Method of images in defect conformal field theories

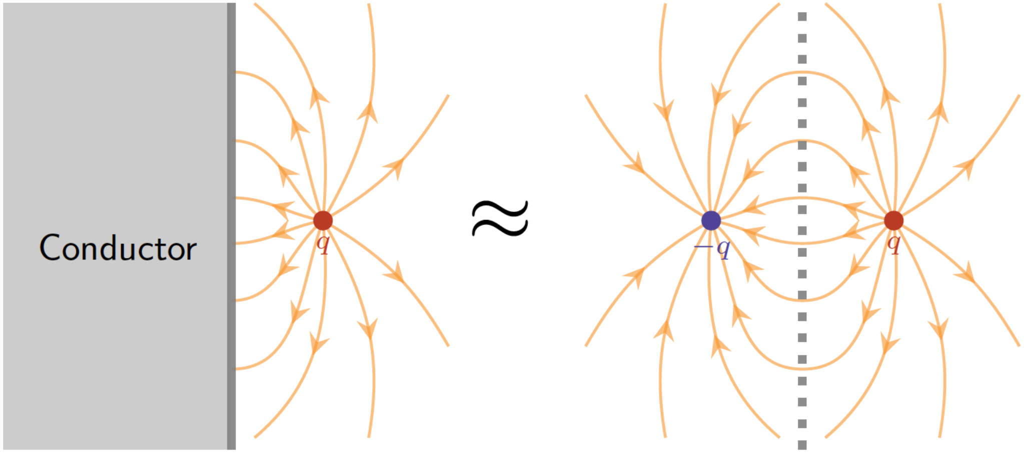
Soichiro Shimamori (Osaka U)

Based on arxiv2205.05370 [hep-th] with T. Nishioka and Y. Okuyama

Seminar at Kyoto U (2022/5/25)

Quick overview of this paper [T. Nishioka, Y.Okuyama, S.S, 2022]

Method of image : tool for solving differential equations



In this work, we find that the image method works in defect CFT.

Our prescription provides an efficient way to calculate correlation functions in defect CFT.

Outline

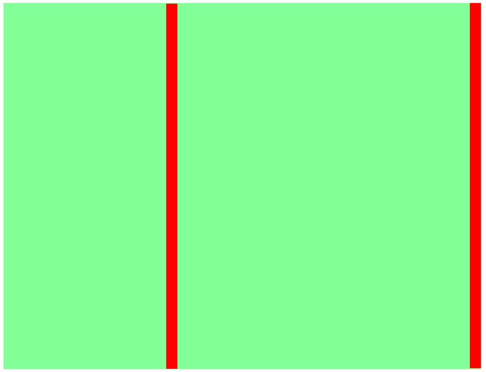
Defect in Quantum Field Theory

Correlation functions in DCFT

Our work

Summary and future work

Defect Conformal Field Theory (DCFT)

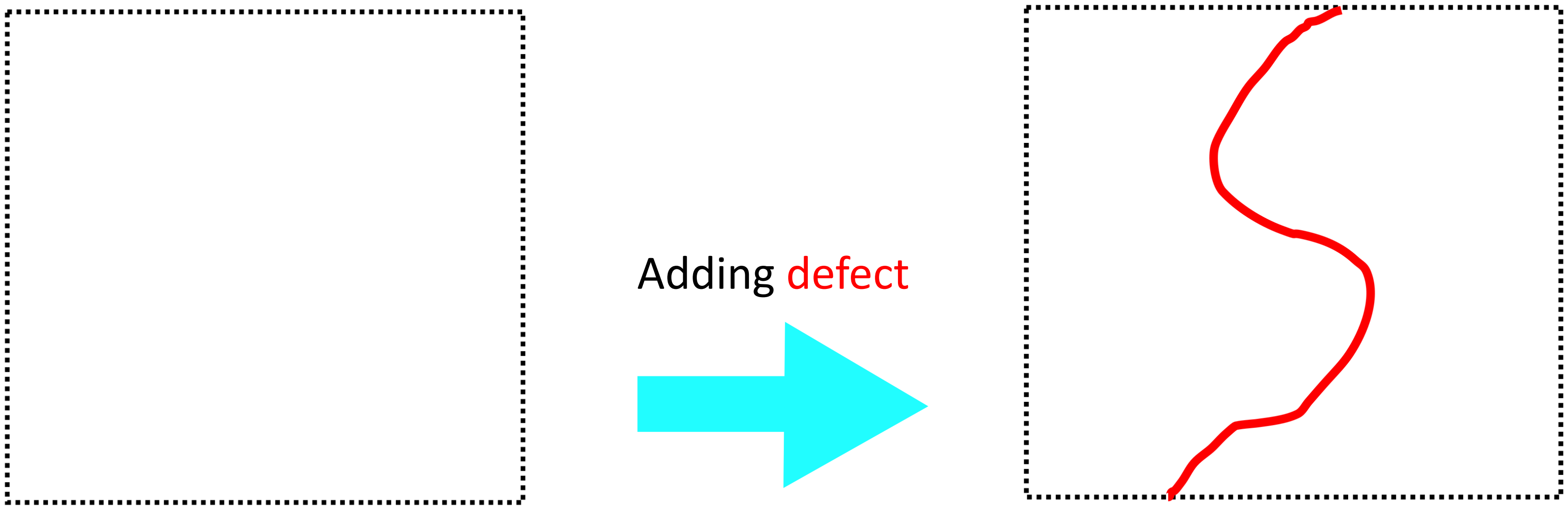


- Defect : non-local objects in QFT
- Many examples in physics
  - Wilson / ‘t Hooft loop
  - Impurities, boundaries (Condensed Matter Physics)
  - D brane
- Recently, defects also play important roles in defining “symmetry”.

[Gaiotto, et,al., 2014] …….

