Vacuum decay with the Lorentzian path integral

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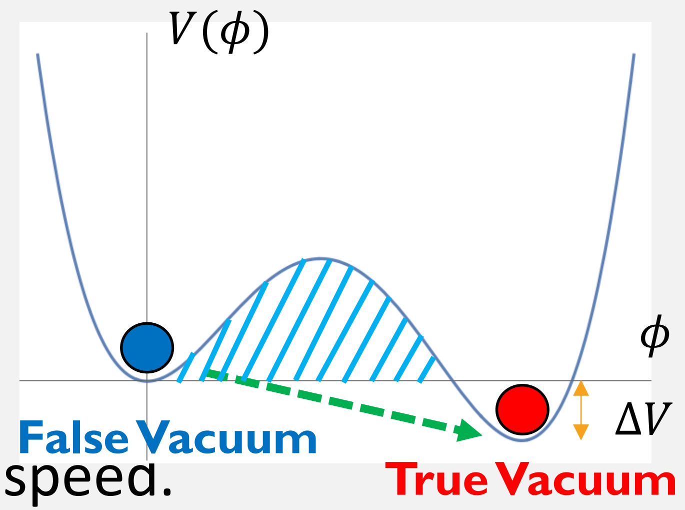
Introduction: Vacuum metastability and quantum tunneling



- Vacuum decay is a quantum tunneling of scalar field

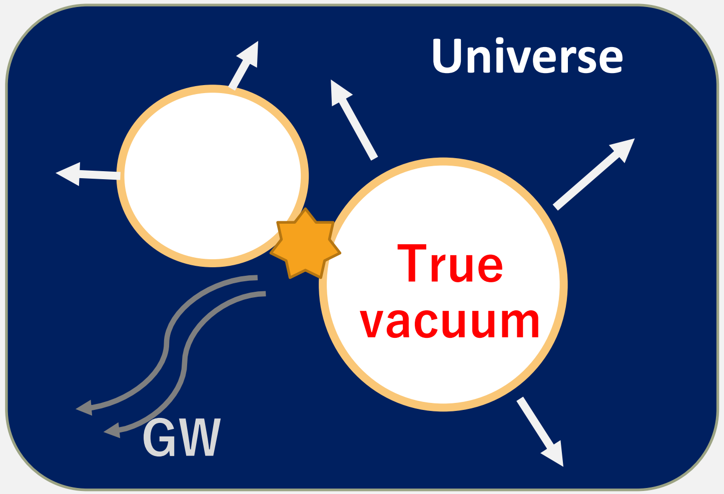
In quantum mechanics, tunneling probability is

$$P_{\mathrm{D}} \sim \exp\left[-\int d\phi\sqrt{2(V(\phi)-E_0)}\right]$$

In QFT, a **bubble** of true vacuum is nucleated and expand at the light speed.

- Rich phenomenology from bubble nucleation

- **GW** production from density perturbation by bubble collision
- **PBH** production
- **Particle Creation** of coupling fields to ϕ



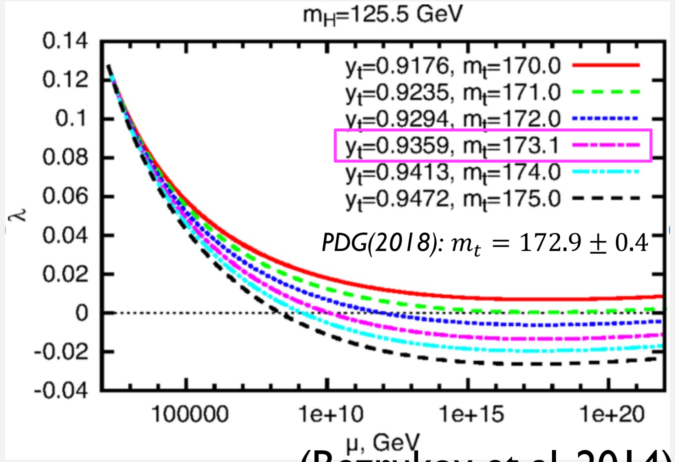
Introduction: Vacuum decay in our Universe

➢Vacuum instability is ubiquitous

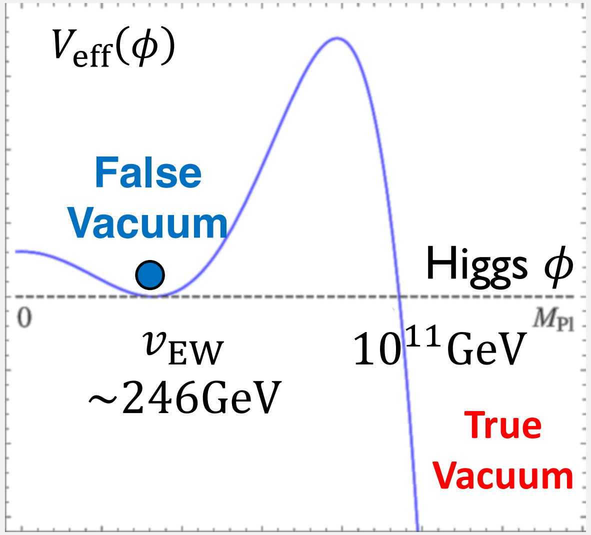
- Higgs instability in SM

$V_{\text{eff}}(\phi) \sim \lambda(\phi)\phi^4, \lambda$: effective coupling

$\lambda(\mu)$ is negative for large energy scale($\phi \sim 10^{11}\text{GeV}$)

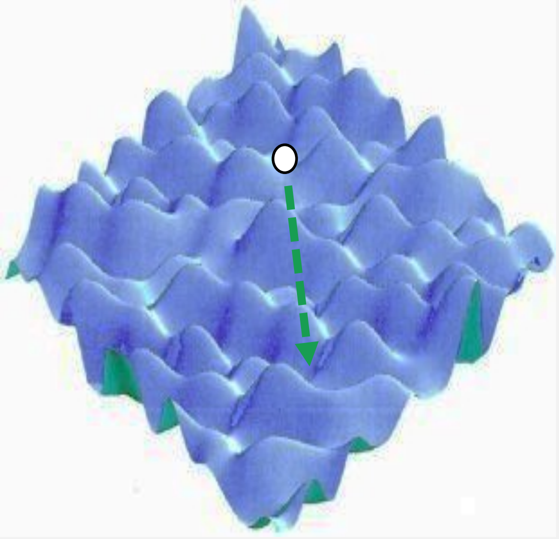


(Bezrukov et al. 2014)



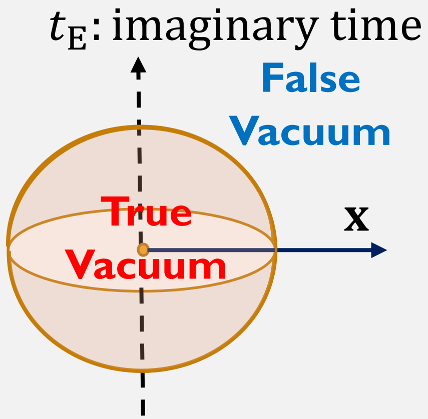
- Many unstable vacua in UV theory

many fields appear → complicated vacuum structure is expected



Introduction: Vacuum decay in Euclidean formalism

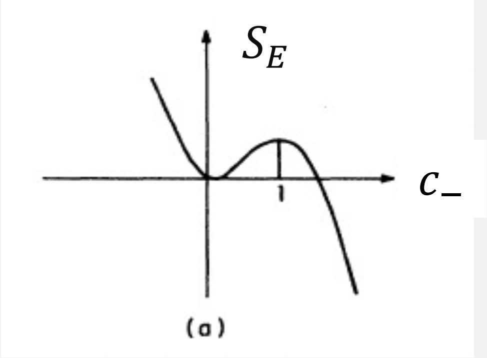
➢ Decay rate is estimated by the action of bounce (Callan, Coleman 1977)



Decay rate is defined as imaginary part of false vacuum energy E_0.

