(x) = E\psi(x)$$

$$P(g^2) = \sum_{q=0}^{\infty} a_q g^{2q}$$ Perturbative series is often divergent factorially $$a_q \propto q!$$

・Construct an **analytic function** from asymptotic series

Borel resummation：Analytic function which has original perturbative series as asymptotic series

Note that the analytic function is not unique for one asymptotic series.

Perturbation and Borel resummation

$$\left[H_0 + g^2 H_{\rm pert}\right] \psi(x) = E\psi(x)$$

$$P(g^2) = \sum_{q=0}^{\infty} a_q g^{2q}$$ Perturbative series is often divergent factorially $$a_q \propto q!$$

$$\Rightarrow \quad BP(t) := \sum_{q=0}^{\infty} \frac{a_q}{q!} t^q$$ Borel transform

$$\Rightarrow \quad \mathbb{B}(g^2) = \int_0^{\infty} \frac{dt}{g^2} e^{-t/g^2} BP(t)$$ Borel resummation

Perturbation and Borel resummation

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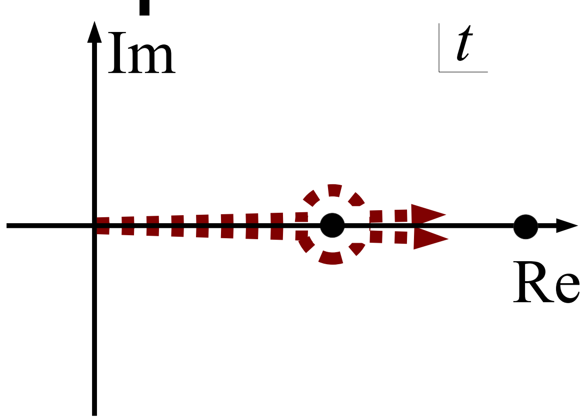
Borel transform can have singularities on positive real axis

$$BP(t) := \sum_{q=0}^{\infty} \frac{a_q}{q!} t^q$$



$$\Rightarrow \quad \mathbb{B}(g^2 e^{\mp i\epsilon}) = \int_0^{\infty e^{\pm i\epsilon}} \frac{dt}{g^2} e^{-\frac{t}{g^2}} BP(t)$$

Singularities on positive real axis leads to ambiguity

Perturbation and Borel resummation

$$\left[H_0 + g^2 H_{\rm pert}\right]\psi(x) = E\psi(x)$$

$$P(g^2) = \sum_{q=0}^{\infty} a_q g^{2q}$$

Perturbative series is often divergent factorially

$$a_q \propto q!$$

$$\mathbb{B}(g^2 e^{\mp i\epsilon}) = \mathrm{Re}[\mathbb{B}(g^2)] \pm i\mathrm{Im}[\mathbb{B}(g^2)]$$

$$\mathrm{Im}[\mathbb{B}(g^2)] \approx e^{-\frac{A}{g^2}}$$

This should be cancelled by that from non-perturbative contribution!

Non-perturbative effect reappears in perturbative calculation through imaginary ambiguity !

Comment on Borel resummation

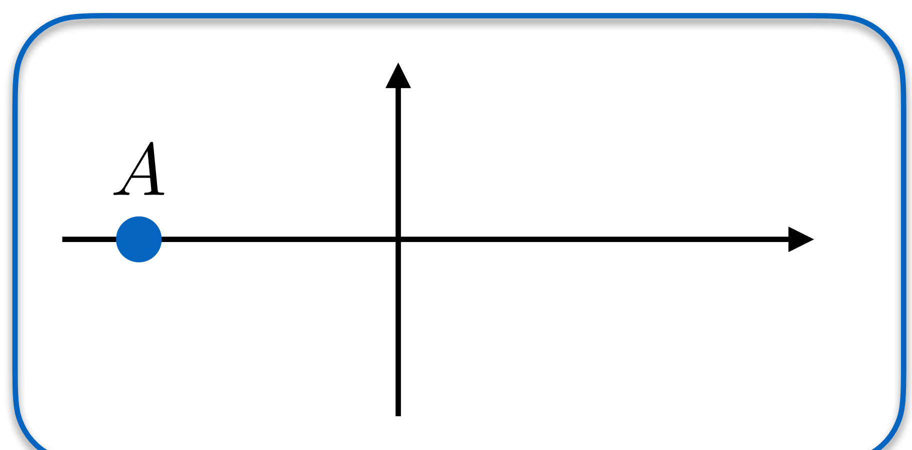
$$\sum_{n=0}^{\infty} a_n \lambda^n = \begin{cases} +\int_0^{+\infty} dt\, e^{-t/\lambda} B(t) & \text{for } \lambda > 0 \\ -\int_{-\infty}^{0} dt\, e^{-t/\lambda} B(t) & \text{for } \lambda < 0 \end{cases} \quad \text{with} \quad B(t) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n)} t^{n-1}$$

When we redefine $t/\lambda \to t$,

Integration should be performed on positive real axis even for $\lambda<0$,

where $B(\lambda t)$ has singularities on real axis.

$$\Rightarrow \sum_{n=0}^{\infty} a_n \lambda^n = \lambda \int_0^{+\infty} dt\, e^{-t} B(t\lambda)$$



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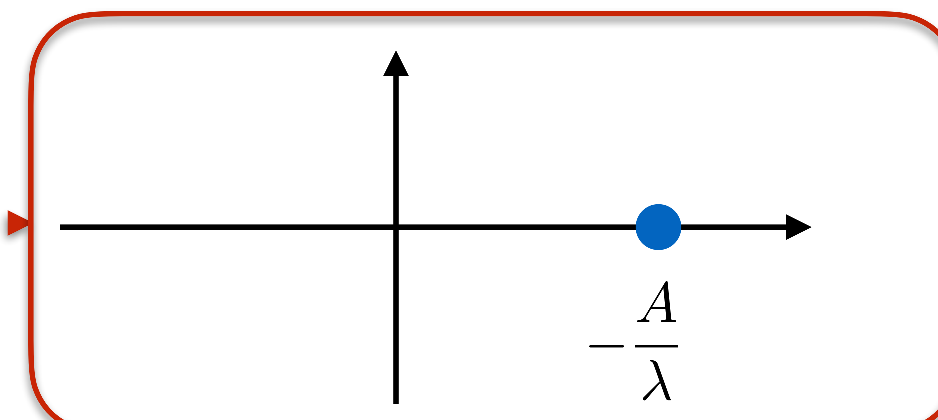


Resurgent structure in integral

For review for physicists,
Cherman, Dorigoni, Unsal(14)

In integral, original contour decomposes into **steepest decent contours** (Lefschetz thimbles) associated with **complex saddles**

⇨ **Thimbles associated with distinct saddles have nontrivial relation via Stokes phenomena**

- **Airy integral**

$$\mathrm{Ai}(g^{-2}) = \int_{-\infty}^{\infty} d\phi \exp\left[-i\left(\frac{\phi^3}{3} + \frac{\phi}{g^2}\right)\right]$$

Resurgent structure in integral

Complex saddle contributions in thimble decomposition
(Steepest descent method)

- **Airy integral**

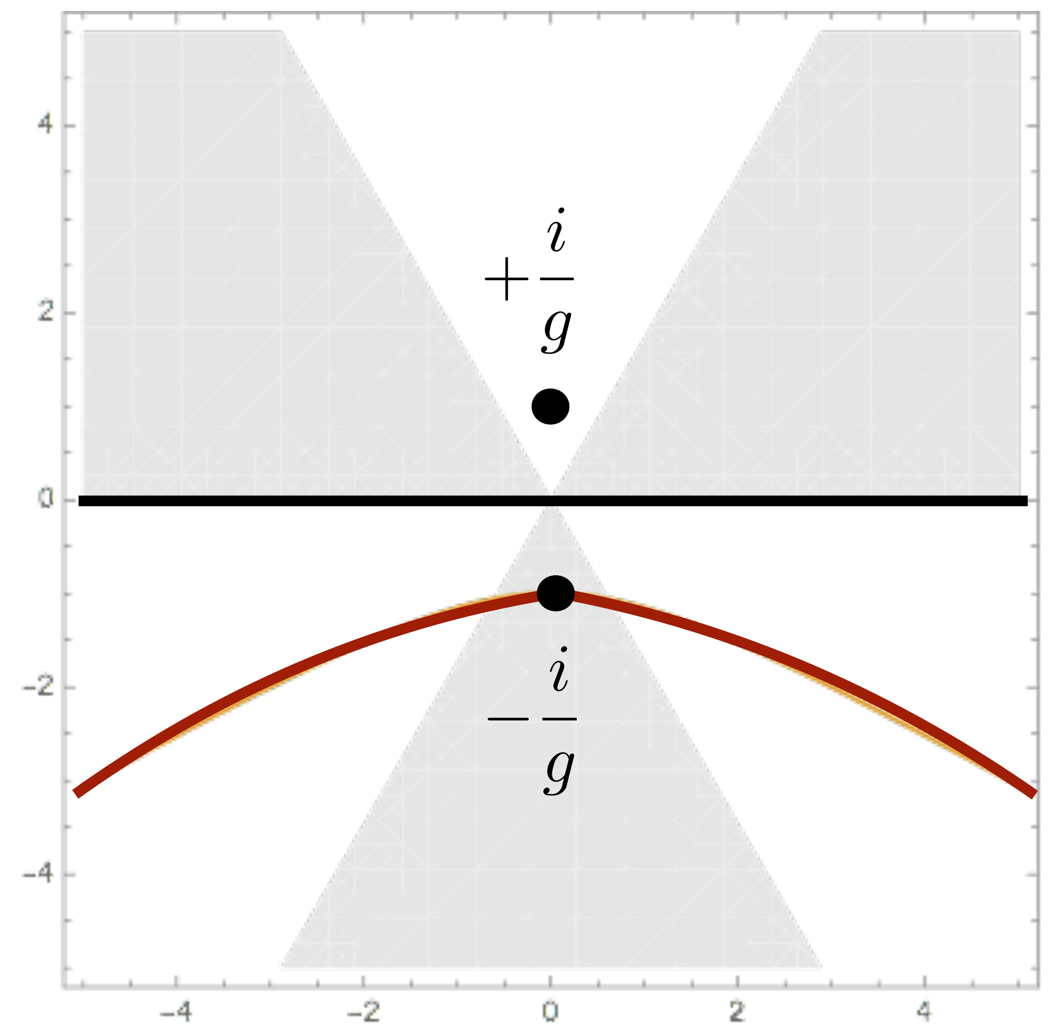
$$\mathrm{Ai}(g^{-2}) = \int_{-\infty}^{\infty} d\phi \exp\left[-i\left(\frac{\phi^3}{3} + \frac{\phi}{g^2}\right)\right]$$

complex saddle points $\phi = \pm\frac{i}{g}$

Steepest descent method :
original contour is decomposed into thimbles associated with saddle points.