However, it is quite hard to understand QFT with defects in general.

main reason : less symmetry



Symmetry

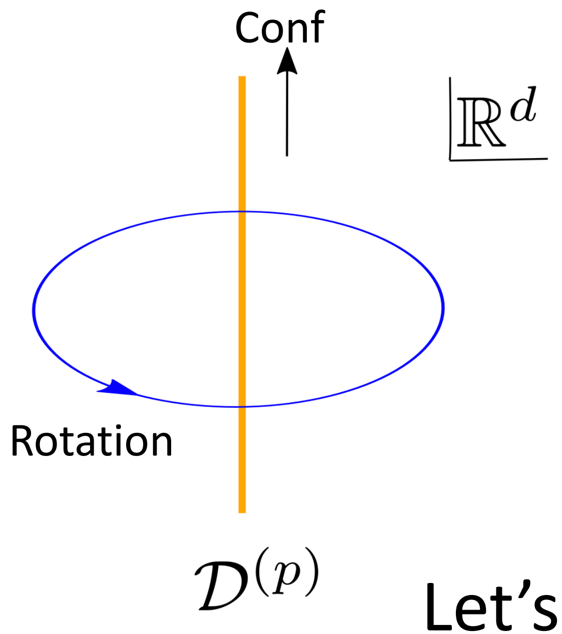
	translation	~~translation~~
	rotation	~~rotation~~

To analyze QFTs with defects, some constraints on defects are necessary.

One way is to impose conformal symmetry on defects.

→ shape of defects : planar or spherical (conformal defect)

Rough picture of DCFT



Let's see planar case more in detail !

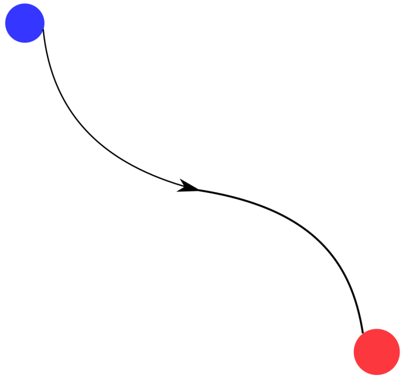
Motivation to study DCFT

- Critical phenomena (defect system)

  [Ludwig, Affleck, 1991] [Affleck, 1995]

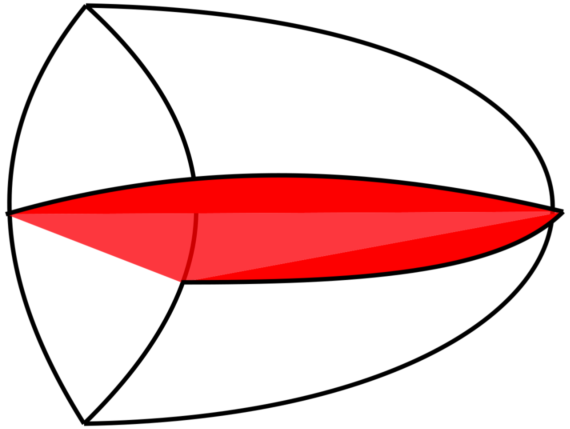
- Fixed points of defect quantum field theories

  [Polchinski, Sully, 2011] [Beccaria, et.al., 2018]



- AdS / DCFT correspondence

  [Karch, Randall, 2001] [DeWolfe, et.al., 2002]



What's interesting about DCFT ?

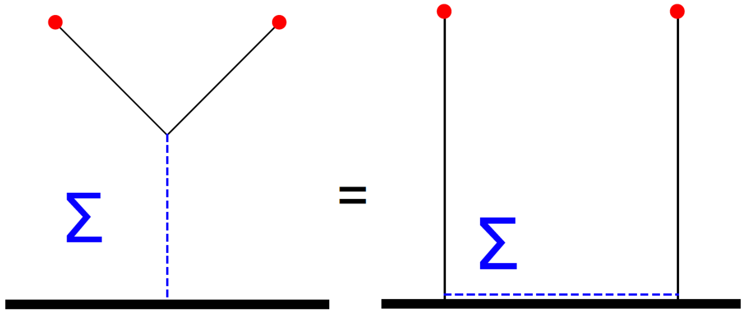
1. Trace anomalies [Herzog, Huang, 2018] [Chalabi, et.al., 2021]

$\langle T^{\mu}_{\mu} \rangle$: invariant under diffeo. & conformal dimension d

extrinsic curvature on a defect can contributes

→ Structure of trace anomalies becomes richer.

2. defect conformal bootstrap [Billo, et.al., 2016] [Isachenkov, et.al., 2018]



and so on……

In d-dimensional Euclidean CFT, the conformal group is $\mathrm{SO}(d+1,1)$.