$$e^{-E_0 T} = \langle \mathrm{fv} | e^{-T\hat{H}} | \mathrm{fv} \rangle = \int_{-\frac{T}{2}<t_{\mathrm{E}}<\frac{T}{2},\, \frac{L}{2}<x_i<\frac{L}{2}} \mathcal{D}\phi e^{-S_{\mathrm{Euclid}}[\phi]} \sim \exp(K e^{-B} T),$$

with $T \to \infty$, $B_{\mathrm{Coleman}} = S_{\mathrm{E}}[\phi_b]$, ϕ_b: $O(4)$ bounce solution of $\frac{\delta S_E}{\delta \phi} = 0$, $K = L^3 \sqrt{\frac{B}{2\pi}} \left(\frac{\det\prime\left(-\Box + V''(\phi_{\mathrm{b}})\right)}{\det\left(-\Box + V''(0)\right)} \right)^{\frac{1}{2}}$

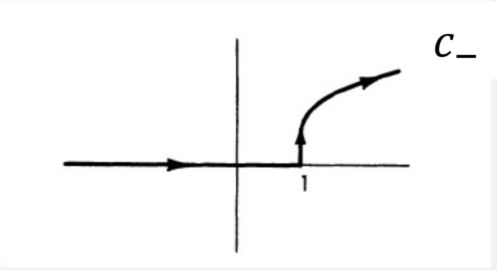


K originates from integral of gaussian fluctuation $\delta\phi = c_i f_i$ around ϕ_{b}

However, the path integral diverges due to negative mode.

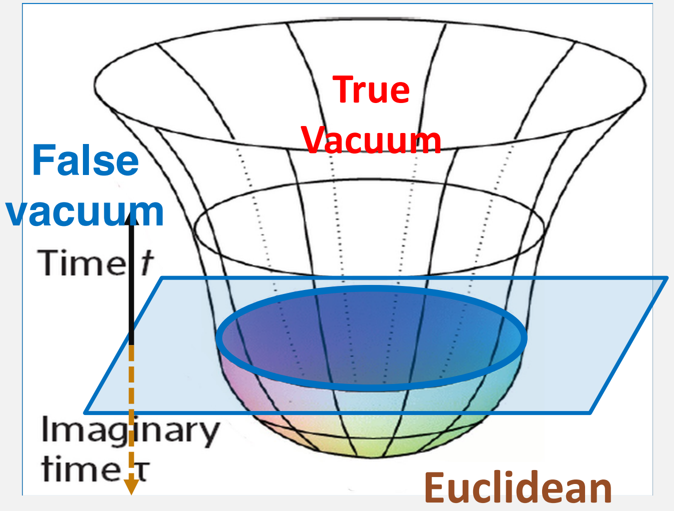
To avoid divergence, we adopt complex integration contour for c_-,

which gives $\mathrm{Im}\, E_0$:



$$\Gamma_{\mathrm{Decay}} = \mathrm{Im}\, E_0 = A e^{-\boldsymbol{B}_{\mathbf{Coleman}}}, \text{with } A = L^3 \frac{1}{2} \left(\frac{B_{\mathrm{Coleman}}}{2\pi} \right)^2 \left| \frac{\det\prime\left(-\Box + V''(\phi_{\mathrm{cl}})\right)}{\det\left(-\Box + V''(0)\right)} \right|^{-\frac{1}{2}}$$

Introduction: Vacuum decay in Euclidean formalism 2



In Lorentzian spacetime, it is **supposed** bubble with bounce radius is nucleated.

And Lorentzian evolution is obtained by analytic continuation $t_{\mathrm{E}} \to -it$, as

$$\phi_b\left(\sqrt{t_{\mathrm{E}}^2 + \mathbf{x}^2}\right) \to \phi_{\mathrm{Lorentz}}(t, x) = \phi_b(\sqrt{-t^2 + \mathbf{x}^2})$$

Questions on Euclidean formalism:

Q. Euclidean formalism just gives decay rate of $|\mathrm{fv}\rangle$. What is the final state after decay?

Q. How to calculate **larger** or **smaller** bubble nucleation probability?

Q. In curved space, how to define decay rate where notion of energy and vacuum is **not** unique?

Q. With the (dynamical) gravity, energy E_0 is **zero** by Hamiltonian constraint. Is it consistent? (future work)

→ **Directly evaluate transition amplitude $\langle \phi_{\mathrm{bubble}} | \mathrm{FV} \rangle$ in Lorentzian path integral !**

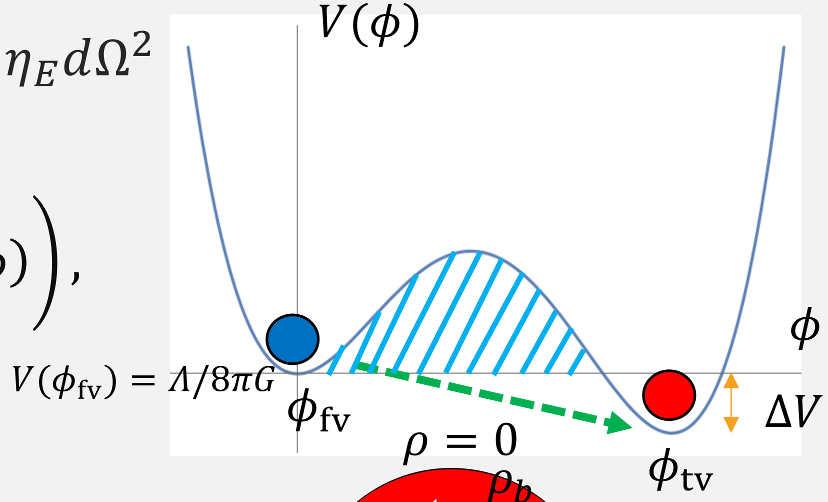
Model: Vacuum decay in de-Sitter Universe (Coleman de-Luccia 1980)

We consider vacuum decay in de-Sitter spacetime with radius $l_{\mathrm{dS}} = \sqrt{3/\Lambda} = \sqrt{3/8\pi G V(\phi_{\mathrm{fv}})}$:

$$ds^2_{\mathrm{dS}_4} = d\xi^2 + \rho(\xi)^2 ds^2_{\mathrm{dS}_3} \ \text{ with } \ \rho = l_{\mathrm{dS}} \sin\frac{\xi}{l_{\mathrm{dS}}}, ds^2_{\mathrm{dS}_3} = -d\eta^2 + \cosh^2\eta\, d\Omega^2$$

In order to go to Euclidean spacetime, we perform analytic continuation $\eta \to -i\left(\eta_E - \frac{\pi}{2}\right)$:

$$ds^2_{S_4} = d\xi^2 + \rho(\xi)^2 ds^2_{S_3} \quad \text{with } \ S_n\text{: n-sphere, } \ ds^2_{S_3} = d\eta_E^2 + \sin^2\eta_E\, d\Omega^2$$



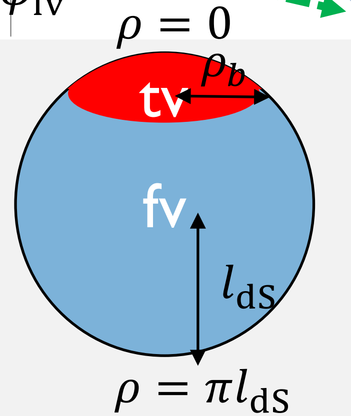
Euclidean action and O(4) symmetric bounce solution is

$$S_E = \int d^4x_E \sqrt{-g_E}\left(\frac{1}{2} g_E^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right),$$

$$\frac{\delta S_E}{\delta\phi} = \ddot{\phi} + 3\frac{\dot{\rho}}{\rho}\dot{\phi} - \frac{dV}{d\phi} = 0,$$

with a boundary conditions:

$$\phi(0) \sim \phi_{\mathrm{tv}}, \quad \phi(\pi l_{dS}) \sim \phi_{\mathrm{fv}}$$



We neglect contribution of E-H action, which is valid when $\frac{\Delta V}{V} \ll 1$ and week gravity.