$$\mathcal{C} = \sum_{\sigma} n_{\sigma} \mathcal{J}_{\sigma}$$

Steepest descent contour
= Thimble



$\arg[g^2] = 0+$

Resurgent structure in integral

Complex saddle contributions in thimble decomposition
(Steepest descent method)

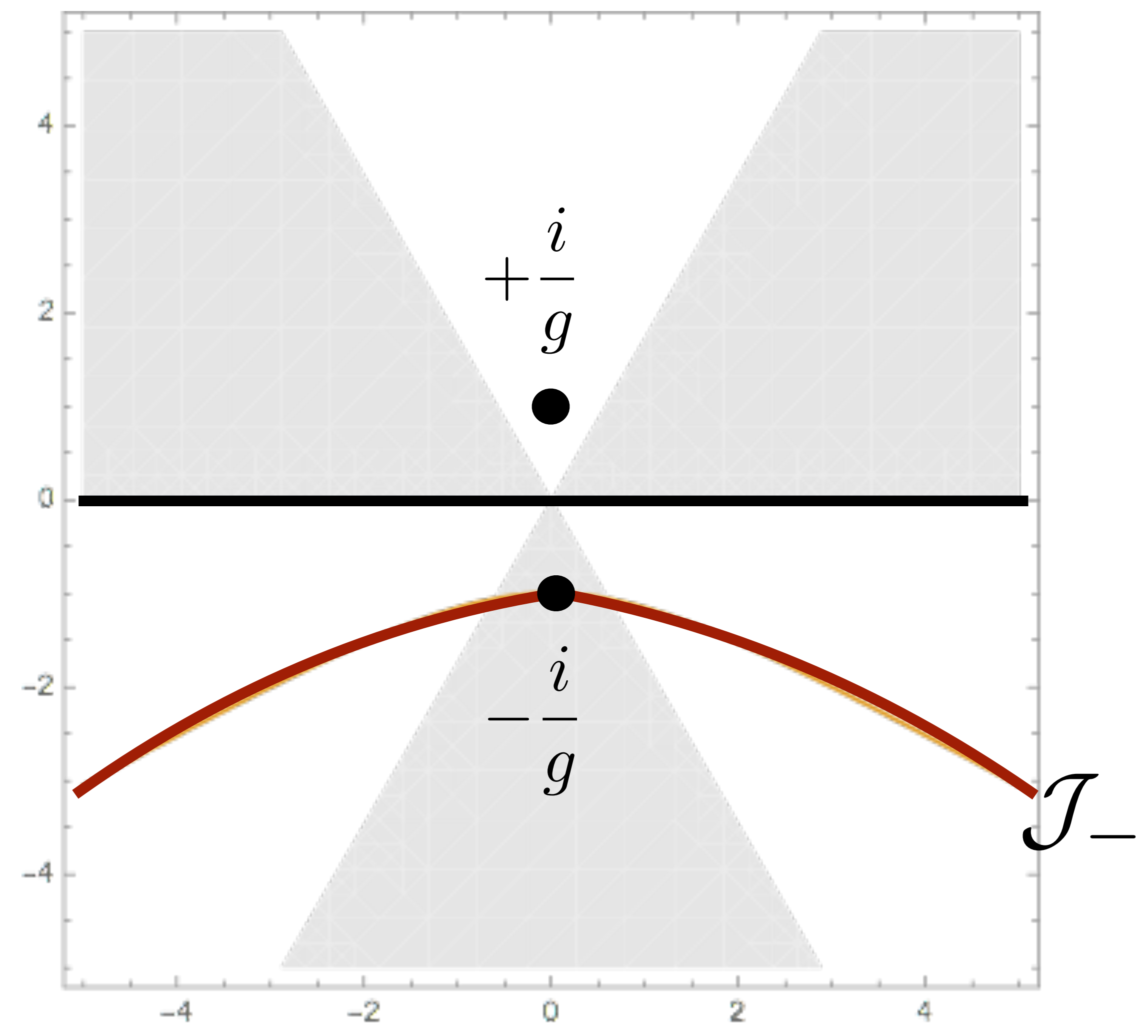
- **Airy integral**

$$\mathrm{Ai}(g^{-2}) = \int_{-\infty}^{\infty} d\phi \exp\left[-i\left(\frac{\phi^3}{3} + \frac{\phi}{g^2}\right)\right]$$

- $\mathcal{J}_\sigma$ $\quad \mathrm{Im}[S] = \mathrm{Im}[S_0]$, $\quad \mathrm{Re}[S] \leq \mathrm{Re}[S_0]$ **Thimble**

- $n_\sigma = \langle \mathcal{K}_\sigma, \mathcal{C} \rangle$ **Intersection number of dual thimble $\mathcal{K}$ and original contour**

$$\Rightarrow \quad \mathcal{C} = \sum_\sigma n_\sigma \mathcal{J}_\sigma$$



$\arg[g^2] = 0+$

Resurgent structure in integral

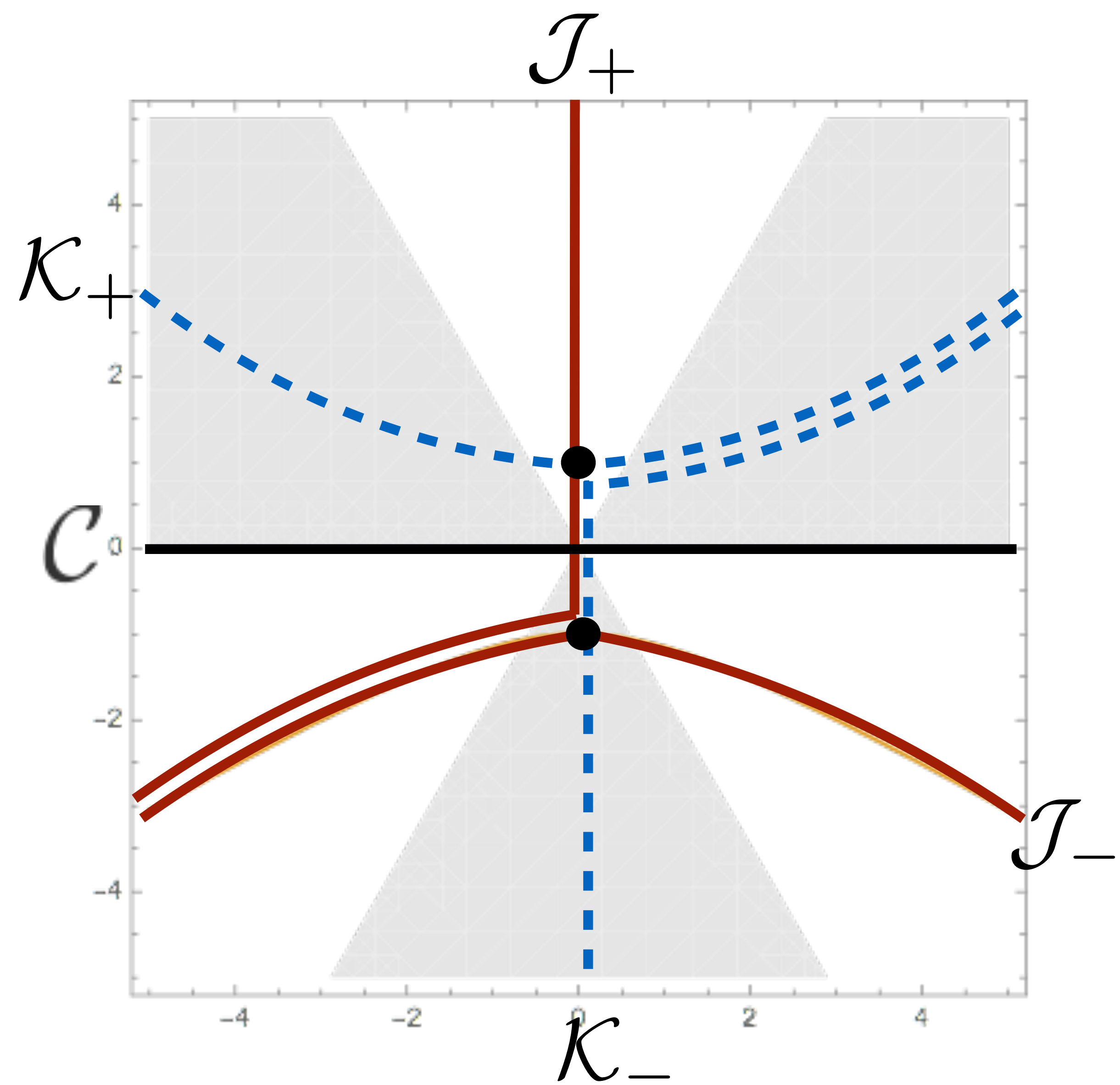
Complex saddle contributions in thimble decomposition (Steepest descent method)

- **Airy integral** $\arg[g^2] = 0+$

$$n_+ = \langle \mathcal{K}_+, \mathcal{C} \rangle = 0$$
$$n_- = \langle \mathcal{K}_-, \mathcal{C} \rangle = 1$$

$$\mathcal{C} = \sum_\sigma n_\sigma \mathcal{J}_\sigma \Rightarrow \boxed{\mathcal{C} = \mathcal{J}_-}$$

valid decomposition till $\arg[g^2] = \frac{2\pi}{3} -$



$\arg[g^2] = 0+$

Resurgent structure in integral

Complex saddle contributions in thimble decomposition
(Steepest descent method)