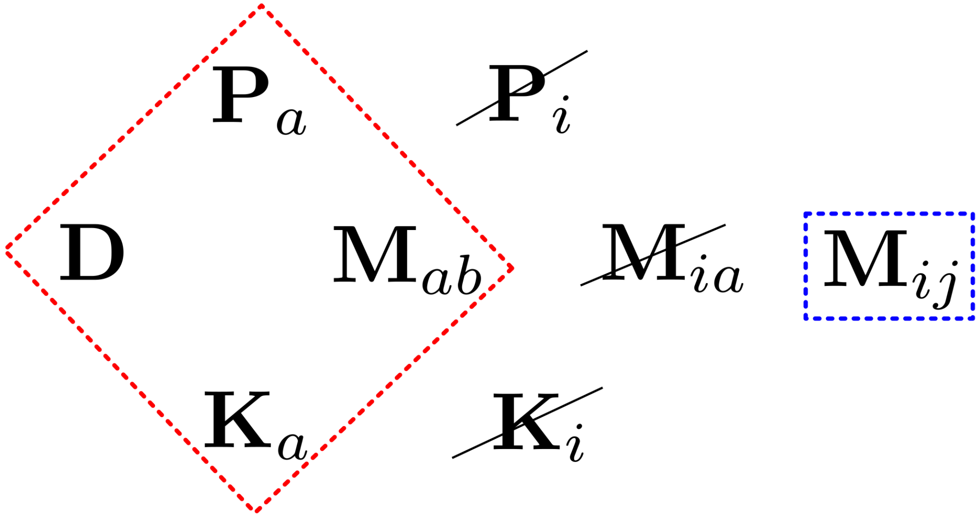
conformal generators

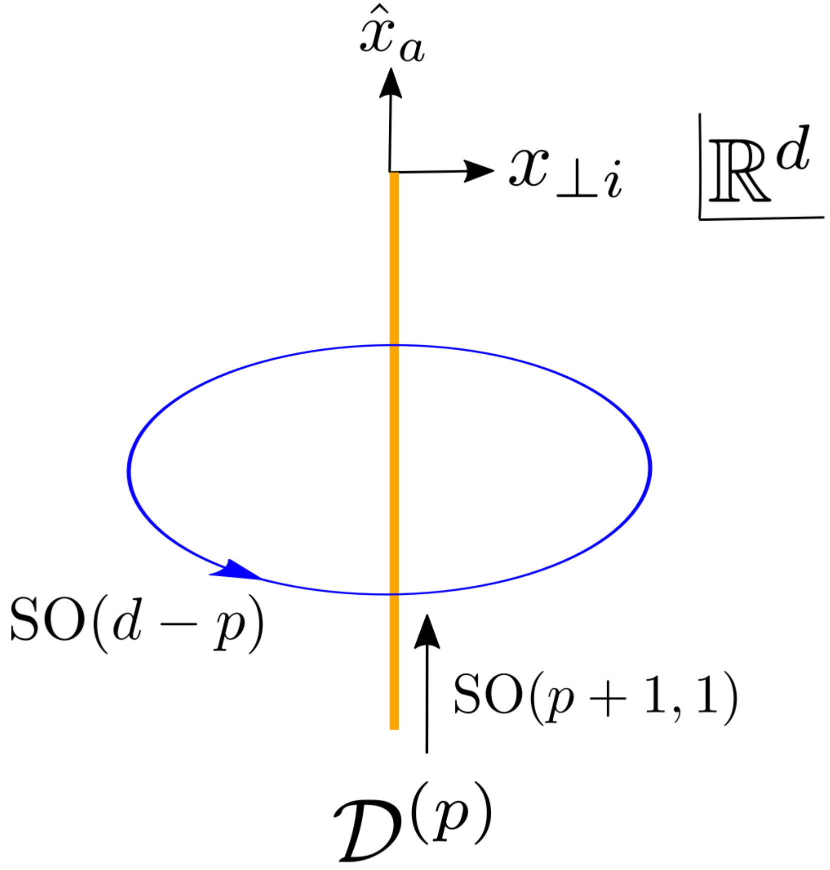
$$\mathbf{P}_\mu \qquad \mathbf{D} \qquad \mathbf{M}_{\mu\nu} \qquad \mathbf{K}_\mu$$

$$\mathbf{J}_{MN} = \begin{array}{c} {\scriptstyle M \backslash N \rightarrow} \\ {\scriptstyle \downarrow} \\ -1 \\ 0 \\ \mu \end{array} \begin{array}{c} \begin{array}{ccc} -1 & 0 & \nu \end{array} \\ \begin{pmatrix} 0 & \mathbf{D} & \frac{1}{2}(\mathbf{P}_\nu - \mathbf{K}_\nu) \\ -\mathbf{D} & 0 & \frac{1}{2}(\mathbf{P}_\nu + \mathbf{K}_\nu) \\ -\frac{1}{2}(\mathbf{P}_\mu - \mathbf{K}_\mu) & -\frac{1}{2}(\mathbf{P}_\mu + \mathbf{K}_\mu) & \mathbf{M}_{\mu\nu} \end{pmatrix} \end{array}$$

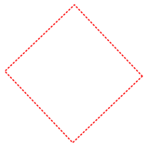
$$[\mathbf{J}_{KL}, \mathbf{J}_{MN}] = \eta_{LM}\mathbf{J}_{KN} - \eta_{KM}\mathbf{J}_{LN} + \eta_{KN}\mathbf{J}_{LM} - \eta_{LN}\mathbf{J}_{KM}$$

Next, insert p-dimensional conformal defect into CFT.





Residual symmetries

 : conformal symmetry on the defect

 : rotation in the transverse directions

$$a = 1 \sim p$$
$$i = p+1 \sim d$$

$$
\mathbf{J}_{MN} = \begin{array}{c|cccc}
{}_{M\downarrow}\backslash^{N\rightarrow} & -1 & 0 & b & j \\
\hline
-1 & 0 & \mathbf{D} & \frac{1}{2}(\mathbf{P}_b - \mathbf{K}_b) & \frac{1}{2}(\mathbf{P}_j - \mathbf{K}_j) \\
0 & -\mathbf{D} & 0 & \frac{1}{2}(\mathbf{P}_b + \mathbf{K}_b) & \frac{1}{2}(\mathbf{P}_j + \mathbf{K}_j) \\
a & -\frac{1}{2}(\mathbf{P}_a - \mathbf{K}_a) & -\frac{1}{2}(\mathbf{P}_a + \mathbf{K}_a) & \mathbf{M}_{ab} & \mathbf{M}_{aj} \\
i & -\frac{1}{2}(\mathbf{P}_i - \mathbf{K}_i) & -\frac{1}{2}(\mathbf{P}_i + \mathbf{K}_i) & \mathbf{M}_{ib} & \mathbf{M}_{ij}
\end{array}
$$

DCFT : Conformal field theory in the presence of a p-dimensional conformal defect. The symmetries are $\mathrm{SO}(p+1,1) \times \mathrm{SO}(d-p)$.

Transformation law for $\mathrm{SO}(p+1,1)\times\mathrm{SO}(d-p)$

- parallel translations $\mathbf{P}_a$: $\hat{x}^a \mapsto \hat{x}^a + \hat{c}^a$.
- parallel rotations $\mathbf{M}_{ab}$: $\hat{x}^a \mapsto L^a{}_b\,\hat{x}^b$, $\qquad L^a{}_b \in \mathrm{SO}(p)$.
- parallel special conformal transformations $\mathbf{K}_a$: $\hat{x}^a \mapsto \frac{\hat{x}^a - \hat{b}^a x^2}{\hat{\gamma}(x)}$ with $\hat{\gamma}(x) = 1 - 2\hat{b}\cdot\hat{x} + \hat{b}^2 x^2$.
  $x_\perp^i \mapsto \frac{x_\perp^i}{\hat{\gamma}(x)}$
- dilatations $\mathbf{D}$: $x^\mu \mapsto \lambda\, x^\mu$.
- transverse rotations $\mathbf{M}_{ij}$: $x_\perp^i \mapsto L^i{}_j\, x_\perp^j$, $\qquad L^i{}_j \in \mathrm{SO}(d-p)$.

Remark: If we set p=d, usual conformal symmetry is restored.

