Tensor renormalization group and the volume independence in 2D U(N) and SU(N) gauge theories

Atis Yosprakob[1], Akira Matsumoto[1], Mitsuaki Hirasawa[3], Jun Nishimura[1,2]

Based on J. High Energy. Phys. 2021, 11 (2021)

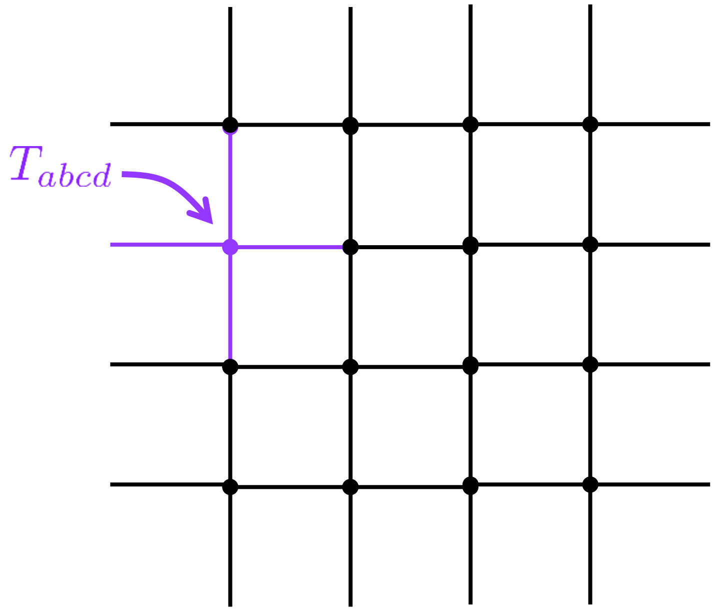
1: The Graduate University for Advanced Studies (SOKENDAI)
2: High Energy Accelerator Research Organization (KEK)
3: INFN (Milano-Bicocca section)

@ Kyoto (15/12/2021)

Tensor renormalization group (TRG)

- Rewriting the partition function as a network of tensors [Levin & Nave,'07]

$$Z = \sum_{\{\text{indices}\}} T_{abcd} T_{defg} \cdots$$



- Non-stochastic = no sign problem!

- Can access large volumes with logarithmic cost

- Can handle Grassmann fields directly

Application of TRG in gauge theories

- There were works on U(1), SU(2), and SU(3) gauge theories in 2d
  [Bazavov et al.,'19; Kuramashi & Yoshimura,'20; Fukuma et al.,'21]

- Our interest: higher rank gauge group in 2d (character expansion)

- Questions to be answered:

  1. How to impose the cutoff on representations?

  2. Interesting large-N behaviors?

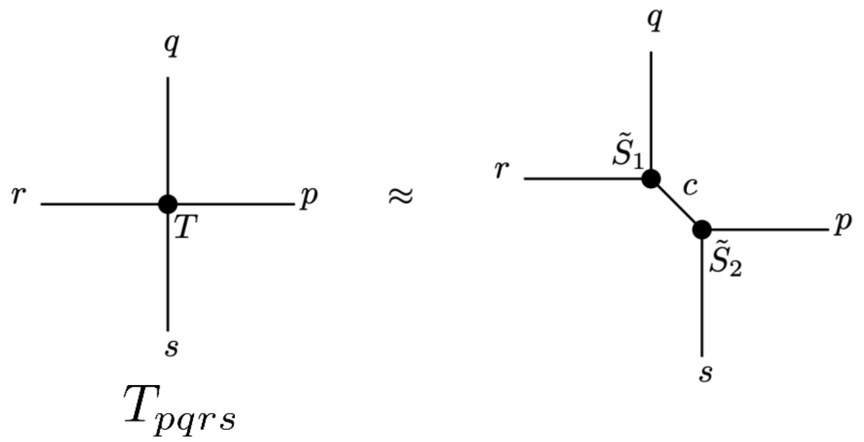
Outline

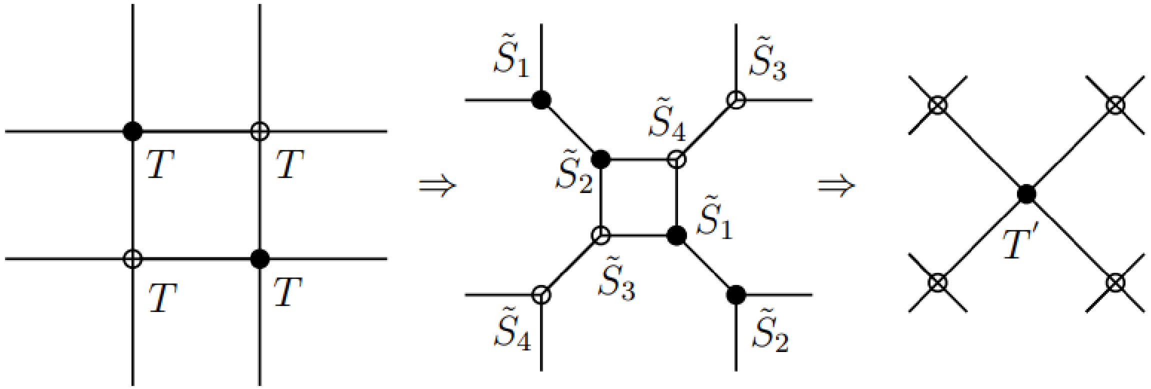
- Introduction
- TRG for 2d gauge theories
  - Efficient cut-off condition for irreps
- Singular value analysis
  - SV vs couplings
  - Large-N expressions of SVs
- Numerical results
  - GWW transition and parity SSB at $\theta = \pi$
  - Eguchi-Kawai reduction
  - New volume reduction at strong coupling
- Summary and discussion

TRG for 2d gauge theories

Brief review of 2d TRG

1. Write the partition function as a network of tensor
2. Decompose the tensor with SVD
3. Recombine the tensors into the coarse-grained network (keep the tensor rank fixed to D_{cut})
4. Repeat at step 2





(1) (2) (3)

2d gauge theories with a theta term

$$P_n = U_{n,1}U_{n+\hat{1},2}U^\dagger_{n+\hat{2},1}U^\dagger_{n,2}$$
$$\lambda = 2Na^2g^2$$

Action: $$S = \frac{1}{4g^2}\int d^2x \mathrm{tr} F_{\mu\nu}^2 - i\theta Q \implies -\frac{N}{\lambda}\sum_n \mathrm{tr}(P_n + P_n^\dagger) - i\theta Q$$

