Peeking into the θ vacuum of 4d SU(2) Yang-Mills theory

Norikazu Yamada (KEK/SOKENDAI)
in collaboration with
Ryuichiro Kitano (KEK/SOKENDAI)
Ryutaro Matsudo (KEK)
Masahito Yamazaki (Kavli IPMU)

Based on 2102.08784 [hep-lat], JHEP02, 073 (2021)

@Kyoto Univ. 2021/06/30

Introduction

- Many interesting subjects related to the θ term or topology of YM theory
  e.g.) strong CP, axion, EDM, fate of $U_A(1)$, …

Focus on the θ dependence of free energy density in 4d SU(2) YM

Free energy density: $f(\theta)$

$$e^{-Vf(\theta)} = \frac{Z(\theta)}{Z(0)}$$

Where $Z(\theta) = \int \mathscr{D}U \, e^{-S_{\rm YM} + i\theta Q}$

$$Q = \int d^4x \, q(x) \quad \text{and} \quad q(x) = \frac{1}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}$$

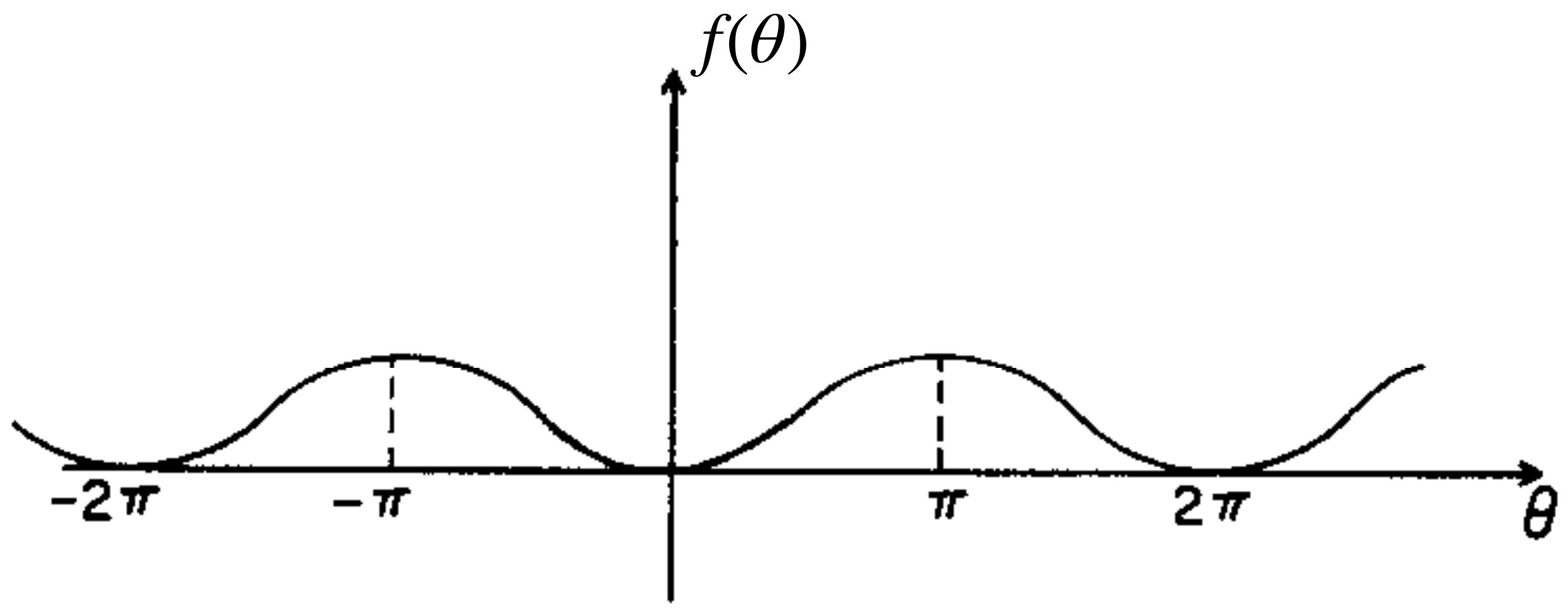
For $SU(N)$ YM theory,

$$\left.\begin{array}{l} Q \in \mathbb{Z} \Rightarrow Z(\theta) = Z(\theta + 2\pi) \Rightarrow f(\theta) = f(\theta + 2\pi) \\ S_{\rm YM} \text{ is CP even} \Rightarrow Z(\theta) = Z(-\theta) \Rightarrow f(\theta) = f(-\theta) \end{array}\right\} \; f(\pi - \theta') = f(\pi + \theta')$$

θ dependence and CP violation

Dilute instanton gas approximation (DIGA)

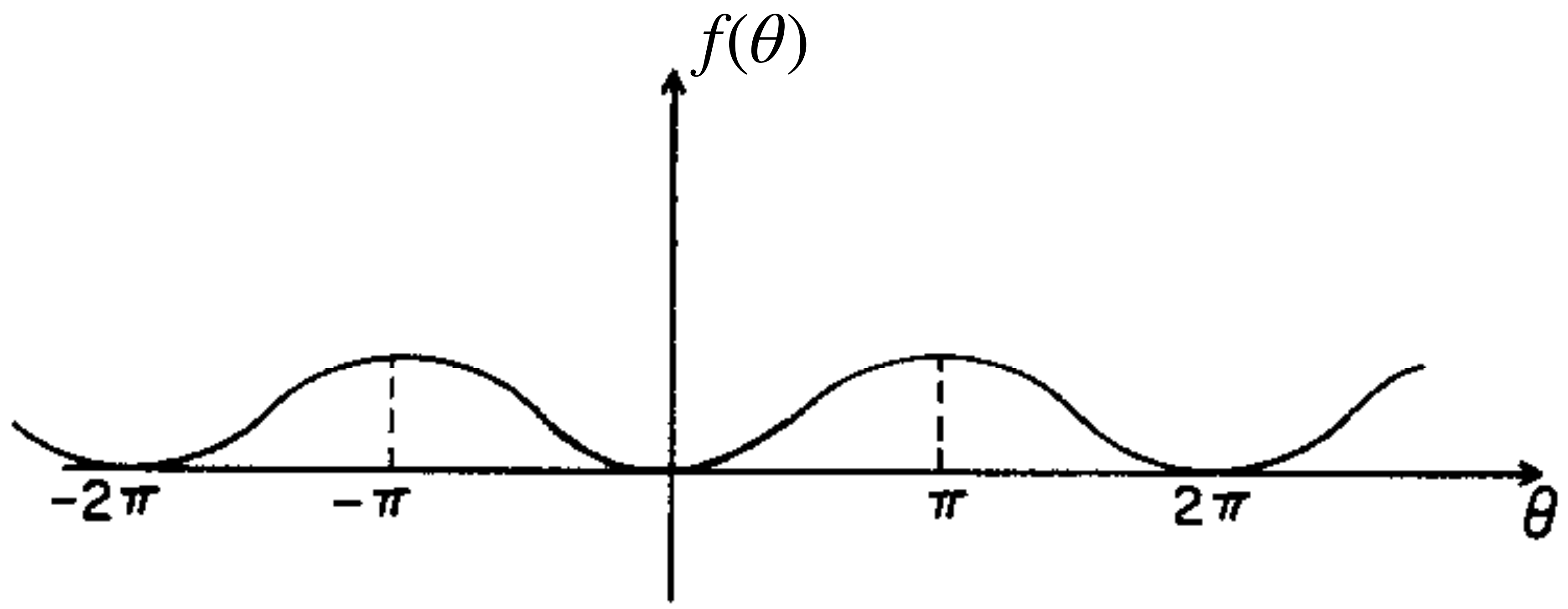
$\Rightarrow f(\theta) = \chi(1 - \cos\theta)$



- a single branch
- smooth everywhere

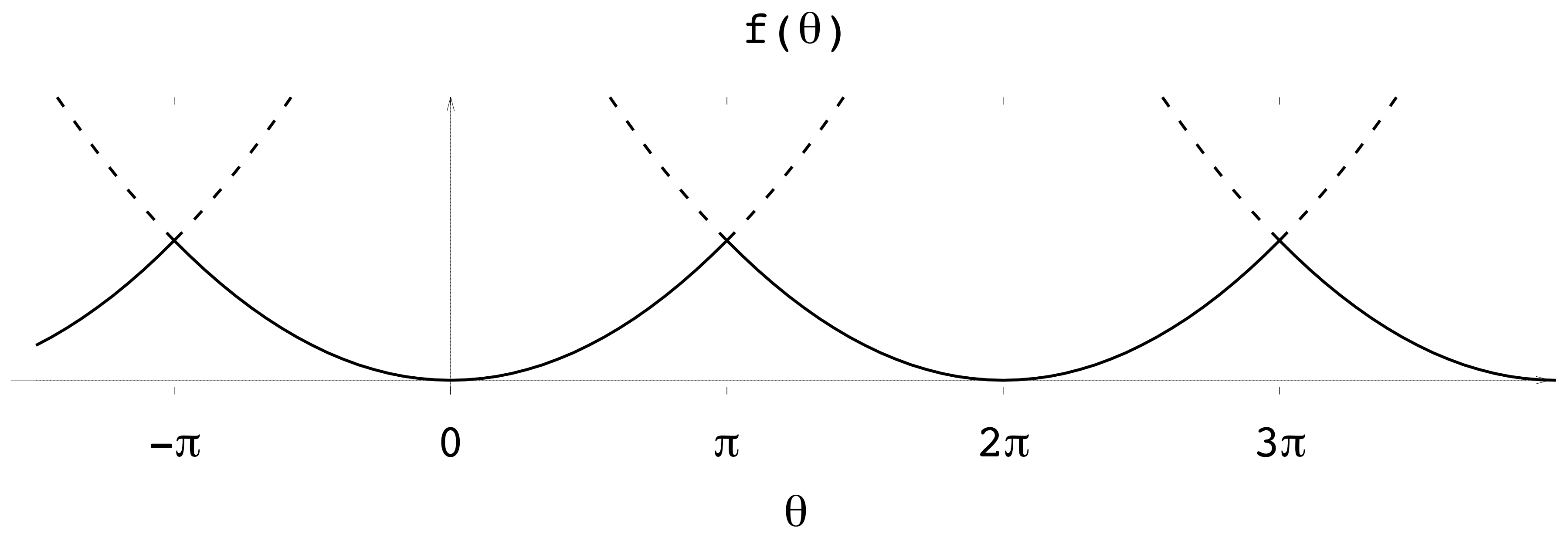
θ dependence and CP violation

Dilute instanton gas approximation (DIGA)
$\Rightarrow f(\theta) = \chi(1 - \cos\theta)$



- a single branch
- smooth everywhere

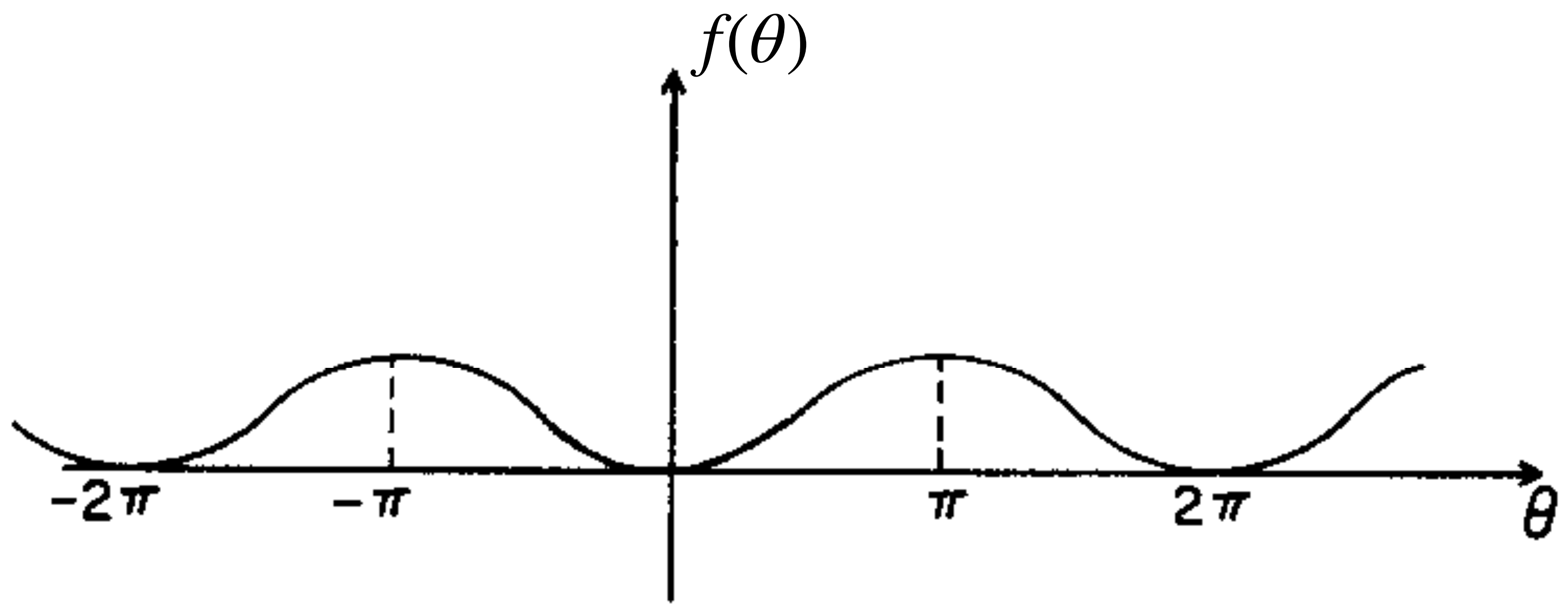
Large N argument [Witten (1980, 1998)]
$\Rightarrow f(\theta) = \chi/2 \min_{k\in\mathbb{Z}}(\theta + 2\pi k)^2 + O(1/N^2)$



- several branches crossing
- spontaneous CPV (1st order PT) at $\theta = \pi$ with the order parameter $df(\theta)/d\theta = -i\langle q(x)\rangle$

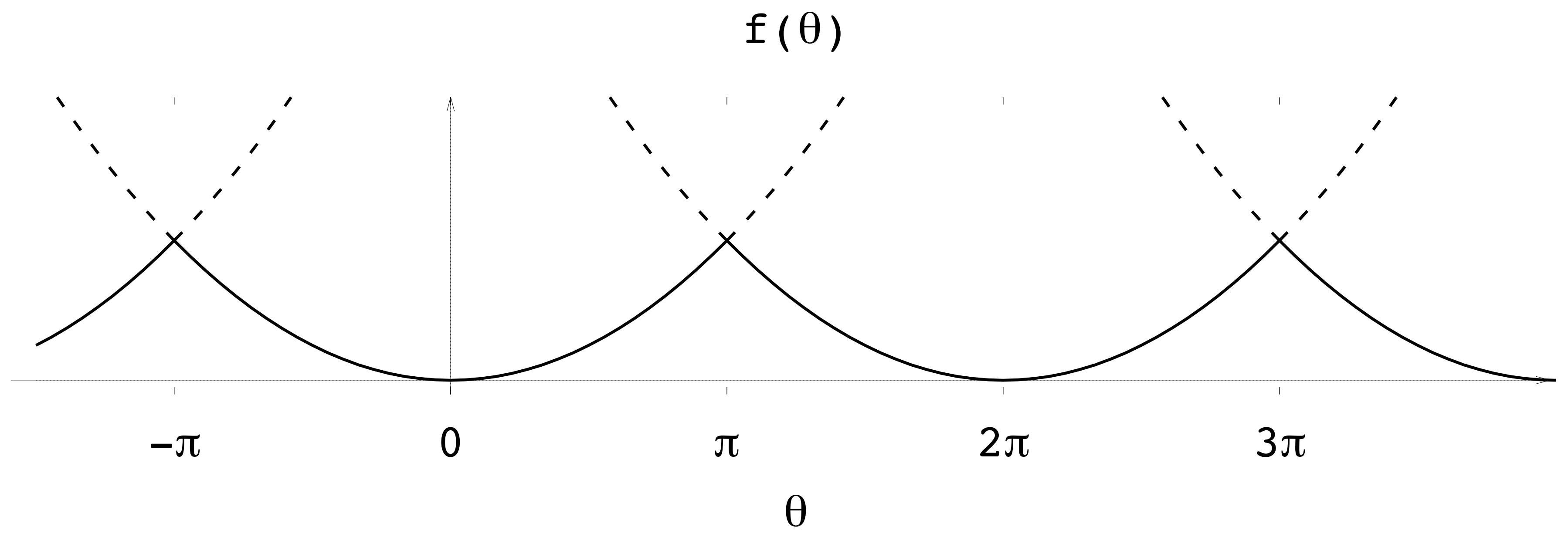
θ dependence and CP violation

Dilute instanton gas approximation (DIGA)
$\Rightarrow f(\theta) = \chi(1 - \cos\theta)$



- a single branch
- smooth everywhere

Large N argument [Witten (1980, 1998)]
$\Rightarrow f(\theta) = \chi/2 \min_{k\in\mathbb{Z}}(\theta + 2\pi k)^2 + O(1/N^2)$



- several branches crossing
- spontaneous CPV (1st order PT) at $\theta = \pi$ with the order parameter $df(\theta)/d\theta = -i\langle q(x)\rangle$

Interested in $f(\theta)$ around $\theta \approx \pi$ in 4d $SU(N)$ YM theory.

Learning from 2d CP^{N-1} model

$$\mathscr{L} = \frac{N}{2g}\overline{D_\mu z}\,D_\mu z - i\theta q$$

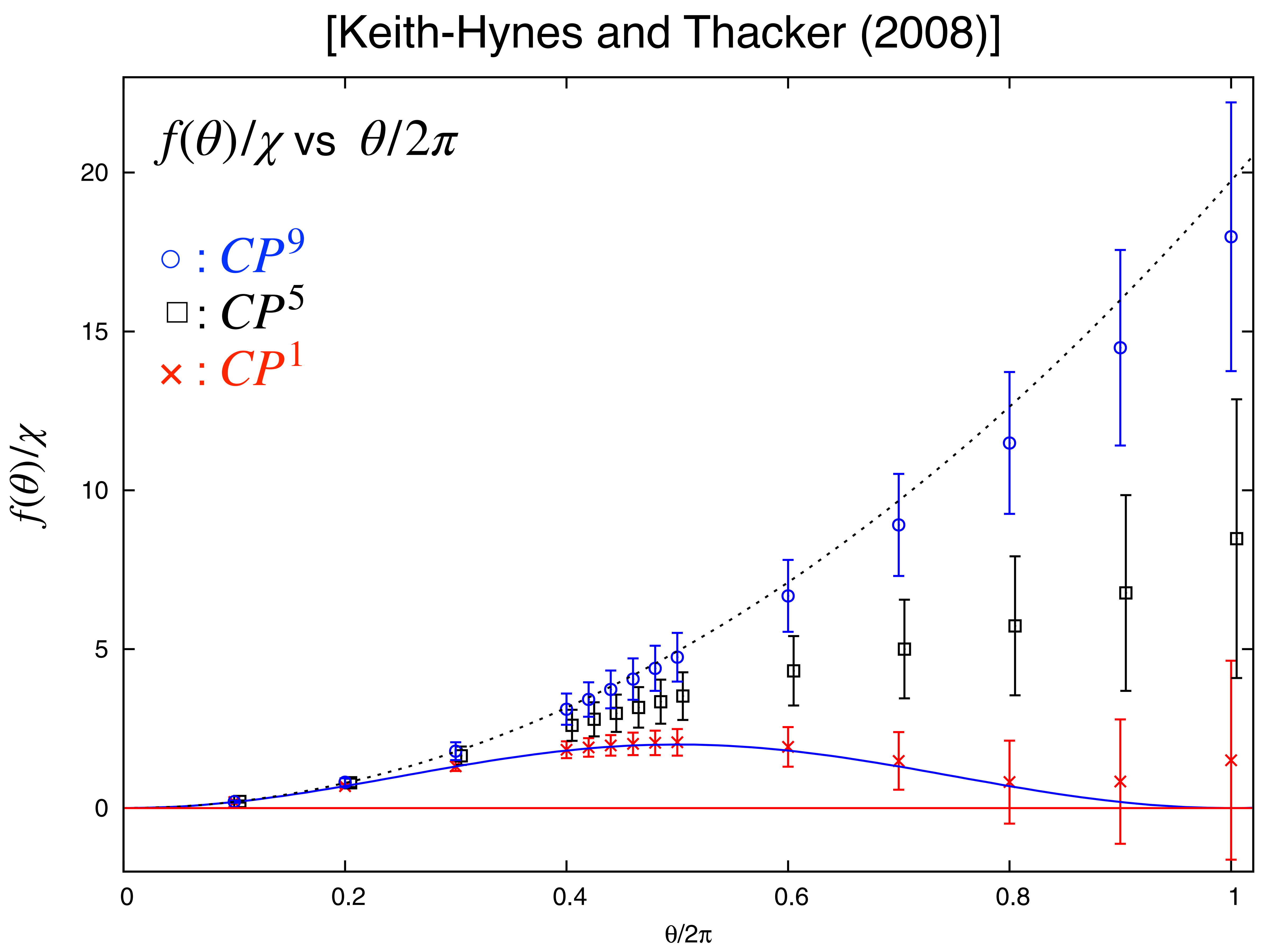
z : N-component complex scalar field with $\bar{z}z = 1$

$$D_\mu = \partial_\mu + iA_\mu \ , \quad A_\mu = i\bar{z}\partial_\mu z$$

$$q(x) = \frac{1}{2\pi}\epsilon_{\mu\nu}\partial_\mu A_\nu = \frac{i}{2\pi}\epsilon_{\mu\nu}\overline{D_\mu z}\,D_\mu z$$

- Good testing ground for 4d $SU(N)$ because of many similarities
  [asymptotic freedom, dynamical mass gap, instanton, $1/N$ expandable, …]
- Gapped and CP broken at $\theta = \pi$ for $N \geq 3$.
- But CP^1 (i.e. $N = 2$) is exceptional !
  $\Rightarrow$ gapless and no CPV at $\theta = \pi$ ($\Longleftrightarrow$Haldane conjecture)

$f(\theta)$ in 2d CP^{N-1} model (lattice results)



$$e^{-V_{\rm sub} f_{\rm sub}(\theta)} = \frac{1}{Z[0]} \int \mathscr{D}z\mathscr{D}\bar{z}\, e^{-S_{CP(N-1)} - i\theta Q_{\rm sub}}$$

$$= \langle e^{-i\theta Q_{\rm sub}} \rangle$$

$$Q_{\rm sub} = \int_{x\in V_{\rm sub}} d^2x\, \frac{1}{2\pi} \epsilon_{\mu\nu}\partial_\mu A_\nu \;\rightarrow\; Q^{\rm lat}_{\rm sub} = \frac{-i}{2\pi} \sum_{x\in V_{\rm sub}} \ln P_x$$

(P_x : plaquette)

CP^1 is indeed consistent with the DIGA,

$$f(\theta) = \chi(1-\cos\theta)$$

while others indicate CPV.