Model: Vacuum decay in de-Sitter Universe 2 (Coleman de-Luccia 1980)

Let's use thin-wall approximation. Action split into bulk and surface term:

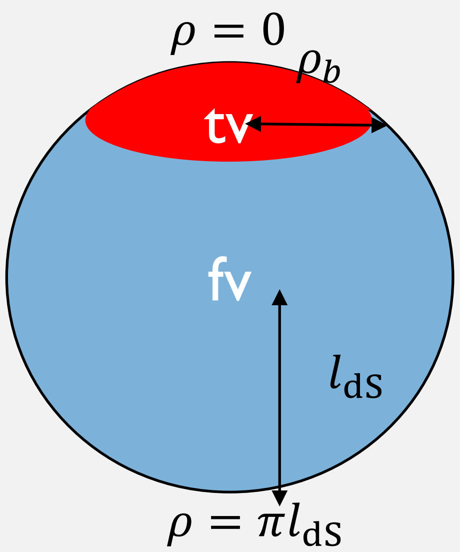
$$B = S_E[\phi] = 2\pi^2\sigma\rho_b^3 - \int_0^{\rho_b} d\rho \frac{\rho^3 \Delta V}{\sqrt{1-\frac{\rho^2}{l_{dS}^2}}}, \qquad \sigma = \int d\phi \sqrt{2(V - V(\phi_{\rm fv}))}\text{: wall tension}$$

Since ϕ is saddle solution, ρ_b can be determined by $\frac{\mathrm{d}B}{\mathrm{d}\rho_b} = 0$:

$$\boldsymbol{\rho_b} = \frac{1}{\sqrt{\rho_0^{-2} + l_{dS}^{-2}}}, \qquad \text{with } \rho_0 = \frac{3\sigma}{\Delta V}\text{: bubble radius in Minkowski spacetime}$$

Then decay probability is

$$\mathrm{P_D} \sim \exp(-\boldsymbol{B}_{\mathbf{Coleman}}), \quad \boldsymbol{B}_{\mathbf{Coleman}} = \frac{2\pi^2\sigma\rho_b^3}{\left(1+\frac{\rho_b}{\rho_0}\right)^2}.$$

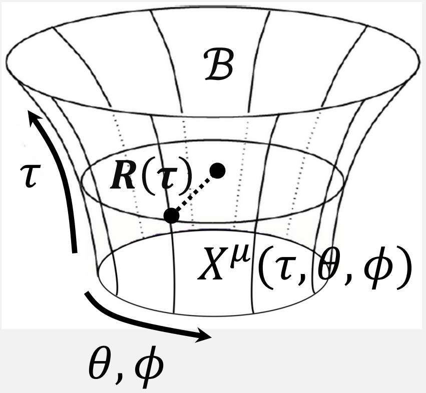


This is the result of the Euclidean formalism in our model.

Model: Vacuum decay as bubble wall nucleation

- Effective theory of a bubble in de-Sitter space (Basu+ 1991)

$$S_{\rm NG}[X^\mu] = \underbrace{\sigma \int_{\partial\mathcal{B}} d^3x \sqrt{-\det\partial_a X^\mu \partial_b X_\mu}}_{\text{Nambu-Goto action}} - \Delta V \int_{\mathcal{B}} d^4X\sqrt{-g}$$



with $ds^2 = f dT^2 + f^{-1}dR^2 + R^2 d\Omega_2^2,\ f = 1 - \frac{R^2}{l_{ds}^2},\ x^a = (\tau,\theta,\phi)$,

$X^\mu(x)$: embedding in bulk spacetime, $\partial_a X^\mu \partial_b X_\mu$: induce metric

DOF $: X^\mu(x^a) = \{T, R, \Theta, \Phi\} \xrightarrow{O(3)\text{ sym.}} \{T(\tau), R(\tau), \theta, \phi\}$

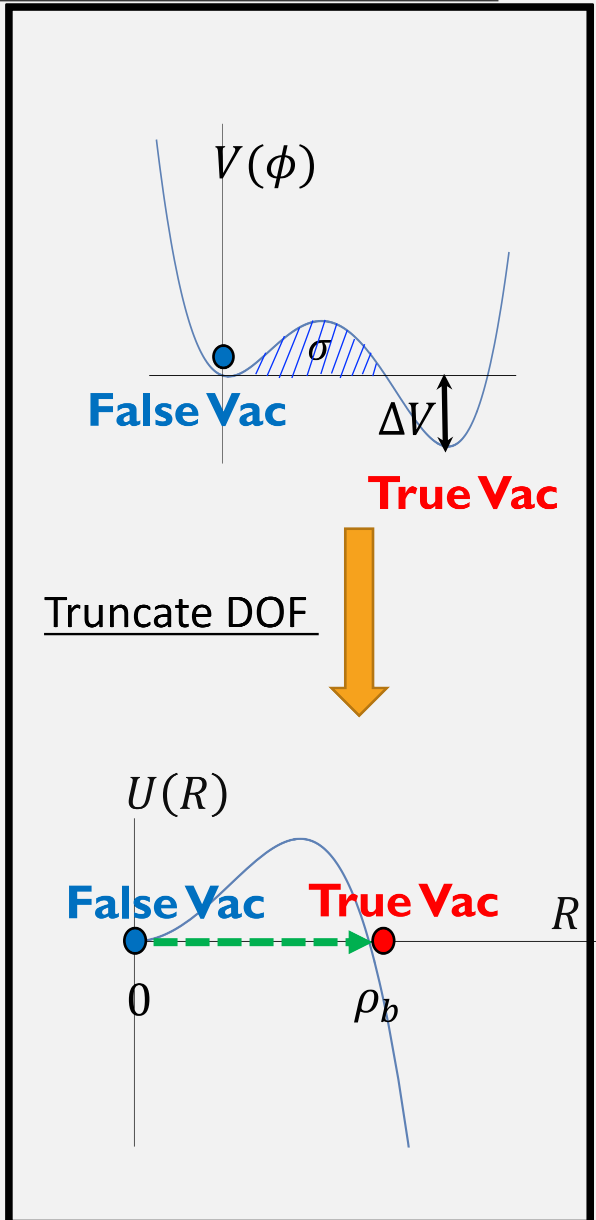
$$S_{\rm NG}[R] = 4\pi\sigma \int dT \left(-R^2\sqrt{f - f^{-1}\left(\frac{dR}{dT}\right)^2} + \frac{R^3}{\rho_0}\right) \quad (\text{gauge fixing: } \tau = T)$$

Analytic continuation $T \to -iT_E$ gives bounce equation:

$$R^2\left(\frac{dR}{dT_E}\right)^2 - 2U(R) = 0,\ \text{with } U(R) = f^2\frac{\rho_0^2}{R^2}\left(1 - \frac{R^2}{\rho_b^2}\right)$$

$$R_{\rm bounce} = \boldsymbol{\rho_b},\ \boldsymbol{B}_{\bf Coleman} = S_{\rm NG,E}[R] = \frac{2\pi^2\sigma\,\rho_b^3}{(1+\rho_b/\rho_0)^2}$$

Same result as field theory



Lorentzian path integral for bubble wall nucleation

- **Quadratic** action suitable for path integral quantization (Polyakov 1981)

$$S_{\rm P}[X^\mu,\gamma^{ab}] = -\sigma\int_{\partial\mathcal{B}} d^3x\,\underline{\sqrt{-\gamma}\,\frac{1}{2}\left[\gamma^{ab}\partial_a X^\mu\partial_b X_\mu - 1\right]} + \Delta V\int_{\mathcal{B}} d^4\tilde{x}\sqrt{-g}$$

with γ_{ab}: metric on wall

modified as quadratic in $\partial_a X^\mu$

on-shell equation $\frac{\delta S_{\rm P}}{\delta\gamma_{ab}} = 0$ gives $\gamma_{ab} \approx \partial_a X^\mu \partial_b X_\mu$ and $S_{\rm P}[X^\mu,\gamma^{ab}] \approx S_{\rm NG}[X^\mu]$.