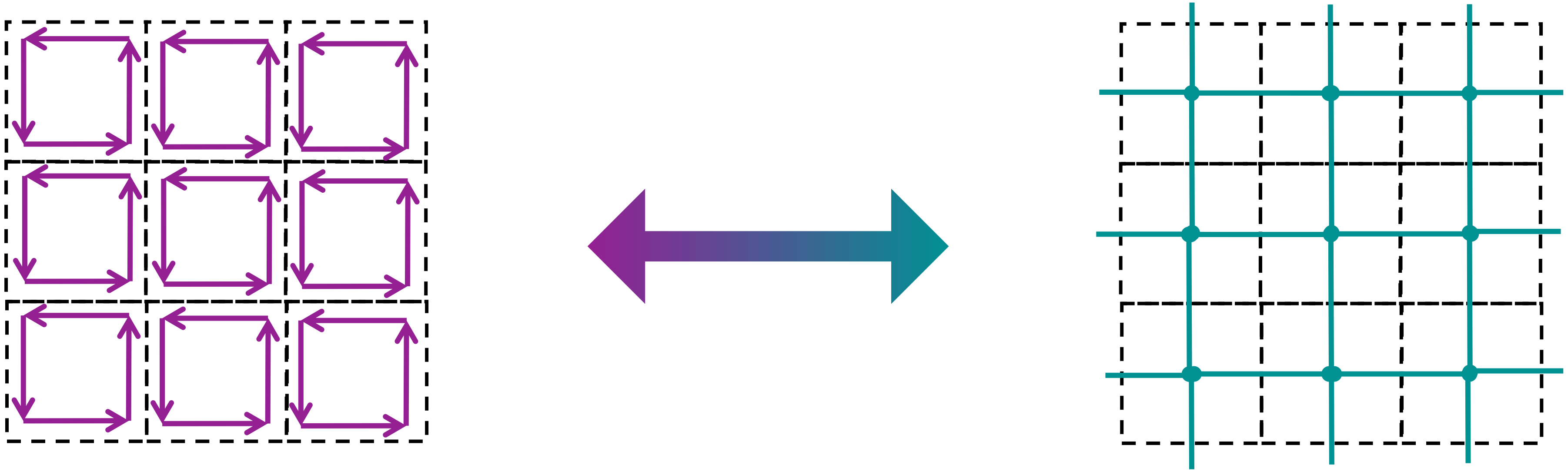
Topological charge: $$Q = \frac{1}{4\pi}\int d^2x \epsilon_{\mu\nu}\mathrm{tr} F_{\mu\nu} \implies \frac{1}{2\pi i}\sum_n \log \det P_n$$

The model can be exactly solved via character expansion.

$$e^{\beta \mathrm{tr}(P+P^\dagger)+i\gamma \mathrm{tr}\log P} = \sum_{r=\mathrm{irrep}} f_r \mathrm{tr}_r P$$

2d gauge theories with a theta term

$$Z = \int d^n U \sum_{\{r\}} \prod_{\text{sites}} f_r \mathrm{tr}_r (UUU^\dagger U^\dagger) = \sum_{\{r\}} \prod_{\text{sites}} T$$



original lattice
integrate over group manifold

dual lattice
sum over group representations

Representations of U(N) and SU(N)

SU(N): $$r^{(\mathrm{SU})} = \{l_1, l_2, .., l_N\}$$

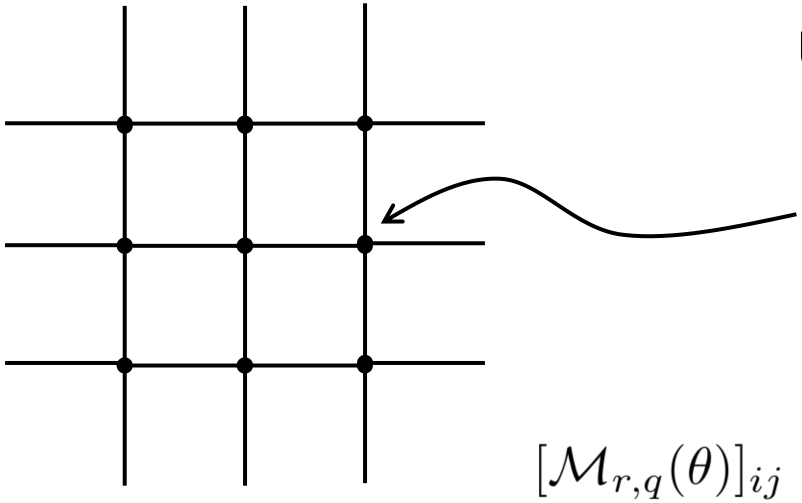
$$l_1 \geq l_2 \geq \cdots \geq l_N$$

$l_N = 0$ for SU(N)

U(N): $$r^{(\mathrm{U})} = (r^{(\mathrm{SU})}, q) = \{l_1 + q, l_2 + q, .., l_N + q\}$$

Dimensionality (matrix size): $$d_r = \prod_{1 \leq i < j \leq N} \left(1 + \frac{l_i - l_j}{j - i}\right)$$

Tensor construction



U(N) :

$$T_{PQRS} = \underbrace{\frac{\det \mathcal{M}_R(\theta)}{d_R}}_{\text{singular value } \sigma_R(\theta)} \delta_{PQRS}$$

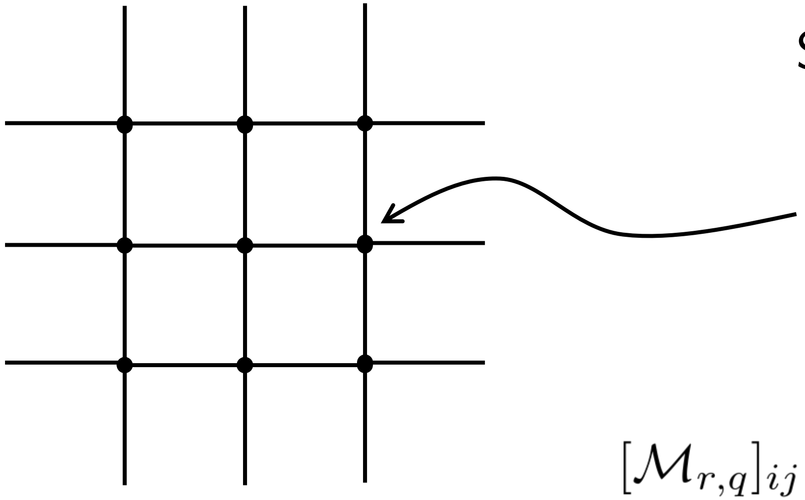
$$[\mathcal{M}_{r,q}(\theta)]_{ij} = \int_{-\pi}^{+\pi} \frac{d\phi}{2\pi} \cos\{(l_j + i - j + q + \tfrac{\theta}{2\pi})\phi\} \exp(\tfrac{2N}{\lambda} \cos\phi)$$

Because the tensor is *diagonal*, singular values have a simple scaling behavior:

$$\sigma_r \xrightarrow{\text{coarse-grain}} \sigma_r^2$$

The partition function can be exactly evaluated $Z = \sum_r \sigma_r(\theta)^V$

Tensor construction



SU(N) :

$$T_{rsmn} = \underbrace{\sum_q \frac{\det \mathcal{M}_{r,q}}{d_r}}_{\text{singular value } \sigma_{r,q}} \delta_{rsmn}$$

$$[\mathcal{M}_{r,q}]_{ij} = \int_{-\pi}^{+\pi} \frac{d\phi}{2\pi} \cos\{(l_j + i - j + q)\phi\} \exp(\tfrac{2N}{\lambda} \cos\phi)$$

Because the tensor is *diagonal*, singular values have a simple scaling behavior:

$$\sigma_r \xrightarrow{\text{coarse-grain}} \sigma_r^2$$

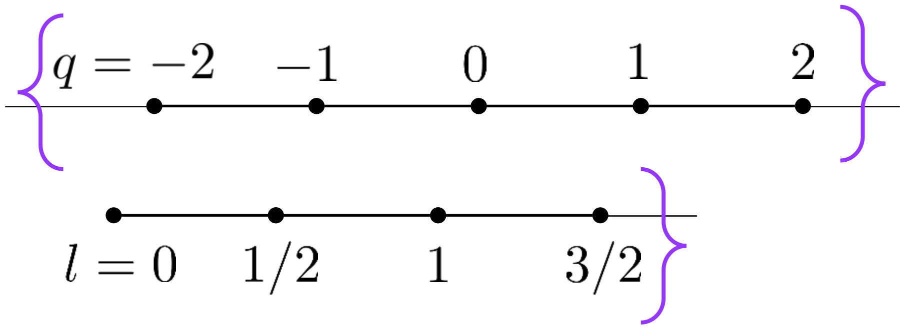
➡ The partition function can be exactly evaluated $Z = \sum_r \sigma_r(\theta)^V$