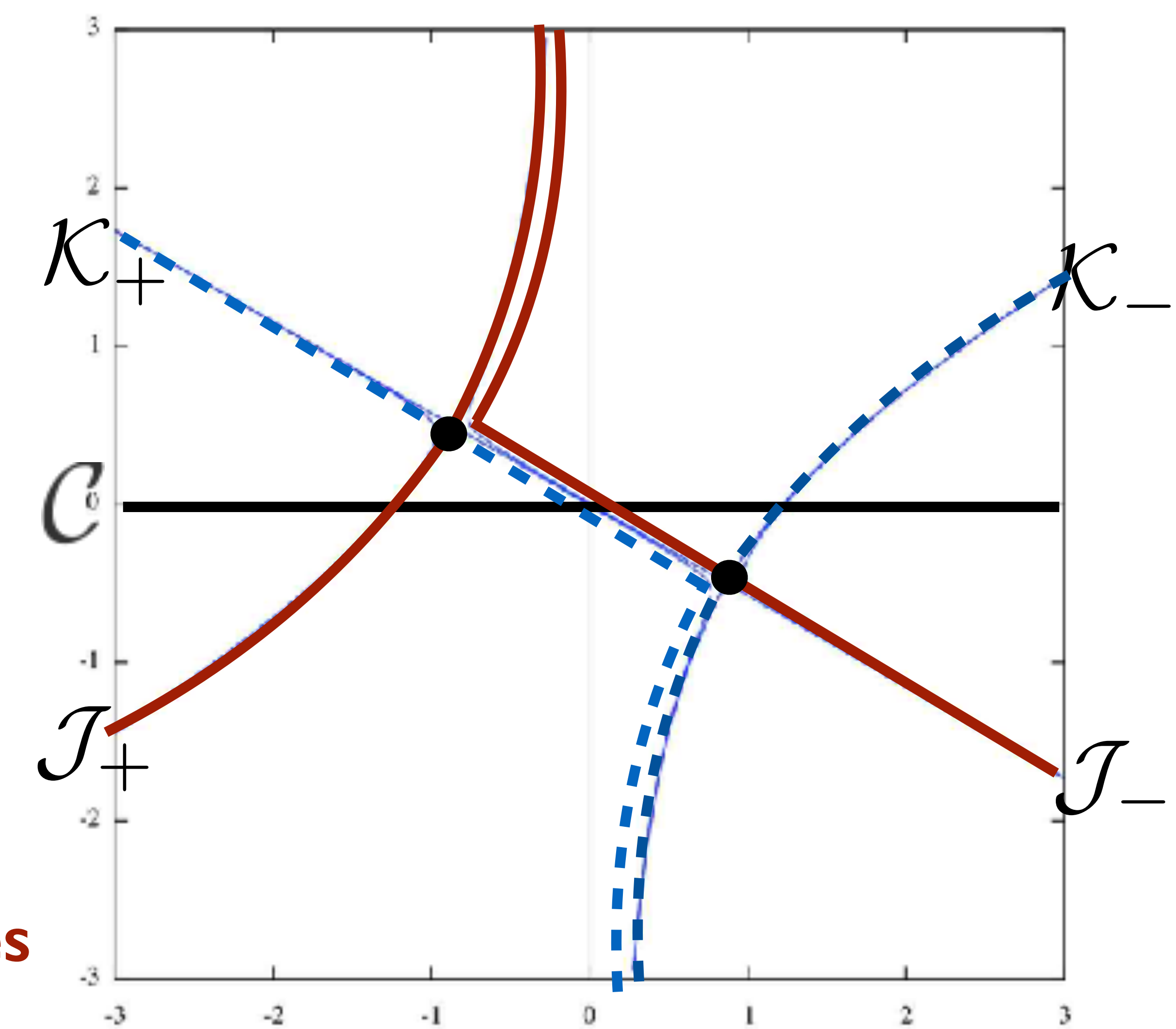
- **Airy integral** $\arg[g^2] = \frac{2\pi}{3}+$

$$n_+ = \langle \mathcal{K}_+, \mathcal{C} \rangle = 1$$

$$n_- = \langle \mathcal{K}_-, \mathcal{C} \rangle = 1$$

$$\mathcal{C} = \sum_\sigma n_\sigma \mathcal{J}_\sigma \Rightarrow \boxed{\mathcal{C} = \mathcal{J}_- + \mathcal{J}_+}$$

Stokes phenomenon : at special arg[g^2], thimble decomposition discretely changes



$\arg[g^2] = \frac{2\pi}{3}+$

Resurgent structure in integral

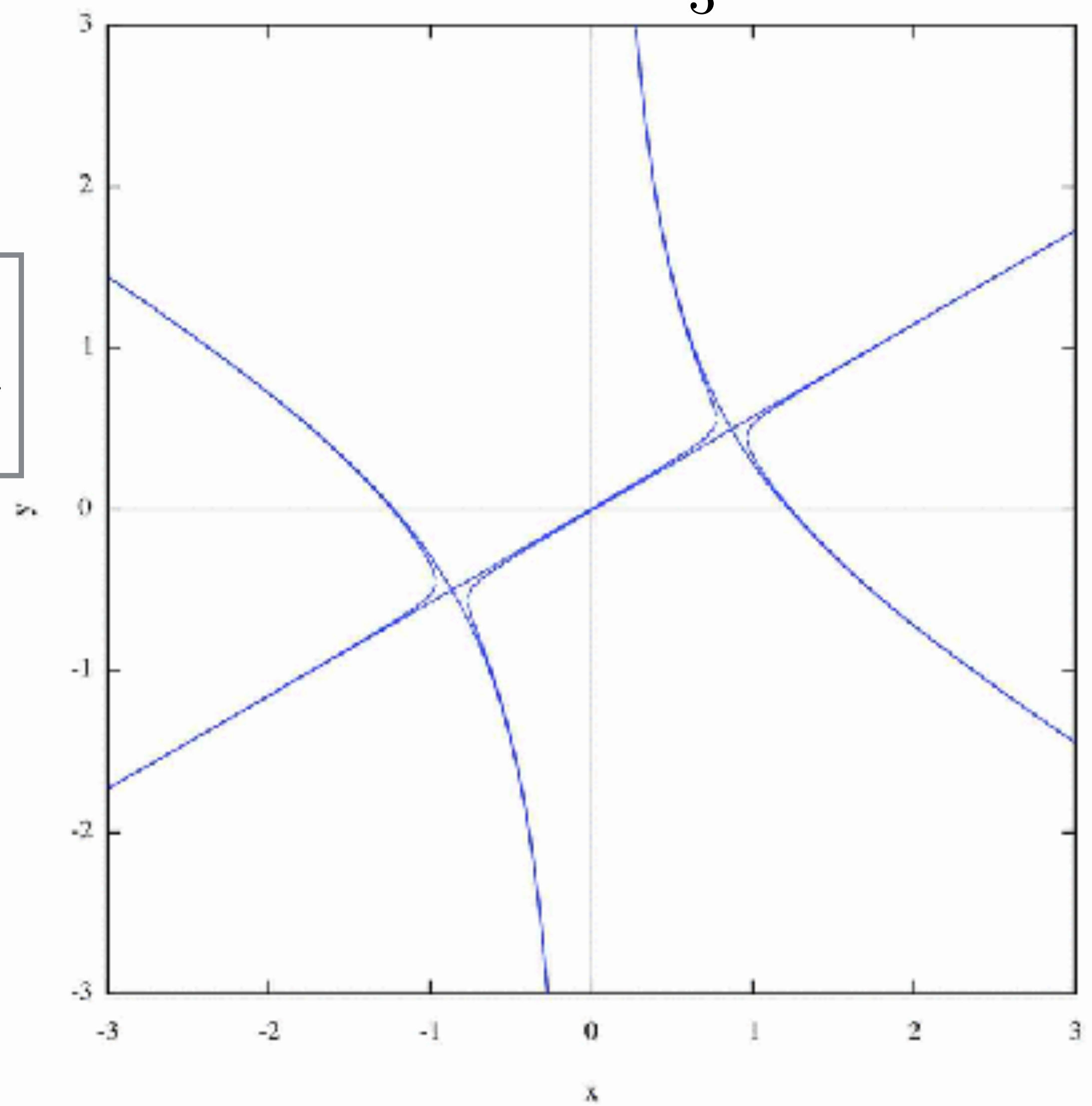
Complex saddle contributions in thimble decomposition (Steepest descent method)

- Airy integral

$$\arg[g^2] = \frac{2\pi}{3}^{-} \qquad \arg[g^2] = \frac{2\pi}{3}^{+}$$

$$\boxed{\mathcal{C} = \mathcal{J}_-} \Leftrightarrow \boxed{\mathcal{C} = \mathcal{J}_- + \mathcal{J}_+}$$

$\arg[g^2] = -\frac{2\pi}{3} \to \pi$



* Thimble decomposition is discretely changed at Stokes line.
* Airy function is continuous even at the Stokes line.

$$\mathcal{J}_-\left[\frac{2\pi}{3}^{-}\right] = \mathcal{J}_-\left[\frac{2\pi}{3}^{+}\right] + \mathcal{J}_+$$

Resurgent structure in integral

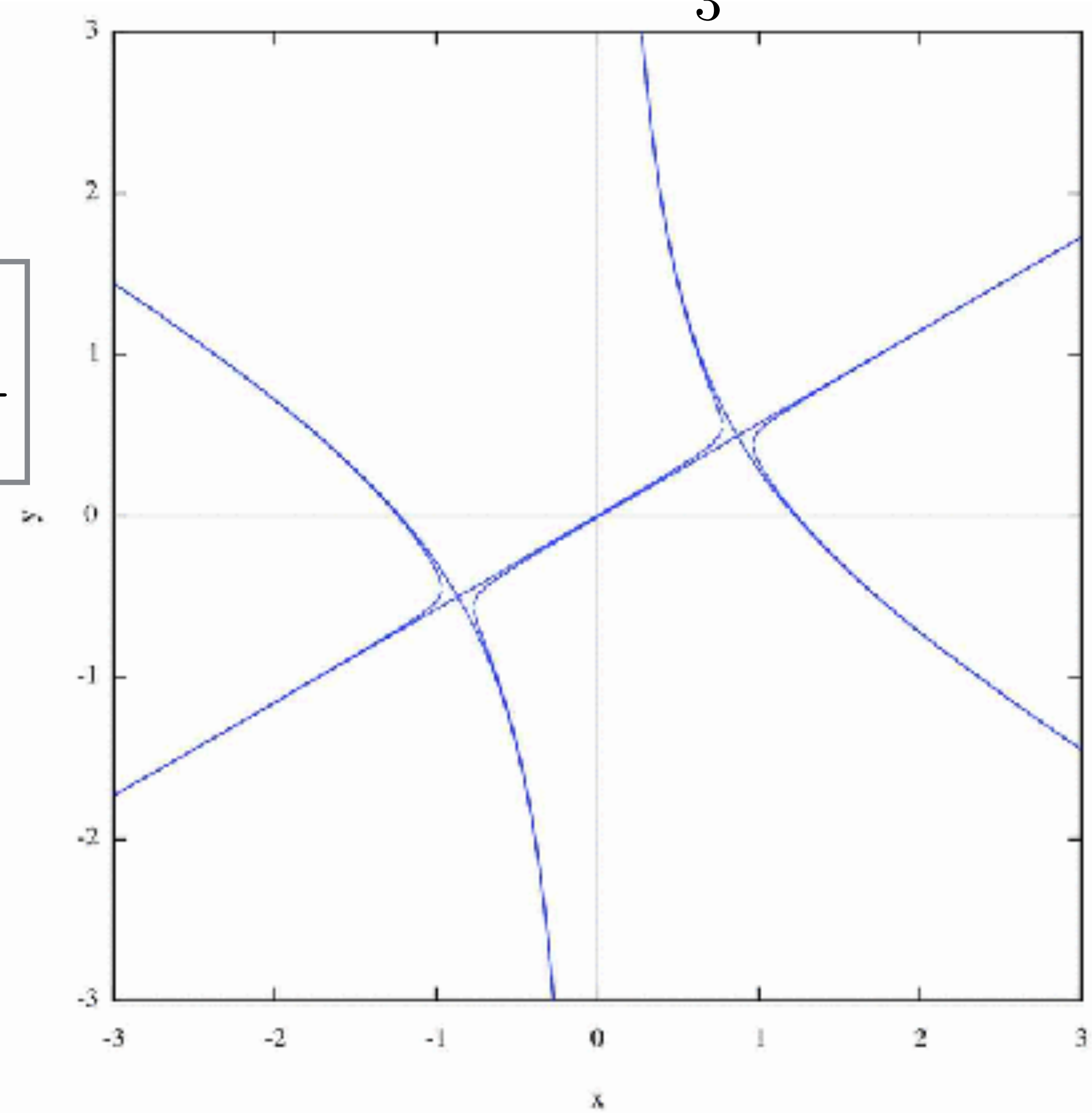
Complex saddle contributions in thimble decomposition (Steepest descent method)

- Airy integral

$$\arg[g^2] = \frac{2\pi}{3}- \qquad \arg[g^2] = \frac{2\pi}{3}+$$

$$\boxed{\mathcal{C} = \mathcal{J}_-} \iff \boxed{\mathcal{C} = \mathcal{J}_- + \mathcal{J}_+}$$

$$\arg[g^2] = -\frac{2\pi}{3} \to \pi$$



* Thimble decomposition is discretely changed at Stokes line.