<u>DOF</u> :$\{X^\mu(x^a)\},\{\gamma_{ab}\} \xrightarrow{O(3)\text{ sym.}} \{T(\tau),R(\tau),\theta,\phi\},\ \gamma_{ab}dx^a dx^b = -N(\tau)^2 d\tau + R(\tau)^2 d\Omega^2$

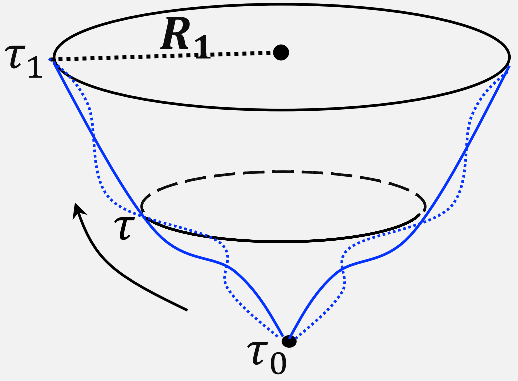
$$S_{\rm P}[T,R,N] = 4\pi\sigma\frac{1}{2\pi}\int d\tau\left\{\frac{1}{2}R^2\left[N^{-1}\left(-f\dot{T}^2 + f^{-1}\dot{R}^2\right) - N\right] + \frac{R^3\dot{T}}{\rho_0}\right\}$$

- transition amplitude for bubble nucleation defined by the **Lorentzian path integral**

$$G(R_1;0) = \int_0^\infty dN \int_{R(0)=0}^{R(1)=R_1} \mathcal{D}T\mathcal{D}R\,\exp(iS_{\rm P}[T,R,N])\ \ (\text{gauge fixing: }\dot{N} = 0)$$

R_1 is a nucleated bubble radius which is **freely chosen.**

nucleated bubble radius
(**Arbitrary choice**)



Lorentzian path integral for bubble wall nucleation 2

Approximate transition amplitude as saddle solution for $\{T, R\}$, except for N,

because no classical solution connecting $R(0) = 0$ and $R(1) = R_1 > 0$ (see figure)

$$G(R_1; 0) = \int_0^\infty dN \int_{R(0)=0}^{R(1)=R_1} \mathcal{D}T\mathcal{D}R \exp(iS_{\mathrm{P}}[T, R, N])$$

$$\simeq \int_0^\infty dN A(N) \exp(iS_{\mathrm{eff}}[N]), \quad S_{\mathrm{eff}}(N) = S_{\mathrm{P}}[\bar{T}, \bar{R}, N]$$

$\bar{T}, \bar{R}$ is a saddle solution which obeys (integrated) equation of motion:

$$-4\pi\sigma \frac{1}{2} R^2 \left[N^{-2}\left(f\dot{T}^2 - f^{-1}\dot{R}^2\right) - 1\right] = H$$

$$4\pi\sigma R^2 f \left[N^{-2}\left(\frac{\dot{T}}{N} - \frac{f^{-1}R}{\rho_0}\right)\right] = E = 0$$

Where $H(\neq 0)$ is an integration constant depending on R_1, and E is a conjugate energy of T (which is zero since we consider tunneling from vacuum) $A(N)$ comes from fluctuation around saddle solution.

The Lorentzian path integral is reduced to **single integral**!
But difficult to evaluate due to the **oscillatory behavior from** $\mathrm{Im}[iS_{\mathrm{eff}}(N)]$

Picard-Lefschetz theory applied to path integral

➢ The oscillatory integral converges with P-L theory (Feldbrugge-Lehners-Turok, 2017)

Let's consider deformation of contour $\mathcal{C}$ to sum of steepest decent contours of $\mathrm{Re}[iS_{\mathrm{eff}}(\mathrm{N})]$, $\mathcal{C}_{\mathrm{new}}$:

$$\int_{\mathcal{C}} dN \exp[iS_{\mathrm{eff}}(N)] \to \int_{\mathcal{C}_{\mathrm{new}}} dN \exp[iS_{\mathrm{eff}}(N)],$$

with N_s: saddle point, $\mathcal{J}_s$: steepest decent contour of from N_s, $\mathcal{C}_{\mathrm{new}} = \sum_s n_s \mathcal{J}_s$,

The weight of decent contours, n_s, is given by

$n_s = \langle C, \mathcal{K}_s \rangle$, $\mathcal{K}_s$: steepest acent contour from N_s, $\langle \cdot,\cdot \rangle$: geometrical intersection number(=0, ± 1)

This is because $\langle \mathcal{J}_s, \mathcal{K}_{s'} \rangle = \delta_{s,s'}$ and intersection number is topological and conserved smooth deformation $\mathcal{C} \to \mathcal{C}_{\mathrm{new}}$.

- On steepest contours $\mathcal{C}_{\mathrm{new}}$, $\mathrm{Im}[iS_{\mathrm{eff}}(N)]$ is stationary, so integration over $\mathcal{C}_{\mathrm{new}}$ is **NOT oscillatory**!

→ saddle point approximation around. $N = N_s$ is valid

Picard-Lefschetz theory applied to path integral 2

Ex.) Airy function:

$$\mathrm{Ai}(x) = \int_{\mathbb{R}} dN \exp[iS_{\mathrm{eff}}(N)], \; S_{\mathrm{eff}}(N) = \frac{N^3}{3} + xN$$

This integral is convergent but NOT absolutely convergent.

Saddle point: $N_{\pm} = \pm i\sqrt{x}$.

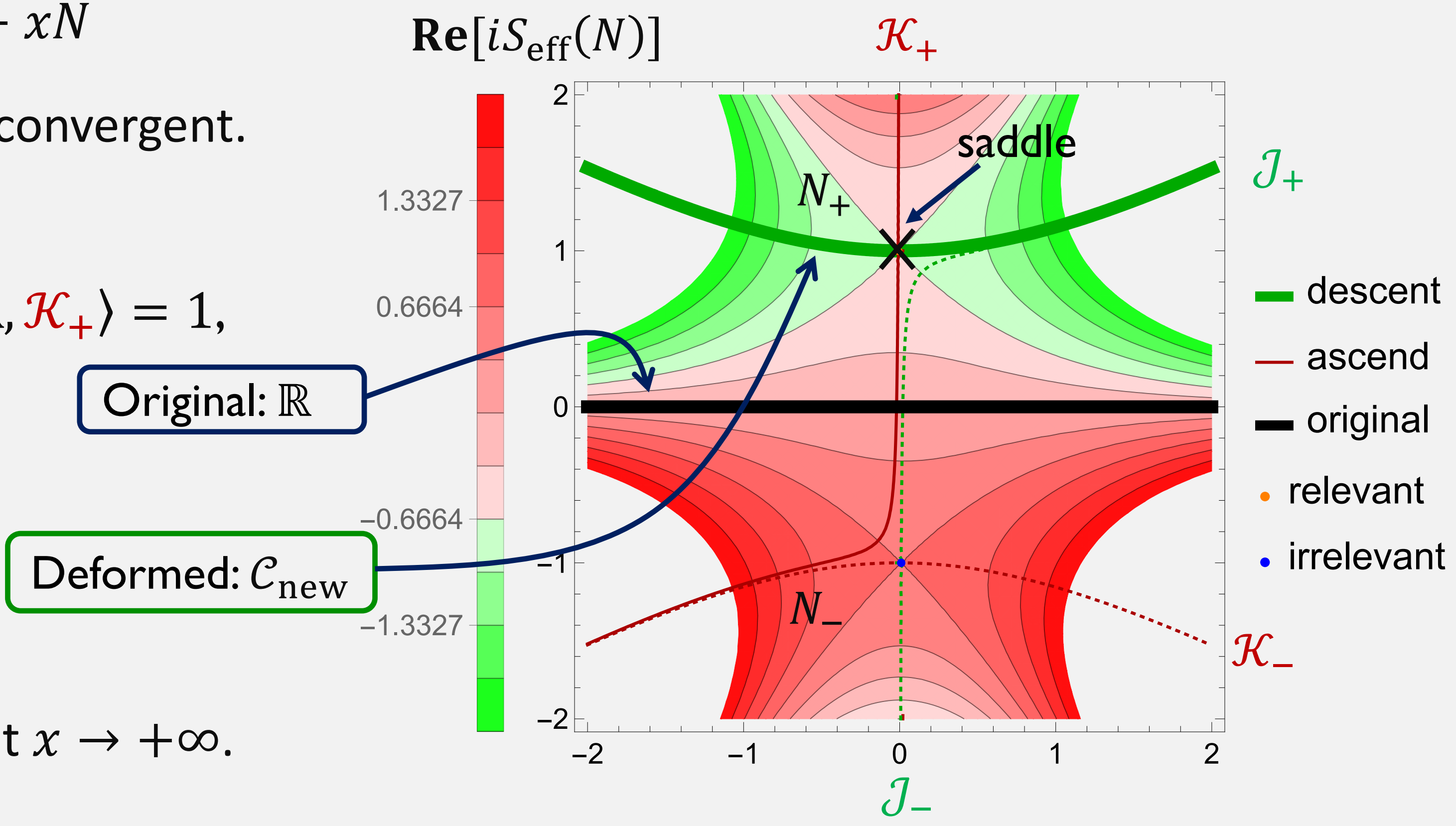
Steepest ascent $\mathcal{K}_+$ intersect $\mathbb{R}$ once: $n_+ = \langle \mathbb{R}, \mathcal{K}_+ \rangle = 1$,