Outline

~~Defect in Quantum Field Theory~~

Correlation functions in DCFT

Our work

Summary and future work

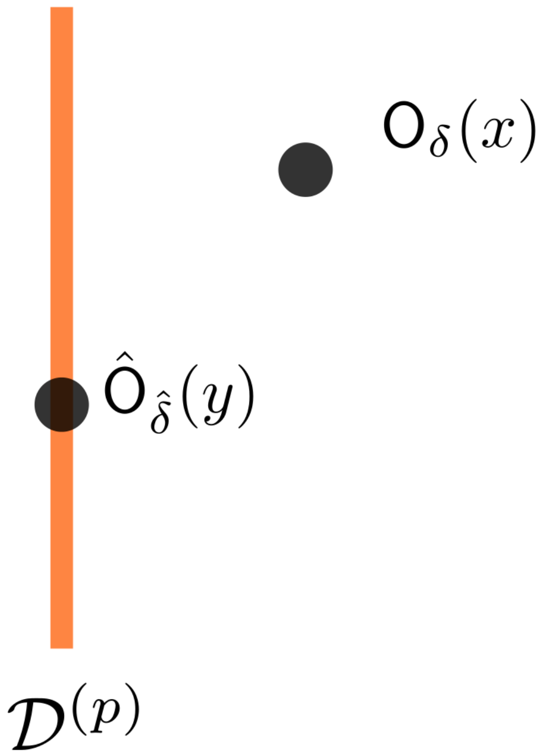
Correlation functions in DCFT

Two types of operators appearing in DCFT

— $\mathrm{O}_{\delta}(x)$: Bulk local operator
w/ conformal dimension δ

— $\widehat{\mathrm{O}}_{\hat{\delta}}(y)$: Defect local operator
w/ conformal dimension $\hat{\delta}$

$y = (\hat{y}^a, y_{\perp}^i = 0)$



DCFT correlator [Billo, et.al., 2016]

$$\langle \mathsf{O}_\delta(x_1)\cdots\widehat{\mathsf{O}}_{\hat{\delta}}(y_1)\cdots\rangle_{\mathsf{DCFT}} \equiv \frac{\langle \mathcal{D}^{(p)}\mathsf{O}_\delta(x_1)\cdots\widehat{\mathsf{O}}_{\hat{\delta}}(y_1)\cdots\rangle_{\mathsf{CFT}}}{\langle \mathcal{D}^{(p)}\rangle_{\mathsf{CFT}}}$$

Symmetries

$\mathrm{SO}(p+1,1)\times\mathrm{SO}(d-p)$

→ Strong constraints on DCFT correlators

Conformal Ward Identities

Warm up : CFT correlator $\langle \mathcal{O}_{\Delta_1}(x_1)\cdots\mathcal{O}_{\Delta_n}(x_n)\rangle_{\text{CFT}}$

Conformal Ward Identities w.r.t $\text{SO}(d+1,1)$

- translations + rotation : $x_\alpha^\mu \mapsto x'^{\,\mu}_\alpha = L^\mu{}_\nu\, x_\alpha^\nu + a^\mu$.

$$\langle \mathcal{O}_{\Delta_1}(x_1)\cdots\mathcal{O}_{\Delta_n}(x_n)\rangle_{\text{CFT}} = \langle \mathcal{O}_{\Delta_1}(x'_1)\cdots\mathcal{O}_{\Delta_n}(x'_n)\rangle_{\text{CFT}} \ .$$

- dilatations : $x_\alpha^\mu \mapsto x'^{\,\mu}_\alpha = \lambda\, x_\alpha^\mu$.

$$\langle \mathcal{O}_{\Delta_1}(x_1)\cdots\mathcal{O}_{\Delta_n}(x_n)\rangle_{\text{CFT}} = \left(\prod_{\alpha=1}^{n} \lambda^{\Delta_\alpha}\right) \langle \mathcal{O}_{\Delta_1}(x'_1)\cdots\mathcal{O}_{\Delta_n}(x'_n)\rangle_{\text{CFT}} \ .$$

- special conformal transformations : $x_\alpha^\mu \mapsto x'^{\,\mu}_\alpha = \frac{x_\alpha^\mu - b^\mu x_\alpha^2}{1-2b\cdot x_\alpha + b^2 x_\alpha^2}$.

$$\langle \mathcal{O}_{\Delta_1}(x_1)\cdots\mathcal{O}_{\Delta_n}(x_n)\rangle_{\text{CFT}} = \left[\prod_{i=1}^{n} \gamma^{-\Delta_i}(x_\alpha)\right] \langle \mathcal{O}_{\Delta_1}(x'_1)\cdots\mathcal{O}_{\Delta_n}(x'_n)\rangle_{\text{CFT}} \ ,$$

$$\gamma(x_\alpha) = 1 - 2b\cdot x_\alpha + b^2 x_\alpha^2 \ .$$

➢ scalar 1-pt function

- Translational and rotational invariance:

$$\langle \mathcal{O}_\Delta(x) \rangle_{\mathsf{CFT}} = \langle \mathcal{O}_\Delta(0) \rangle_{\mathsf{CFT}} = \text{const.}$$

- Covariance under scale transformation:

$$\langle \mathcal{O}_\Delta(x) \rangle_{\mathsf{CFT}} = \lambda^{\Delta} \langle \mathcal{O}_\Delta(\lambda x) \rangle_{\mathsf{CFT}} \ .$$

$$\langle \mathcal{O}_\Delta(x) \rangle_{\mathsf{CFT}} = \begin{cases} \text{const.} & \text{if } \mathcal{O}_\Delta(x) \text{ is the identity operator: } (\Delta = 0) \\ 0 & \text{otherwise} \end{cases}$$

➢ scalar 2-pt function

- Translational and rotational invariance:

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle_{\text{CFT}} = \langle \mathcal{O}_{\Delta_1}(x_1 - x_2)\mathcal{O}_{\Delta_2}(0)\rangle_{\text{CFT}} = f(|x_1 - x_2|) \ .$$

- Covariance under scale transformation:

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle_{\text{CFT}} = \lambda^{\Delta_1+\Delta_2} \langle \mathcal{O}_{\Delta_1}(\lambda x_1)\mathcal{O}_{\Delta_2}(\lambda x_2)\rangle_{\text{CFT}} \ ,$$

$$\longrightarrow \langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle_{\text{CFT}} = f(|x_1 - x_2|) \propto \frac{1}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \ .$$

Combining with special conformal transformations ,

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle_{\text{CFT}} = \frac{\delta_{\Delta_1,\Delta_2}}{|x_1 - x_2|^{2\Delta_1}}$$