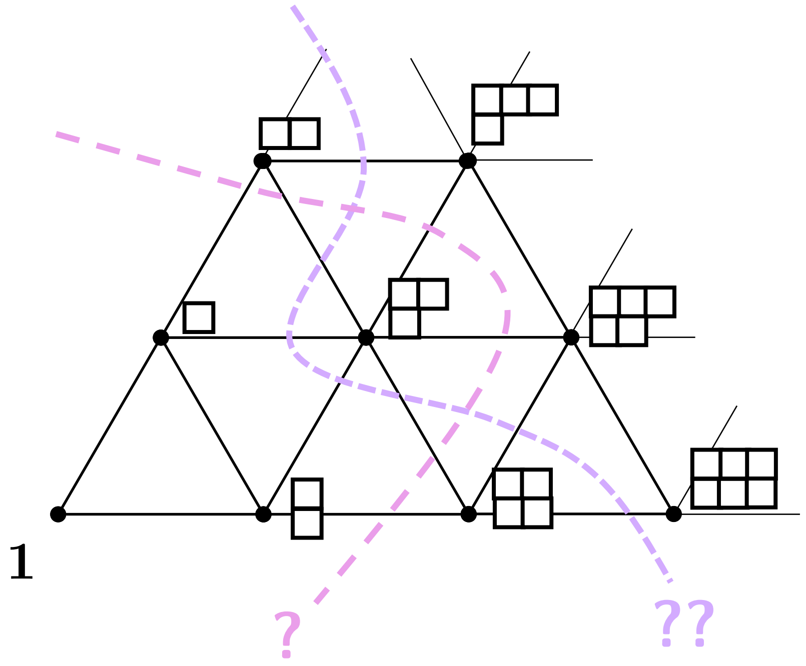
Representation cut-off

There are infinitely many irreps! → need a cut-off

Examples

- U(1): cut-off on the charge $|q| < q_{\text{max}}$
- SU(2): cut-off on the spin $l < l_{\text{max}}$
- SU(3) ?





Cut-off becomes nontrivial!

Representation cut-off

Our general strategy :

1. Calculate SVs of all irreps that is within the cut-off condition
   This number is usually larger than D_{cut} → Larger number = less efficiency
2. Keep only D_{cut} irreps in the calculation
3. Extend the cut-off until the calculation is unchanged (D_{cut} kept fixed)
4. The `U(1) charge' for U(N) can be cut off independently and straightforwardly

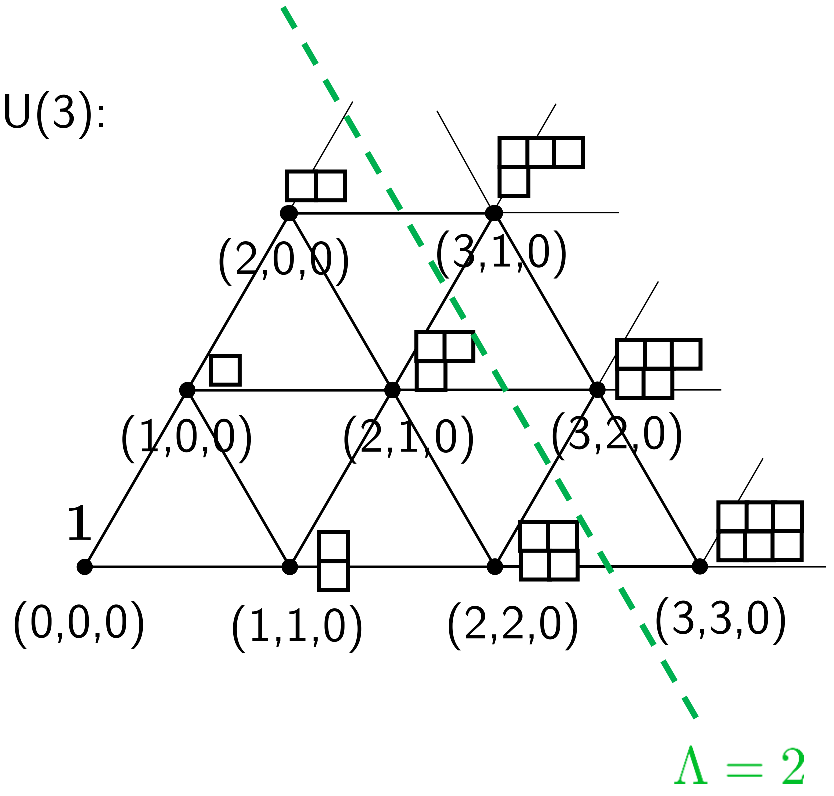
Question: what is the most efficient cut-off condition?

Representation cut-off

Example 1: using l_1 as the cut-off condition

$$r^{(\mathrm{SU})} = \{l_1, l_2, .., l_N\} \quad l_1 \geq l_2 \geq \cdots \geq l_N = 0 \Bigg\} \quad l_1 \leq \Lambda$$

SU(3):



Problem: the number of irreps grows like Λ^{N-1}

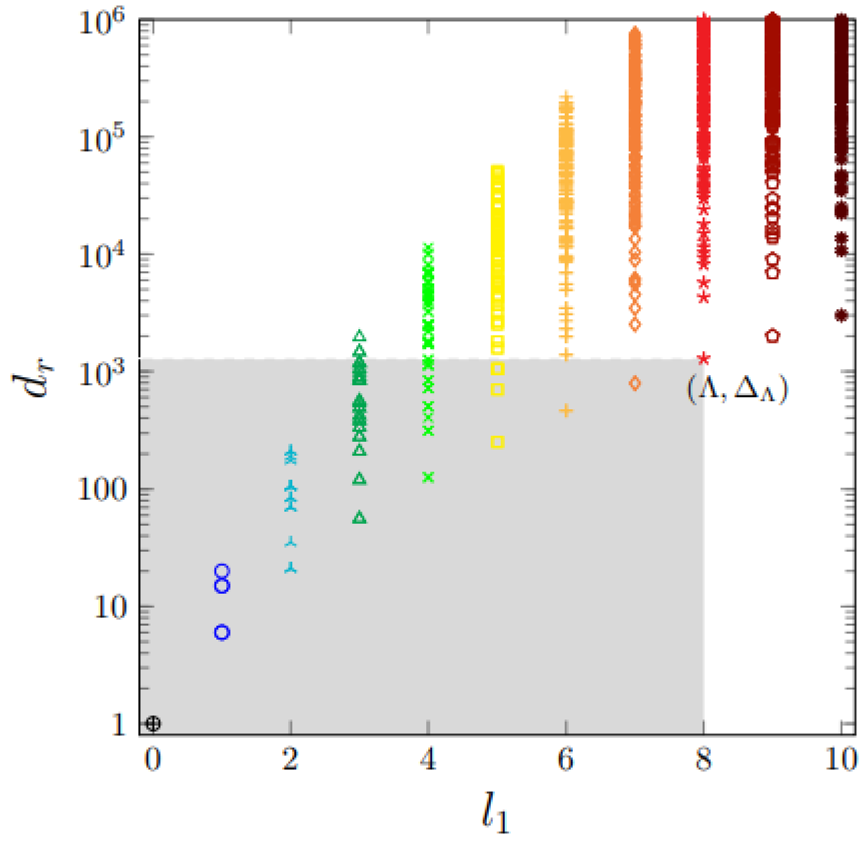
(for fixed *N*)

Too large too quickly for large *N* !