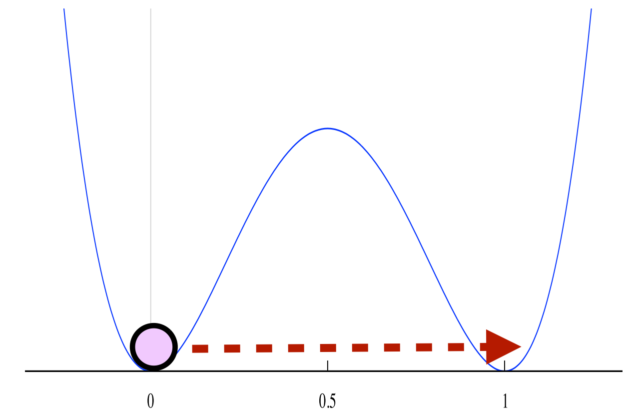
* Airy function is continuous even at the Stokes line.

Two thimbles have resurgent relation via ambiguity due to Stokes phenomena !

Example in Double-well QM

Bender-Wu(73)
Bogomolny(77) Zinn-Justin(81)

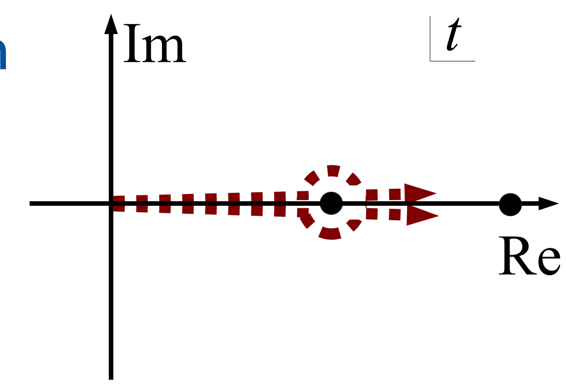
- Perturbation in Double-well QM



$$H = \frac{p^2}{2} + \frac{x^2(1-gx)^2}{2}$$

$$E_{0,pert} = \sum_{q=0} a_q g^{2q} \qquad a_q = -\frac{3^{q+1}}{\pi} q! \qquad (\, q \gg 1 \,)$$

coming from # of Feynman diagram



$$\Rightarrow \quad BP_{pert}(t) = \frac{1}{\pi} \frac{1}{t - 1/3}$$

Singularity on positive real axis

$$\Rightarrow \quad \mathrm{Im}[\mathbb{B}_{pert}(g^2 e^{\mp i\epsilon})] = \mathrm{Im}\left[\int_0^{\infty e^{\pm i\epsilon}} \frac{dt}{g^2} \frac{e^{-\frac{t}{g^2}}}{\pi} \frac{1}{t-1/3} \right] = \mp \frac{e^{-\frac{1}{3g^2}}}{g^2}$$

Instanton solution $\quad x(\tau) = \frac{1}{2g}\left(1 + \tanh \frac{\tau - \tau_{\mathcal{I}}}{2}\right) \qquad \boxed{S_I = \frac{1}{6g^2}}$

twice

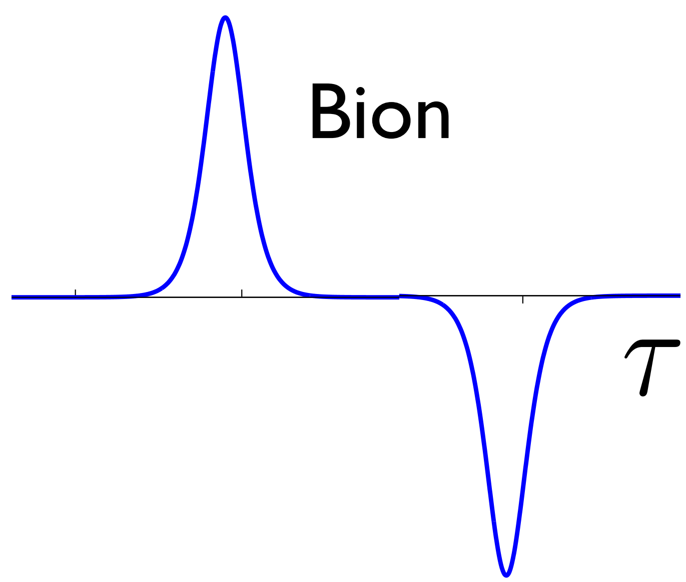
Instanton-antiinstanton configuration = Bion

Unsal (07)
Dunne, Unsal (12)

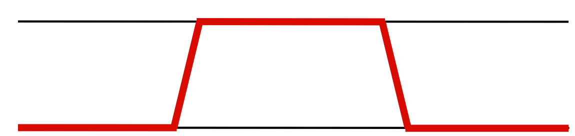
$$x_{\mathcal{I}\bar{\mathcal{I}}} = \frac{1}{2g}\left[\tanh\frac{\tau-\tau_{\mathcal{I}}}{2} - \tanh\frac{\tau-\tau_{\bar{\mathcal{I}}}}{2}\right]$$

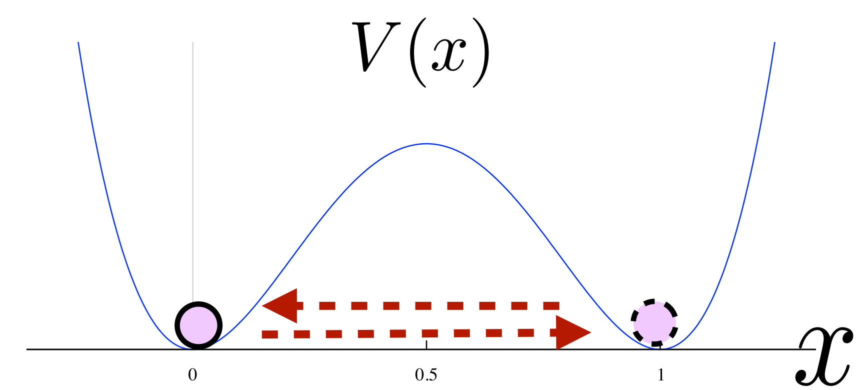
Precisely, bions are complex euclidean solutions

Behtash, et.al. (15)



$[\mathcal{I}\bar{\mathcal{I}}]$

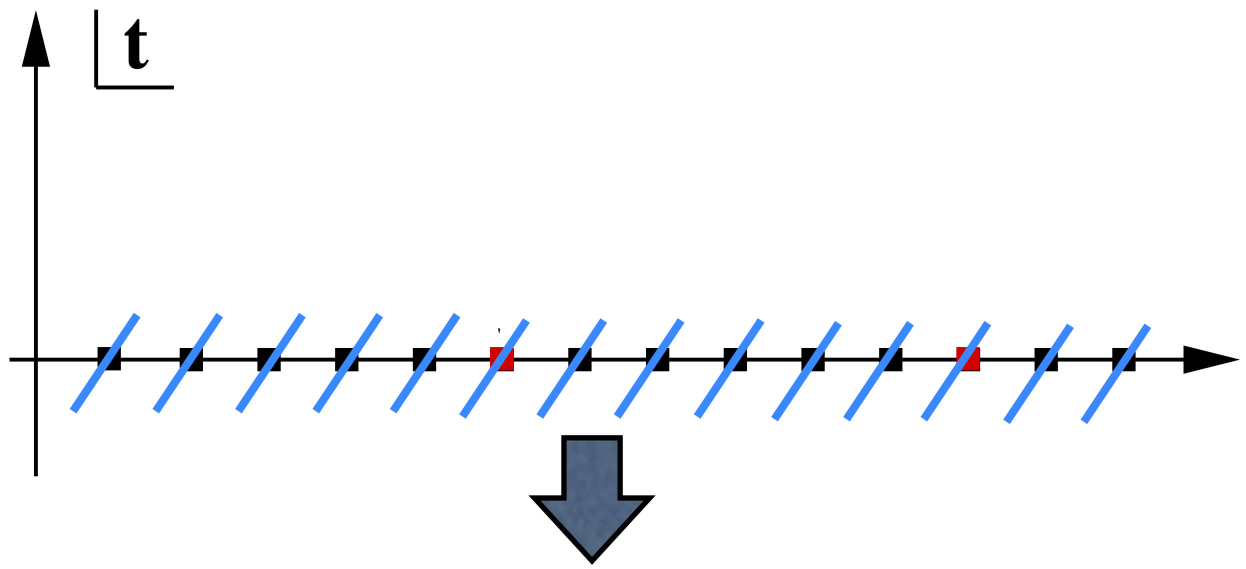




$$\boxed{E_b \approx -\lim_{\beta\to\infty}\frac{1}{\beta}\frac{Z_b}{Z_0} = \frac{e^{-2S_I}}{\pi g^2}\left[\gamma + \log\frac{2}{g^2} \pm i\pi\right]}$$

cancels the imaginary ambiguity $\mp\dfrac{e^{-2S_I}}{g^2}$ in perturbation

Resurgent trans-series in quantum mechanics



- **All ambiguities are cancelled in trans-series of complex solutions**

$$0 = \mathrm{Im}\left(\mathbb{B}_{[0,0]} + \mathbb{B}_{[2,0]}[\mathcal{I}\bar{\mathcal{I}}] + \mathbb{B}_{[4,0]}[\mathcal{I}\mathcal{I}\bar{\mathcal{I}}\bar{\mathcal{I}}] + \ldots\right)$$

Cancellation of imaginary ambiguity

Fujimori, Kamata, TM, Nitta, Sakai (16)(17)
Sueishi, Kamata, Misumi, Unsal (20)

- **Exact result is given as the trans-series of saddle contributions**

$$F(g^2) \approx \sum_{q=0}^{\infty} c_{(0,q)} g^{2q} + \sum_{n=1}^{\infty} e^{-nA/g^2} \sum_{q=0}^{\infty} c_{(n,k)} g^{2q}$$

Exact result as trans-series

Resurgent structure

$$\mathcal{S}_{+}\Phi_0(z) - \mathcal{S}_{-}\Phi_0(z) \approx \mathfrak{s} e^{-Az} \mathcal{S}\Phi_1(z)$$

Perturbative imaginary ambiguity

Non-perturbative effect

Perturbative series include nonpert. information !