➢ scalar 3-pt function

- Translation + rotation + scale transformation:

$$\longrightarrow \langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle_{\text{CFT}}$$

$$\propto \sum_{a+b+c=\Delta_1+\Delta_2+\Delta_3} \frac{C(\Delta_1,\Delta_2,\Delta_3)}{|x_1-x_2|^a|x_2-x_3|^b|x_3-x_1|^c} .$$

- Covariance under special conformal transformations completely fixes the form

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle_{\text{CFT}} = \frac{c_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}} .$$

➢ scalar 4-pt function

$$u = \frac{|x_1 - x_2|^2 |x_3 - x_4|^2}{|x_1 - x_3|^2 |x_2 - x_4|^2}, \qquad v = \frac{|x_1 - x_4|^2 |x_2 - x_3|^2}{|x_1 - x_3|^2 |x_2 - x_4|^2}$$: conformal invariant

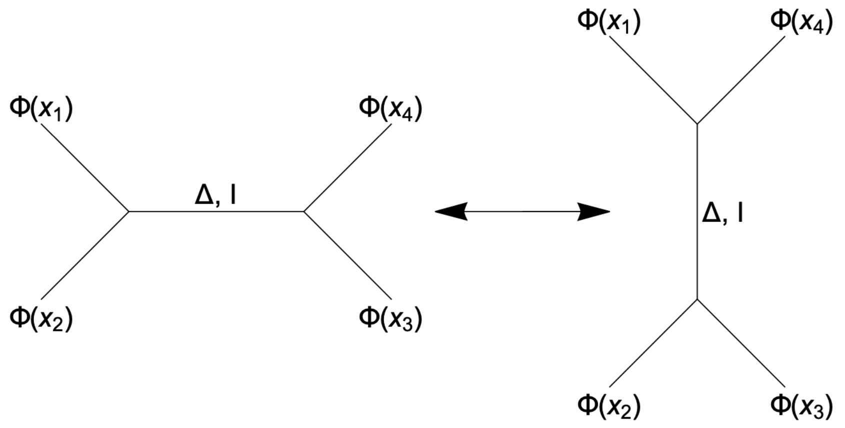
➡ 4-pt function is not completely fixed by conformal invariance.

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4)\rangle_{\text{CFT}} = \frac{F(u,v)}{\prod_{i<j}|x_{ij}^2|^{\delta_{ij}}} \qquad \sum_{j\neq i}\delta_{ij} = \Delta_i$$

OPE associativity

➡ new constraint on $F(u,v)$

Conformal Bootstrap



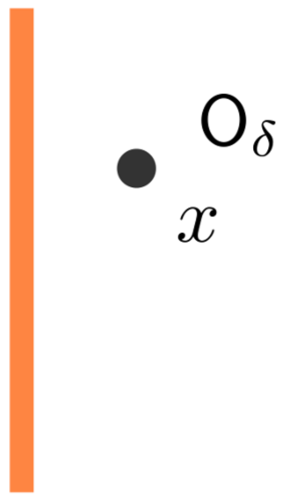
Next, we will consider DCFT correlators.

Correlator including only defect local operators $\langle \widehat{\mathrm{O}}_{\hat{\delta}_1}(y_1)\cdots\widehat{\mathrm{O}}_{\hat{\delta}_n}(y_n)\rangle_{\mathrm{DCFT}}$

$\approx$ CFT correlator in $\mathbb{R}^p$ $\langle \widehat{\mathrm{O}}_{\hat{\delta}_1}(y_1)\cdots\widehat{\mathrm{O}}_{\hat{\delta}_n}(y_n)\rangle_{\mathrm{CFT}}$

→ We consider DCFT correlators in which one or more bulk local operators are inserted.

➢ Bulk 1-pt function $\langle \mathrm{O}_\delta(x) \rangle_{\mathrm{DCFT}}$



- parallel translations and transverse rotations,

$$\langle \mathrm{O}_\delta(x) \rangle_{\mathrm{DCFT}} = \langle \mathrm{O}_\delta(\hat{x}^a = 0, x_\perp^i) \rangle_{\mathrm{DCFT}} = f(|x_\perp|)$$

- scale transformation

$$\boxed{\langle \mathrm{O}_\delta(x) \rangle_{\mathrm{DCFT}} = \frac{a_{\mathrm{O}}}{|x_\perp|^\delta}}$$

consistent w/ other symmetries

similar to 2-pt function in CFT $\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \rangle_{\mathrm{CFT}} = \frac{\delta_{\Delta_1, \Delta_2}}{|x_1 - x_2|^{2\Delta_1}}$

➢ Bulk-defect 2-pt function $\langle \mathrm{O}_\delta(x)\widehat{\mathrm{O}}_{\hat\delta}(y)\rangle_{\mathrm{DCFT}}$

- parallel translations and rotations

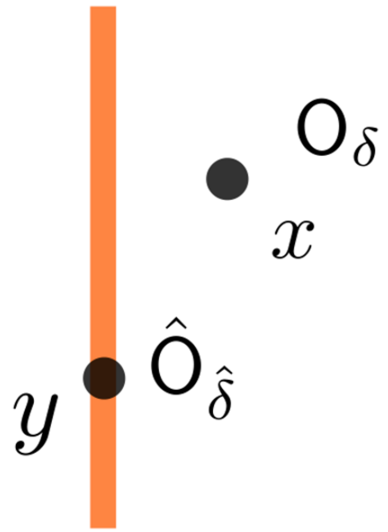
$$\langle \mathrm{O}_\delta(x)\widehat{\mathrm{O}}_{\hat\delta}(y)\rangle_{\mathrm{DCFT}} = \langle \mathrm{O}_\delta(\hat{x}^a-\hat{y}^a, x_\perp^i)\widehat{\mathrm{O}}_{\hat\delta}(0)\rangle_{\mathrm{DCFT}} = f(|\hat{x}-\hat{y}|, |x_\perp|)$$

- scale transformation

$$\langle \mathrm{O}_\delta(x)\widehat{\mathrm{O}}_{\hat\delta}(y)\rangle_{\mathrm{DCFT}} = \sum_{m+n=\delta+\hat\delta} \frac{C(m,n)}{|\hat{x}-\hat{y}|^n |x_\perp|^m}$$