whereas $\mathcal{K}_-$ does NOT. : $n_- = \langle \mathbb{R}, \mathcal{K}_- \rangle = 0$

$\rightarrow \mathcal{C}_{\mathrm{new}} = \mathcal{J}_+$

$$\mathrm{Ai}(x) \simeq \frac{1}{2\sqrt{\pi} x^{\frac{1}{4}}} \exp\left(-\frac{2}{3} x^{\frac{3}{2}}\right), \; (N \sim N_+)$$

This is asymptotic expression of Airy function at $x \to +\infty$.



Case 1:Transition amplitude for **critical size** bubble

➢ The transition amplitude is consistent with Euclidean decay rate

$$\frac{\delta S_{\mathrm{P}}}{\delta \bar{R}} = \frac{\delta S_{\mathrm{P}}}{\delta \bar{T}} = 0, \quad \text{with } \bar{R}(\tau_0) = 0, \bar{R}(\tau_1) = \underline{R_1 = \boldsymbol{\rho_b}}$$

$$\rightarrow \bar{R}^2(\tau) = \frac{\rho_b^2}{\rho_0 + \rho_b} \mathrm{csch}^2 z \sinh(z\tau)\{\rho_b \sinh(z\tau) + \rho_0 \sinh[z(2-\tau)]\}, \quad z = N/\rho_b$$

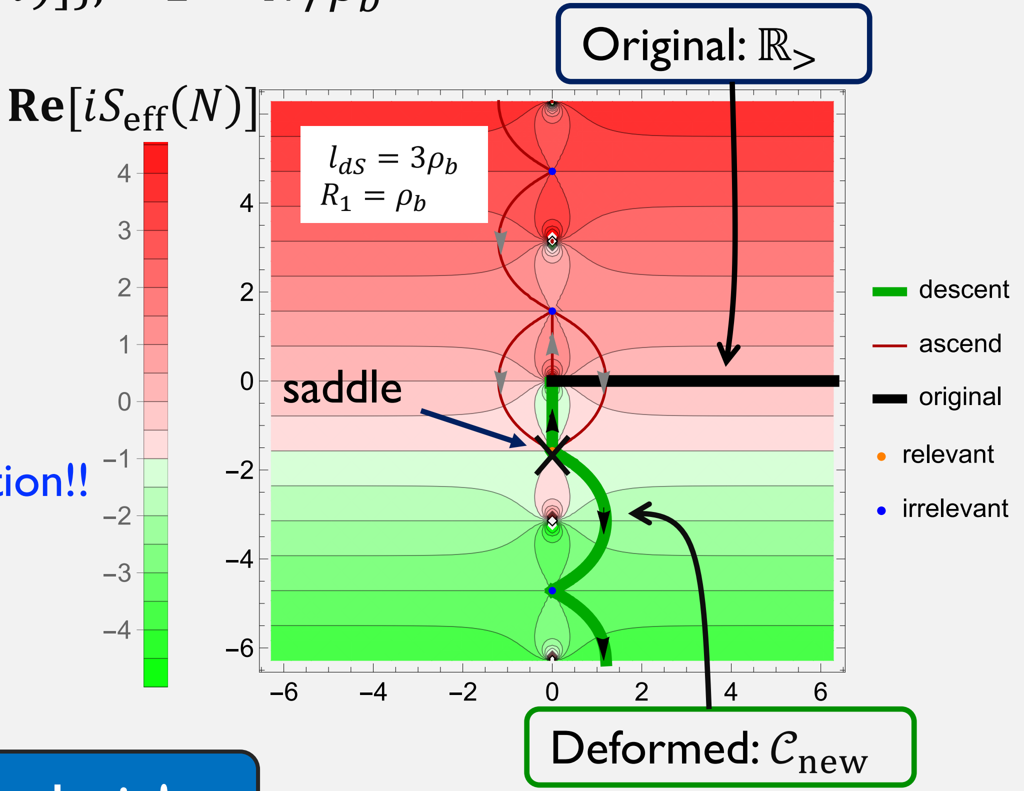
$$S_{\mathrm{eff}}(N) = S_{\mathrm{P}}[\bar{T}, \bar{R}, N] = \frac{2\pi\sigma\rho_b^3}{(1+\rho_b/\rho_0)^2}\left[\coth\frac{N}{\rho_b} - \frac{N}{\rho_b}\right]$$

- Transition amplitude for **critical size** bubble:

$$G(\rho_b; 0) \sim \int_0^\infty dN \exp[iS_{\mathrm{eff}}(N)] \overset{\text{saddle}}{\sim} \exp\left[-\frac{\pi^2\sigma\rho_b^3}{(1+\rho_b/\rho_0)^2}\right]$$

$= B/2$, bounce action!!

$$P_D \sim |G(\rho_b; 0)|^2 \sim \exp(-\boldsymbol{B}_{\mathbf{Coleman}})$$



Lorentzian path integral is consistent with Euclidean analysis!

Case 2: Large bubble nucleation

➢ Final state is a **large bubble** with radius R_1 $(R_1 > \rho_b)$

$$\frac{\delta S_{\mathrm{P}}}{\delta \bar{R}} = \frac{\delta S_{\mathrm{P}}}{\delta \bar{T}} = 0, \quad \text{with } \bar{R}(\tau_0) = 0, \bar{R}(\tau_1) = \underline{R_1 > \boldsymbol{\rho_b}}$$

$S_{\mathrm{eff}}(N) =$

$$\frac{2\pi\sigma l_{dS}^4}{\rho_b}\left\{\left(1+\frac{\rho_b^2}{\rho_0^2}\right)(\coth z - z) - \left(\frac{R_1^2}{\rho_b^2}-1\right)\frac{\rho_b^2}{l_{dS}^2}\coth z - 2\left[\sqrt{\cosh^2 z - \left(\frac{R_1^2}{\rho_b^2}-1\right)\frac{\rho_0^2}{l_{dS}^2}} - \frac{\rho_b}{\rho_0}\mathrm{Arccoth}\left(\operatorname{csch} z\sqrt{\cosh^2 z - \left(\frac{R_1^2}{\rho_b^2}-1\right)\frac{\rho_0^2}{l_{dS}^2}}\right)\right]\right\}$$