Representation cut-off

Example 2: using dimensionality as the cut-off condition

$$d_r \leq \Delta_\Lambda$$



$$\Delta_\Lambda = \frac{(\Lambda + N - 1)!}{\Lambda!(N-1)!}$$

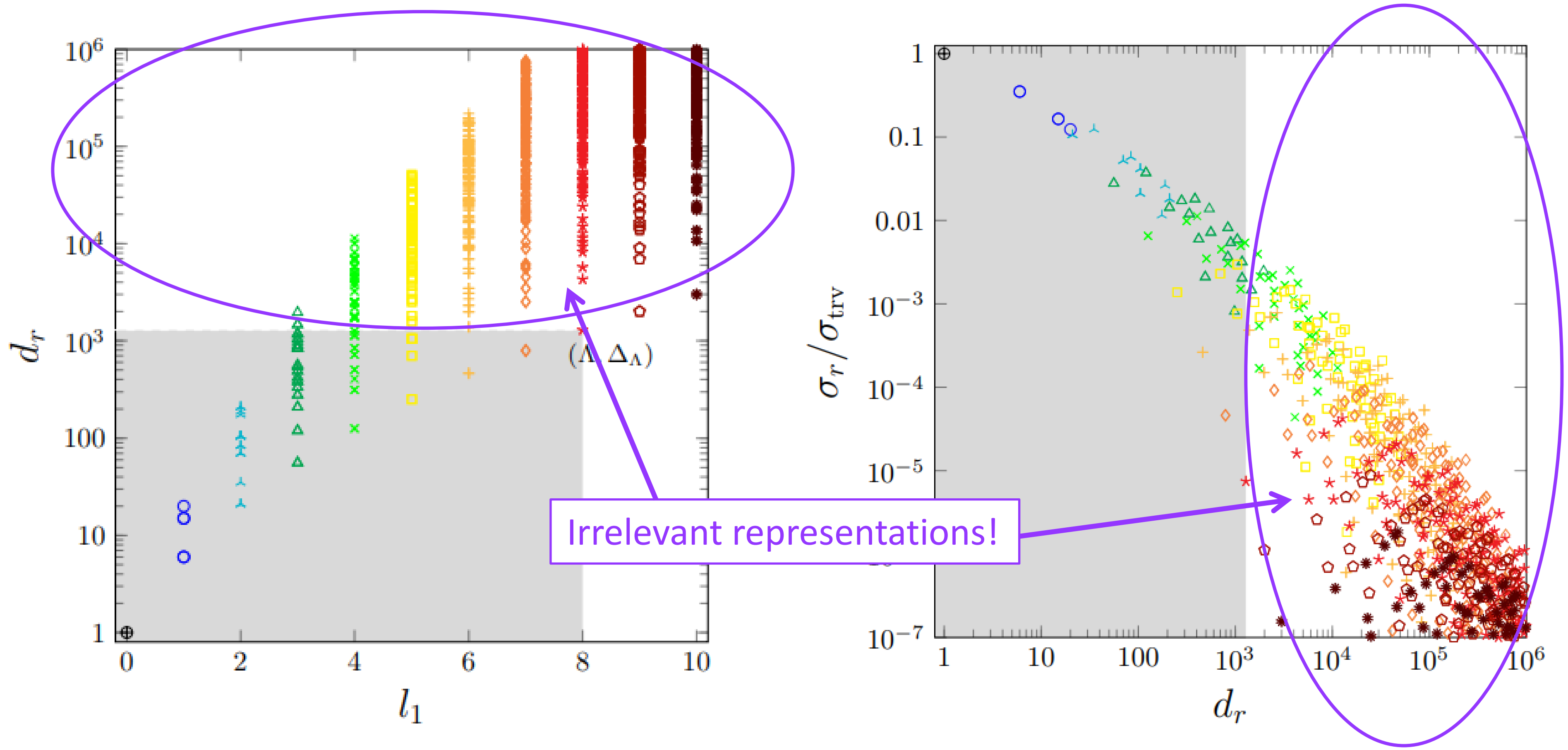
= smallest dim. of irreps with $l_1 = \Lambda$

For each Λ , consider only those with $d_r \leq \Delta_\Lambda$

Representation cut-off

Example 2: using dimensionality as the cut-off condition

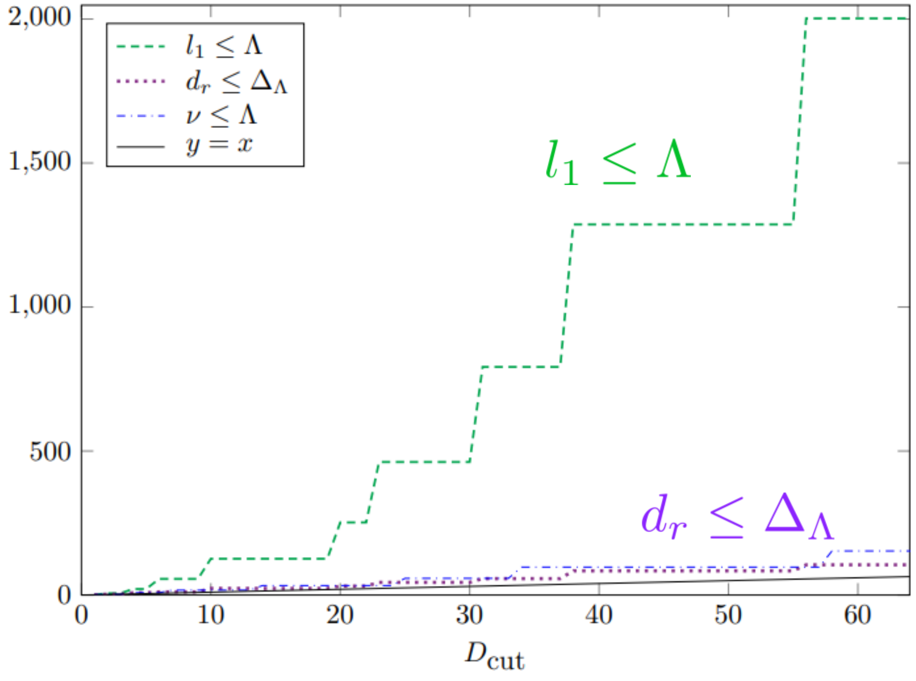
$$d_r \leq \Delta_\Lambda$$



Representation cut-off

Question: what is the most efficient cut-off condition?