- parallel special conformal transformations

$$\boxed{\langle \mathrm{O}_\delta(x)\widehat{\mathrm{O}}_{\hat\delta}(y)\rangle_{\mathrm{DCFT}} = \frac{b_{\mathrm{O}\widehat{\mathrm{O}}}}{|x_\perp|^{\delta-\hat\delta}\left(|\hat{x}-\hat{y}|^2+|x_\perp|^2\right)^{\hat\delta}}}$$



$$\langle \mathsf{O}_\delta(x)\widehat{\mathsf{O}}_{\hat{\delta}}(y)\rangle_{\mathsf{DCFT}} = \frac{b_{\mathsf{O}\widehat{\mathsf{O}}}}{|x_\perp|^{\delta-\hat{\delta}}\left(|\hat{x}-\hat{y}|^2+|x_\perp|^2\right)^{\hat{\delta}}}$$

similar to 3-pt function in CFT

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle_{\mathsf{CFT}} = \frac{c_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}}$$

$$\langle \mathsf{O}_{\delta_1}(x_1)\mathsf{O}_{\delta_2}(x_2)\rangle_{\mathsf{DCFT}} \qquad \langle \mathsf{O}_{\delta}(x)\widehat{\mathsf{O}}_{\hat{\delta}_1}(y_1)\widehat{\mathsf{O}}_{\hat{\delta}_2}(y_2)\rangle_{\mathsf{DCFT}}$$

two $\mathrm{SO}(p+1,1)\times\mathrm{SO}(d-p)$ invariant quantities

→ These are not completely fixed by $\mathrm{SO}(p+1,1)\times\mathrm{SO}(d-p)$.

similar to 4-pt function in CFT

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4)\rangle_{\mathsf{CFT}} = \frac{F(u,v)}{\prod_{i<j}|x_{ij}^2|^{\delta_{ij}}}$$

Outline

~~Defect in Quantum Field Theory~~

~~Correlation functions in DCFT~~

Our work

Summary and future work

Our work

- Similarity between DCFT and CFT correlators

DCFT side		CFT side
defect n-pt functions	$\Longleftrightarrow$	n-pt function
bulk 1-pt functions	$\Longleftrightarrow$	2-pt functions
bulk-to-defect 2-pt functions	$\Longleftrightarrow$	3-pt functions
bulk 2-pt functions bulk-defect-defect 2-pt func	$\Longleftrightarrow$	4-pt functions

➢ Naive question

DCFT correlator
w/ n bulk op.
w/ m defect op.

⟷

CFT correlator
w/ $(2n+m)$ op.

Is there any correspondence ??

We find out such correspondence and prove

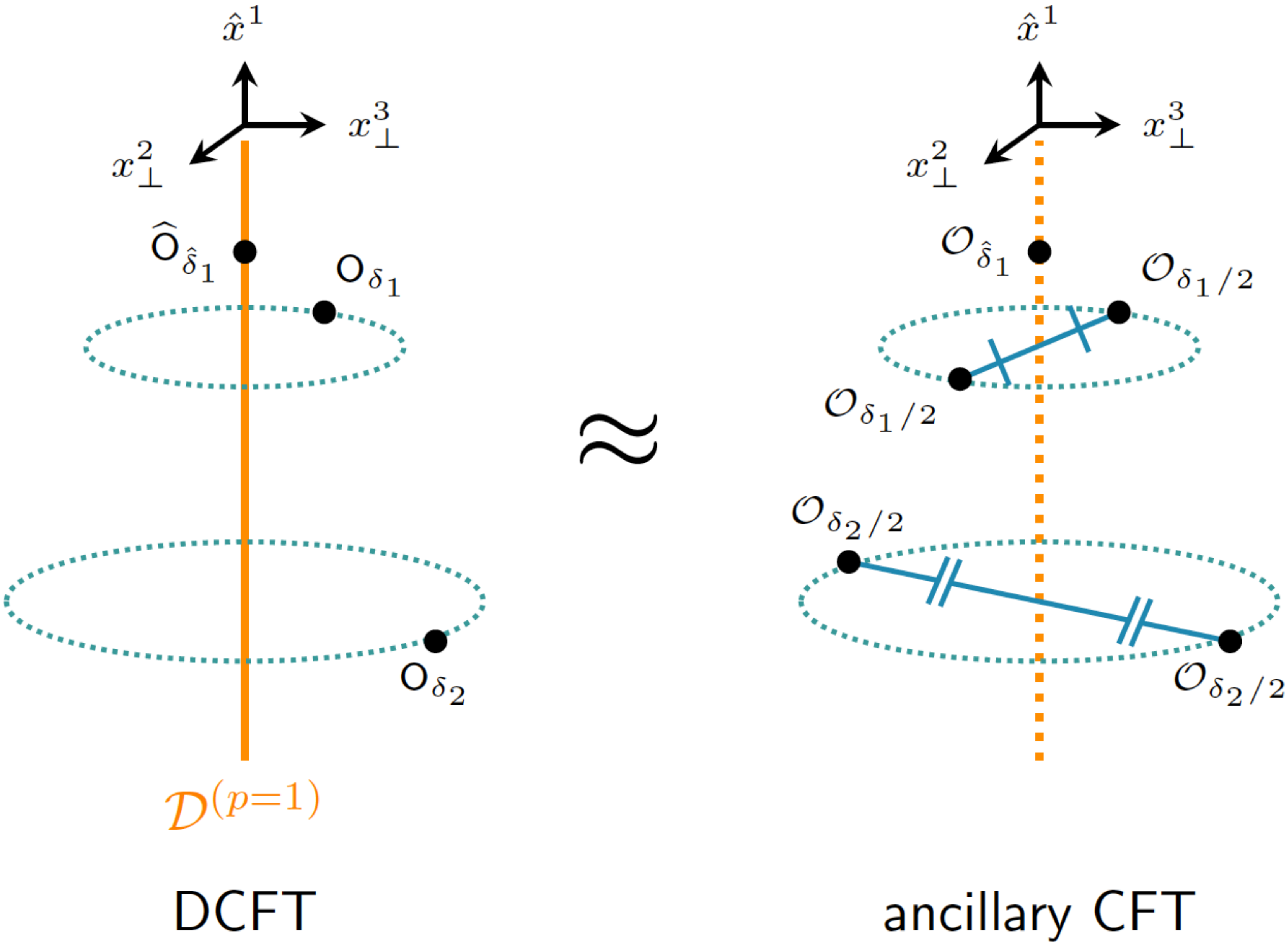
$$\left\langle \prod_{\alpha=1}^{n} \mathsf{O}_{\delta_\alpha}(x_\alpha) \prod_{\hat{\alpha}=1}^{m} \widehat{\mathsf{O}}_{\hat{\delta}_{\hat{\alpha}}}(y_{\hat{\alpha}}) \right\rangle_{\mathsf{DCFT}} \approx \left\langle \prod_{\alpha=1}^{n} \left[\mathcal{O}_{\delta_\alpha/2}(x_\alpha)\, \mathcal{O}_{\delta_\alpha/2}(\bar{x}_\alpha)\right] \prod_{\hat{\alpha}=1}^{m} \mathcal{O}_{\hat{\delta}_{\hat{\alpha}}}(y_{\hat{\alpha}}) \right\rangle_{\mathsf{CFT}} ,$$

where $\approx$ means that both sides satisfy the same differential equations dictated by conformal symmetry.