Previous Lattice calculations of 4d $SU(N)$

$$\mathscr{L}_\theta = \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} - i\theta q$$

$$q(x) = \frac{g^2}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}$$

- Sign problem makes direct lattice calculation difficult/impossible.

Previous Lattice calculations of 4d $SU(N)$

$$\mathscr{L}_\theta = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} - i\theta q$$

$$q(x) = \frac{g^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}$$

- Sign problem makes direct lattice calculation difficult/impossible.

- Relies on Taylor expansion around $\theta = 0$

$$f(\theta) = \frac{\chi}{2}\theta^2(1 + b_2\theta^2 + b_4\theta^4 + \cdots)$$

and determines each coefficient on the lattice by

$$\chi = \frac{\langle Q^2 \rangle_{\theta=0}}{V}$$

$$b_2 = -\frac{\langle Q^4 \rangle_{\theta=0} - 3\langle Q^2 \rangle^2_{\theta=0}}{12\langle Q^2 \rangle_{\theta=0}}$$

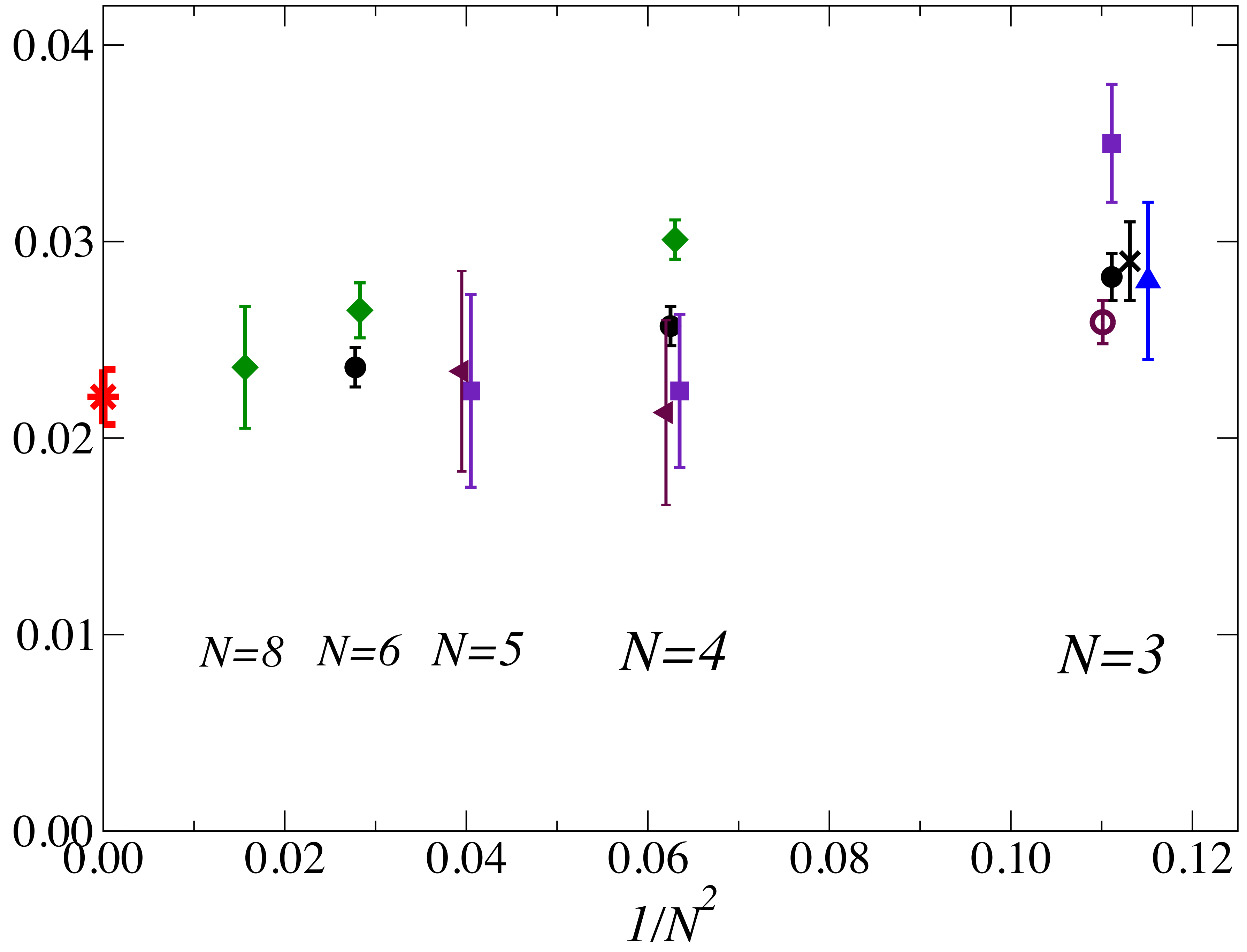
$$b_4 = \frac{\langle Q^6 \rangle_{\theta=0} - 15\langle Q^2 \rangle_{\theta=0}\langle Q^4 \rangle_{\theta=0} + 30\langle Q^2 \rangle^3_{\theta=0}}{360\langle Q^2 \rangle_{\theta=0}}$$

$$\vdots$$

First two coefficients

[Review by Vicari and Panagopoulos (2018)]

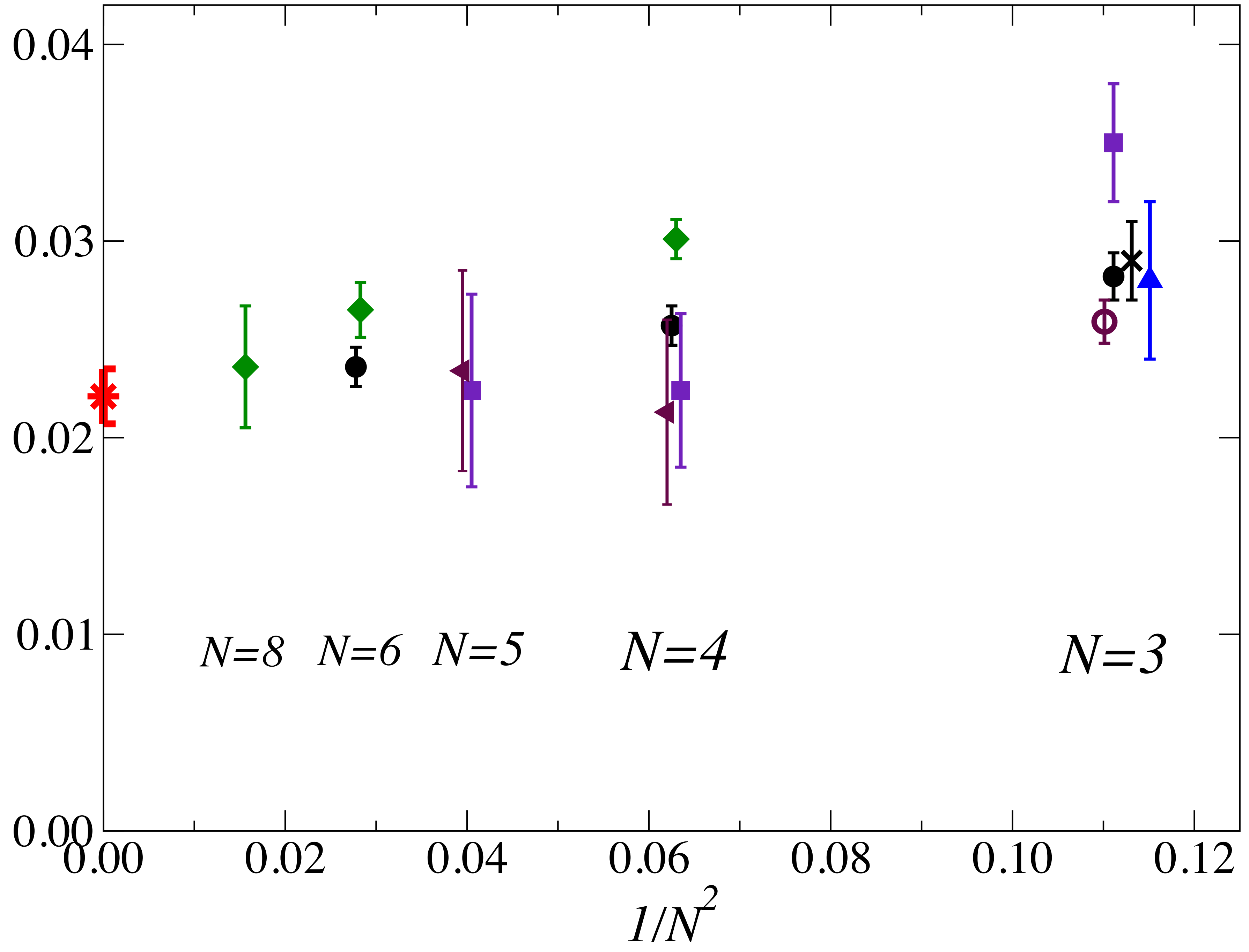
$$\chi/\sigma^2 = C_\infty + \frac{c_2}{N^2} + O(1/N^4)$$



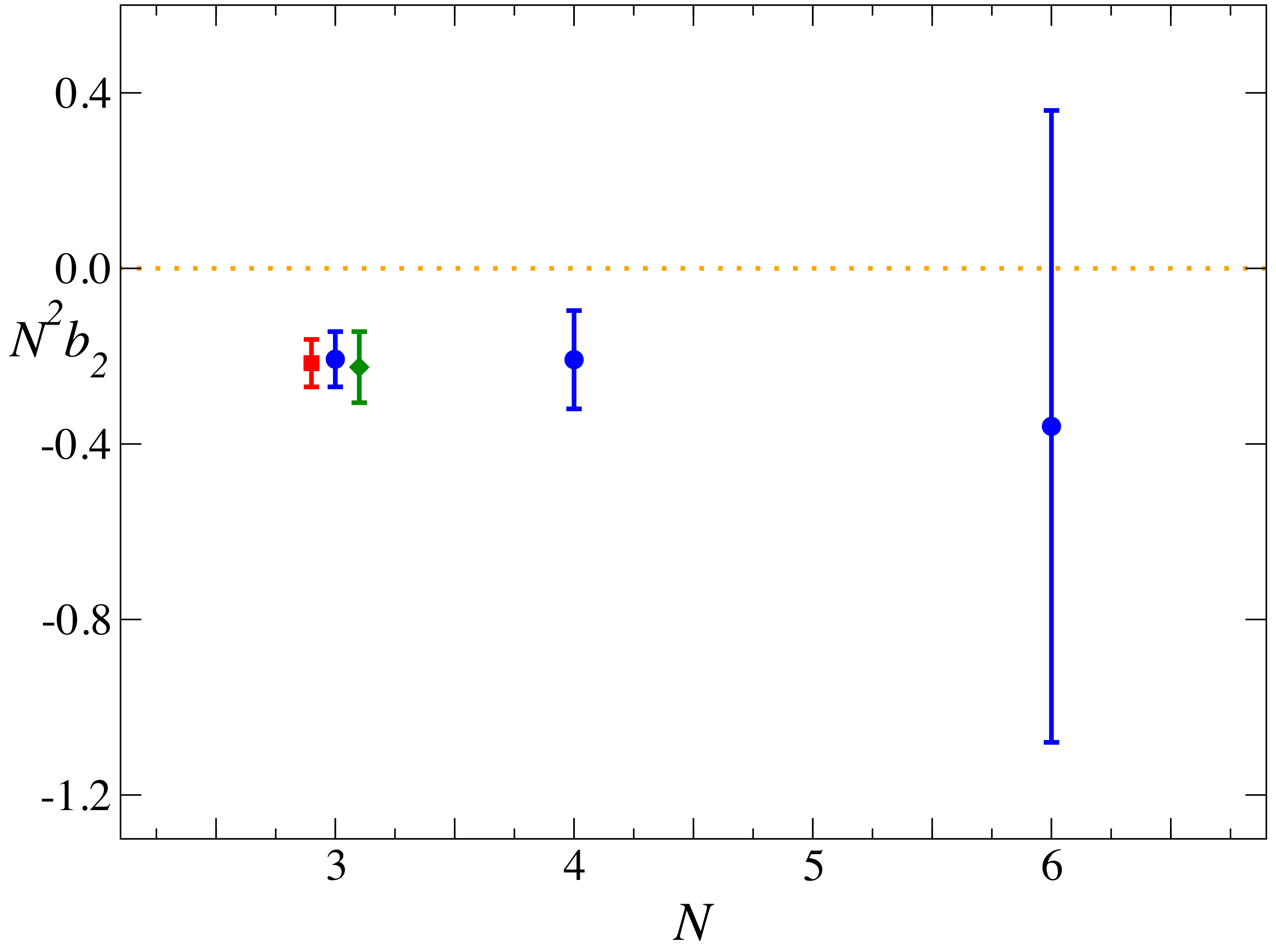
First two coefficients

[Review by Vicari and Panagopoulos (2018)]

$$\chi/\sigma^2 = C_\infty + \frac{c_2}{N^2} + O(1/N^4)$$



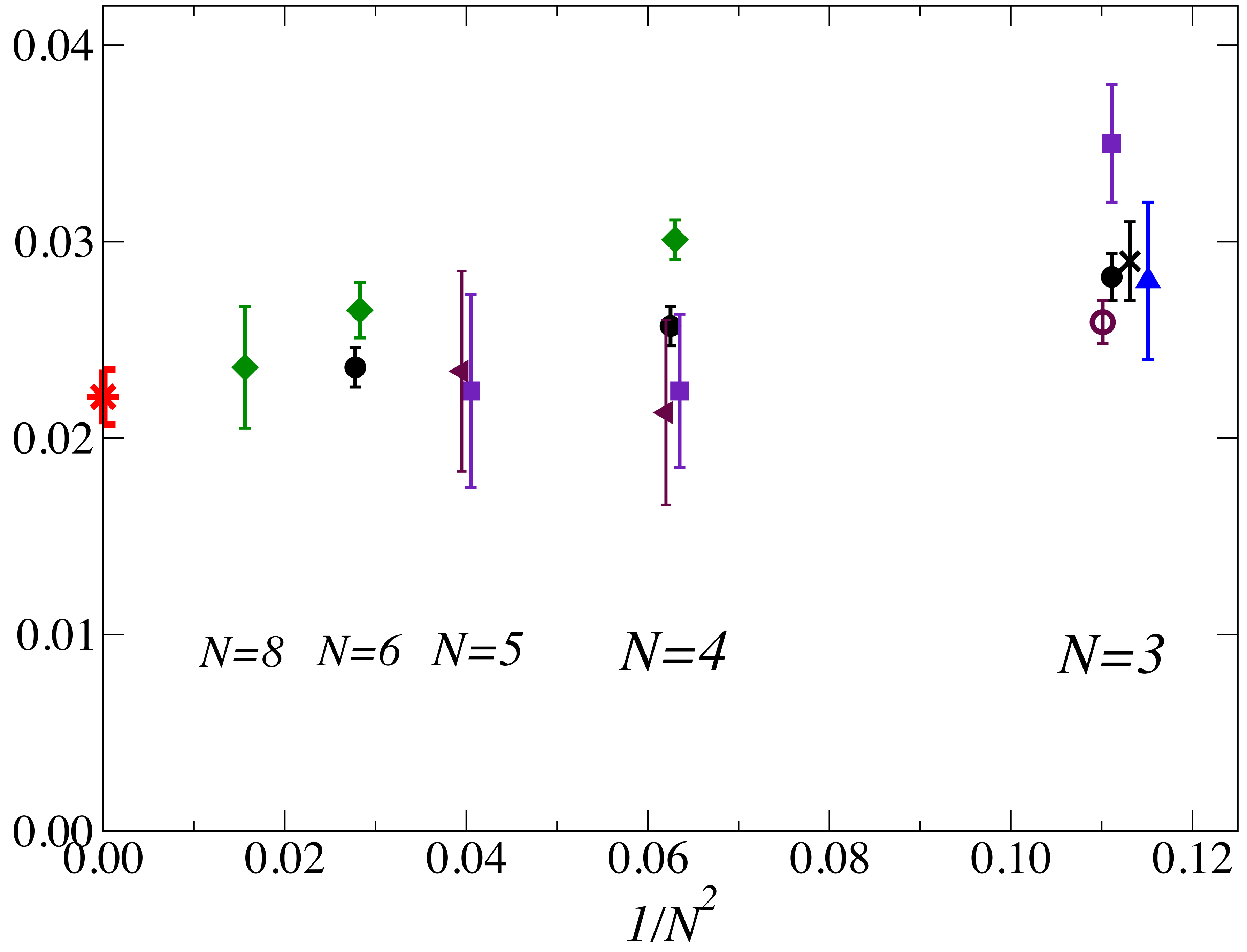
$$b_2 = \frac{\bar{b}_2}{N^2} + O(1/N^4)$$



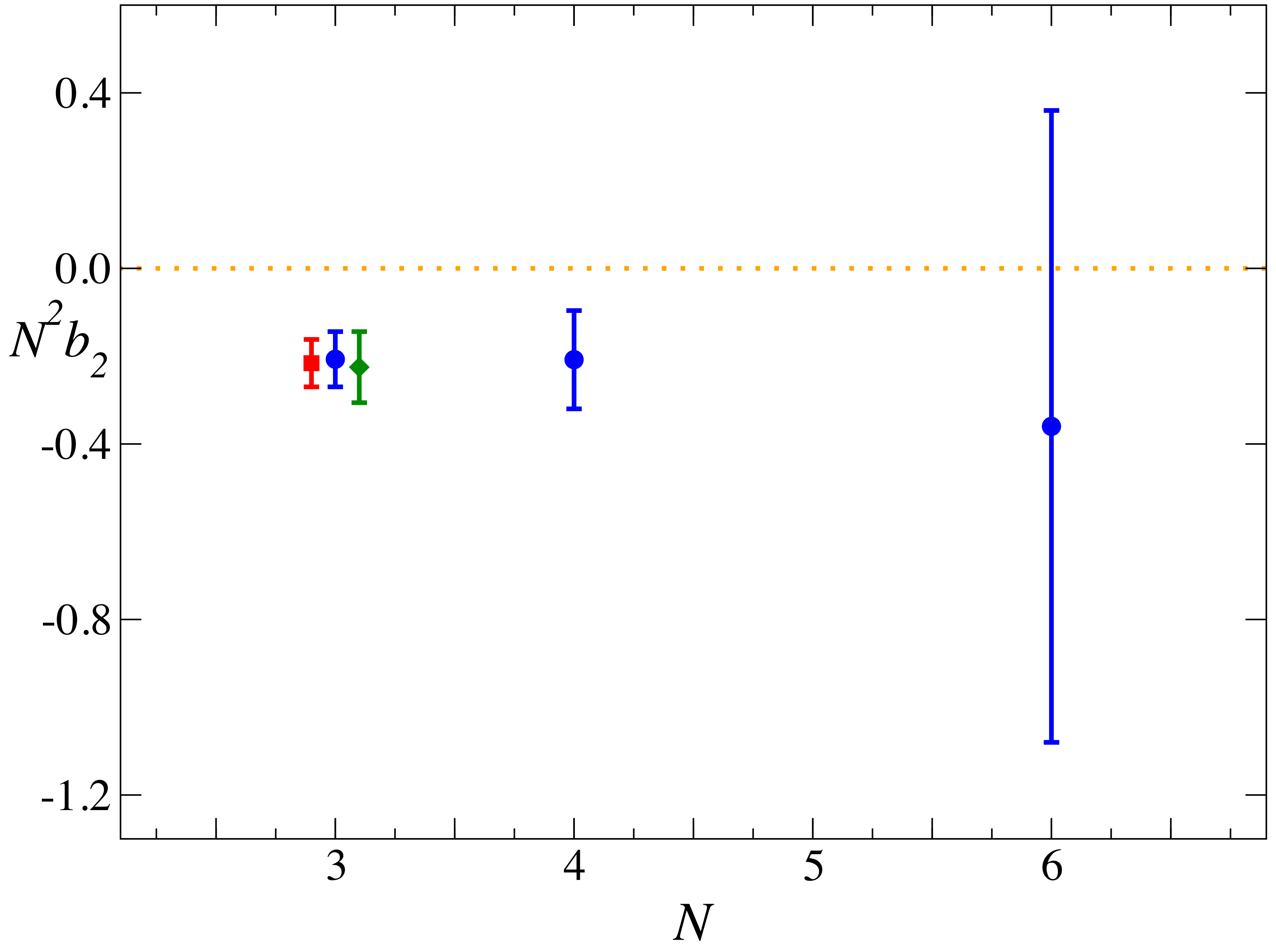
First two coefficients

[Review by Vicari and Panagopoulos (2018)]

$$\chi/\sigma^2 = C_\infty + \frac{c_2}{N^2} + O(1/N^4)$$



$$b_2 = \frac{\bar{b}_2}{N^2} + O(1/N^4)$$



only small corrections to the large N limit indicates CPV for $N \geq 3$.
(No $SU(2)$ calculation)