1. We could derive non-perturbative physics from perturbative theory.
2. We could define QFT through perturbative series and semiclassical (trans-series) expansion.

Applications

- Aymptotically-free field theory Dunne,Unsal(12~), Cherman, Dorigoni, Sulejmanpasic, Tanizaki… TM, Nitta, Sakai(14~),…
- SUSY and String theories Marino(08~), Schiappa, Aniceto, Honda,…
- Integrable models & High-T_c superconductor Marino, Reis(19~),….
- Hydrodynamics, Fluid dynamics Behtash, Kamata, Martinez, Shi(19),…
- Schwinger mechanism Taya, Fujimori, TM, Nitta, Sakai (20), Enomoto, Matsuda(20)
- Quantization conditions via exact-WKB, TBA equations Ito, Shu (19) Sueishi, Kamata, TM, Unsal(20~)…
- Phase transition Yoda, Honda, Fujimori, TM, Sakai (21)…

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Infrared renormalon in QCD

't Hooft(79)

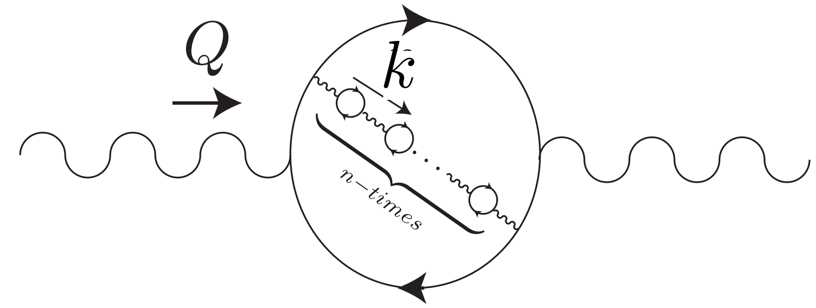
In asymptotically free QFT, another source of Im ambiguity exists.

◆Adler function and renormalon

$$D(Q^2) = 4\pi^2 \frac{d\Pi(Q^2)}{dQ^2}$$

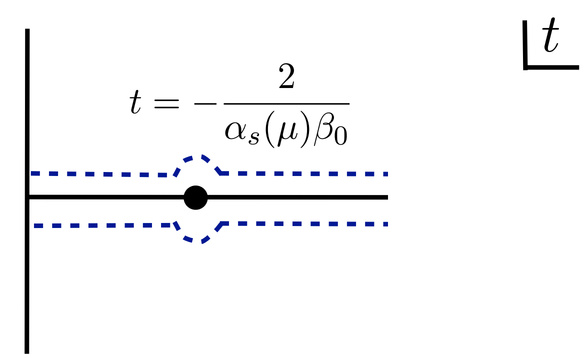
$$D(Q^2) = \sum_{n=0}^{\infty} \alpha_s \int_0^{\infty} d\hat{k}^2 \frac{F(\hat{k}^2)}{\hat{k}^2} \left[\beta_0 \alpha_s \log \frac{\hat{k}^2 Q^2}{\mu^2}\right]^n$$

$$\approx \alpha_s \sum_n \left(-\frac{\alpha_s \beta_0}{2}\right)^2 n! + \text{UV contr.}$$



$$\Rightarrow \quad BP(t) = \alpha_s(\mu) \sum_n \left(-\frac{\alpha_s(\mu)\beta_0 t}{2}\right)^n = \frac{\alpha_s(\mu)}{1 + \alpha_s(\mu)\beta_0 t/2}$$

$$\Rightarrow \quad t = -\frac{2}{\alpha_s(\mu)\beta_0}$$ Singularity on real axis ($\beta_0 = -\frac{11N_c - 2N_f}{12\pi}$)



$$\Rightarrow \quad \mathbf{B}(\alpha_s) = \mathrm{Re}\mathbf{B} \pm \frac{i\pi}{\beta_0} e^{\frac{2}{\alpha_s \beta_0}} \approx \left(\frac{\Lambda_{QCD}}{Q}\right)^4$$

QCD scale & low-energy physics
→Renormalon (survive in large N)

How is the renormalon ambiguity cancelled?

How is IR-renormalon ambiguity cancelled ?

1. There have been intensive studies on this subject.

   Dunne, Unsal (12) TM, Nitta, Sakai (14) Anber, Sulejmanpasic(14) Fujimori, Kamata, TM, Nitta, Sakai (18) Ishikawa, Morikawa, Nakayama, Shibata, Suzuki, Takaura (19) Yamazaki, Yonekura(18)(19) Morikawa, Takaura (20) Fujimori, TM, Nishimura, Nitta, Sakai, (21)

2. There is big difference of renormalon properties between uncompactified and Z_N-twisted compactified theories.

   Ashie, Morikawa, Suzuki, Takaura (20) Morikawa, Takaura (20)

3. Renormalon imaginary ambiguities are cancelled by combined imaginary ambiguities from different semiclassical orders.

   Fujimori, TM, Nishimura, Nitta, Sakai, (21)

We will see the cancellation mechanism.

Large-N sigma model

Fujimori, TM, Nishimura, Nitta, Sakai, (21)

Large-N O(N) model

- Action of O(N) model

$$S=\frac{1}{2g^2}\int d^2x\left[(\partial_i\phi^a)^2+D\left\{(\phi^a)^2-1\right\}\right] \qquad a=1\dots N \qquad (\phi^a)^2=1$$

- Effective potential in large N

$$V_{\rm eff}(D)=\frac{N}{2}\left[\int\frac{d^2p}{(2\pi)^2}\log\left(p^2+D\right)-\frac{D}{\lambda}\right]$$

't Hooft coupling : $\lambda=g^2N$

UV subtraction and renormalizing coupling

$$V_{\rm eff}(D)=-\frac{N}{8\pi}D\left(\log\frac{D}{\Lambda^2}-1\right)$$

Dynamical scale : $\Lambda=\mu\exp\left(-\frac{2\pi}{\lambda_\mu}\right)$

$$\langle D\rangle=\Lambda^2$$

it works as a dynamical mass of ϕ

Large-N O(N) model

- Fluctuation of D $$D(x) = \Lambda^2 + \frac{\delta D(x)}{\sqrt{N}}$$

- 2-point function of Fluctuation of D

$$\langle \delta D(x) \delta D(0) \rangle = \int \frac{d^2p}{(2\pi)^2} e^{ip\cdot x} \Delta(p) \quad \xrightarrow{x \to 0} \quad \langle \delta D^2 \rangle_{\tilde{a}} \equiv \int_{|p|<\tilde{a}} \frac{d^2p}{(2\pi)^2} \Delta(p)$$

$\boxed{\text{UV cutoff } \tilde{a}}$

$$\Delta(p) \equiv \frac{8\pi\sqrt{p^2(p^2+4\Lambda^2)}}{s_p} \qquad s_p = 4\log\left(\sqrt{\frac{p^2}{4\Lambda^2}+1} + \sqrt{\frac{p^2}{4\Lambda^2}}\right)$$

- Exact result of the condensate Novikov, Shifman, Vainshtein, Zakharov (84)

$$\langle \delta D^2 \rangle_{\tilde{a}} = 2\Lambda^4 \int_0^{s_{\tilde{a}}} ds \, \frac{\cosh s - 1}{s} = 2\Lambda^4 \, \mathrm{Chin}(s_{\tilde{a}}) \qquad \mathrm{Chin}(s_{\tilde{a}}) = \mathrm{Chi}(s_{\tilde{a}}) - \log(s_{\tilde{a}}) - \gamma_E$$

No ambiguous and IR convergent

How to derive trans-series

We have two ways to study resurgent structure:

(1) Expand $\Delta(p)$ w.r.t. $(\Lambda/p)^2$ for $|p| \gg \Lambda$, leading to trans-series, and analytically continue to $|p| < \Lambda$ with IR cutoff a

Nonperturbative exponential : $\Lambda^2/p^2 = \exp(-4\pi/\lambda_p)$

$$\Delta(p) \equiv \frac{8\pi\sqrt{p^2\left(p^2+4\Lambda^2\right)}}{s_p} \qquad s_p = 4\log\left(\sqrt{\frac{p^2}{4\Lambda^2}+1}+\sqrt{\frac{p^2}{4\Lambda^2}}\right)$$

⇨ we can imitate semiclassical expansion

(2) Extract trans-series expression from exact result

$$\left\langle \delta D^2 \right\rangle_{\tilde{a}} = \Lambda^4 F\left(s_{\tilde{a}}\right) = 2\Lambda^4 \int_0^{s_{\tilde{a}}} ds\, \frac{\cosh s - 1}{s}$$

$$\left\langle \delta D^2 \right\rangle_{\tilde{a},a} = \Lambda^4 \left\{F\left(s_{\tilde{a}}\right) - F\left(s_a\right)\right\}$$

How to derive trans-series

(1) Expand $\Delta(p)$ w.r.t. $(\Lambda/p)^2$ for $|p| \gg \Lambda$

- Expansion w.r.t. $(\Lambda/p)^2$ and λ_p

$$\boxed{\Lambda^2/p^2 = \exp(-4\pi/\lambda_p) \qquad \lambda_p \equiv \frac{2\pi}{\log(p/\Lambda)}}$$

$$s_p = 4\log\left(\sqrt{\frac{p^2}{4\Lambda^2}+1}+\sqrt{\frac{p^2}{4\Lambda^2}}\right) = \frac{8\pi}{\lambda_p} + \frac{4\Lambda^2}{p^2} - \frac{6\Lambda^4}{p^4} + \mathcal{O}(\Lambda^6)$$

$\Rightarrow$ Trans-series (semiclassical) expansion of $\Delta(p)$

$$\Delta(p) = p^2 \sum_{l=0}^{\infty} \left(\frac{\Lambda}{p}\right)^{2l} f_l(\lambda_p) \qquad f_l(\lambda_p) = P_l(\Lambda\partial_\Lambda)\,\lambda_p.$$: polynomial of λ_p

$$P_l(t) \equiv \frac{(-1)^l}{l!}\left[(t+l+1)^{(l)} - 4l(t+l)^{(l-1)}\right] \quad \text{with} \quad (a)^{(l)} = \frac{\Gamma(a+l)}{\Gamma(a)}$$

How to derive trans-series

Trans-series (semiclassical) expansion of $<\delta D^2>$

we here introduce $\boxed{\text{IR cutoff } a}$ to regulate IR divergence

$$\langle \delta D^2 \rangle_{\tilde{a},a} \underset{\text{s.c.}}{=} \sum_{l=0}^{\infty} \Lambda^{2l} C_{2l}, \qquad C_{2l} = \int_{a<|p|<\tilde{a}} \frac{d^2p}{(2\pi)^2} p^{2-2l} f_l(\lambda_p),$$