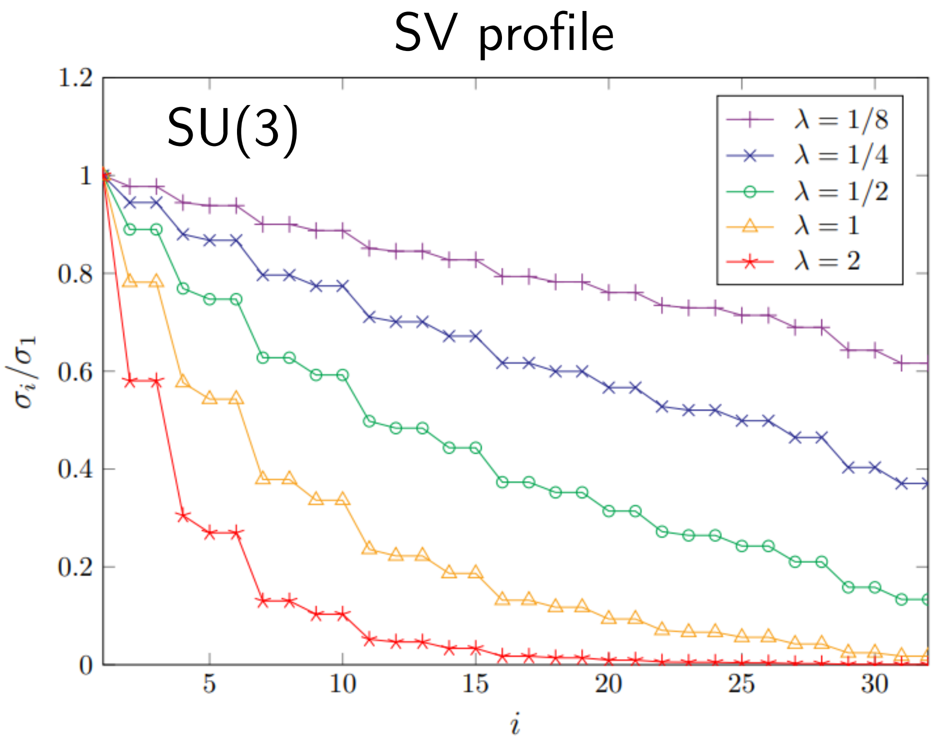
number of irreps
within the cut-off

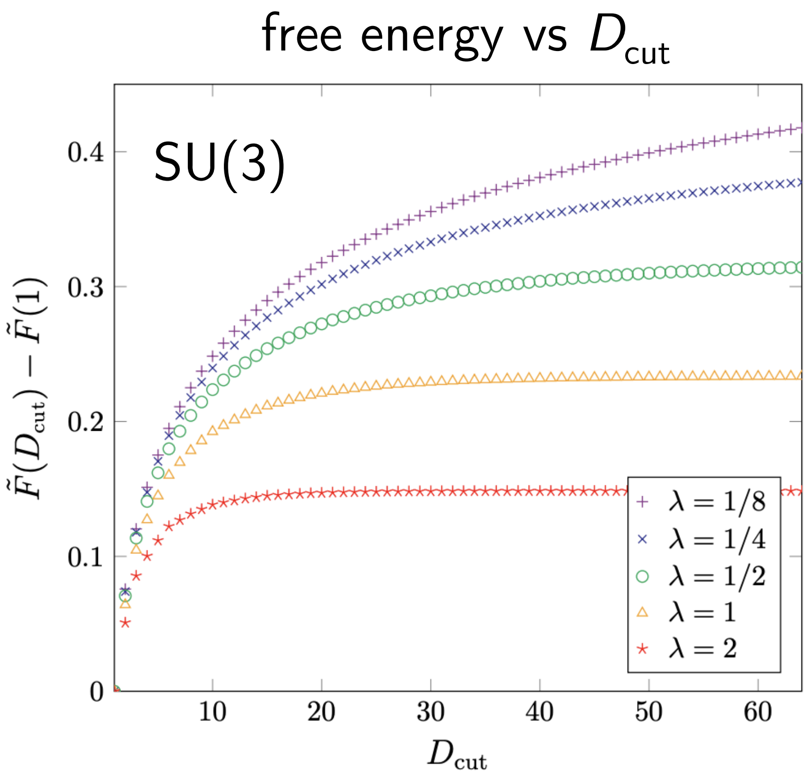


cutting by dimensionality is the most efficient condition so far

Singular value analysis

Singular values vs 't Hooft couplings





SV decays faster for larger 't Hooft couplings

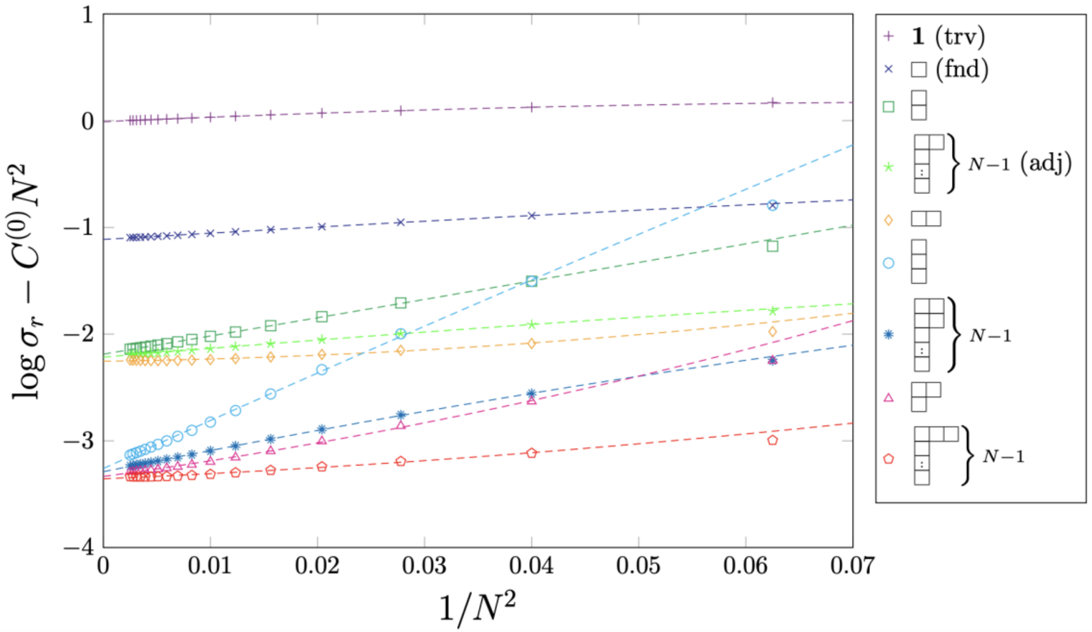
This is true for both U(N) and SU(N)

Singular value profile at large N

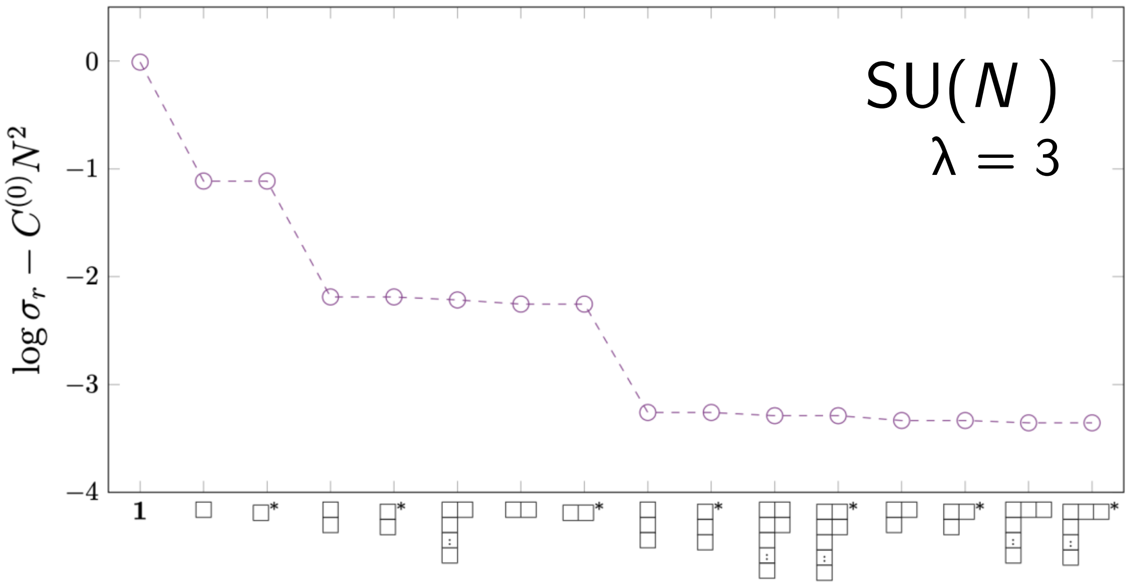
Questions:

- Does large N pose any problem for TRG calculation?
- Any dominant representation?
- Is there a definite profile?