Finite temperature

DIGA : $f(\theta) \sim T^4 \exp\left(-\frac{8\pi^2}{g^2(T)}\right)(1-\cos\theta) \sim T^{4-\frac{11N}{3}}\Lambda^{\frac{11N}{3}}(1-\cos\theta)$

$\propto \chi(T)$

$= \theta^2/2(1+b_2\theta^2+b_4\theta^4+\cdots)$

$\Rightarrow b_2 = -1/12, \quad \cdots$

Finite temperature

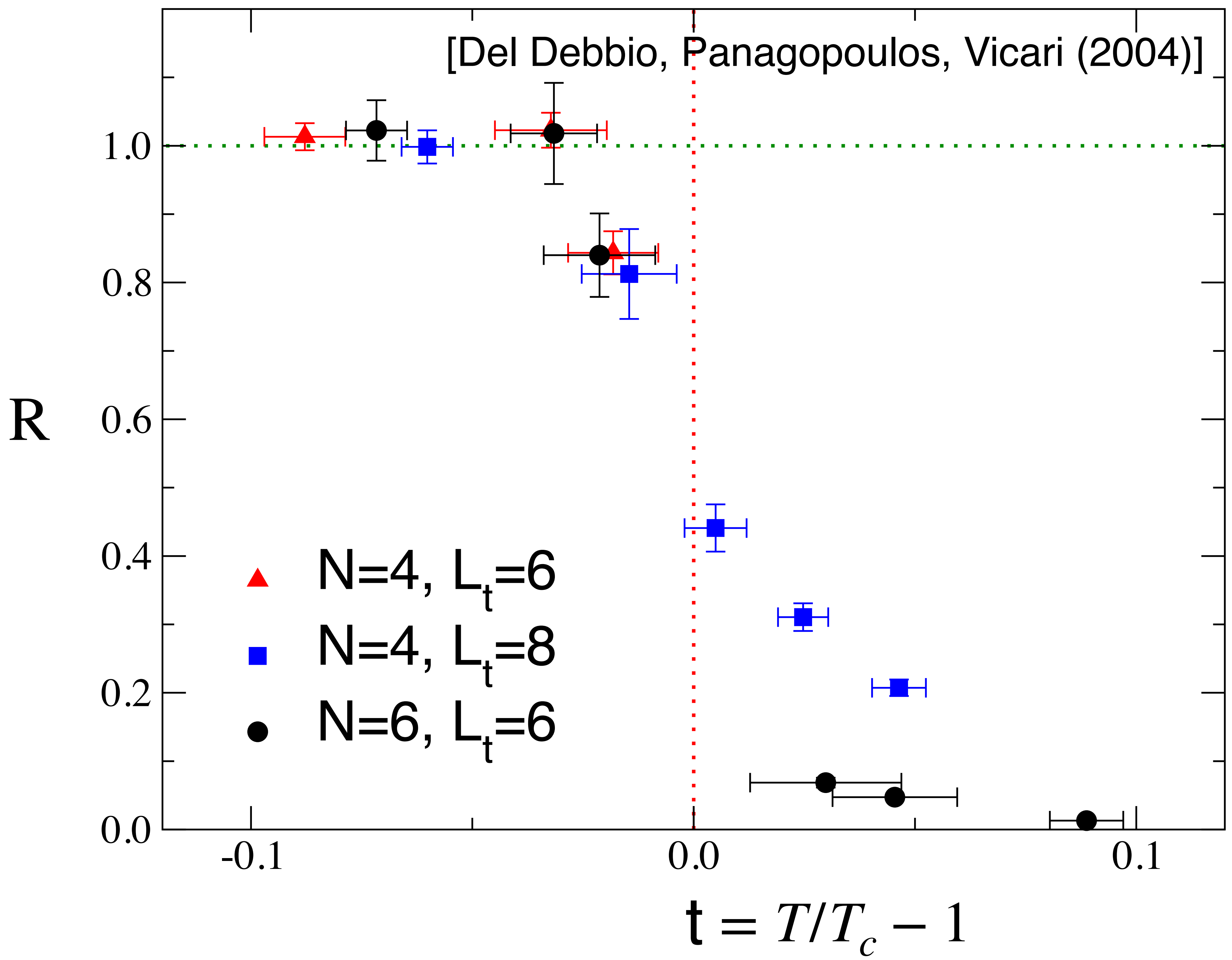
DIGA : $f(\theta) \sim T^4 \exp\left(-\frac{8\pi^2}{g^2(T)}\right)(1-\cos\theta) \sim T^{4-\frac{11N}{3}}\Lambda^{\frac{11N}{3}}(1-\cos\theta)$

$\propto \chi(T)$

$= \theta^2/2(1 + b_2\theta^2 + b_4\theta^4 + \cdots)$

$\Rightarrow b_2 = -1/12, \quad \cdots$

$R(T) = \chi(T)/\chi(T=0)$



Finite temperature

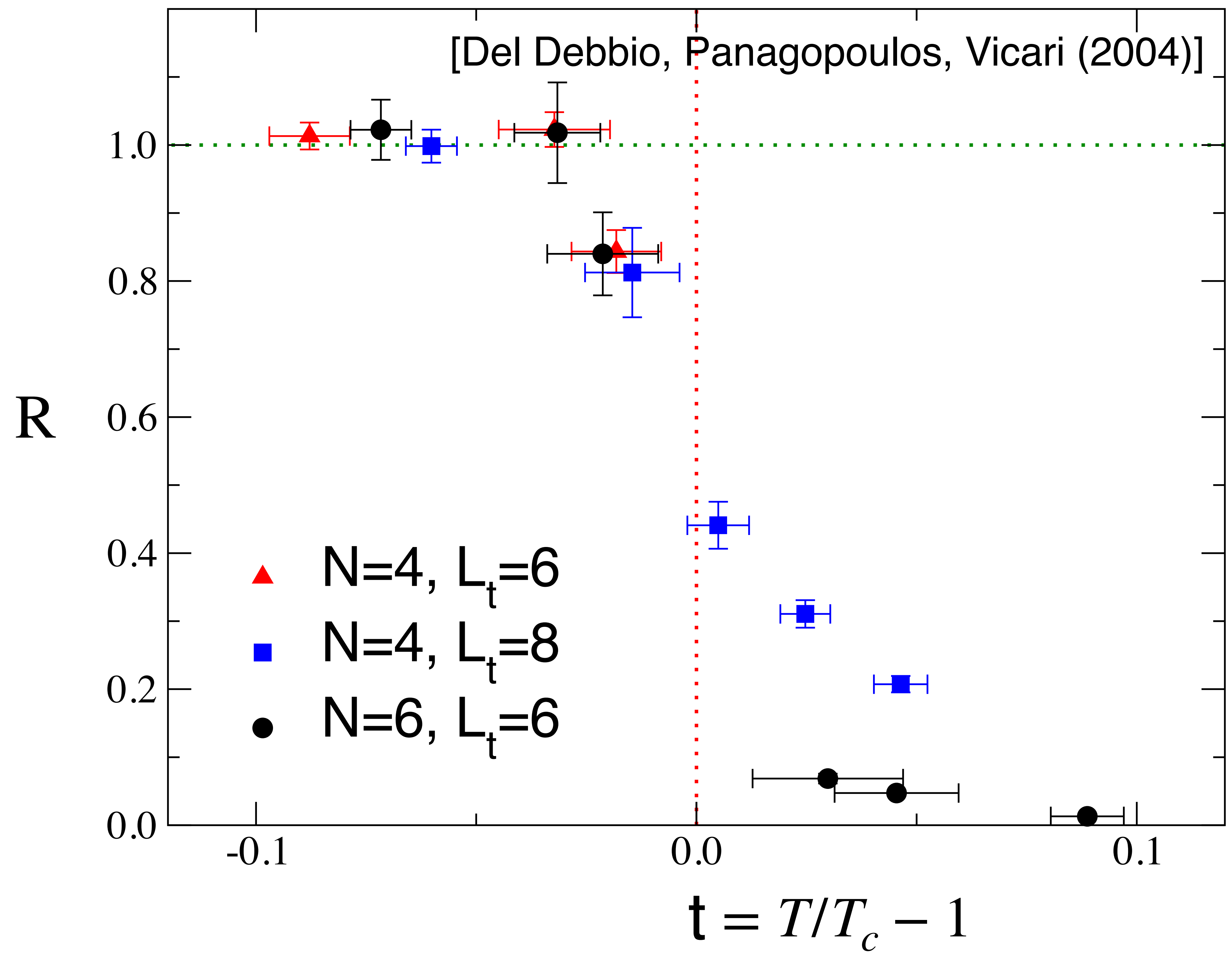
DIGA : $f(\theta) \sim T^4 \exp\left(-\frac{8\pi^2}{g^2(T)}\right)(1-\cos\theta) \sim T^{4-\frac{11N}{3}}\Lambda^{\frac{11N}{3}}(1-\cos\theta)$

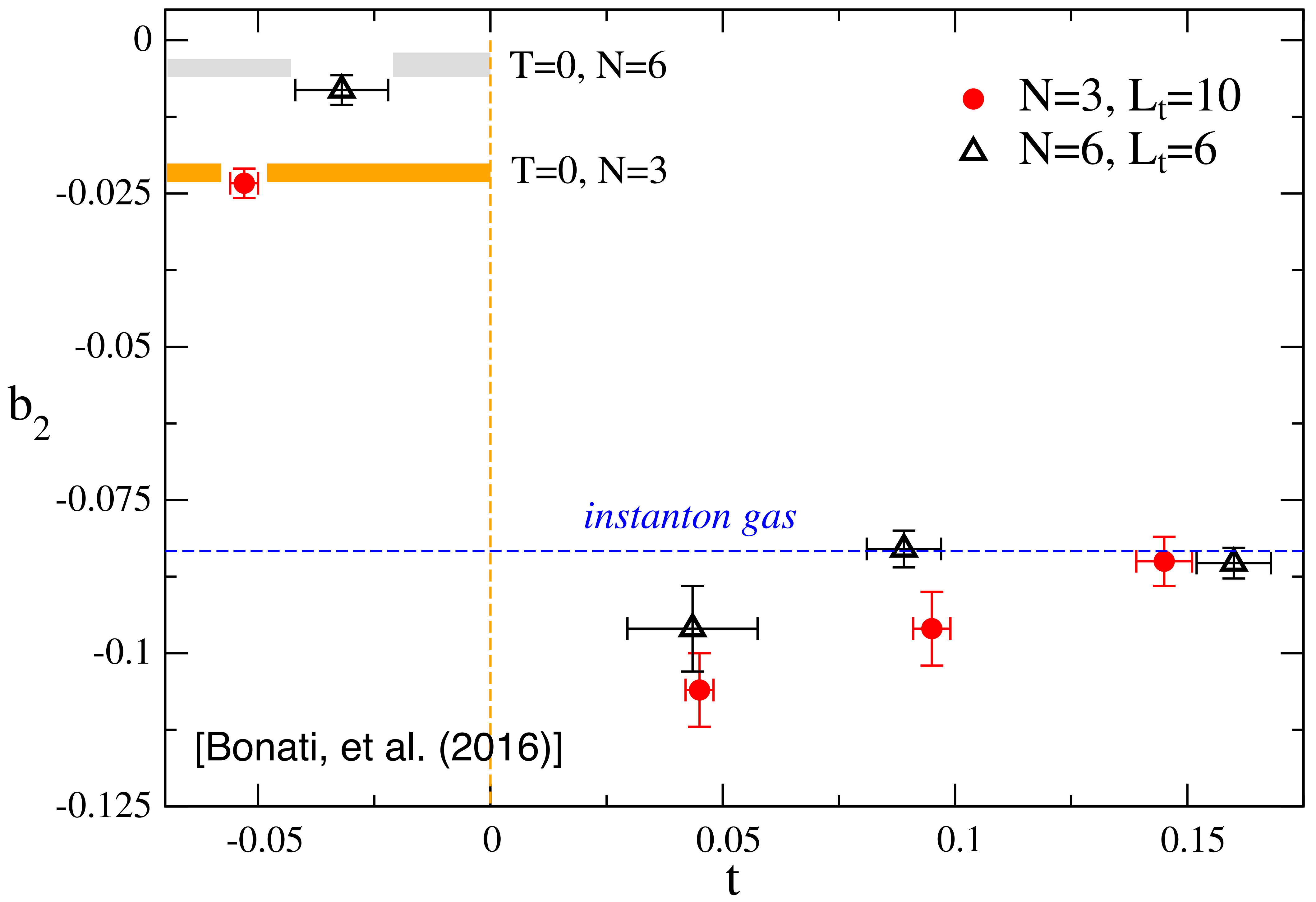
$\propto \chi(T)$

$= \theta^2/2(1 + b_2\theta^2 + b_4\theta^4 + \cdots)$

$\Rightarrow b_2 = -1/12, \quad \cdots$

$R(T) = \chi(T)/\chi(T=0)$





Finite temperature

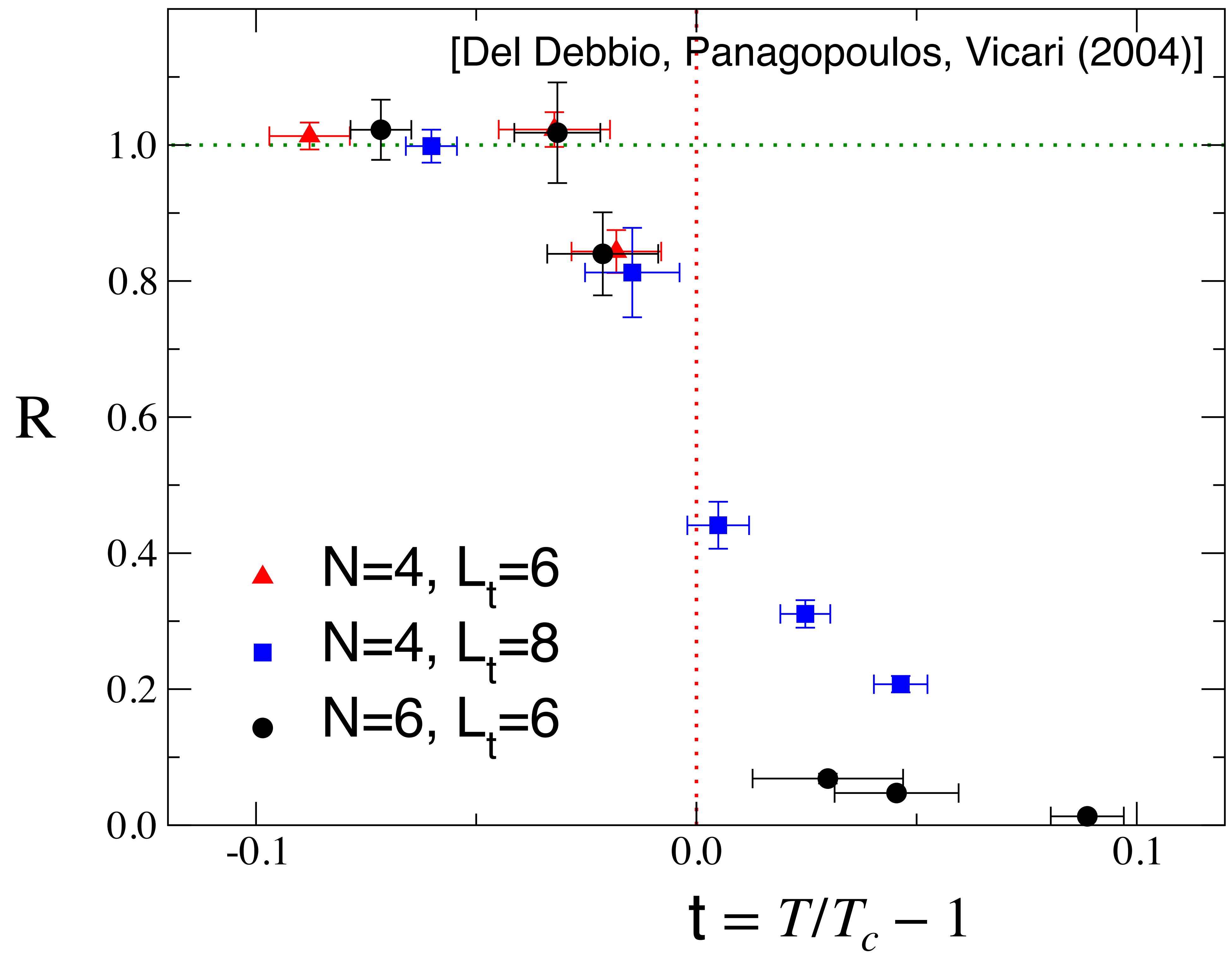
DIGA : $f(\theta) \sim T^4 \exp\left(-\frac{8\pi^2}{g^2(T)}\right)(1-\cos\theta) \sim T^{4-\frac{11N}{3}}\Lambda^{\frac{11N}{3}}(1-\cos\theta)$