Coupling expansion of each trans-series coefficient

$$C_{2l} = \sum_{n=0}^{\infty} \lambda_{\tilde{a}}^{n+1} c_{(2l,n)} \qquad \frac{\lambda_p}{4\pi} = \left[\frac{4\pi}{\lambda_{\tilde{a}}} + \log\left(\frac{p^2}{\tilde{a}^2}\right)\right]^{-1} = \sum_{n=0}^{\infty} \left(\frac{\lambda_{\tilde{a}}}{4\pi}\right)^{n+1} \left[-\log\left(\frac{p^2}{\tilde{a}^2}\right)\right]^n$$

Borel-resummation-like expression $\quad \tilde{t} = \log(\tilde{a}^2/p^2)$

$$C_{2l} = \frac{1}{4\pi} \int_0^{\log(\tilde{a}^2/a^2)} d\tilde{t} \, \left(\tilde{a}^2 e^{-\tilde{t}}\right)^{2-l} f_l\left(\frac{4\pi}{4\pi/\lambda_{\tilde{a}} - \tilde{t}}\right)$$

we need to complexify the coupling as $\lambda_{\tilde{a}} \to \lambda_{\tilde{a}} \pm i\epsilon$

How to derive trans-series

Result of imaginary ambiguities

$$\Lambda^2/p^2 = \exp(-4\pi/\lambda_p)$$

$$\mathrm{Im}\left\langle \delta D^2 \right\rangle_{\tilde{a},a} \underset{\text{s.c.}}{=} \pm\pi\left[\left(\tilde{a}^2 e^{-\frac{4\pi}{\lambda_{\tilde{a}}}}\right)^2 \Lambda^0 - 2\Lambda^4 + \left(\tilde{a}^2 e^{-\frac{4\pi}{\lambda_{\tilde{a}}}}\right)^{-2} \Lambda^8\right]\theta(\Lambda - a) = 0.$$

How to derive trans-series

$\boxed{\text{Result of imaginary ambiguities}}$

$$\Lambda^2/p^2 = \exp(-4\pi/\lambda_p)$$

$$\mathrm{Im}\left\langle \delta D^2 \right\rangle_{\tilde{a},a} \underset{\text{s.c.}}{=} \pm\pi \left[\underbrace{\left(\tilde{a}^2 e^{-\frac{4\pi}{\lambda_{\tilde{a}}}}\right)^2}_{\Lambda^4} \Lambda^0 - 2\Lambda^4 + \left(\tilde{a}^2 e^{-\frac{4\pi}{\lambda_{\tilde{a}}}}\right)^{-2} \Lambda^8 \right] \theta(\Lambda - a) = 0.$$

Known IR renormalon !

How to derive trans-series

Result of imaginary ambiguities

$\Lambda^2/p^2 = \exp(-4\pi/\lambda_p)$

$$\mathrm{Im}\,\langle \delta D^2 \rangle_{\tilde{a},a} \underset{\text{s.c.}}{=} \pm\pi \left[\underbrace{\left(\tilde{a}^2 e^{-\frac{4\pi}{\lambda_{\tilde{a}}}}\right)^2}_{\Lambda^4} \Lambda^0 - 2\Lambda^4 + \underbrace{\left(\tilde{a}^2 e^{-\frac{4\pi}{\lambda_{\tilde{a}}}}\right)^{-2}}_{\Lambda^{-4}} \Lambda^8 \right] \theta(\Lambda - a) = 0.$$

(1) Renormalon ambiguity on the trivial vacuum (order Λ^0) is cancelled not only by order Λ^4, but by combination of order Λ^4 and Λ^8 !

(2) The ambiguities emerge only for $a<\Lambda$!

How to derive trans-series

Result of imaginary ambiguities

$$\Lambda^2/p^2 = \exp(-4\pi/\lambda_p)$$

$$\mathrm{Im}\,\langle \delta D^2 \rangle_{\tilde{a},a} \underset{\text{s.c.}}{=} \pm\pi \left[\underbrace{\left(\tilde{a}^2 e^{-\frac{4\pi}{\lambda_{\tilde{a}}}}\right)^2}_{\Lambda^4} \Lambda^0 - 2\Lambda^4 + \underbrace{\left(\tilde{a}^2 e^{-\frac{4\pi}{\lambda_{\tilde{a}}}}\right)^{-2}}_{\Lambda^{-4}} \Lambda^8 \right] \theta(\Lambda - a) = 0.$$

(1) Renormalon ambiguity on the trivial vacuum (order Λ^0) is cancelled not only by order Λ^4, but by combination of order Λ^4 and Λ^8 !

(2) The ambiguities emerge only for $a<\Lambda$!

How does the term at order Λ^8 cancel the renormalon ambiguity?

Cancellation mechanism

- separate UV and IR contributions

$$C_{2l} = \int_a^{\tilde{a}} \frac{dp}{2\pi}\, p^{3-2l}\, f_l(\lambda_p) = \mathcal{C}_{2l}(p)\big|_a^{\tilde{a}} = \mathcal{C}_{2l}(\tilde{a}) - \mathcal{C}_{2l}(a),$$

ex.) l=0 (order Λ^0)

$$c_{(0,n)} = \int_{a<|p|<\tilde{a}} \frac{d^2p}{(2\pi)^2}\, p^2 \left(\frac{1}{4\pi}\log\frac{\tilde{a}^2}{p^2}\right)^n = \frac{\tilde{a}^4}{(8\pi)^{n+1}}\left[\underbrace{\Gamma(n+1)}_{\mathcal{C}_0(\tilde{a})} - \underbrace{\Gamma\left(n+1, 2\log\frac{\tilde{a}^2}{a^2}\right)}_{\mathcal{C}_0(a)}\right]$$

$$\Rightarrow \quad \mathcal{C}_0(p) = \tilde{a}^4 \sum_{n=0}^{\infty} \left(\frac{\lambda_{\tilde{a}}}{8\pi}\right)^{n+1} \Gamma\left(n+1, 2\log\frac{\tilde{a}^2}{p^2}\right)$$

Let us look into how the ambiguity emerges in each order.

Cancellation mechanism

Order Λ^0

$$\mathcal{C}_0(p) = \tilde{a}^4 \sum_{n=0}^{\infty} \left(\frac{\lambda_{\tilde{a}}}{8\pi}\right)^{n+1} \Gamma\left(n+1, 2\log\frac{\tilde{a}^2}{p^2}\right)$$

$$= -p^4 \int_0^\infty dt\, \frac{e^{-t}}{t - \frac{8\pi}{\lambda_p}} = p^4 e^{-8\pi/\lambda_p} \left[\gamma_E + \boxed{\log\left(-\frac{8\pi}{\lambda_p}\right)} - \mathrm{Ein}\left(-\frac{8\pi}{\lambda_p}\right)\right]$$

$\Rightarrow$ $\mathrm{Im}\, C_0 = \mathrm{Im}\, \mathcal{C}_0(\tilde{a}) - \mathrm{Im}\, \mathcal{C}_0(a) = \pm\{\pi - \pi\theta(a-\Lambda)\}\Lambda^4 = \underline{\pm\pi\Lambda^4\theta(\Lambda - a)}$

The ambiguities emerge only for $a < \Lambda$!
$\lambda_a < 0$

Known IR renormalon !

Order Λ^4

$$\mathcal{C}_4(p) = \boxed{-2\log\left(\frac{4\pi}{\lambda_p}\right)} - \frac{\lambda_p^2 - 2\pi\lambda_p}{8\pi^2}$$

$\Rightarrow$ $\mathrm{Im}\, C_4 = \mathrm{Im}\, \mathcal{C}_4(\tilde{a}) - \mathrm{Im}\, \mathcal{C}_4(a) = \underline{\mp 2\pi\theta(\Lambda - a)}.$

The ambiguities emerge only for $a < \Lambda$!
$\lambda_a < 0$

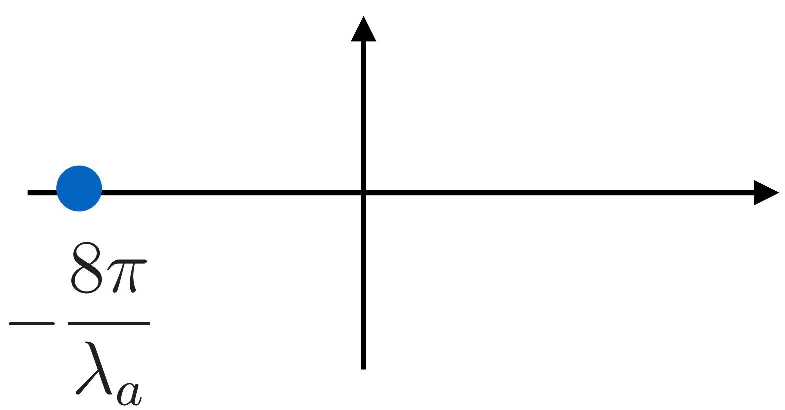
Cancellation mechanism

Order Λ^8

$$
\mathcal{C}_8(p) \supset \frac{1}{\tilde{a}^4} \sum_{n=0}^{\infty} \left(-\frac{\lambda_{\tilde{a}}}{8\pi} \right)^{n+1} \Gamma\left(n+1, -2\log\frac{\tilde{a}^2}{p^2} \right) = -\frac{1}{p^4} \underline{\int_0^\infty dt \, \frac{e^{-t}}{t + \frac{8\pi}{\lambda_p}}}.
$$

$$
= \frac{1}{\Lambda^4} \left[-\mathrm{Ein}\left(\frac{8\pi}{\lambda_p} \right) + \boxed{\log\left(\frac{8\pi}{\lambda_p} \right)} + \gamma_E \right]
$$

$$
\Rightarrow \quad \mathrm{Im}\, C_8 = \pm\theta(\Lambda - a)\frac{\pi}{\Lambda^4}.
$$



$a > \Lambda$

$\lambda_a > 0$ $\Rightarrow$ No ambiguity

- The ambiguity emerge only for $a < \Lambda$
- It is accompanied by $\exp(+8\pi/\lambda_a) \propto 1/\Lambda^4$

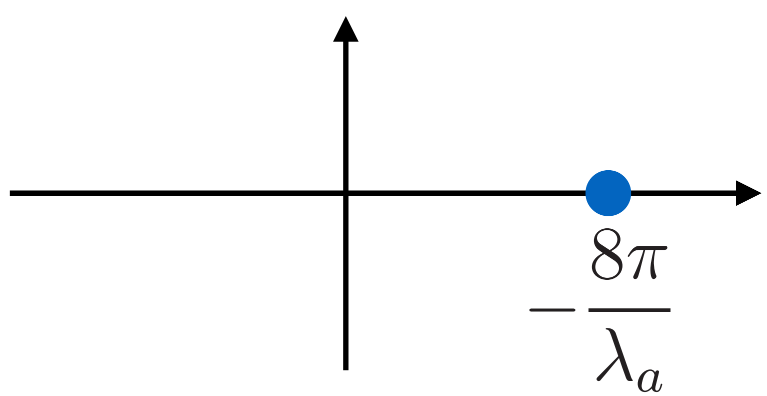
Cancellation mechanism

Order Λ^8

$$\mathcal{C}_8(p) \supset \frac{1}{\tilde{a}^4}\sum_{n=0}^{\infty}\left(-\frac{\lambda_{\tilde{a}}}{8\pi}\right)^{n+1}\Gamma\left(n+1, -2\log\frac{\tilde{a}^2}{p^2}\right) = \underline{-\frac{1}{p^4}\int_0^\infty dt\,\frac{e^{-t}}{t+\frac{8\pi}{\lambda_p}}}.$$

$$= \frac{1}{\Lambda^4}\left[-\mathrm{Ein}\left(\frac{8\pi}{\lambda_p}\right) + \boxed{\log\left(\frac{8\pi}{\lambda_p}\right)} + \gamma_E\right]$$

$$\Rightarrow \quad \mathrm{Im}\, C_8 = \pm\theta(\Lambda - a)\frac{\pi}{\Lambda^4}.$$