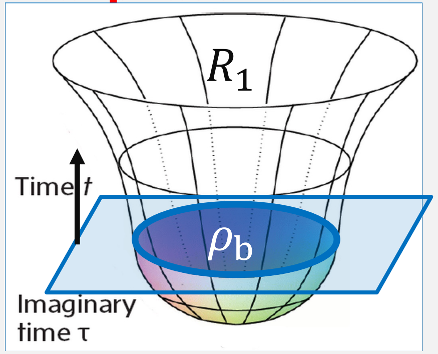
- Transition amplitude: $, z = \frac{N}{\rho_b}$

$$G(R_1;0) \sim \int_0^\infty dN \exp[iS_{\mathrm{eff}}(N)]$$

$$\overset{\text{saddle}}{\sim} \exp\left\{-\underbrace{\frac{\pi^2\sigma\rho_b^3}{(1+\rho_b/\rho_0)^2}}_{= B/2} + \boldsymbol{i}\,\underbrace{\frac{2\pi\sigma\rho_0^2\rho_b^3}{(\rho_0^2-\rho_b^2)^2}\left[-\sqrt{\frac{R_1^2}{\rho_b^2}\left(\frac{R_1^2}{\rho_b^2}-1\right)}(\rho_0^2-\rho_b^2) - (\rho_0^2+\rho_b^2)\operatorname{arctanh}\sqrt{\frac{R_1^2-\rho_b^2}{R_1^2}} + 2\rho_0\rho_b\operatorname{arctanh}\sqrt{\frac{R_1^2-\rho_b^2}{R_1^2}\frac{\rho_0}{\rho_b}}\right]}\right\}$$

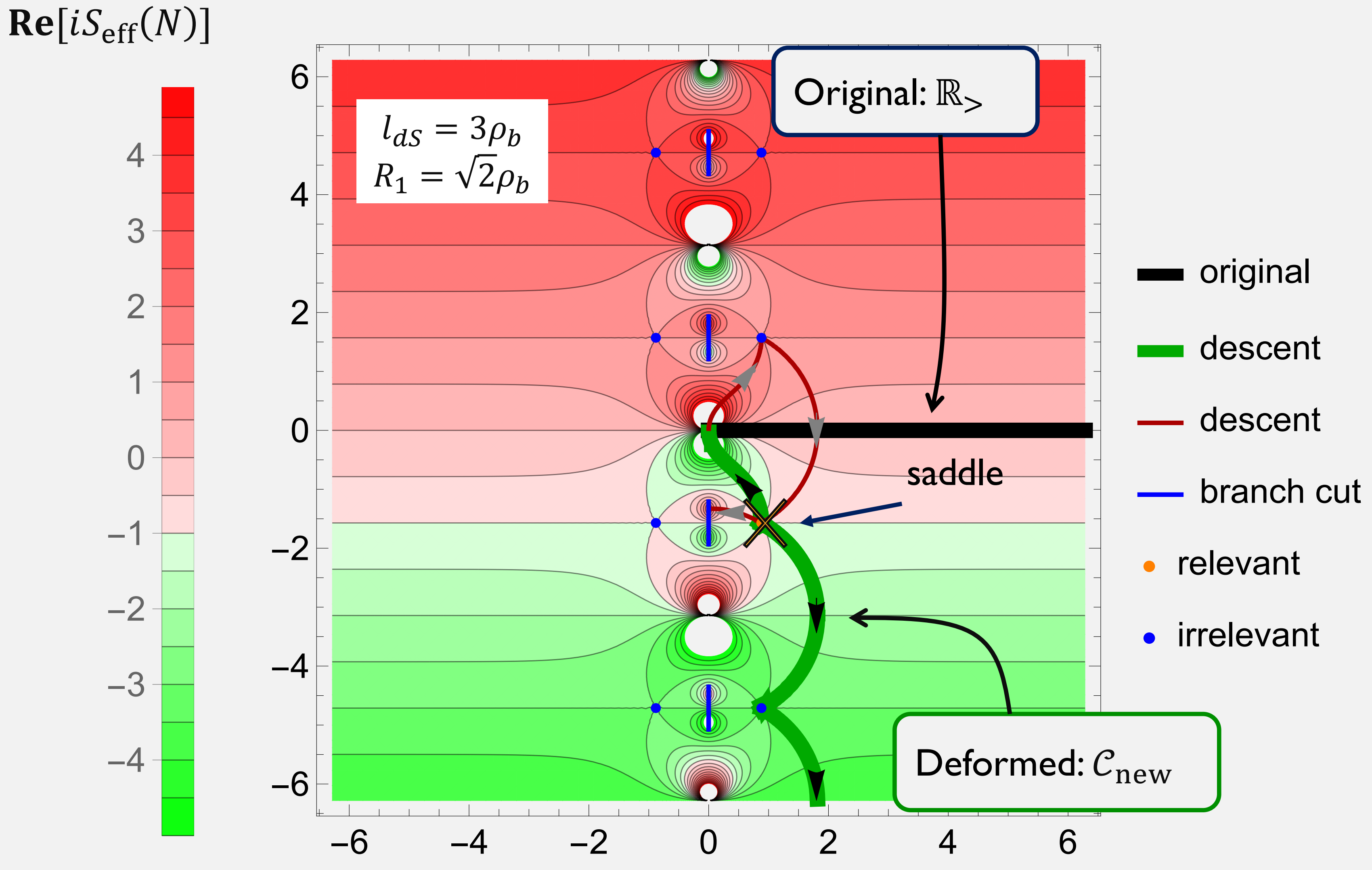
$= S_{NG}[\rho_b \to R_1]$ **classical phase rotation**

$$\boxed{P_D \sim |G(r_b;0)|^2 \sim \exp(-\boldsymbol{B}_{\mathbf{Coleman}})}$$



Critical size bubble is nucleated and classically expand to large bubble!

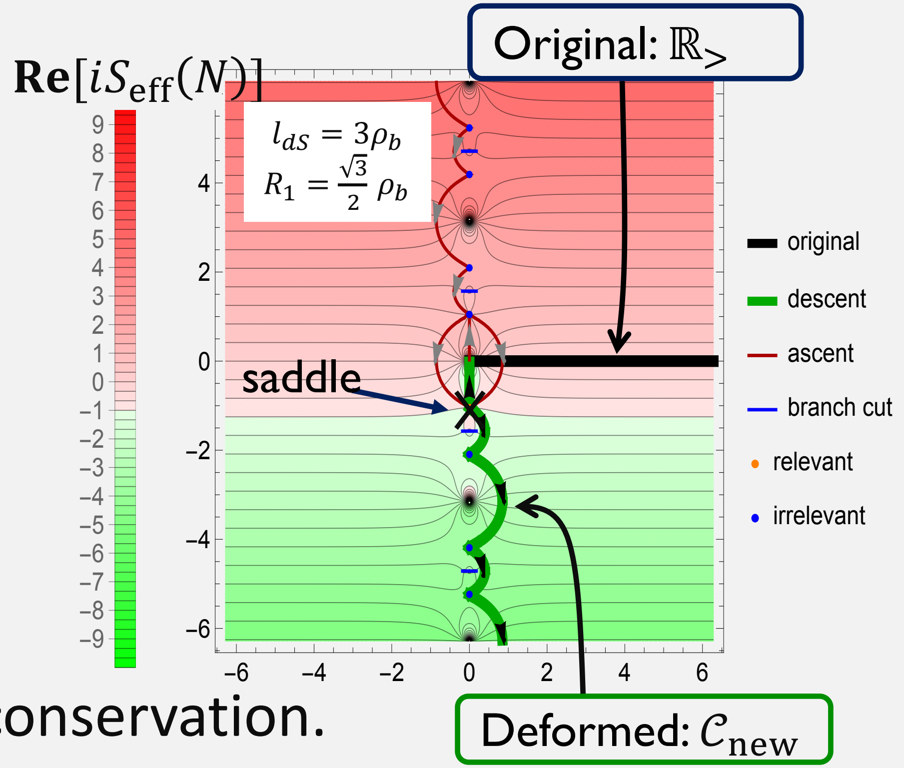
Case 2: Large bubble nucleation



Case 3: Small bubble nucleation

The transition amplitude for **small bubble** nucleation $(R_1 < \rho_b)$

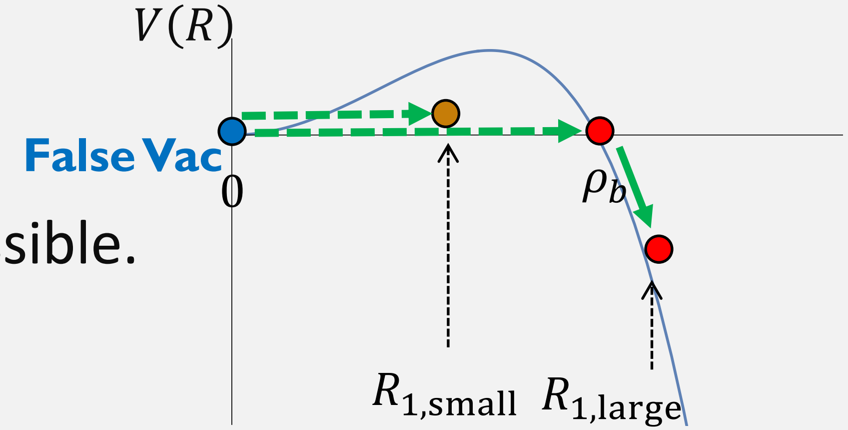
$$G(R_1;0) \sim \int_0^\infty dN \exp[iS_{\text{eff}}(N)] \sim \exp[-B/2 + (\text{positive real})]$$