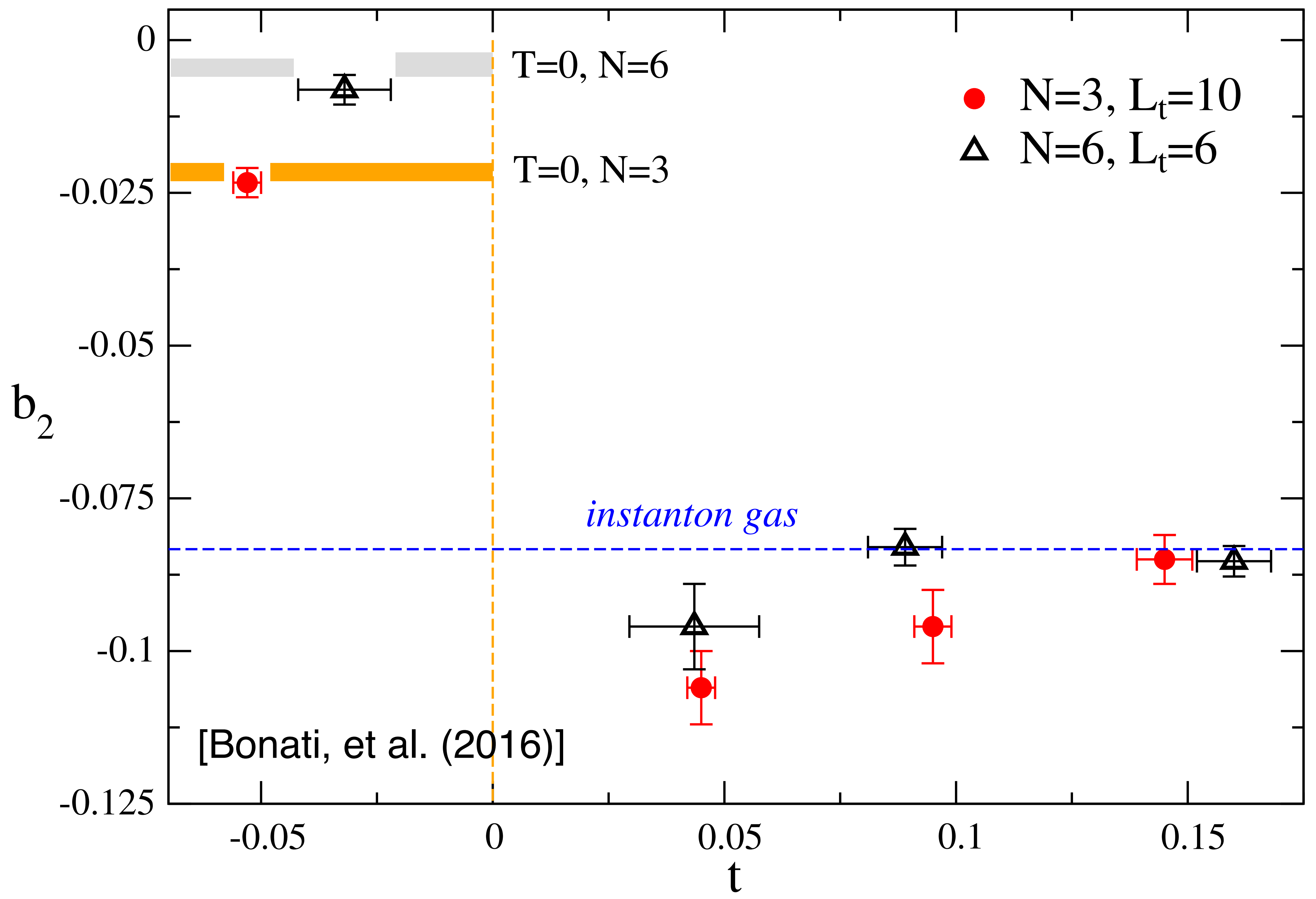
$\propto \chi(T)$

$= \theta^2/2(1 + b_2\theta^2 + b_4\theta^4 + \cdots)$

$\Rightarrow b_2 = -1/12, \quad \cdots$

$R(T) = \chi(T)/\chi(T=0)$





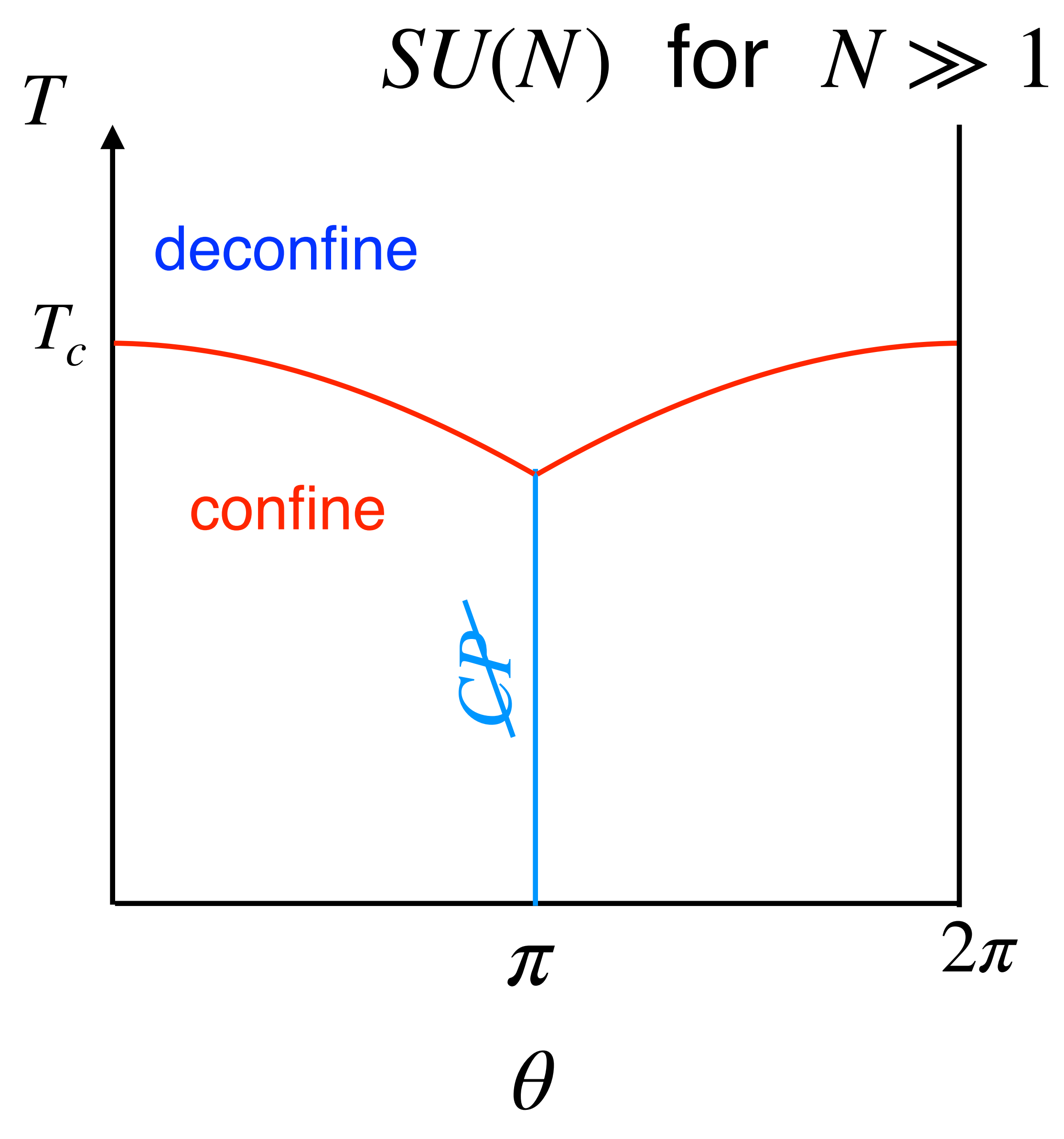
$-\frac{1}{12}$

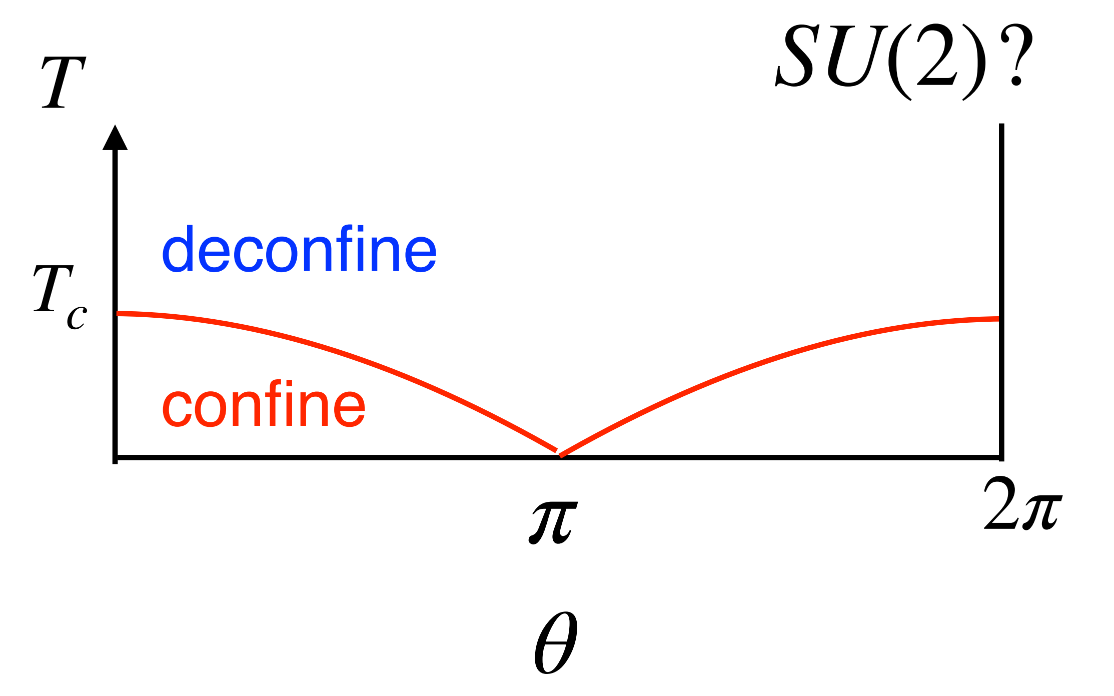
DIGA works for $T > 1.15\, T_c$ (No $SU(2)$ calculation)

Summary of previous results on $f(\theta)$

- Large N argument seems robust ⇒ CPV at $\theta = \pi$ and large N
- Formal arguments tell "For general N, CP has to be broken at $\theta = \pi$ if the vacuum is in the confining phase." [Gaiotto, et al.(2017)], [Kitano, Suyama, NY(2017)]
- Numerical evidences of CPV for $N \geq 3$
- What happens to the possible smallest N, i.e. $SU(2)$? Is it like "large N" or "2d CP^1" ?

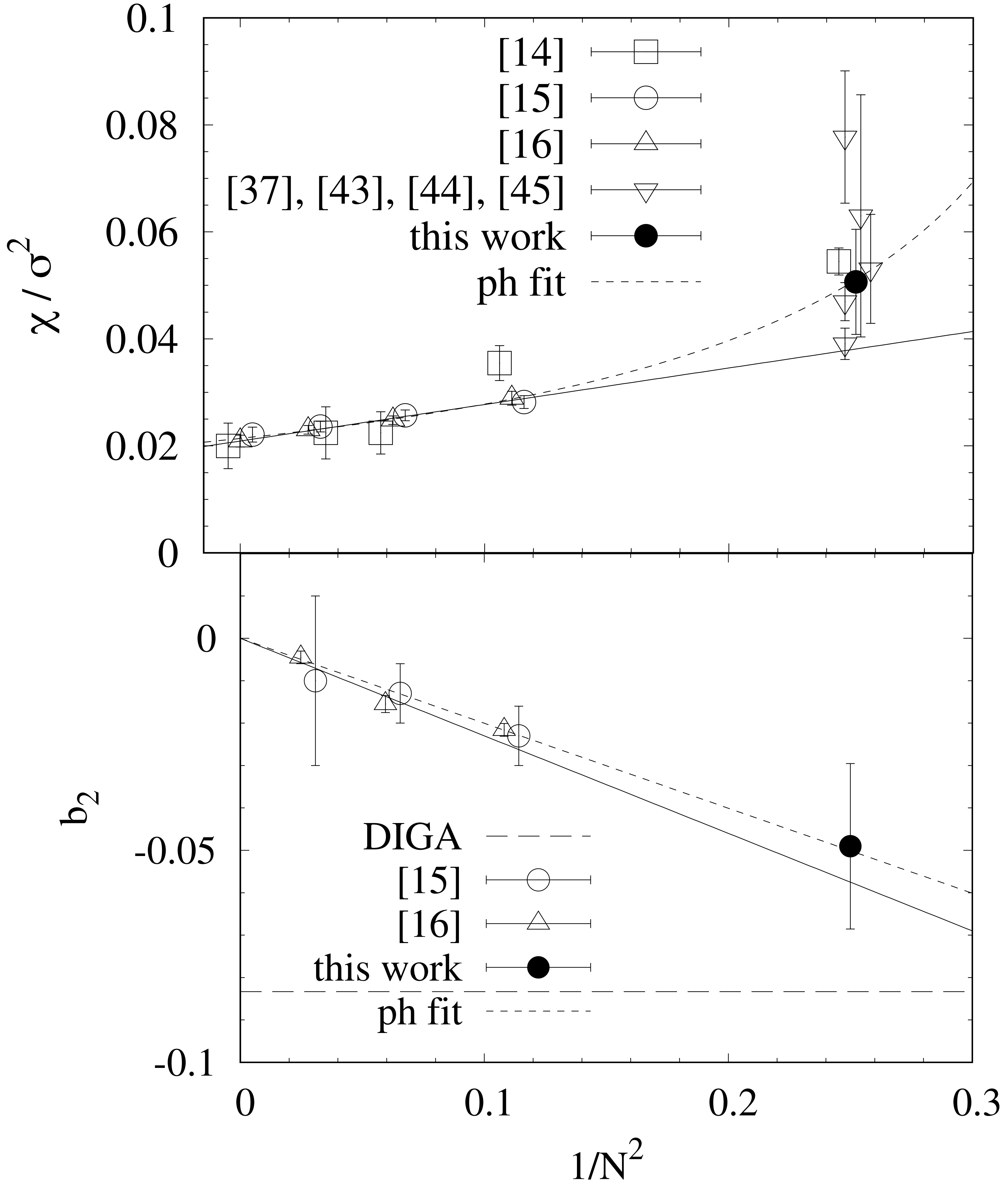
⇒ Lattice numerical simulations





χ and b_2 in $SU(2)$ at $T = 0$

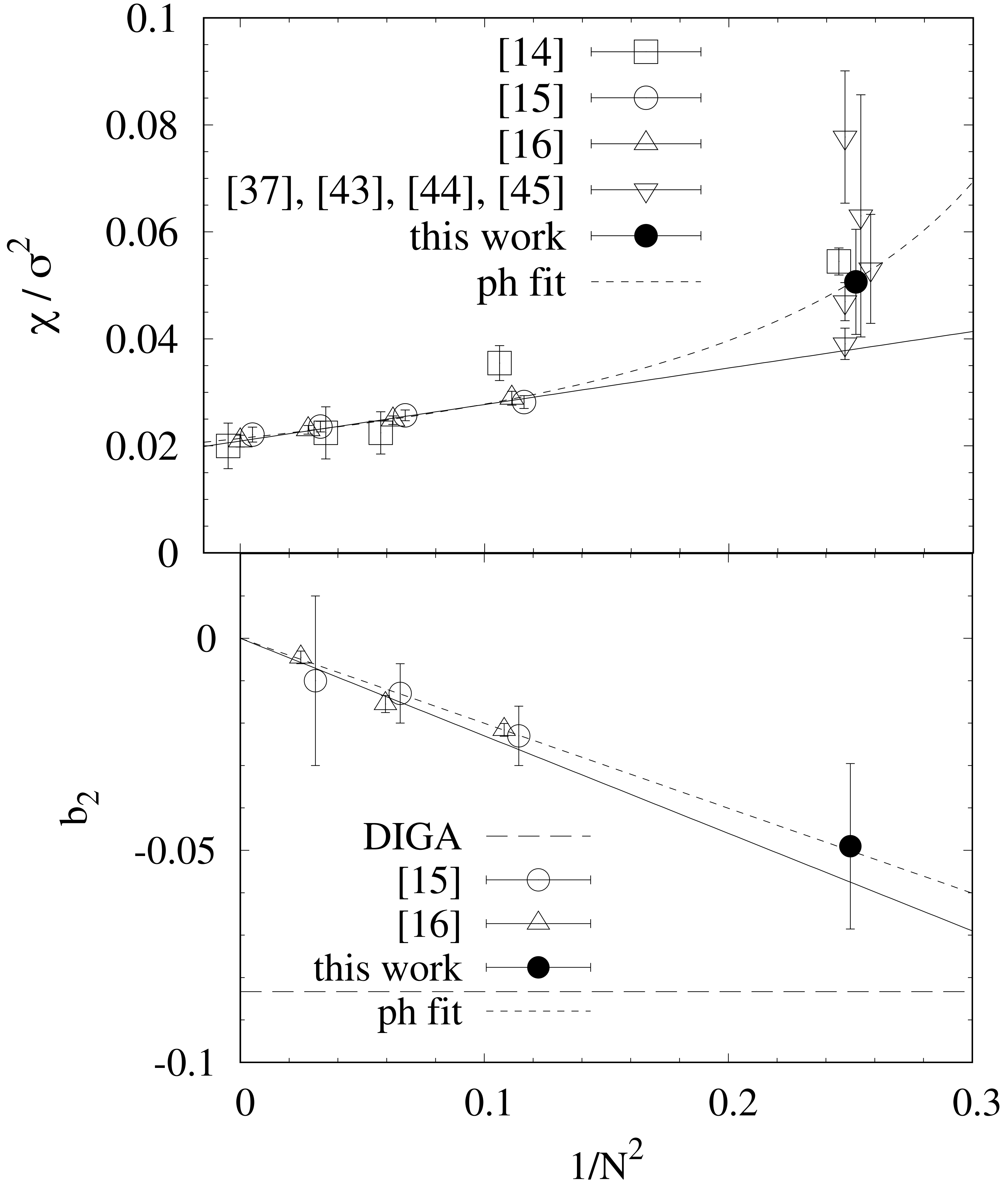
[Kitano, NY, Yamazaki (2021)]



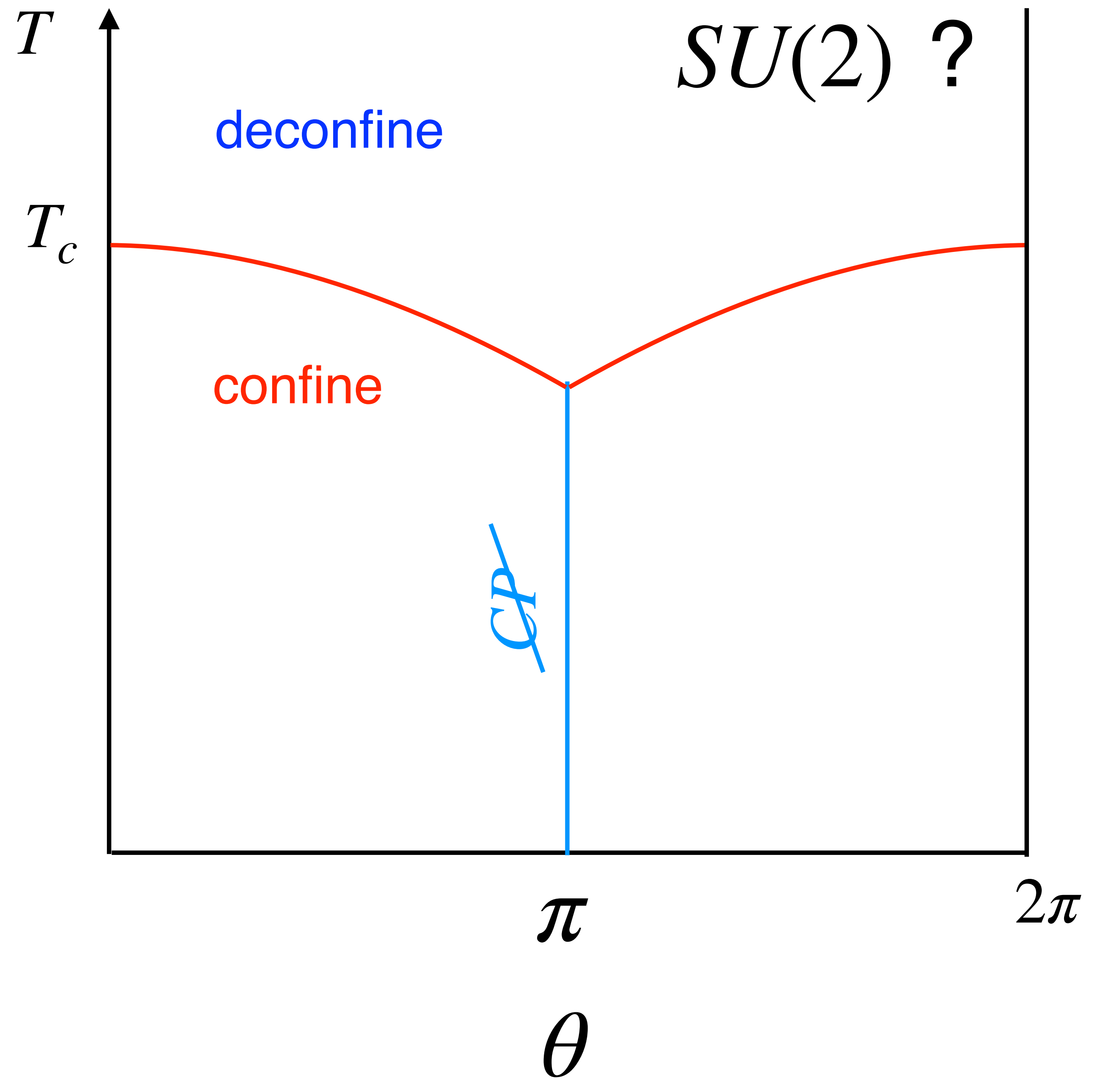
- smoothly connected to large N limit
  $\Rightarrow N \in \mathbb{Z}$ can be analytically continued to $\mathbb{R}$
  $\Rightarrow f(\theta)$ would be smooth function of N
- $b_2 \neq -\dfrac{1}{12}$ (i.e. not instanton-like)
  $\Rightarrow$ conjectured that SU(2) belongs to large N class and
  CPV takes places at $\theta = \pi$.

χ and b_2 in $SU(2)$ at $T = 0$

[Kitano, NY, Yamazaki (2021)]



- smoothly connected to large N limit
  $\Rightarrow N \in \mathbb{Z}$ can be analytically continued to $\mathbb{R}$
  $\Rightarrow f(\theta)$ would be smooth function of N
- $b_2 \neq -\dfrac{1}{12}$ (i.e. not instanton-like)
  $\Rightarrow$ conjectured that SU(2) belongs to large N class and
  CPV takes places at $\theta = \pi$.