SU(N)
$\lambda = 3$



Singular value profile at large N



- U(N) and SU(N) have the same profile for $\lambda > 2$ (strong coupling)
- But they have different profiles for $\lambda < 2$ (weak coupling)

Large -N expansions

SU(N): $\log \sigma_r = C^{(0)} N^2 + C_r^{(1)} + \mathcal{O}(N^{-2})$

U(N) $\lambda < 2$: $\log \sigma_{r,q} = C^{(0)} N^2 + C_{r,q}^{(1)} + \mathcal{O}(N^{-1})$

U(N) $\lambda > 2$: $\log \sigma_{r,q} = C^{(0)} N^2 - a_q N + b_q \chi_N + C_{r,q}^{(1)} + \mathcal{O}(N^{-1})$ $\quad \chi_N \sim \frac{1}{2\pi^2} \log N$

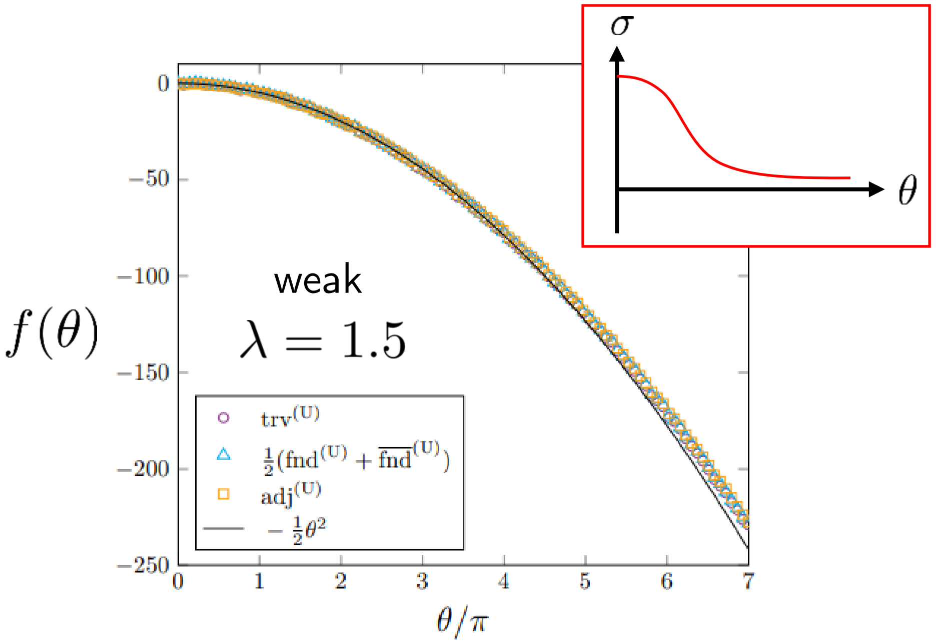
this term suppresses contributions
from most of the charges
(only one q survives at large N)

U(1) d.o.f. is 'trivialized' ⟶ U(N) ~ SU(N)

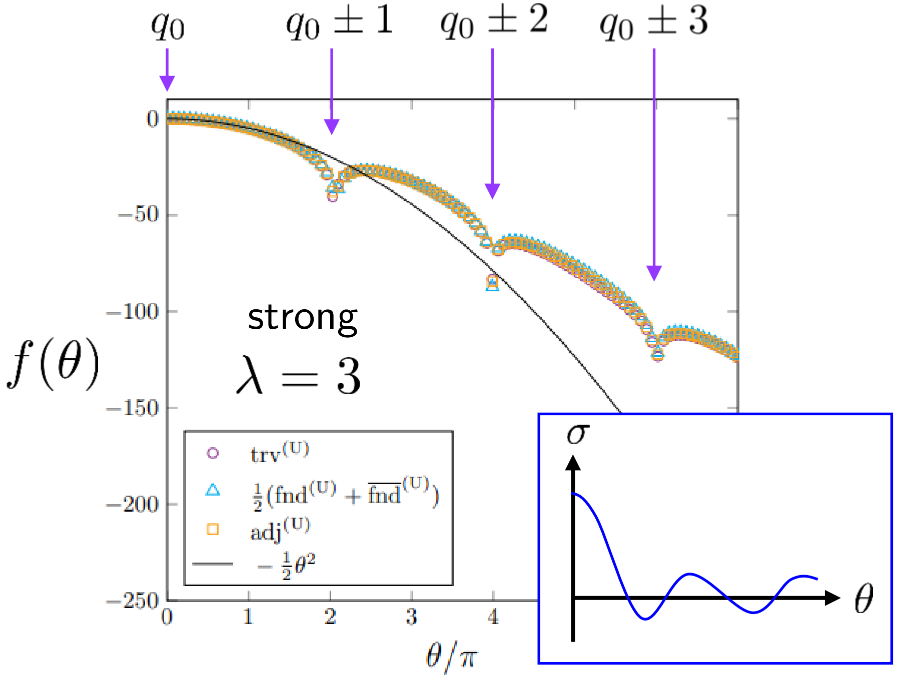
Singular values with nonzero θ

- U(1) charge is related to the theta term! $\sigma_{(r,q+1)}(\theta) = \sigma_{(r,q)}(\theta + 2\pi)$
- Large -N expansion $\log \sigma_{(r,q_0)}(\theta) = C^{(0)}N^2 + \chi_N f(\theta) + C_r^{(1)} + \mathcal{O}(1/N)$