$$P \sim |G(R_1;0)|^2 \sim \exp[-B + (\textbf{positive real})]$$

→<u>**higher probablility** to find small bubble!</u>

But the small bubble is not classical, since it breaks energy conservation.

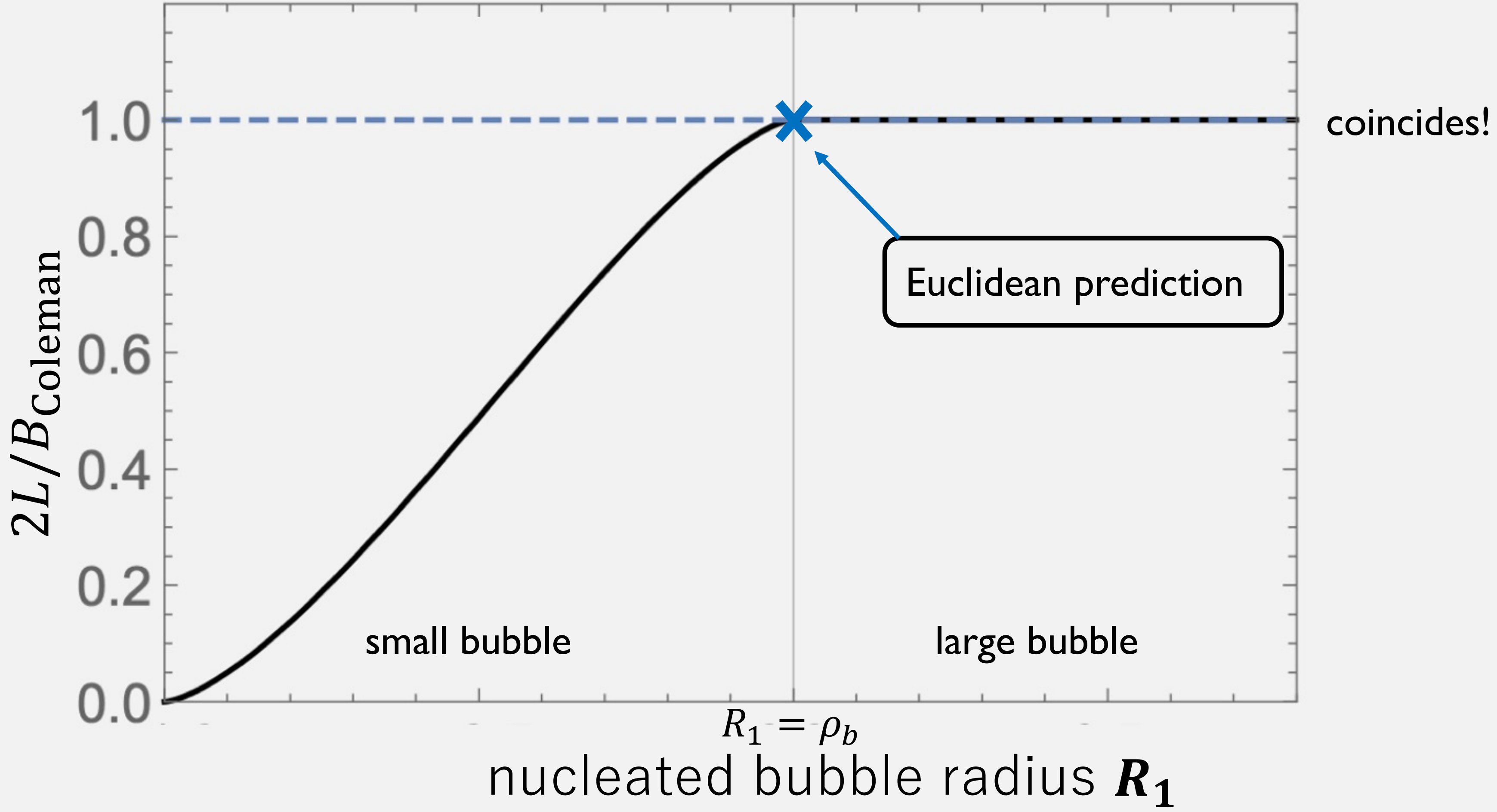


If there are additional energy sources(e.g. external field,

coupled particle), nucleation of small bubbles could be possible.

Case 1-3: Bubble nucleation probability

Summary of Case 1-3, nucleation probability: $P \sim \exp[-2L]$



Summary

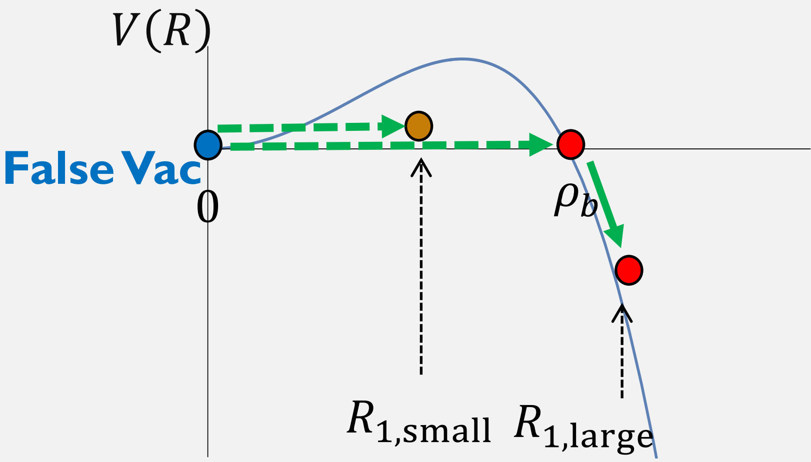
✓We formulate vacuum decay in **Lorentzian path integral** in de-Sitter space with the direct interpretation as the nucleation of the bubble wall.

✓We find

(1)**consistent probability** for bubble nucleation with Euclidean analysis,

(2)large bubble nucleation is understood as bubble **nucleation + classical expansion**,