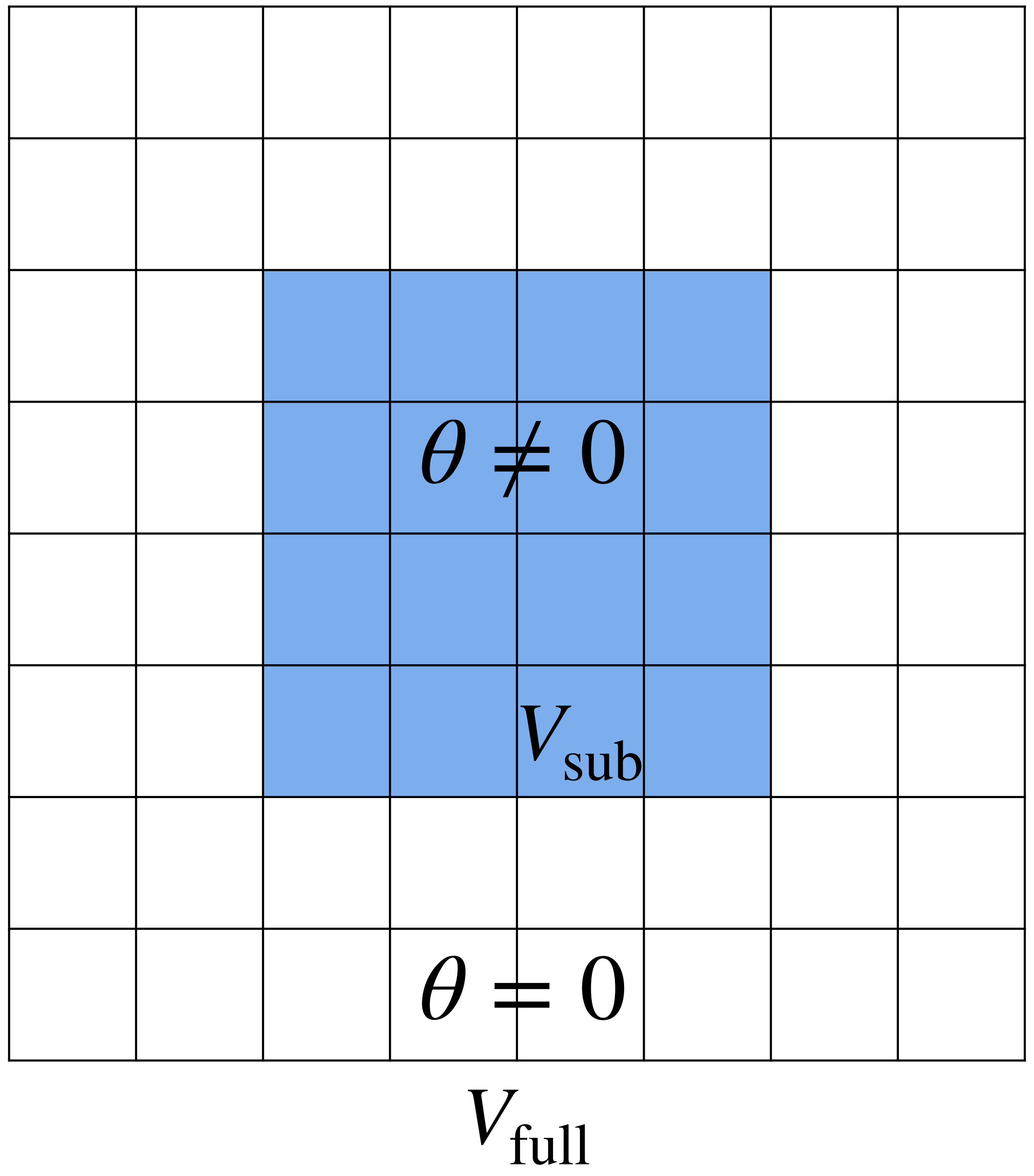
New method without any expansion

[Kitano,Matsudo,NY,Yamazaki(2021)]

Introduce sub-volume $V_{\rm sub} = l^4$ and

$$Q_{\rm sub} = \sum_{x \in V_{\rm sub}} q(x) \quad \notin \mathbb{Z}$$

“sub-volume method”



$$e^{-V_{\rm sub} f_{\rm sub}(\theta)} = \frac{Z(\theta)}{Z(0)} = \frac{1}{Z(0)} \int \mathscr{D}U \; e^{-S_g + i\theta Q_{\rm sub}} = \langle e^{i\theta Q_{\rm sub}} \rangle_{\theta=0}$$

$$f_{\rm sub}(\theta) = -\frac{1}{V_{\rm sub}} \ln \langle \cos(\theta Q_{\rm sub}) \rangle_{\theta=0}$$

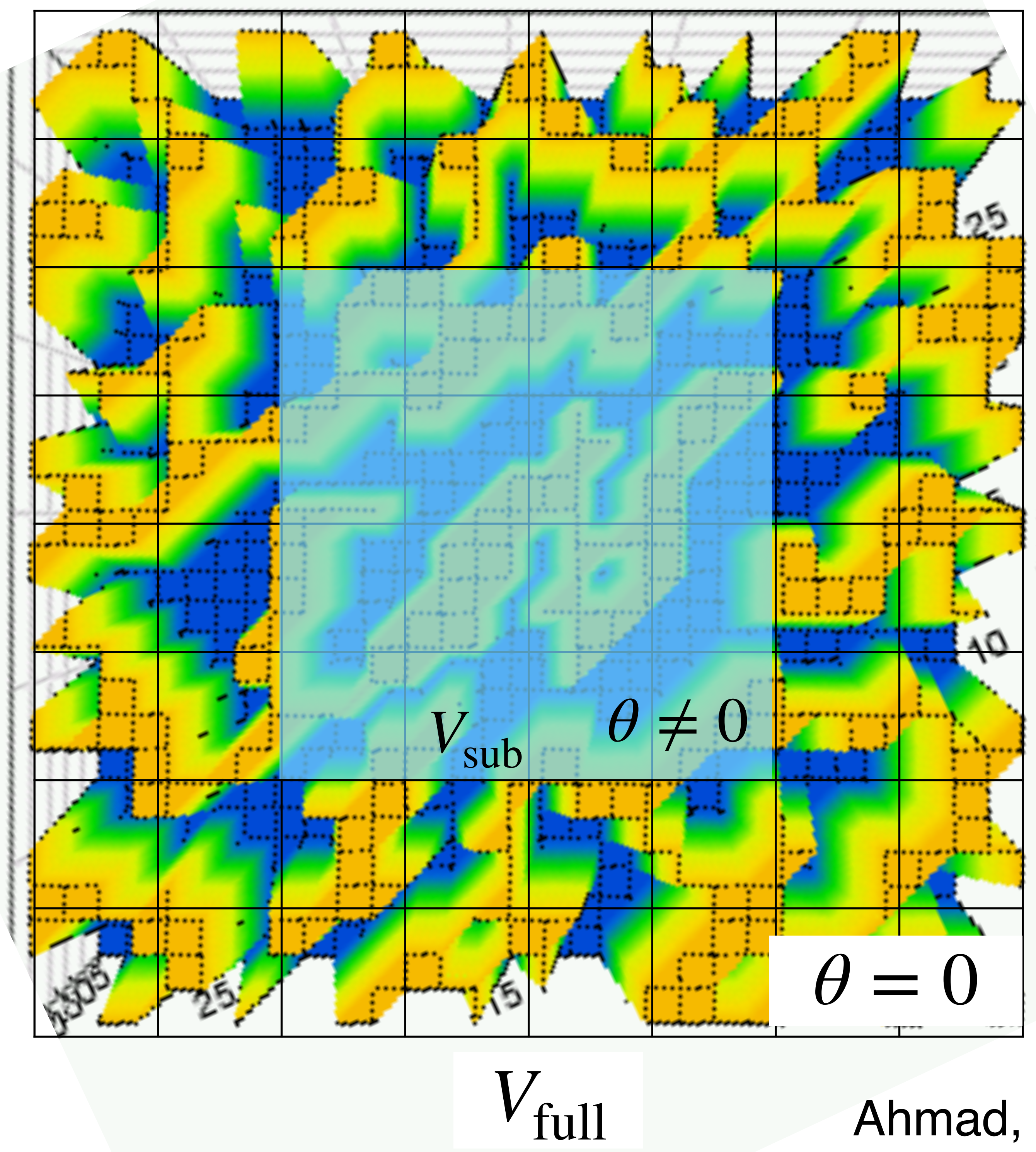
$$f(\theta) = \lim_{V_{\rm sub}\to\infty} f_{\rm sub}(\theta) = \lim_{l\to\infty} \left\{ f(\theta) + \frac{s(\theta)}{l} + O(1/l^2) \right\}$$

with $l_{\rm dyn}^4 \ll V_{\rm sub} \ll V_{\rm full}$ ($l_{\rm dyn}$: dynamical length scale)

$s(\theta)$: surface tension

Expected behavior of $f_{\rm sub}(\theta)$ as a function of $V_{\rm sub}$

- $V_{\rm sub} \gg l_{\rm dyn}^4$ would have to be satisfied.
- As long as $V_{\rm sub} \gg l_{\rm dyn}^4$, $f_{\rm sub}(\theta)$ is expected to show the scaling behavior, $f_{\rm sub}(\theta) = f(\theta) + \frac{s(\theta)}{l} + O(1/l^2)$.
- Buch a behavior will end as $V_{\rm sub} \to V_{\rm full}$, where $Q_{\rm sub} \to Q_{\rm full} \in \mathbb{Z}$. Thus, $V_{\rm sub} \ll V_{\rm full}$ is required.
- On the other hand, the method fails when $|\theta Q_{\rm sub}| \sim \pi$ because $f_{\rm sub}(\theta) \propto \ln\langle \cos(\theta Q_{\rm sub}) \rangle$ becomes ill-defined.
- Crucial question:
  $V_{\rm sub}$ satisfying $l_{\rm dyn}^4 \ll V_{\rm sub} \ll V_{\rm full}$ and $|\theta Q_{\rm sub}| < \pi$ exists ?



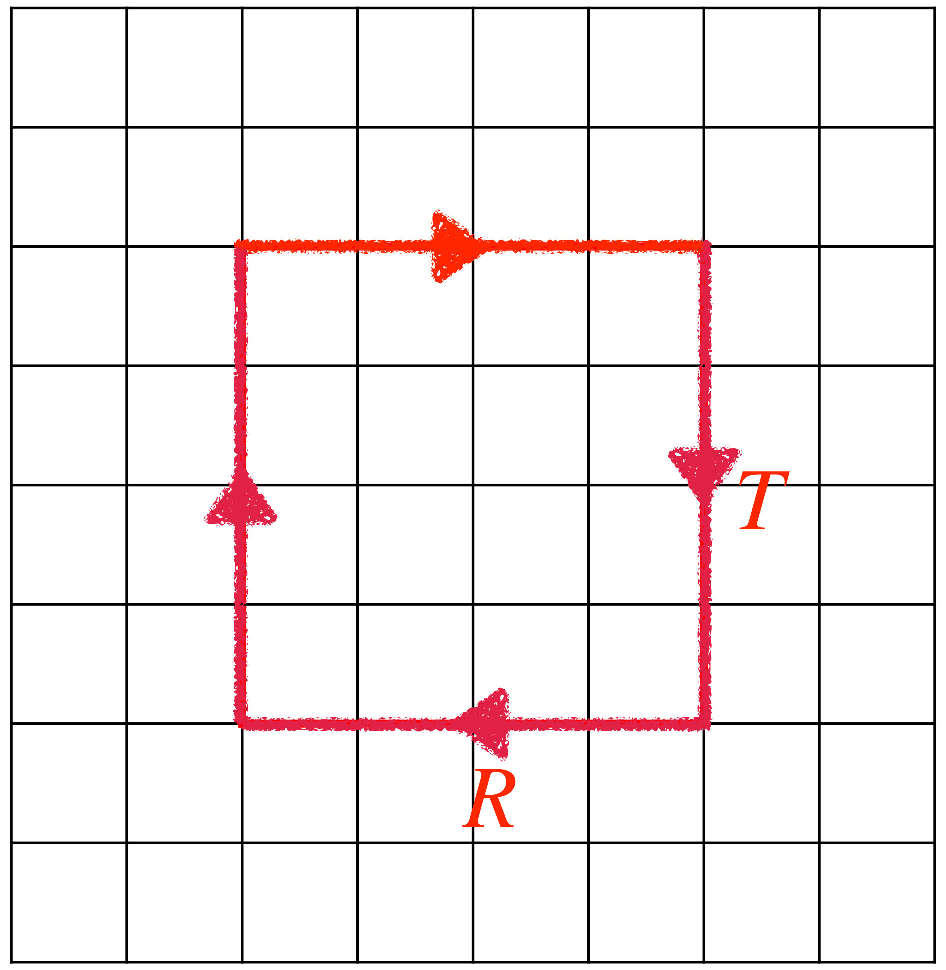
Ahmad, et al. (2005)

Similarity to the static potential calculation

In the static potential calculation, Wilson loop is inserted.

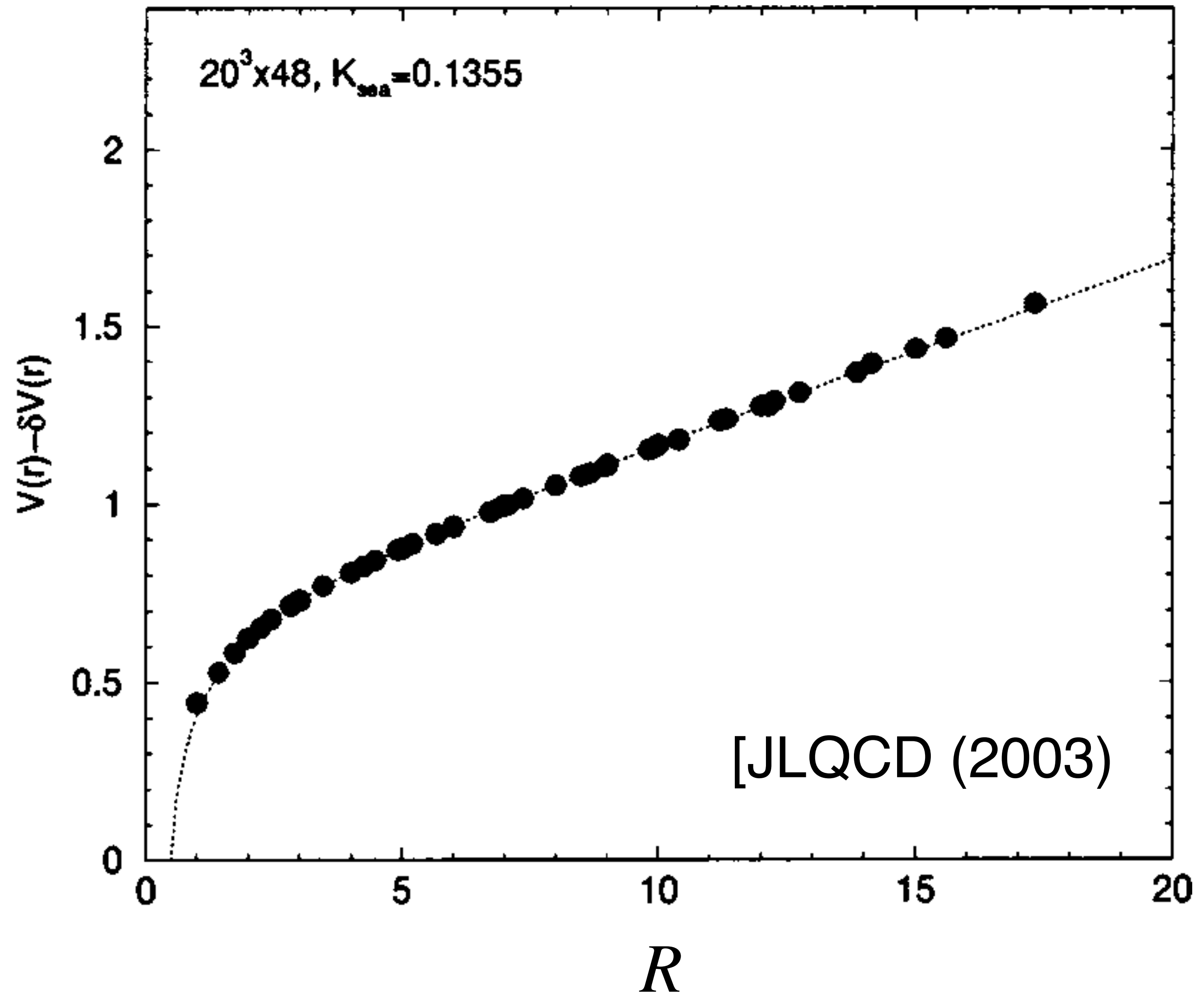
$$\frac{Z(\square)}{Z(1)} = \frac{1}{Z(1)} \int \mathscr{D}U \ \mathrm{Tr}\left[e^{i\oint A}\right] e^{-S_{\mathrm{QCD}}} = \langle \mathrm{Tr}\left[e^{i\oint A}\right] \rangle \to e^{-V(\mathscr{A})}$$

$$V(\mathscr{A}) = -\lim_{\mathscr{A}\to\infty} \ln\langle \mathrm{Tr}\left[e^{i\oint A}\right] \rangle = \sigma\mathscr{A} + \cdots$$



In sub-volume method, instead a operator extending over subvolume is inserted.

$f(\theta)$ is analogous to σ in the static potential.



About smearing

- Need to numerically calculate $q(x) = \frac{1}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}$ on the lattice
- Raw configurations are contaminated by local lumps.
- Smearing (= smoothing a configuration) removes such short-distance artifacts.
- However, at the same time, smearing may alter relevant topological excitations, too.
- We studied this point and developed the procedure to restore relevant information. [Kitano, NY, Yamazaki (2021)]
  - calculate an observable every 5 steps of the smearing
  - extrapolate those back to $n_{\mathrm{APE}} \to 0$, $\langle O \rangle = \lim_{n_{\mathrm{APE}}\to 0} \langle O(n_{\mathrm{APE}}) \rangle$

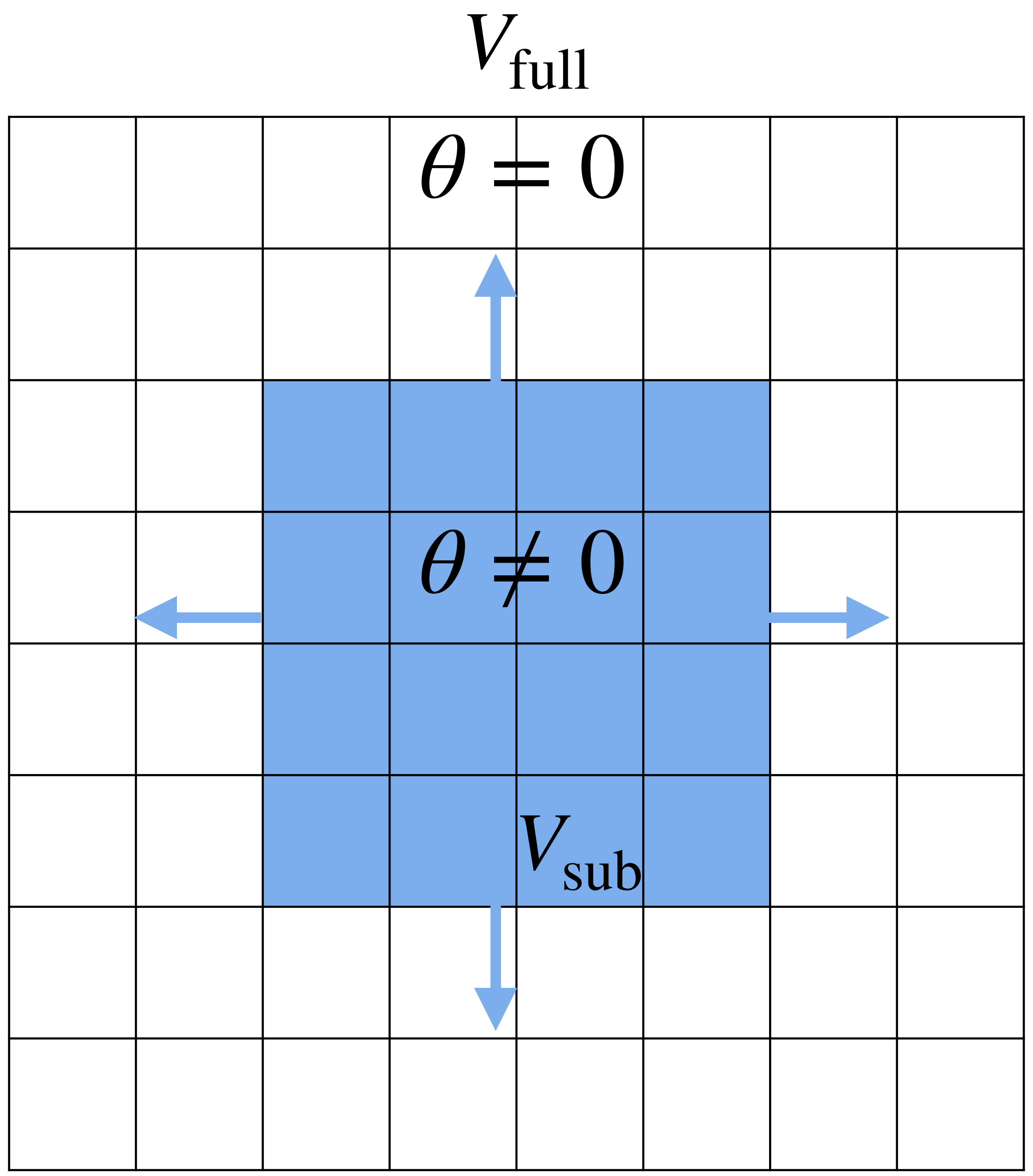
Lattice parameters and observables

- $SU(2)$ YM theory by Symanzik improved gauge action
- $\beta = \dfrac{4}{g^2} = 1.975$ (relatively fine: $1/(aT_c) = 9.50$)
- $V_{\rm full} = 24^3 \times \{48,\ 6,\ 8\}$ $(T = 0,\ 1.2T_c,\ 1.6T_c)$
- Periodic boundary condition in all directions
- # of configs = { 68000 , 5000 , 5000 }
- Calculate $Q_{\rm sub} = \sum_{x \in V_{\rm sub}} q(x)$ and estimate