$a < \Lambda$

$\lambda_a < 0$

$$\Rightarrow \quad \pm\exp\left(+\frac{8\pi}{\lambda_a}\right) \propto \pm\frac{1}{\Lambda^4}$$

- The ambiguity emerge only for $a < \Lambda$
- It is accompanied by $\exp(+8\pi/\lambda_a) \propto 1/\Lambda^4$

Cancellation mechanism

- Resurgent structure in trans-series

$$
\begin{aligned}
\left\langle \delta D^2 \right\rangle_{\tilde{a},a} \underset{\text{s.c.}}{=} & \sum_{l=0}^{\infty} \Lambda^{2l} \Big[\{\mathcal{C}_{2l}(\tilde{a})\} - \{\mathcal{C}_{2l}(a)\} \Big] \\
= \; & \Lambda^0 \left[\tilde{a}^4 \left\{ e^{-8\pi/\lambda_{\tilde{a}}} \mathrm{Ei}\left(\frac{8\pi}{\lambda_{\tilde{a}}}\right) \right\} - a^4 \left\{ e^{-8\pi/\lambda_a} \mathrm{Ei}\left(\frac{8\pi}{\lambda_a}\right) \right\} \boxed{\pm i\pi\Lambda^4\theta(\Lambda - a)} \right] \\
& + \Lambda^2 \left[\tilde{a}^2 \left\{ \frac{\lambda_{\tilde{a}}}{2\pi} \right\} - a^2 \left\{ \frac{\lambda_a}{2\pi} \right\} \right] \\
& + \Lambda^4 \left[\tilde{a}^0 \left\{ \frac{\lambda_{\tilde{a}}}{4\pi} - \frac{\lambda_{\tilde{a}}^2}{8\pi^2} - 2\log\left(\frac{4\pi}{\lambda_{\tilde{a}}}\right) \right\} - a^0 \left\{ \frac{\lambda_a}{4\pi} - \frac{\lambda_a^2}{8\pi^2} - 2\log\left|\frac{4\pi}{\lambda_a}\right| \right\} \boxed{\mp 2\pi i\theta(\Lambda - a)} \right] \\
& + \Lambda^6 \left[\frac{1}{\tilde{a}^2} \left\{ -\frac{\lambda_{\tilde{a}}}{\pi} + \frac{\lambda_{\tilde{a}}^2}{24\pi^2} + \frac{\lambda_{\tilde{a}}^3}{24\pi^3} \right\} - \frac{1}{a^2} \left\{ -\frac{\lambda_a}{\pi} + \frac{\lambda_a^2}{24\pi^2} + \frac{\lambda_a^3}{24\pi^3} \right\} \right] \\
& + \Lambda^8 \left[\frac{1}{\tilde{a}^4} \left\{ e^{8\pi/\lambda_{\tilde{a}}} \mathrm{Ei}\left(-\frac{8\pi}{\lambda_{\tilde{a}}}\right) + \frac{11\lambda_{\tilde{a}}}{8\pi} + \frac{13\lambda_{\tilde{a}}^2}{96\pi^2} - \frac{\lambda_{\tilde{a}}^3}{16\pi^3} - \frac{\lambda_{\tilde{a}}^4}{64\pi^4} \right\} \right. \\
& \qquad \left. - \frac{1}{a^4} \left\{ e^{8\pi/\lambda_a} \mathrm{Ei}\left(-\frac{8\pi}{\lambda_a}\right) + \frac{11\lambda_a}{8\pi} + \frac{13\lambda_a^2}{96\pi^2} - \frac{\lambda_a^3}{16\pi^3} - \frac{\lambda_a^4}{64\pi^4} \right\} \boxed{\pm \frac{i\pi}{\Lambda^4}\theta(\Lambda - a)} \right] \\
& + \mathcal{O}(\Lambda^{10}),
\end{aligned}
$$

Three ambiguities are cancelled !

Comparison to exact result

(2) Extract trans-series expression from exact result

$$\left\langle \delta D^2 \right\rangle_{\tilde{a}} = \Lambda^4 F\left(s_{\tilde{a}}\right) = 2\Lambda^4 \int_0^{s_{\tilde{a}}} ds \, \frac{\cosh s - 1}{s}$$

$$s_{\tilde{a}} = \frac{8\pi}{\lambda_{\tilde{a}}} + u_{\tilde{a}}, \quad u_{\tilde{a}} = 4\log\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\Lambda^2}{\tilde{a}^2}}\right)$$

Introduce IR cutoff a to compare with the trans-series result

$$\left\langle \delta D^2 \right\rangle_{\tilde{a},a} = \Lambda^4 \left\{ F\left(s_{\tilde{a}}\right) - F\left(s_a\right) \right\}$$

$$F\left(\frac{8\pi}{\lambda_{\tilde{a}}}\right) = -\frac{\tilde{a}^4}{\Lambda^4}\int_0^\infty dt \frac{e^{-t}}{t - \frac{8\pi}{\lambda_{\tilde{a}}} \pm i0} + \left[2\log\left(\frac{\lambda_{\tilde{a}}}{8\pi}\right) - 2\gamma_{\mathrm{E}} \mp i\pi\right] - \frac{\Lambda^4}{\tilde{a}^4}\int_0^\infty dt \frac{e^{-t}}{t + \frac{8\pi}{\lambda_{\tilde{a}}}}$$

Let us look into why the ambiguities emerge for $a < \Lambda$

Comparison to exact result

Ambiguous

$$\Lambda^4 F(s_{\tilde a}) = \Lambda^0 \tilde a^4 \left\{ \boxed{-\int_0^\infty dt \frac{e^{-t}}{t - \frac{8\pi}{\lambda_{\tilde a}} \pm i0}} \right\} + \Lambda^2 \tilde a^2 \left\{ \frac{\lambda_{\tilde a}}{2\pi} \right\}$$

Ambiguous $\mp$

$$+\Lambda^4 \left\{ \frac{\lambda_{\tilde a}}{4\pi} - \frac{\lambda_{\tilde a}^2}{8\pi^2} + \boxed{2\log\left(\frac{\lambda_{\tilde a}}{8\pi}\right) - 2\gamma_{\mathrm{E}} \mp i\pi} \right\} + \frac{\Lambda^6}{\tilde a^2} \left\{ -\frac{\lambda_{\tilde a}}{\pi} + \frac{\lambda_{\tilde a}^2}{24\pi^2} + \frac{\lambda_{\tilde a}^3}{24\pi^3} \right\}$$

$$+\frac{\Lambda^8}{\tilde a^4} \left\{ \boxed{-\int_0^\infty dt \frac{e^{-t}}{t + \frac{8\pi}{\lambda_{\tilde a}}}} + \frac{11\lambda_{\tilde a}}{8\pi} + \frac{13\lambda_{\tilde a}^2}{96\pi^2} - \frac{\lambda_{\tilde a}^3}{16\pi^3} - \frac{\lambda_{\tilde a}^4}{64\pi^4} \right\} + \mathcal{O}\left(\frac{\Lambda^{10}}{\tilde a^6}\right).$$

unambiguous

Ambiguous

$a \gg \Lambda \quad \lambda_a > 0$

$$\Lambda^4 F(s_a) = \Lambda^0 a^4 \left\{ \boxed{-\int_0^\infty dt \frac{e^{-t}}{t - \frac{8\pi}{\lambda_a}}} \right\} + \Lambda^2 a^2 \left\{ \frac{\lambda_a}{2\pi} \right\}$$

Ambiguous $\mp$

$$+\Lambda^4 \left\{ \frac{\lambda_a}{4\pi} - \frac{\lambda_a^2}{8\pi^2} + \boxed{2\log\left(\frac{-\lambda_a}{8\pi}\right) - 2\gamma_{\mathrm{E}} \pm i\pi} \right\} + \frac{\Lambda^6}{a^2} \left\{ -\frac{\lambda_a}{\pi} + \frac{\lambda_a^2}{24\pi^2} + \frac{\lambda_a^3}{24\pi^3} \right\}$$

$$+\frac{\Lambda^8}{a^4} \left\{ \boxed{-\int_0^\infty dt \frac{e^{-t}}{t + \frac{8\pi}{\lambda_a} \mp i0}} + \frac{11\lambda_a}{8\pi} + \frac{13\lambda_a^2}{96\pi^2} - \frac{\lambda_a^3}{16\pi^3} - \frac{\lambda_a^4}{64\pi^4} \right\} + \mathcal{O}\left(\frac{\Lambda^{10}}{a^6}\right).$$

unambiguous

No ambiguity at each Λ order in $\langle \delta D^2 \rangle_{\tilde a, a} = \Lambda^4 \{ F(s_{\tilde a}) - F(s_a) \}$

Comparison to exact result

Ambiguous

$$\Lambda^4 F(s_{\tilde a}) = \Lambda^0 \tilde a^4 \left\{ \boxed{-\int_0^\infty dt \frac{e^{-t}}{t-\frac{8\pi}{\lambda_{\tilde a}} \pm i0}} \right\} + \Lambda^2 \tilde a^2 \left\{ \frac{\lambda_{\tilde a}}{2\pi} \right\}$$

Ambiguous $\mp$

$$+\Lambda^4 \left\{ \frac{\lambda_{\tilde a}}{4\pi} - \frac{\lambda_{\tilde a}^2}{8\pi^2} + \boxed{2\log\left(\frac{\lambda_{\tilde a}}{8\pi}\right) - 2\gamma_{\mathrm{E}} \mp i\pi} \right\} + \frac{\Lambda^6}{\tilde a^2} \left\{ -\frac{\lambda_{\tilde a}}{\pi} + \frac{\lambda_{\tilde a}^2}{24\pi^2} + \frac{\lambda_{\tilde a}^3}{24\pi^3} \right\}$$

$$+\frac{\Lambda^8}{\tilde a^4} \left\{ \boxed{-\int_0^\infty dt \frac{e^{-t}}{t+\frac{8\pi}{\lambda_{\tilde a}}}} + \frac{11\lambda_{\tilde a}}{8\pi} + \frac{13\lambda_{\tilde a}^2}{96\pi^2} - \frac{\lambda_{\tilde a}^3}{16\pi^3} - \frac{\lambda_{\tilde a}^4}{64\pi^4} \right\} + \mathcal{O}\left(\frac{\Lambda^{10}}{\tilde a^6}\right).$$

unambiguous

unambiguous

<u>$a < \Lambda \qquad \lambda_a < 0$</u>

$$\Lambda^4 F(s_a) = \Lambda^0 a^4 \left\{ \boxed{-\int_0^\infty dt \frac{e^{-t}}{t-\frac{8\pi}{\lambda_a}}} \right\} + \Lambda^2 a^2 \left\{ \frac{\lambda_a}{2\pi} \right\}$$