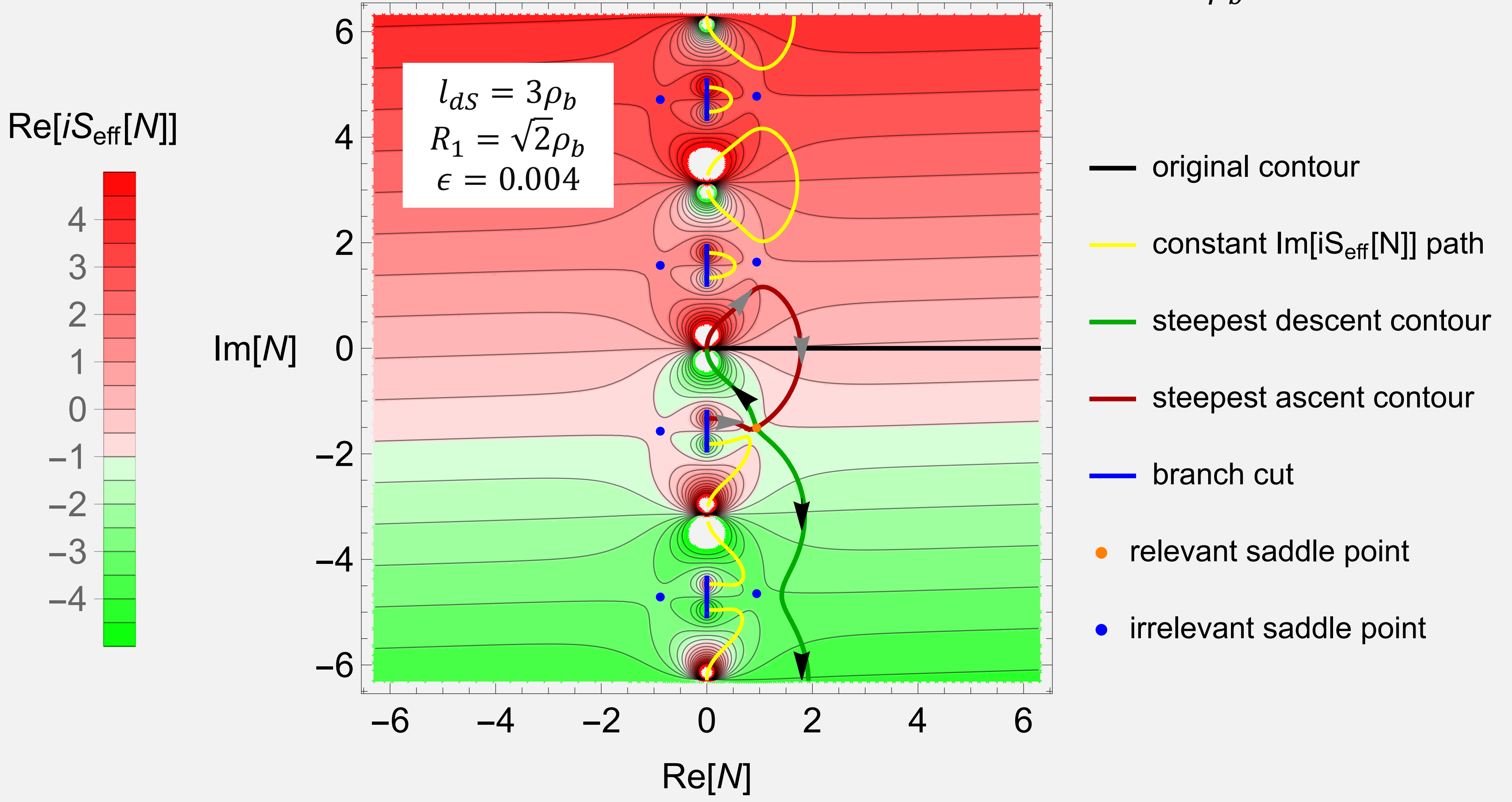
(3)**higher** probability to find small bubble.



Appendix: breaking degeneracy

In our model, saddle points and steepest contours are degenerate.

To break degeneracy, we can add small complex term as $S_{\text{eff}}(N) \to S_{\text{eff}}(N) + i\epsilon \frac{N}{\rho_b} B_{\text{Coleman}}$



Appendix: Contour deformation

➢Integration around infinity (=$\mathcal{J}$) has no contribution

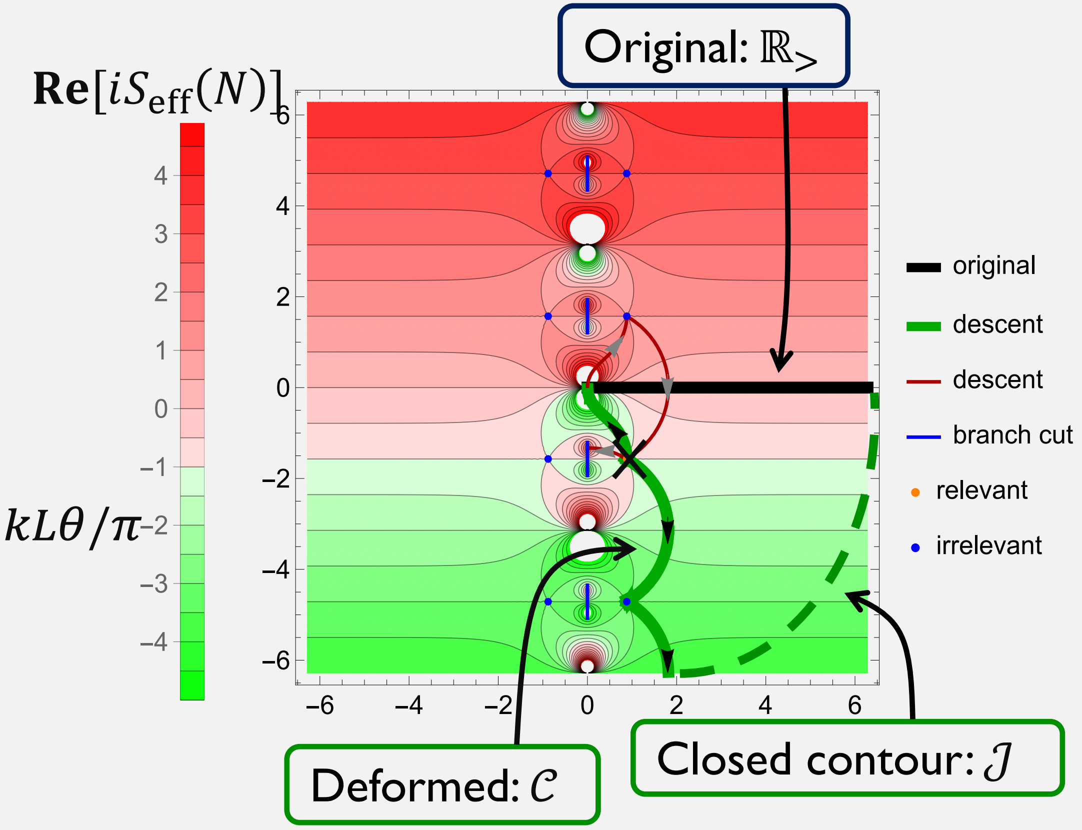
$$S_{\text{eff}}(N) = -kN + O(N^0)$$

- Jordan's lemma

change variable: $N = L\, e^{-i\theta}$

$$\to \left|\int_{\mathcal{J}} dN\, g(N) e^{iS_{\text{eff}}[N]}\right| < 2L \sup\{g(\mathcal{J})\} \int_0^{\frac{\pi}{2}} d\theta\, e^{-2kL\theta/\pi}$$

$$< \frac{\pi}{k} \sup\{g(\mathcal{J})\}$$



We can chance contour from $\mathbb{R}_{>0}$ to $\mathcal{C}$, as pre−factor $g(N) \to 0$

Appendix: Off-shell contribution to effective action

$$S_P[T,R,N] = 4\pi\sigma \int d\tau \left\{ \frac{1}{2} R^2 \left[N^{-1}\left(-f(R)\dot{T}^2 + f(R)^{-1}\dot{R}^2\right) - N \right] + R^3 \dot{T}/\rho_0 \right\}$$

T: free particle on time dependent background

- off-shell contribution of T

$$G(R_1;0) = \int_0^\infty dN \int_{R(\tau_0)=0}^{R(\tau_1)=R_1} \mathcal{D}R\mathcal{D}T \exp(iS_\mathrm{P})$$

$$\overset{R\sim\bar{R}}{\sim} \int_0^\infty dN \exp(iS_\mathrm{P}[\bar{T},\bar{R},N]) \underbrace{\int \mathcal{D}T\mathcal{D}p_T \exp\left[i\int d\tau\left(p_T\dot{T} - p_T^2/2m[\bar{R}]\right)\right]}_{=\langle T_0 | e^{-i\int d\tau\, \widehat{p_T}^2/2m} | T_0\rangle}$$

$$= \int_0^\infty dN \underbrace{\sqrt{\frac{\bar{m}}{2\pi i N}}}_{\boxed{\to 0,\ \text{as}\ N\to\infty}} \exp(iS_\mathrm{P}[\bar{T},\bar{R},N])$$