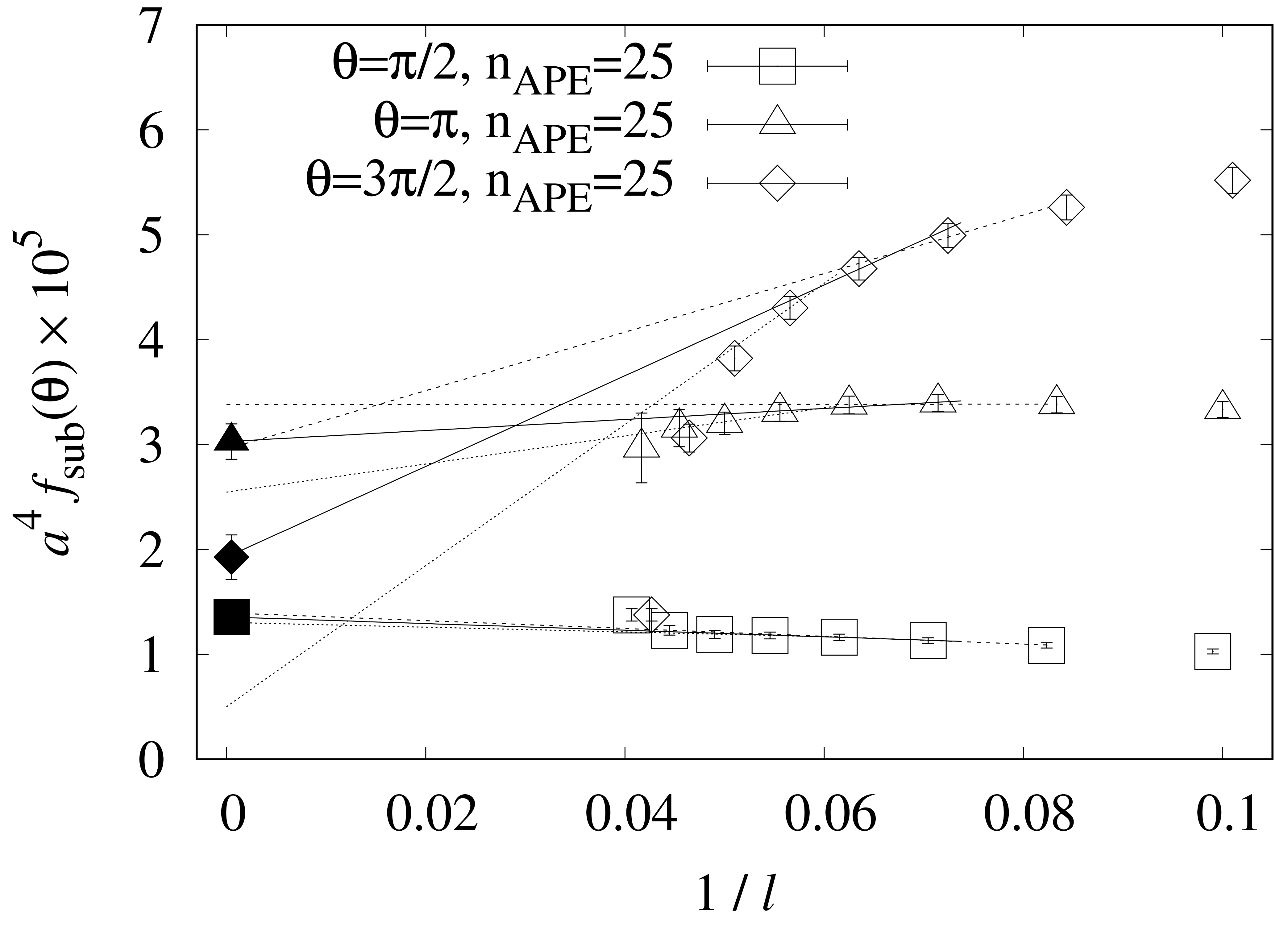
✓ $f(\theta) = -\lim_{V_{\rm sub}\to\infty} \dfrac{1}{V_{\rm sub}} \ln\langle \cos(\theta Q_{\rm sub}) \rangle$

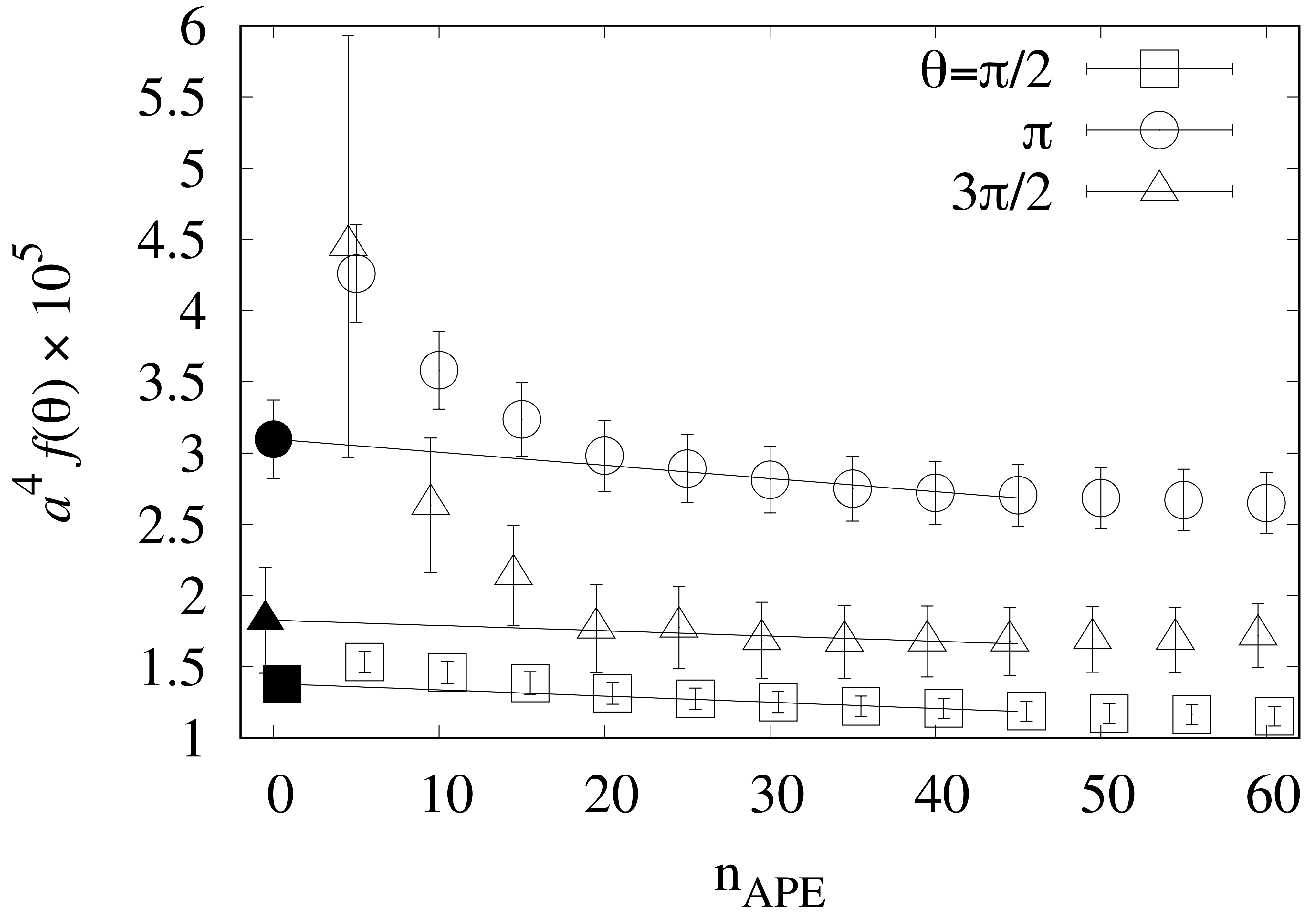
✓ $\dfrac{df(\theta)}{d\theta} = \lim_{V_{\rm sub}\to\infty} \dfrac{1}{V_{\rm sub}} \dfrac{\langle Q_{\rm sub} \sin(\theta Q_{\rm sub}) \rangle}{\langle \cos(\theta Q_{\rm sub}) \rangle}$

which are used to crosscheck each other



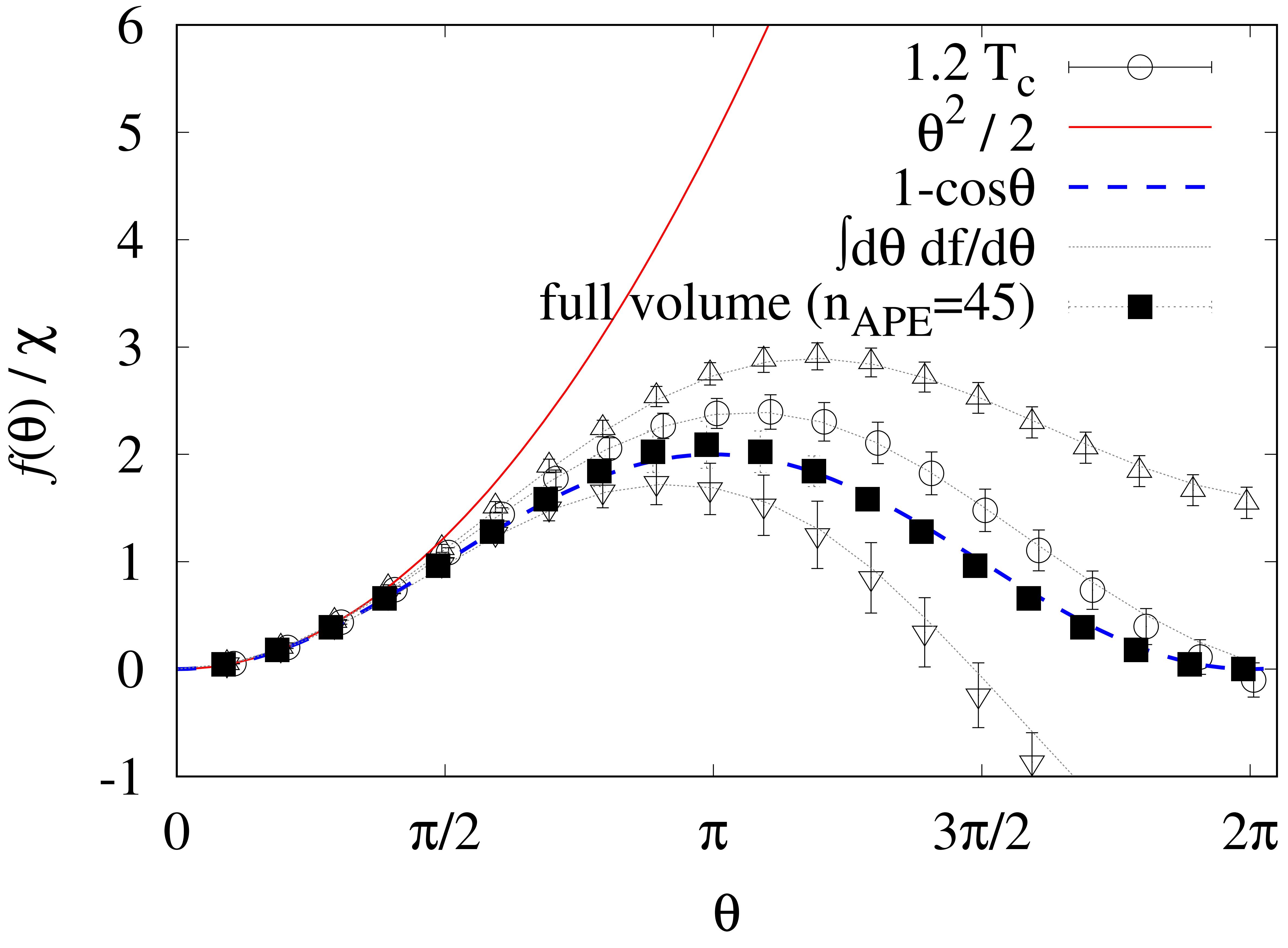
Finite temperature ($T = 1.2\,T_c$)





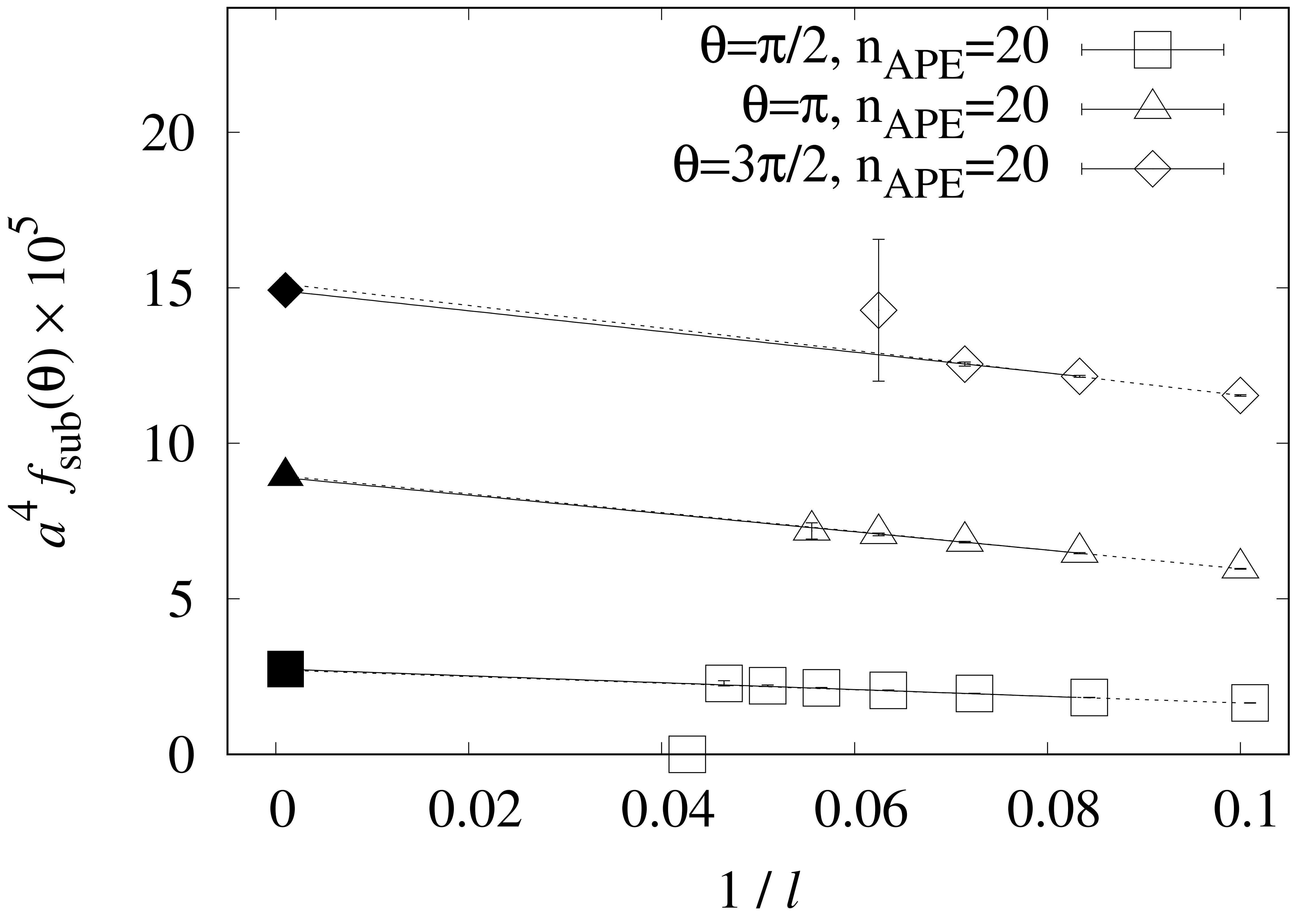
- $V_{\rm sub} = l^3 \times 6$ with $l \in \{12, \cdots, 20\}$
- Linear fit works well in either extrapolations.
- Not monotonic function, $f(\pi) > f(3\pi/2)$

θ dependence of $f(\theta)$ at $T = 1.2T_c$



- Systematic error due to ambiguity of the scaling region is large for $\theta > \pi$
- Within large uncertainty, consistent with the DIGA.
- $df(\theta)/d\theta\Big|_{\theta=\pi} \approx 0 \Rightarrow$ no CPV above T_c
- Numerical consistency with $\int d\theta \frac{df}{d\theta}$
- Similar results at $T = 1.6\,T_c$

$l \to \infty$ limit at $T = 0$

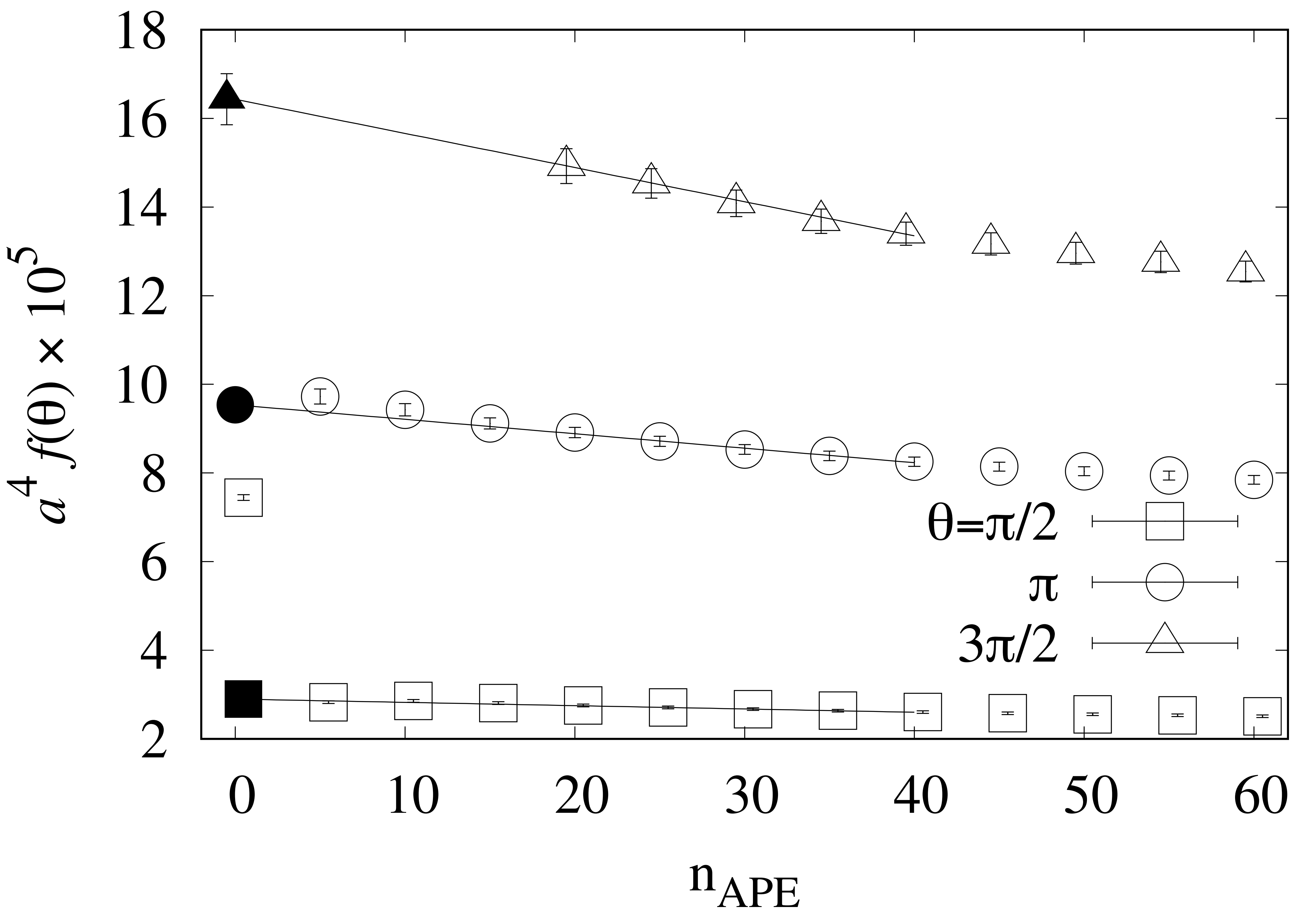


- $V_{\mathrm{sub}} = l^4$ with $l \in \{10,\ 12,\ \cdots,\ 20\}$
- Data in the range of $l_{\mathrm{dyn}}^4 \ll V_{\mathrm{sub}} \ll V_{\mathrm{full}}$ are fitted to