Ambiguous $\pm$

$$+\Lambda^4 \left\{ \frac{\lambda_a}{4\pi} - \frac{\lambda_a^2}{8\pi^2} + \boxed{2\log\left(\frac{-\lambda_a}{8\pi}\right) - 2\gamma_{\mathrm{E}} \pm i\pi} \right\} + \frac{\Lambda^6}{a^2} \left\{ -\frac{\lambda_a}{\pi} + \frac{\lambda_a^2}{24\pi^2} + \frac{\lambda_a^3}{24\pi^3} \right\}$$

$$+\frac{\Lambda^8}{a^4} \left\{ \boxed{-\int_0^\infty dt \frac{e^{-t}}{t+\frac{8\pi}{\lambda_a} \mp i0}} + \frac{11\lambda_a}{8\pi} + \frac{13\lambda_a^2}{96\pi^2} - \frac{\lambda_a^3}{16\pi^3} - \frac{\lambda_a^4}{64\pi^4} \right\} + \mathcal{O}\left(\frac{\Lambda^{10}}{a^6}\right).$$

Ambiguous

$\Rightarrow$ Ambiguous at each $\Lambda^{0,4,8}$ order in $\langle \delta D^2 \rangle_{\tilde a, a} = \Lambda^4 \left\{ F(s_{\tilde a}) - F(s_a) \right\}$

What we found

- Renormalon ambiguity on the trivial vacuum (order Λ^0) is cancelled not only by order Λ^4, but by combination of order Λ^4 and Λ^8 .

- Ambiguities emerge only for $a<\Lambda$ region (a:IR cutoff).

- Borel resummation produces imaginary ambiguity with Λ^{-4} factor at Λ^8 order, leading to complete cancellation of the ambiguities.

Large-N CP^{N-1} sigma model

$$\mathcal{L} = \frac{1}{g^2}\left[\sum_{a=1}^{N} |\mathcal{D}_i\phi^a|^2 + D\left(|\phi^a|^2 - 1\right)\right] \qquad \mathcal{D}_i\phi^a \;=\; (\partial_i + iA_i)\phi^a$$

$$\langle F_{\mu\nu}^2\rangle_{\tilde{a},a} = -\frac{1}{2N}\sum_{l=0}^{\infty}\Lambda^{2l}\int_0^\infty dt\,\Lambda^t\left[\tilde{a}^{2\eta_l(t)} - a^{2\eta_l(t)}\right]\frac{\tilde{P}_l(t)}{\eta_l(t)}$$

condensate of field strength

- on R^2

$$\mathrm{Im}\,\langle F_{\mu\nu}^2\rangle_{\tilde{a},a} = \frac{\pm\pi}{N}\left[\boxed{\left(\tilde{a}e^{-\frac{2\pi}{\lambda\tilde{a}}}\right)^4} - 4\left(\tilde{a}e^{-\frac{2\pi}{\lambda\tilde{a}}}\right)^2\Lambda^2 + 6\Lambda^4 - 4\left(\tilde{a}e^{-\frac{2\pi}{\lambda\tilde{a}}}\right)^{-2}\Lambda^6 + \left(\tilde{a}e^{-\frac{2\pi}{\lambda\tilde{a}}}\right)^{-4}\Lambda^8\right]\theta(\Lambda - a) \;= 0$$

- on $\mathrm{R}^1 \times \mathrm{S}^1$ $\quad L\Lambda \ll 1 \quad N L\Lambda \gg 1$

$$\mathrm{Im}\,\langle F_{\mu\nu}^2\rangle_{\tilde{a},a}^{\mathbb{R}\times S^1} = \frac{\pm\pi}{\boxed{NL}}\left[2\boxed{\left(\tilde{a}e^{-\frac{2\pi}{\lambda\tilde{a}}}\right)^3} - 6\left(\tilde{a}e^{-\frac{2\pi}{\lambda\tilde{a}}}\right)\Lambda^2 + 6\left(\tilde{a}e^{-\frac{2\pi}{\lambda\tilde{a}}}\right)^{-1}\Lambda^4 - 2\left(\tilde{a}e^{-\frac{2\pi}{\lambda\tilde{a}}}\right)^{-3}\Lambda^6\right]\theta(\Lambda - a) = 0$$

In both cases there are resurgent structures similar to O(N).
However, compactification changes the structure.

Large-N CP^{N-1} sigma model

During decompactification, the resurgent structure changes or Stokes phenomena occur every time one of Kaluza Klein masses n/R becomes smaller than the scale Λ !

$$\mathrm{Im}\,\langle \delta D(x)\delta D(0)\rangle_a\Big|_l = \pm\pi \sum_{n\in\mathbb{Z}} \Lambda^{2l} P_l(\Lambda\partial_\Lambda)\left[\frac{\Lambda^{3-2l}}{R}\frac{e^{-i\frac{n}{R}x}}{\sqrt{1-\frac{n^2}{R^2\Lambda^2}}}\,\theta\!\left(\Lambda^2-\frac{n^2}{R^2}\right)\right]$$

Infinitely many Stokes phenomena during decompactification change ambiguity of perturbative part from $O(\Lambda^3/R)$ to $O(\Lambda^4)$.

$$\mathrm{Im}\,\langle \delta D(x)\delta D(0)\rangle_a\Big|_{l=0} = \pm \begin{cases} \Lambda^3/R & \text{for } R<\Lambda^{-1} \\ \Lambda^4+\cdots & \text{for } R\to\infty \end{cases}$$

Summary

1. Renormalon ambiguity in large-N O(N) model is cancelled by combination of order Λ^4 and Λ^8.
2. Ambiguities emerge only for $a<\Lambda$ region (a:IR cutoff).
3. Borel resummation produces imaginary ambiguity with Λ^{-4} factor at Λ^8 order, leading to complete cancellation of the ambiguities.
4. The similar resurgent structure exists in CP^{N-1} model
5. During compactification, the resurgent structure and renormalon property changes due to infinite Stokes phenomena.

Discussion

1. Are there semiclassical configurations (solutions) corresponding to $\exp(-4\pi/\lambda_a) \propto \Lambda^2$?
2. Bion configuration is such a candidate?
3. Yang-Mills theory and QCD have the same resurgent structure?
4. Trans-series expansion can define QFT?
5. If YM and QCD have the same resurgent structure and trans-series structure, does it mean that we can obtain nonperturbative physics from perturbation?

Backup slides

Series expansion for Large-N O(N)

$$\mathrm{Im}\,\langle \delta D(x)\delta D(0)\rangle_a = \pm\pi\Lambda^4 \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} A_{l,n}\left(\frac{\Lambda^2 x^2}{4}\right)^n \Rightarrow 0$$

$$A_{l,n} = (-1)^{l+n}\frac{1}{(n!)^2}\left[\binom{2n+4}{l} - 4\binom{2n+2}{l-1}\right]$$

$n\backslash l$	0	1	2	3	4	5	6	7	$\cdots$
0	-1	0	2	0	-1	0	0	0	$\cdots$
1	1	-2	-1	4	-1	-2	1	0	$\cdots$
$\vdots$									

$$\mathrm{Im}\,\langle \delta D(x)\delta D(0)\rangle_a = \pm\pi\sum_{l=0}^{\infty}\Lambda^{2l}P_l(\Lambda\partial_\Lambda)\left[\Lambda^{4-2l}J_0(\Lambda x)\right] \Rightarrow 0$$

$$P_l(t) = \frac{(-1)^l}{l!}\left[(t+l+1)^{(l)} - 4l(t+l)^{(l-1)}\right]$$

Series expansion for Large-N O(N)

$$
\begin{aligned}
c_{(0,n)}(p) &= \int dp \frac{2p^3 t_p^n}{(4\pi)^{n+1}} \\
&= \frac{\tilde{a}^4}{(8\pi)^{n+1}} \Gamma(n+1, 2t_p) \\
c_{(2,n)}(p) &= \int dp \frac{4p^2 \left(t_p^n - n t_p^{n-1}\right)}{(4\pi)^{n+1}} \\
&= \frac{2p^2 t_p^n}{(4\pi)^{n+1}} \\
c_{(4,n)}(p) &= \int dp \frac{2\left(-2t_p^n - n t_p^{n-1} + 2\,(n)_2\, t_p^{n-2}\right)}{(4\pi)^{n+1}\, p} \\
&= \frac{2t_p^{n+1}}{(4\pi)^{n+1}\,(n+1)} + \frac{t_p^n - 2n t_p^{n-1}}{(4\pi)^{n+1}} \\
c_{(6,n)}(p) &= \int dp \frac{4\left(6t_p^n + 5n t_p^{n-1} - 3\,(n)_2\, t_p^{n-2} - 2\,(n)_3\, t_p^{n-3}\right)}{3\,(4\pi)^{n+1}\, p^3} \\
&= \frac{2\left(-6t_p^n + n t_p^{n-1} + 2\,(n)_2\, t_p^{n-2}\right)}{3\,(4\pi)^{n+1}\, p^2} \\
c_{(8,n)}(p) &= \int dp \frac{-60t_p^n - 59n t_p^{n-1} + 11\,(n)_2\, t_p^{n-2} + 20\,(n)_3\, t_p^{n-3} + 4\,(n)_4\, t_p^{n-4}}{3\,(4\pi)^{n+1}\, p^5} \\
&= \frac{(-1)^{n+1}}{\tilde{a}^4\,(8\pi)^{n+1}} \Gamma(n+1, -2t_p) + \frac{33t_p^n + 13n t^{n-1} - 12\,(n)_2\, t_p^{n-2} - 4\,(n)_3\, t_p^{n-3}}{6\,(4\pi)^{n+1}\, p^4},
\end{aligned}
$$

Beta function

$$\mu \frac{\partial}{\partial \mu} \frac{2\pi}{\lambda_\mu} = \beta(\lambda_\mu)$$

$$\beta(\lambda_\mu) = \beta_0 + \beta_1 \lambda_\mu + \beta_2 \lambda_\mu^2 + \cdots$$

$$\lambda_p\,(p) = \lambda'_p - \frac{\beta_1}{\beta_0} {\lambda'_p}^2 \log \frac{4\pi}{\lambda'_p} + \cdots \quad \text{with} \quad \lambda'_p = \frac{2\pi}{\beta_0 \log p/\Lambda}$$

In large-N, this difference is subleading effect.