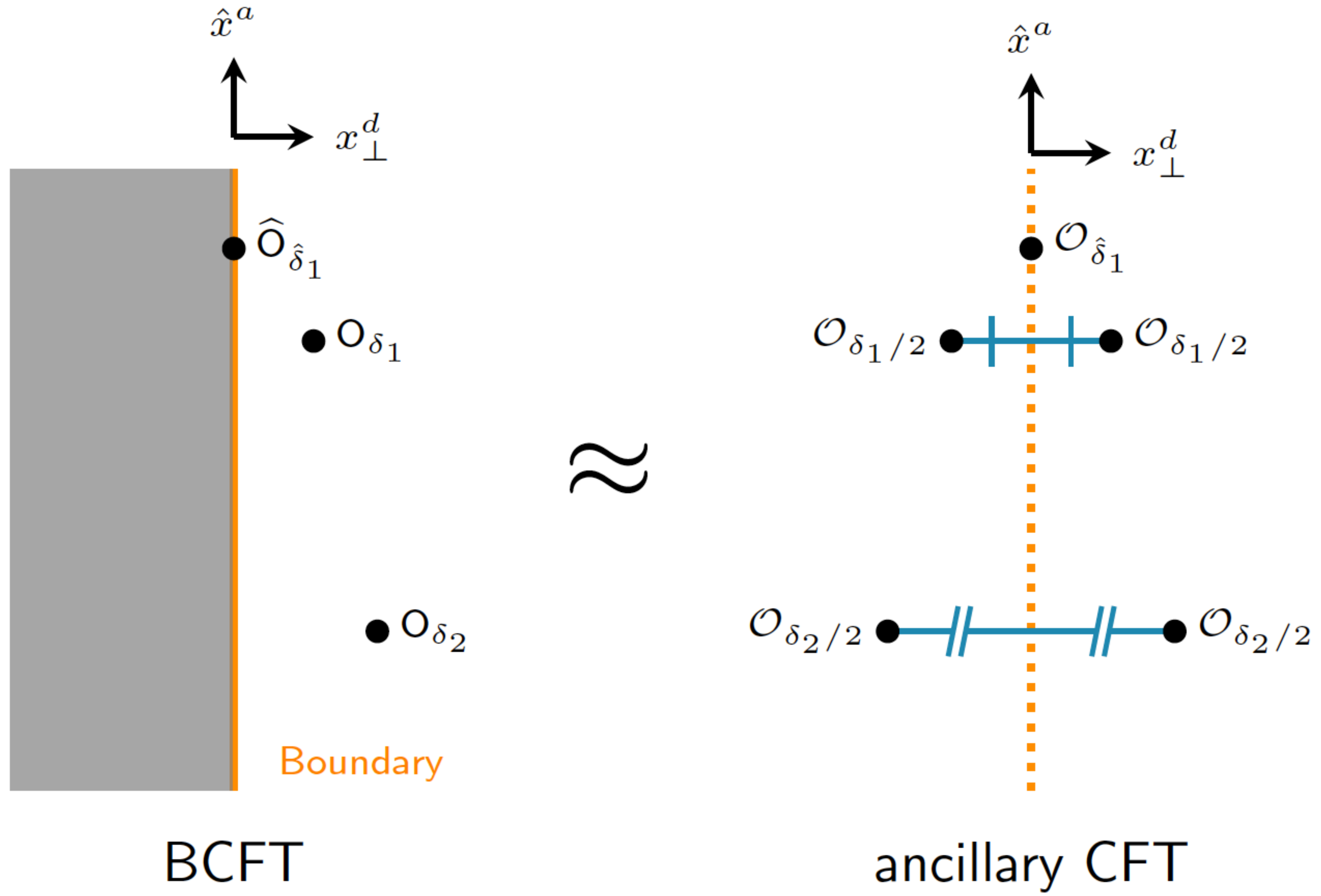
- $\bar{x}$: the anti-podal point of x along the transverse direction to the boundary/defect.
- $y = (\hat{y}^a, y_\perp^i = 0)$: the coordinate on the defect.

Method of image is valid also for defect CFT !!

(Generalization of Cardy's method in BCFT2)
[Cardy, 1984]



$\hat{x}^1$
$x_\perp^3$
$x_\perp^2$
$\widehat{O}_{\hat{\delta}_1}$
O_{δ_1}
O_{δ_2}
$\mathcal{D}^{(p=1)}$
$\approx$
$\mathcal{O}_{\hat{\delta}_1}$
$\mathcal{O}_{\delta_1/2}$
$\mathcal{O}_{\delta_1/2}$
$\mathcal{O}_{\delta_2/2}$
$\mathcal{O}_{\delta_2/2}$
DCFT
ancillary CFT

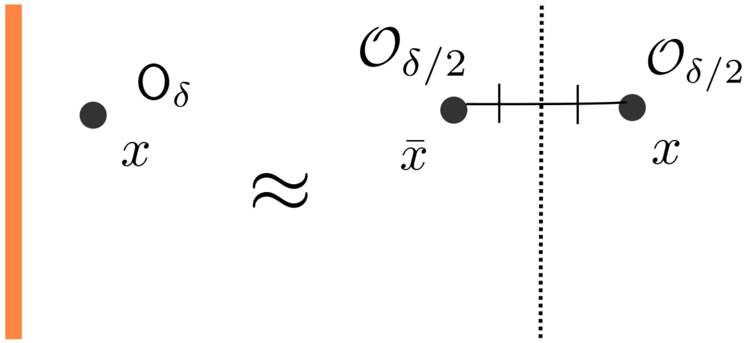


$\hat{x}^a$
$x_\perp^d$
$\widehat{O}_{\hat{\delta}_1}$
O_{δ_1}
O_{δ_2}
Boundary
$\approx$
$\hat{x}^a$
$x_\perp^d$
$\mathcal{O}_{\hat{\delta}_1}$
$\mathcal{O}_{\delta_1/2}$
$\mathcal{O}_{\delta_1/2}$
$\mathcal{O}_{\delta_2/2}$
$\mathcal{O}_{\delta_2/2}$
BCFT
ancillary CFT