$$f_{\mathrm{sub}}(\theta) = f(\theta) + \frac{as(\theta)}{l}$$

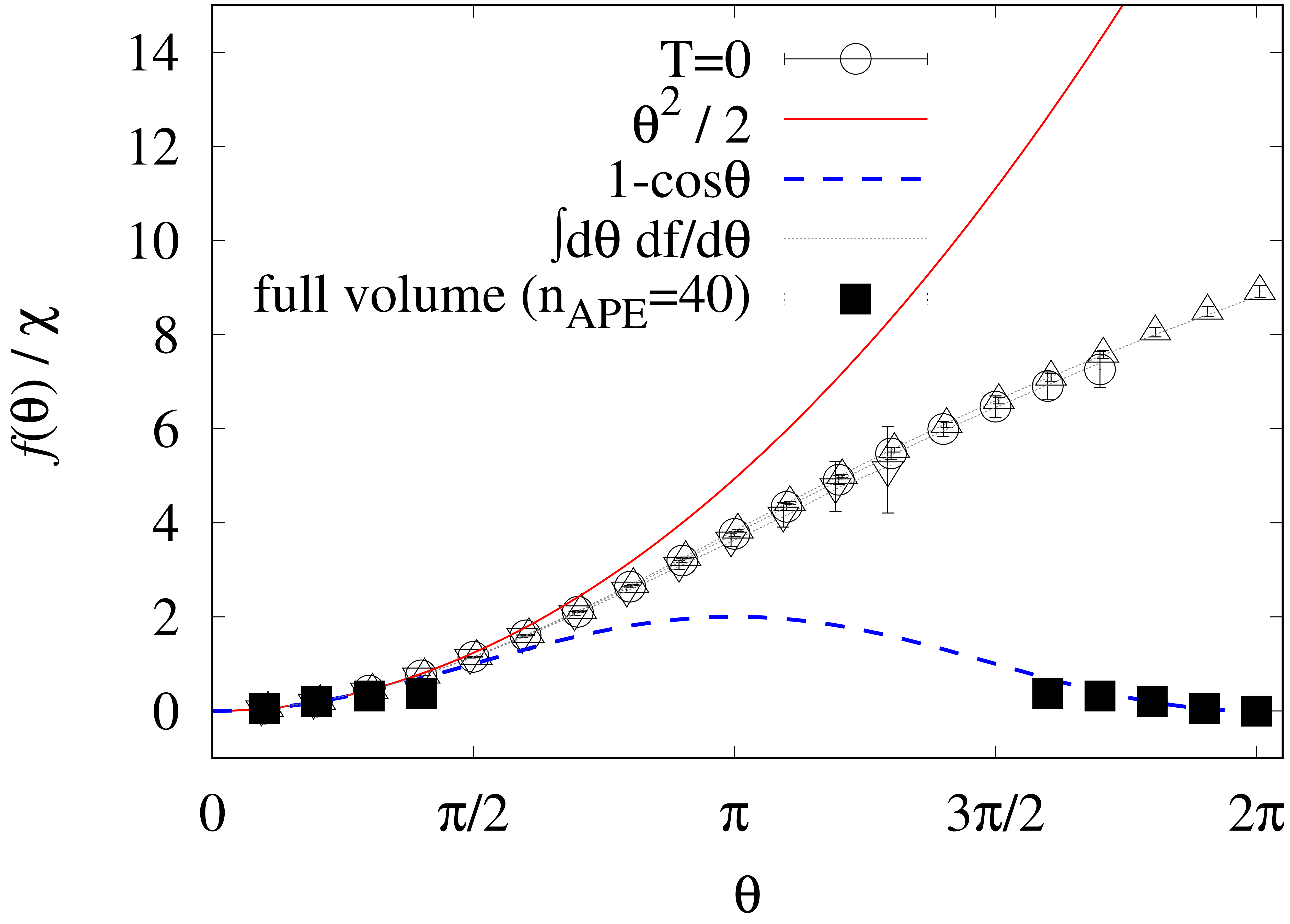
- Linear extrapolation works well.

$n_{\mathrm{APE}} \to 0$ limit at $T = 0$



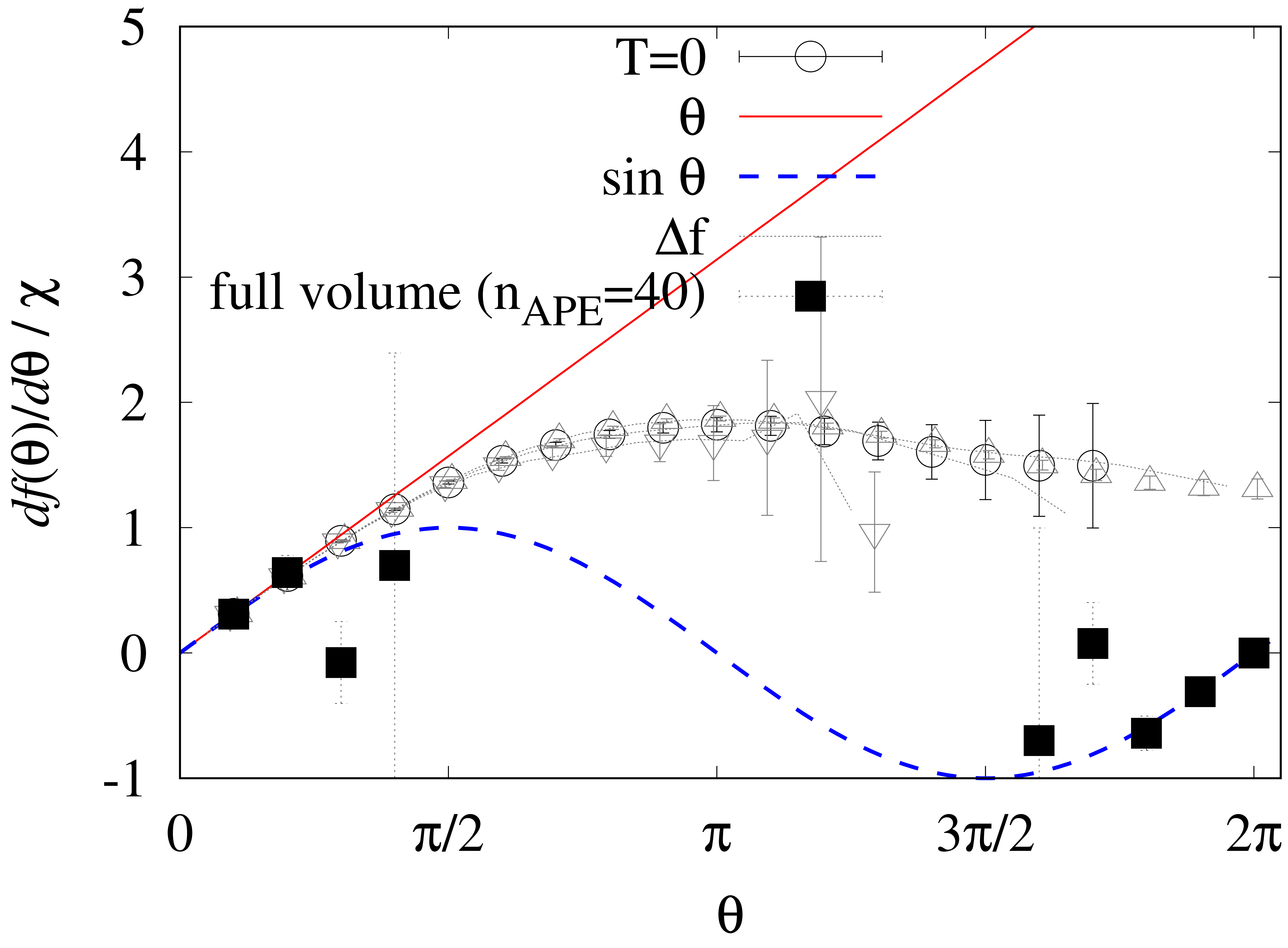
- Fit range $n_{\mathrm{APE}} = [20, 40]$ determined in [Kitano, NY, Yamazaki (2021)].
- Linear fit works well.
- Monotonic function $f(\pi) < f(3\pi/2)$

θ dependence of $f(\theta)$ at $T = 0$



- Succeed to calculate up to $\theta \sim 3\pi/2$
- Monotonically increasing function
- Inconsistent with DIGA
- $f(\pi - \theta) \neq f(\pi + \theta)$ requires explanation.
- Re-weighting (=full volume) method works only around $\theta = 0$.
- Numerical consistency with $\int d\theta \, \frac{df}{d\theta}$

$df(\theta)/d\theta$ at $T = 0$



- Order parameter is non-zero

$$\left. df(\theta)/d\theta \right|_{\theta=\pi} = -i\langle\, q(x)\,\rangle_{\theta=\pi} \neq 0$$

$\Rightarrow$ spontaneous CPV at $\theta = \pi$

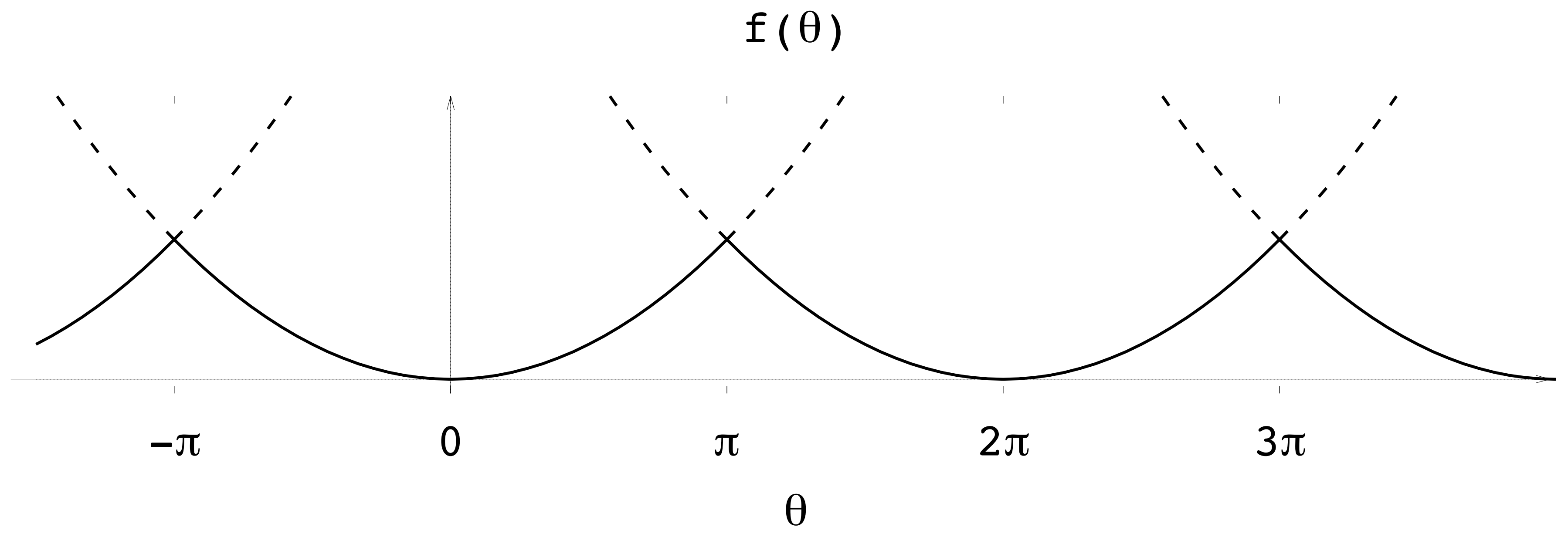
Discussion

- At $T > T_c$,

consistent with $1 - \cos\theta$ and no CPV as expected (though non-trivial).

- At $T = 0$,

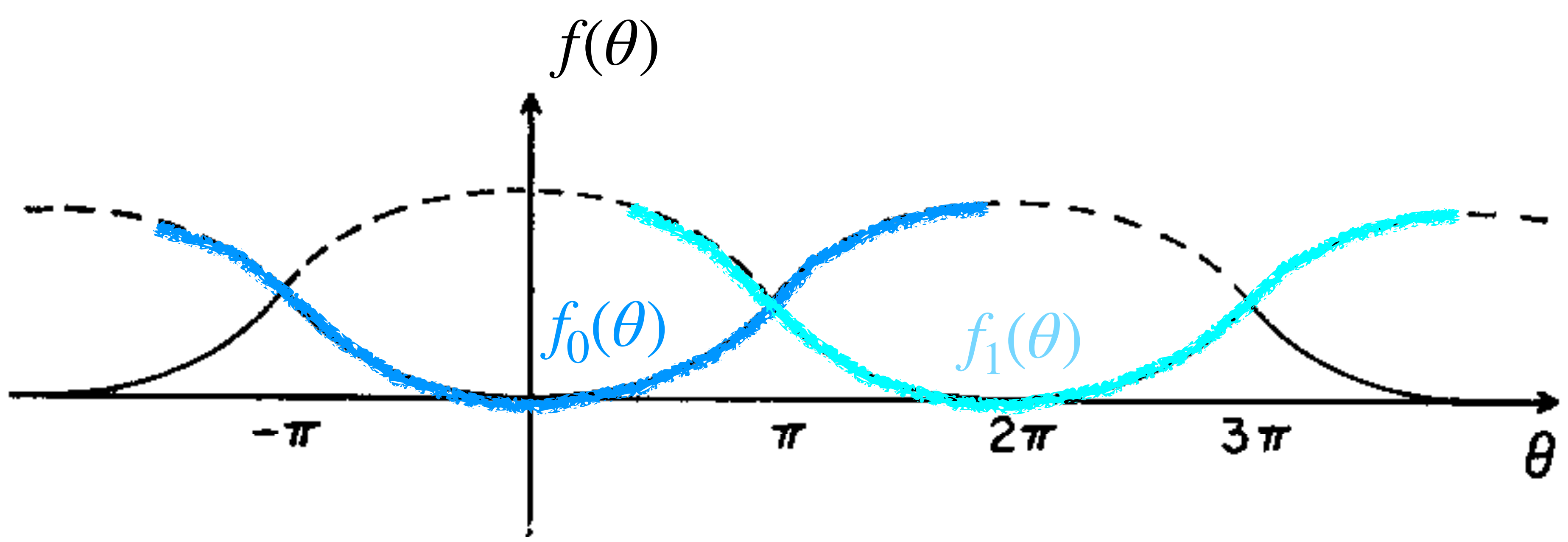
$f(\pi - \theta) \neq f(\pi + \theta)$ and it is not like



Why ?

Interpretation

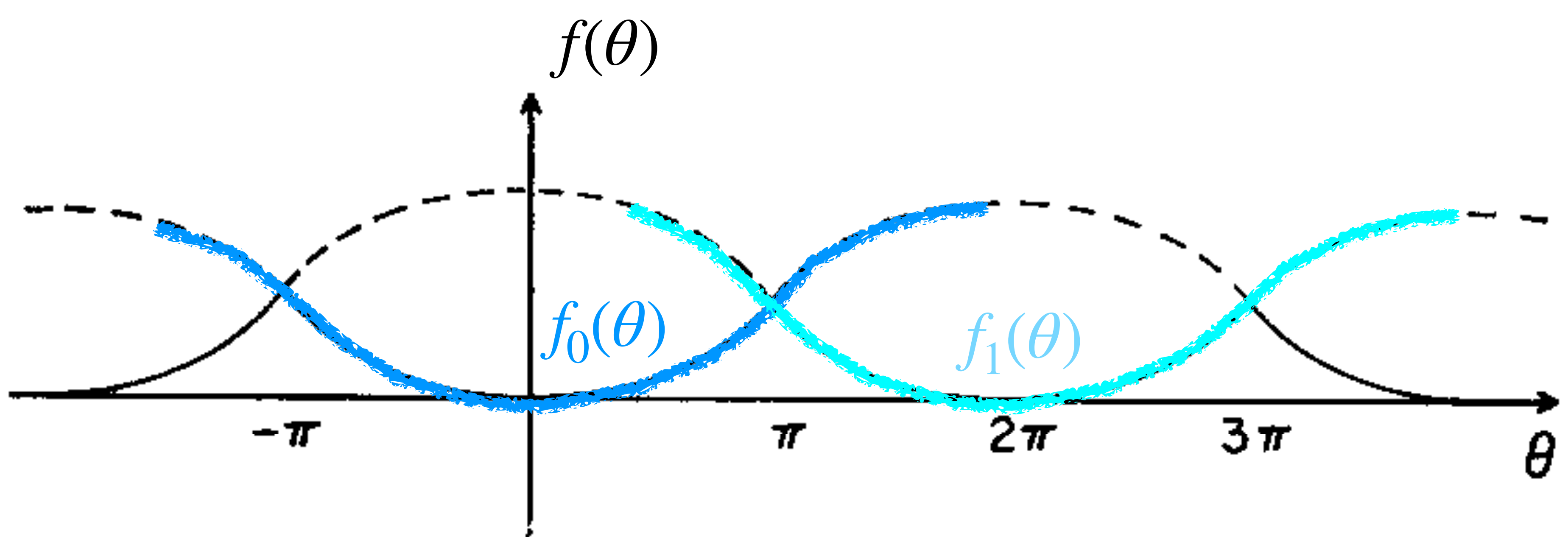
- Sub-volume method seems to trace an original branch even after the crossing point is passed.

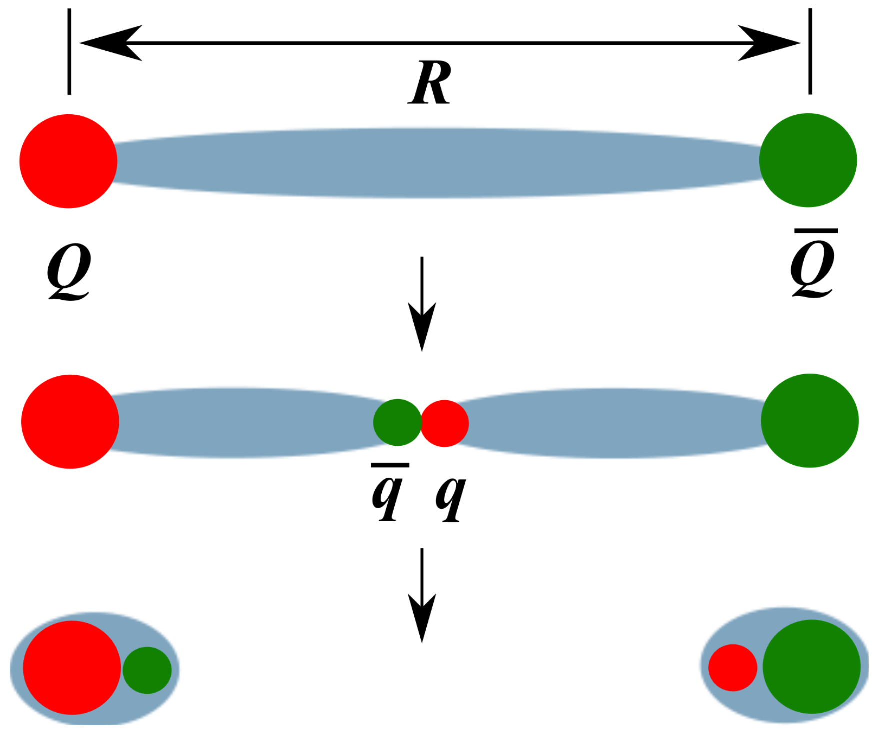


Interpretation

- Sub-volume method seems to trace an original branch even after the crossing point is passed.

- Similar to the calculation of the static potential, where“string breaking” should happen but never occurs.