q_0: U(1) charge that gives the largest SV



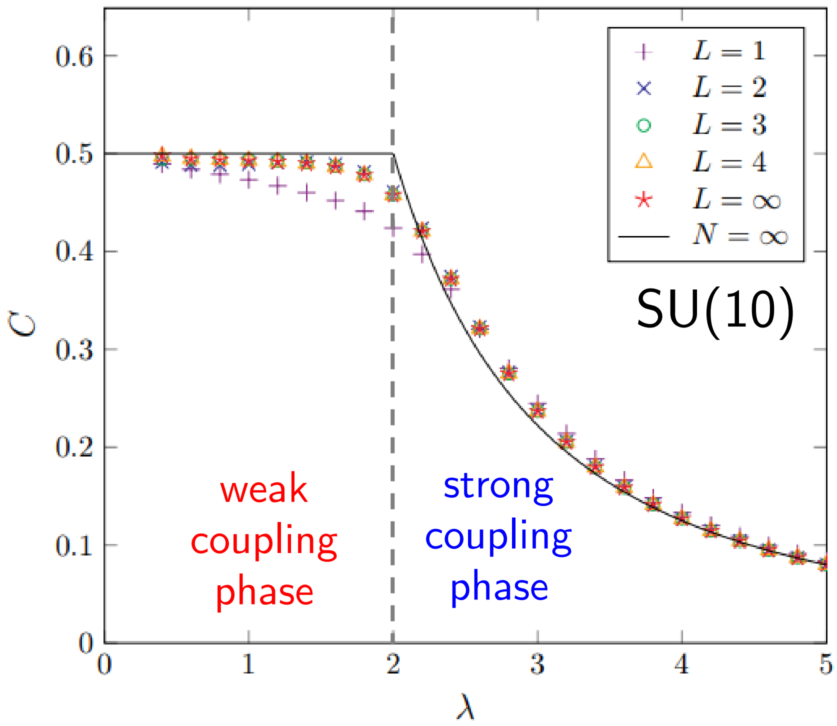
Only $q = q_0$ survives at large N



Numerical results

Basic results

Gross-Witten-Wadia 3rd-order transition

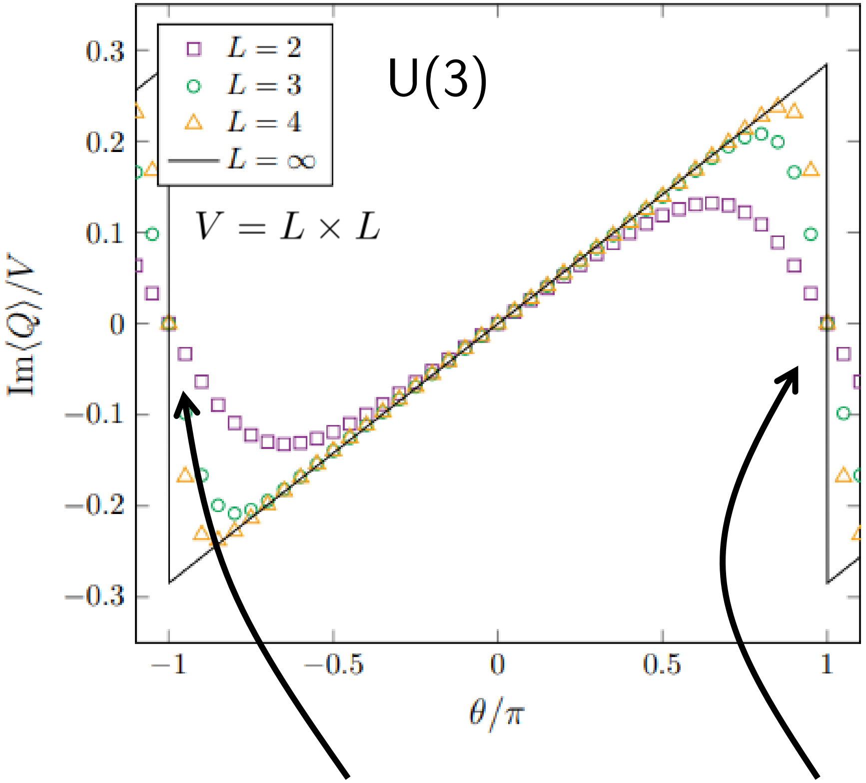


The transition starts to appear at $N = 10$

$$C = \frac{\partial}{\partial\lambda}\left(\lambda^2 \frac{\partial F}{\partial\lambda}\right) \Rightarrow \begin{cases} \frac{1}{2} & ; \lambda < 2, \\ \frac{2}{\lambda^2} & ; \lambda \geq 2 \end{cases}$$

Basic results

1^{st}-order transition at $\theta = \text{odd} \times \pi$



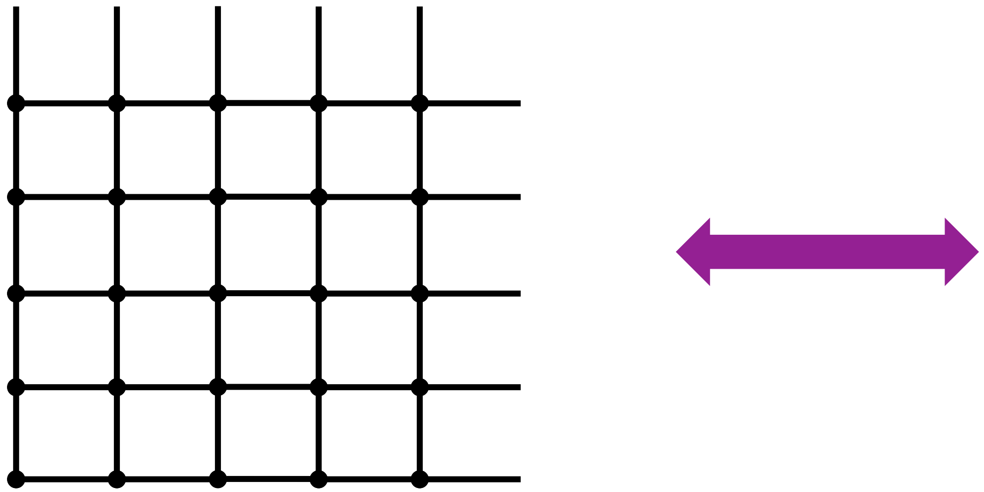
parity spontaneously broken at large volume

*very difficult in Monte Carlo simulations

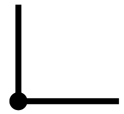
Alternative interpretation for Eguchi-Kawai reduction

EK-reduction: certain properties of large-N theories are independent of volume

*if center sym. is unbroken



original model

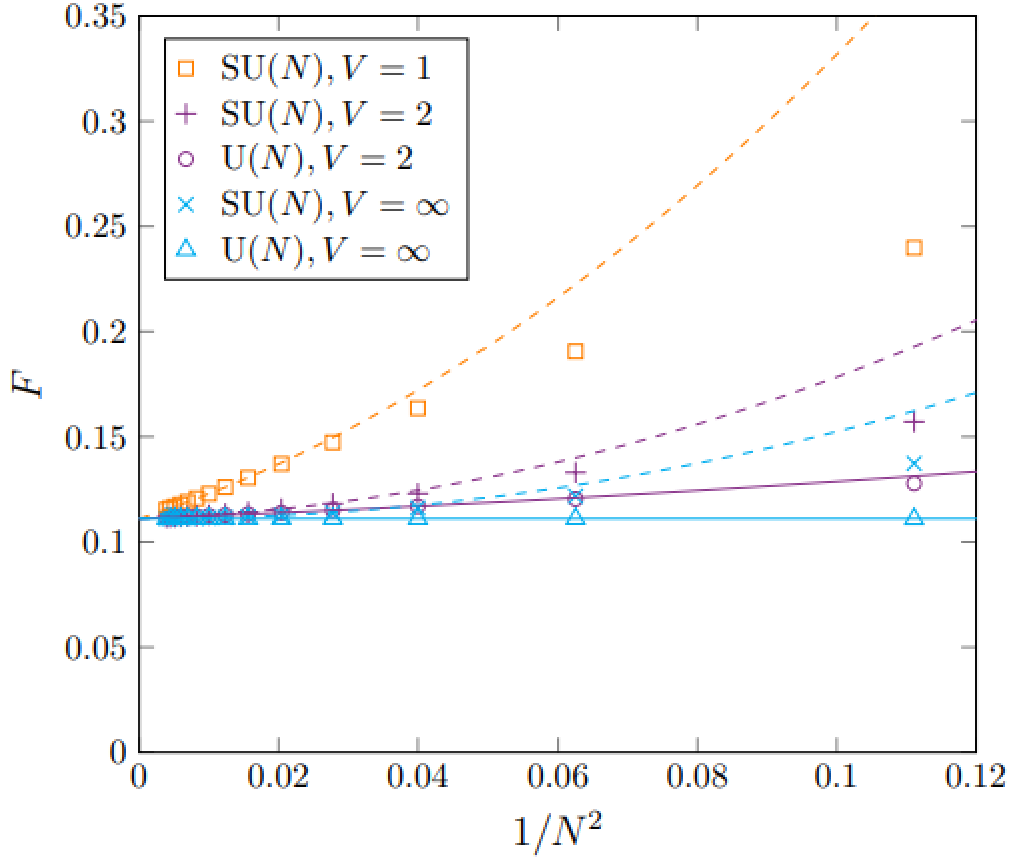


reduced model

Original proof: demonstrating the equivalence of S-D equations for Wilson loops

[Eguchi&Kawai,'92]