Some working examples of our prescription

1-pt bulk DCFT correlator

$$\langle \mathrm{O}_\delta(x) \rangle_{\mathrm{DCFT}} = \frac{a_{\mathrm{O}}}{|x_\perp|^\delta}$$



match

2-pt CFT correlator in the mirror symmetric configuration

$$\langle \mathcal{O}_{\delta/2}(x)\ \mathcal{O}_{\delta/2}(\bar{x}) \rangle_{\mathrm{CFT}} = \frac{1}{|x-\bar{x}|^\delta} = \frac{1}{2^\delta\, |x_\perp|^\delta}$$

2-pt bulk-defect DCFT correlator

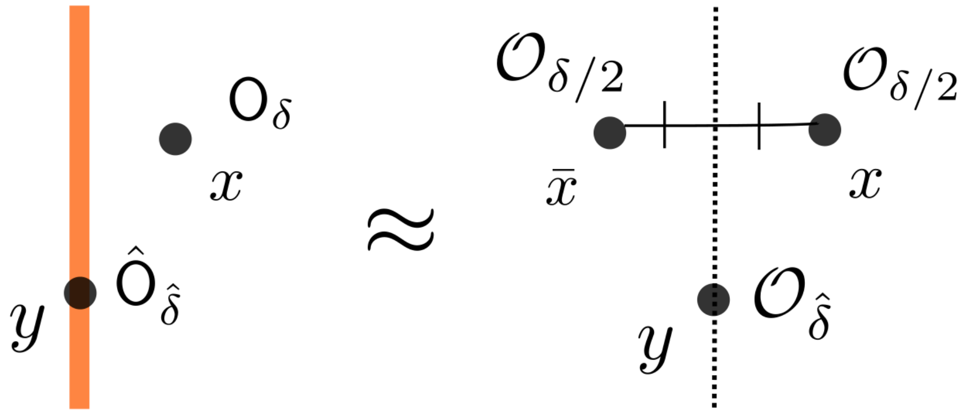
$$\langle\, \mathrm{O}_{\delta}(x)\, \widehat{\mathrm{O}}_{\hat{\delta}}(y)\,\rangle_{\mathrm{DCFT}} = \frac{b_{\mathrm{O}\widehat{\mathrm{O}}}}{|x_{\perp}|^{\delta-\hat{\delta}}\left(|\hat{x}-\hat{y}|^2+|x_{\perp}|^2\right)^{\hat{\delta}}}$$

match

3-pt CFT correlator in the mirror symmetric configuration

$$\langle\, \mathcal{O}_{\delta/2}(x)\, \mathcal{O}_{\delta/2}(\bar{x})\, \mathcal{O}_{\hat{\delta}}(y)\,\rangle_{\mathrm{CFT}} = \frac{1}{|x-y|^{\hat{\delta}}\,|y-\bar{x}|^{\hat{\delta}}\,|x-\bar{x}|^{\delta-\hat{\delta}}}$$

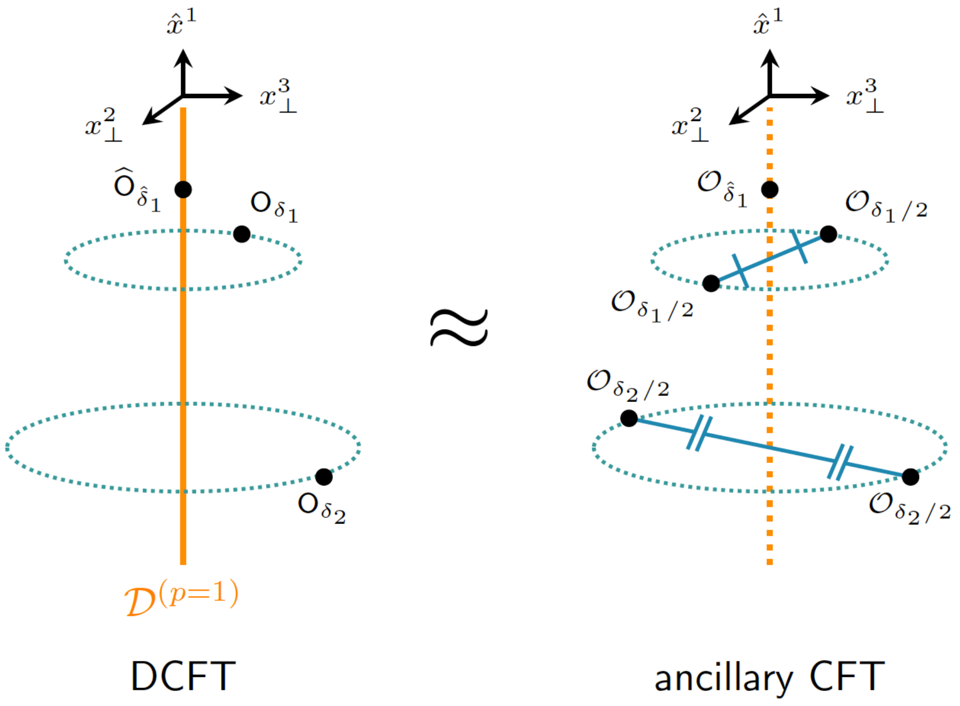
$$= \frac{1}{2^{\delta-\hat{\delta}}\,|x_{\perp}|^{\delta-\hat{\delta}}\left(|\hat{x}-\hat{y}|^2+|x_{\perp}|^2\right)^{\hat{\delta}}}$$



Summary and future work

- We generalize the Cardy's method to higher-dimensional DCFTs.

- Using method of images, we can calculate kinematical parts of DCFT correlators in terms of CFT ones.



- Is there complete dictionary including dynamical CFT data ?

  e.g. OPE coefficients

- Can our method be applied to defect conformal bootstrap ?