Alternative interpretation for Eguchi-Kawai reduction



$$F \equiv \frac{1}{N^2 V} \log Z = \frac{1}{N^2 V} \log \sum_r \sigma_r^V = C^{(0)} + \frac{1}{V N^2} \log(\cdots)$$

volume-independent at large N

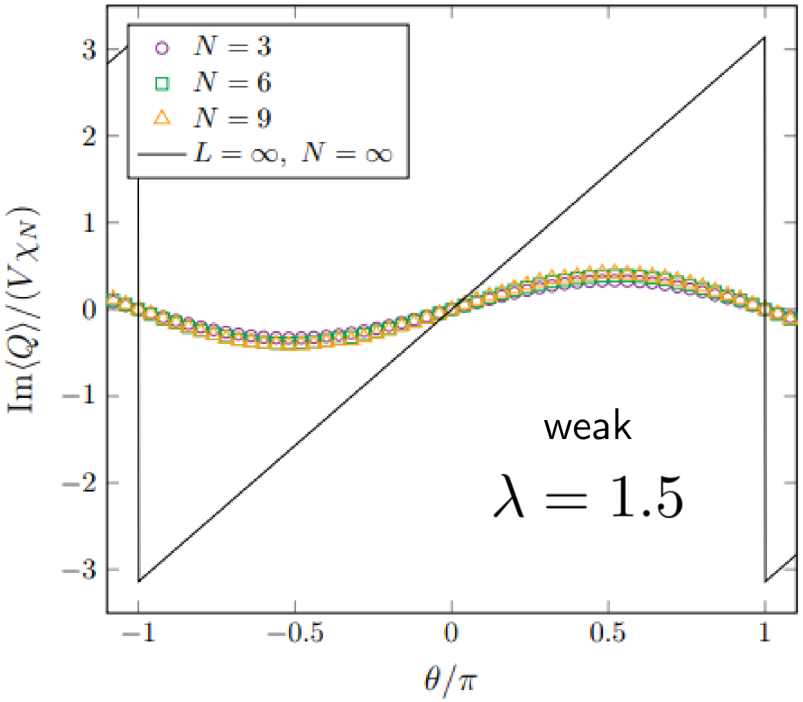
TRG's perspective: the theory is **volume-independent** when the SVs have a **scaling behavior**.

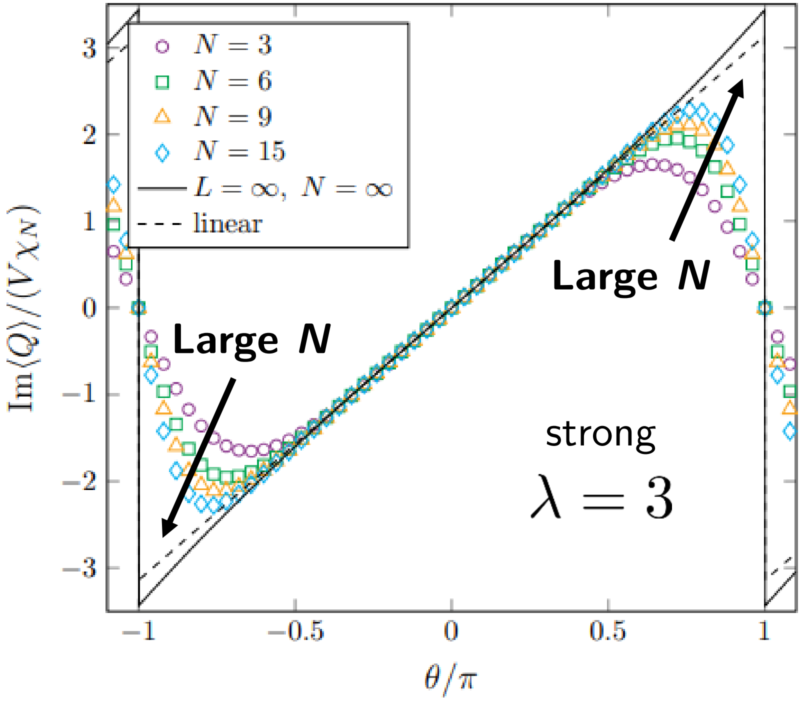
$$\sigma_r \xrightarrow{\text{coarse-grain}} \sigma_r^2$$

Large N and nonzero θ

Recall: parity spontaneously broken at $\theta =$ odd x π at large volume

calculation at small volume (V=2x2)



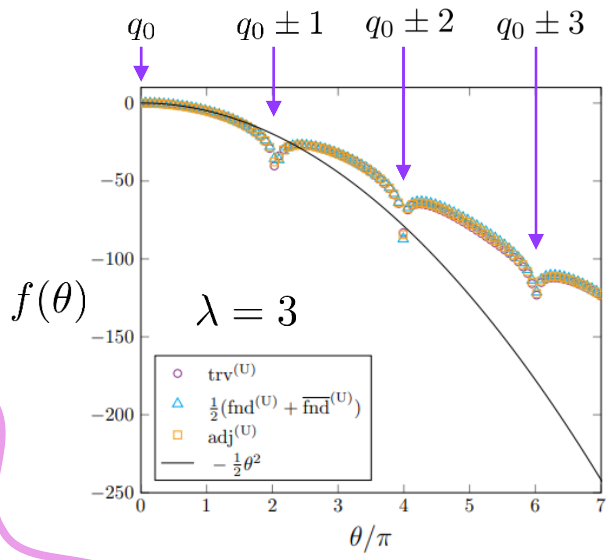


- parity remains unbroken at large N

- parity becomes broken even at small vol
- small vol ~ large vol at large N ??

New volume reduction

Recall: only $q = q_0$ survives at large N and strong coupling



can be pulled out

$$\frac{1}{VN^2}\log Z = C^{(0)} + \frac{\chi}{N^2} f(\theta) + \underbrace{\frac{1}{VN^2}\log\left(\sum_{r^{(\mathrm{SU})}} e^{V \times (C^{(1)}_{r^{(\mathrm{SU})}} + \mathcal{O}(1/N))}\right)}_{\to 0 \text{ as } N \to \infty}$$

sub-leading volume independence;
can be observed via
topological charge density

$$\frac{1}{V}\langle Q\rangle = \frac{1}{iV}\frac{d}{d\theta}\log Z = -i\chi\frac{d}{d\theta}f(\theta) + \mathcal{O}(1/N)$$

A new kind of volume independence different from EK reduction

Summary and Discussions

- We consider using TRG based on character expansion to study 2d non-abelian gauge theories.
- For that, we propose an efficient way to cut-off the irreps:
  that is based on dimensionality.
- Various known results are reproduced: GWW transition and parity SSB at $\theta = \pi$
- By looking at the behavior of singular values, Eguchi-Kawai reduction can be explained.
  A new kind of volume independent at strong coupling is also observed.

Summary and Discussions

- We expect that our method to cut-off the irreps can be useful for more general non-Abelian theories (d>2, or with matter fields, etc.)

- Since Large-N reduction at $\theta \neq 0$ persists even at volume as small as 2x2 where the notion of topology is ambiguous, how is the topological information stored in the large-N matrix?

Thank you!