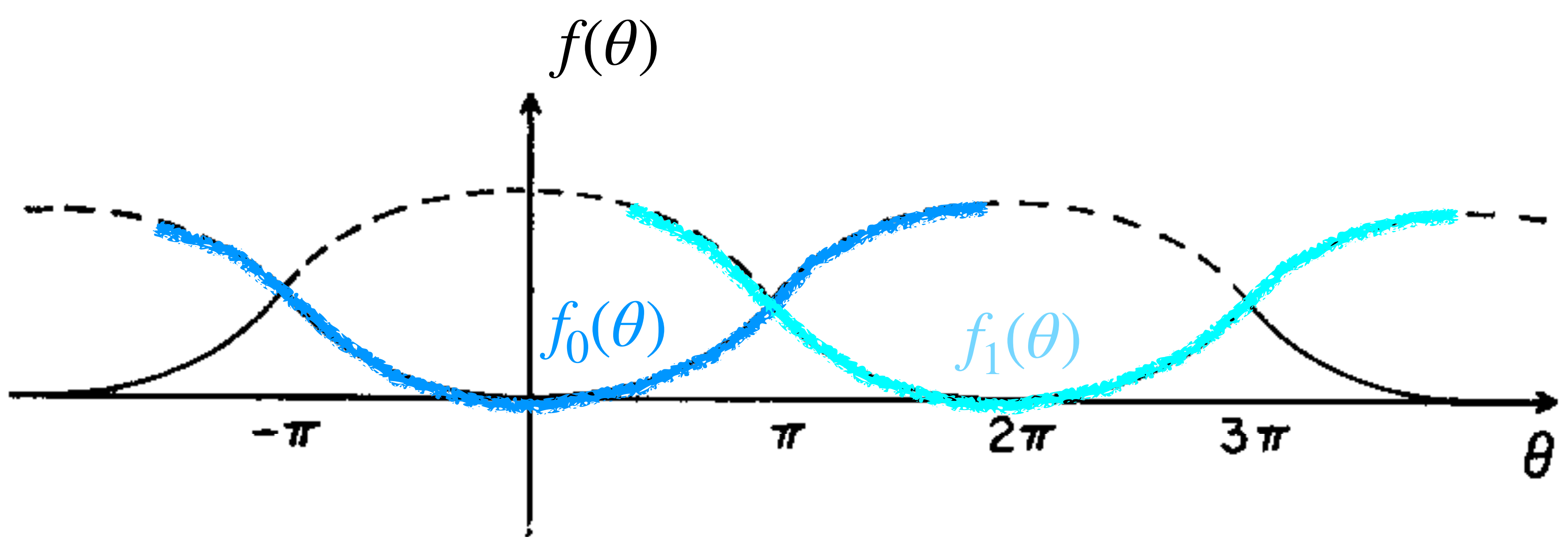




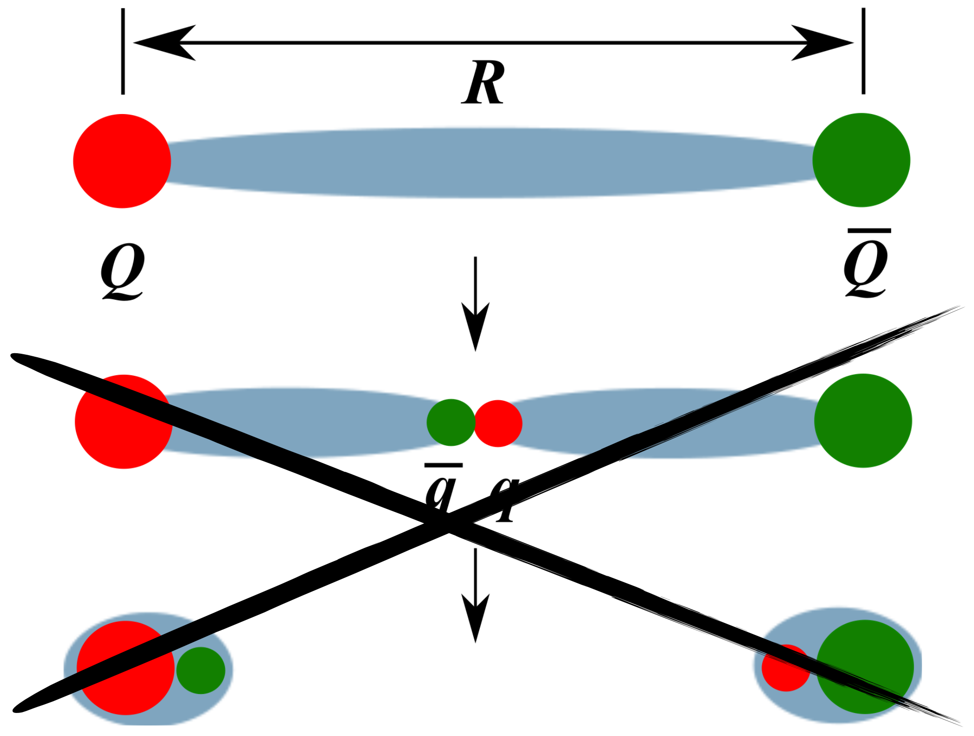
Chernodub (2010)

Interpretation

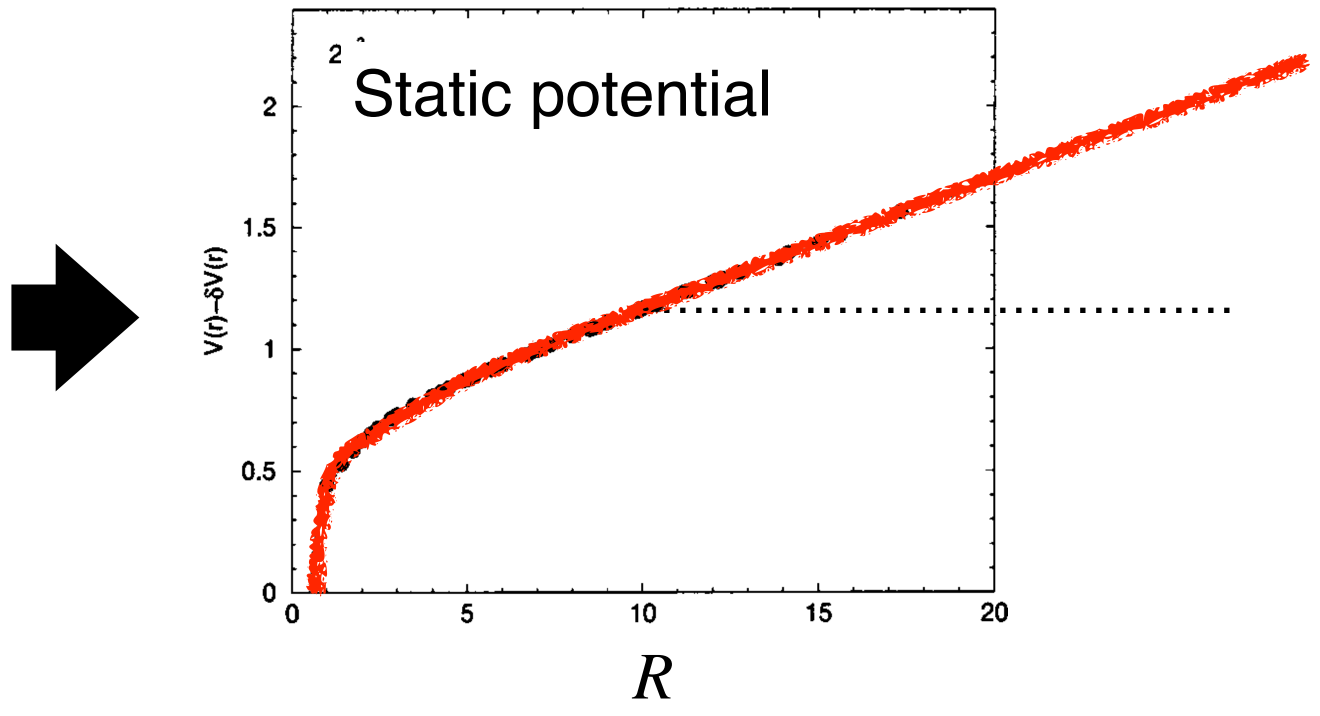
- Sub-volume method seems to trace an original branch even after the crossing point is passed.



- Similar to the calculation of the static potential, where“string breaking” should happen but never occurs.



Chernodub (2010)

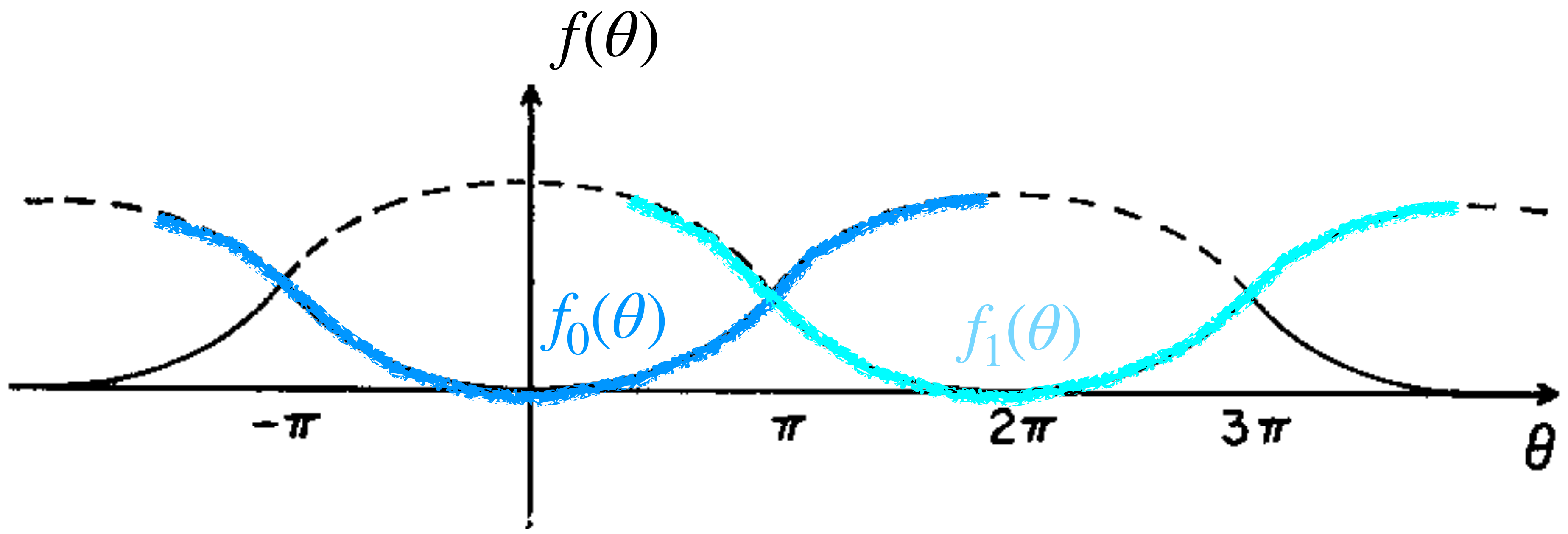


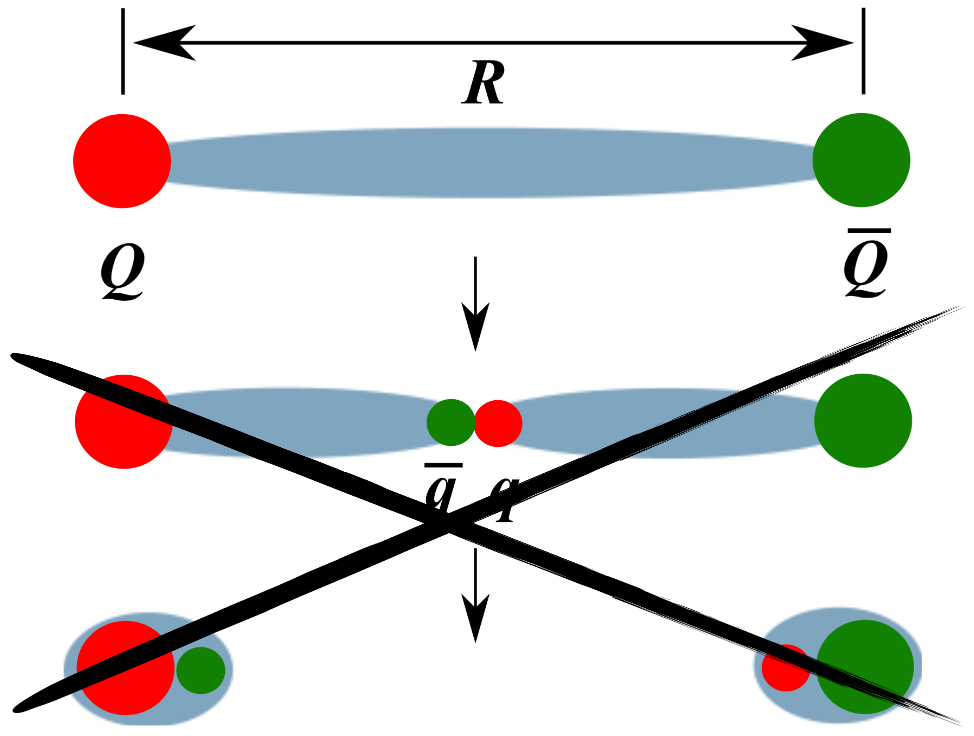
Interpretation

- Sub-volume method seems to trace an original branch even after the crossing point is passed.

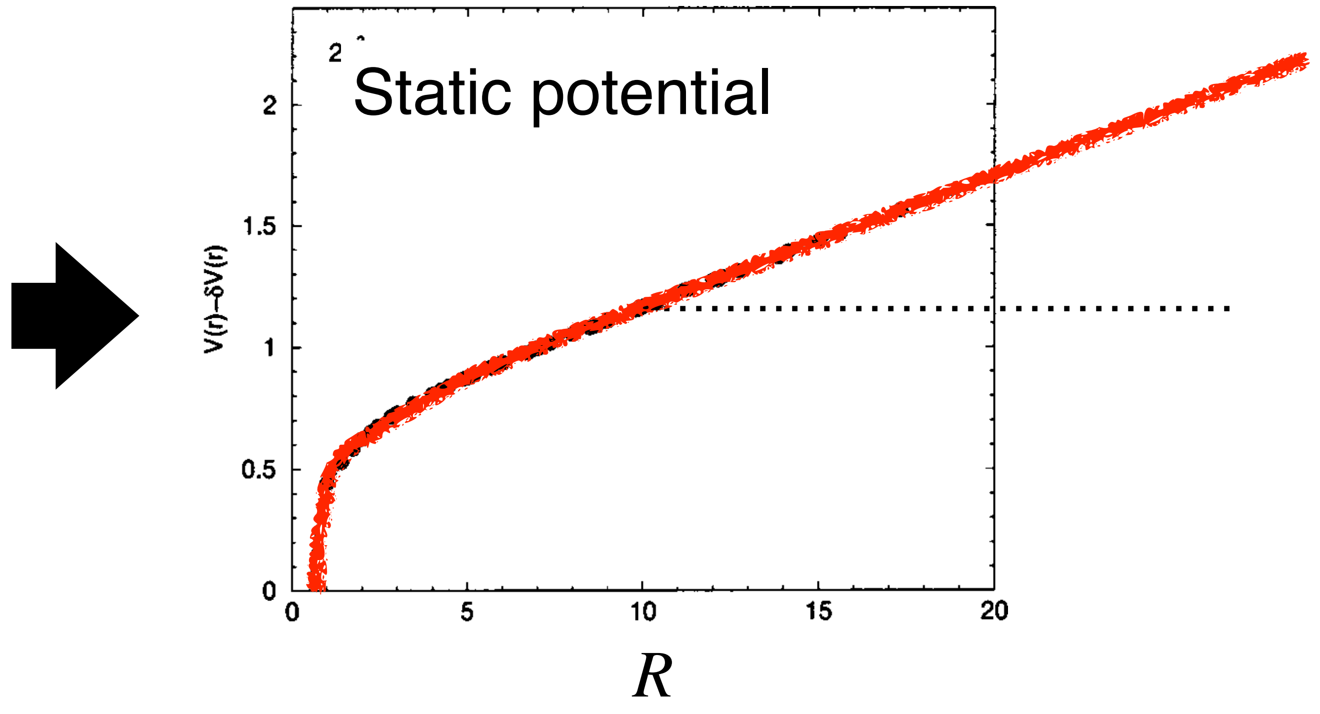
- Similar to the calculation of the static potential, where“string breaking” should happen but never occurs.

- In the present case, the domain to domain-wall transition should occur but does not in this method.

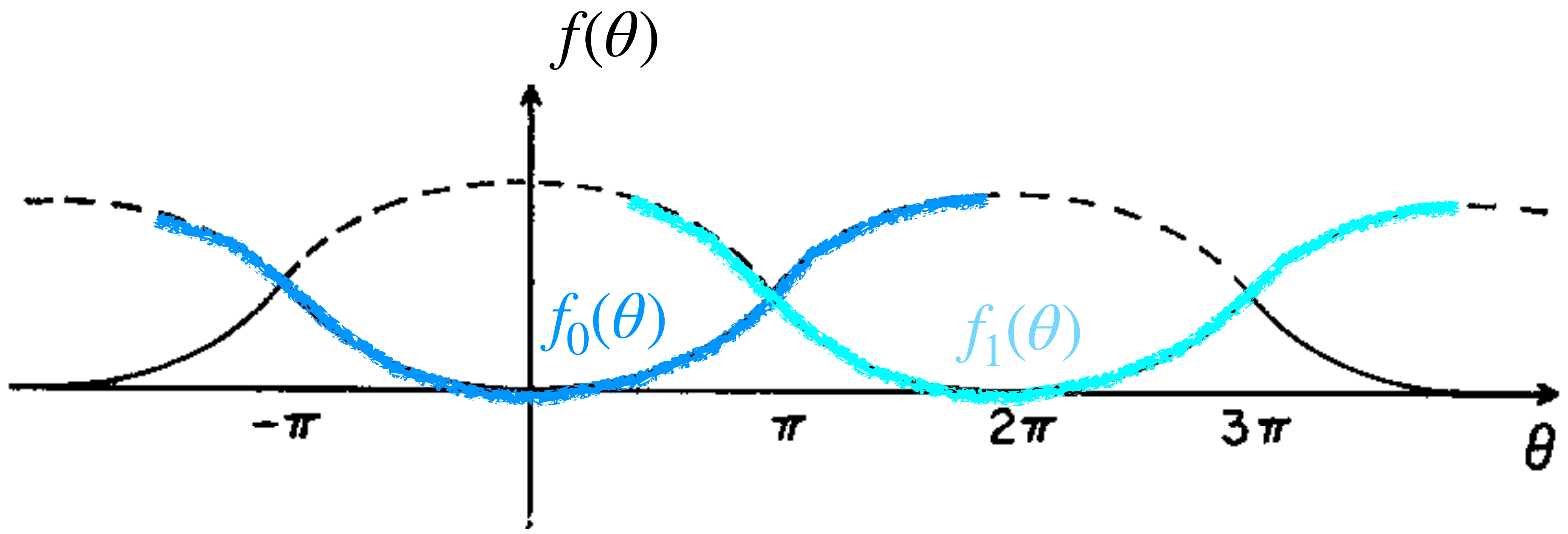




Chernodub (2010)

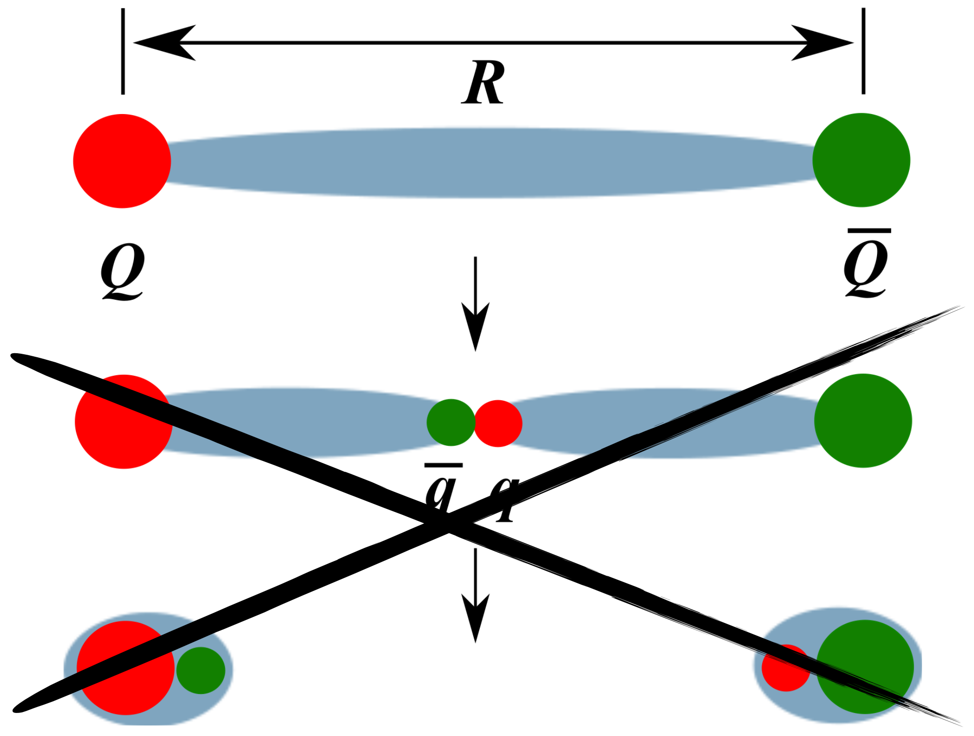


Interpretation

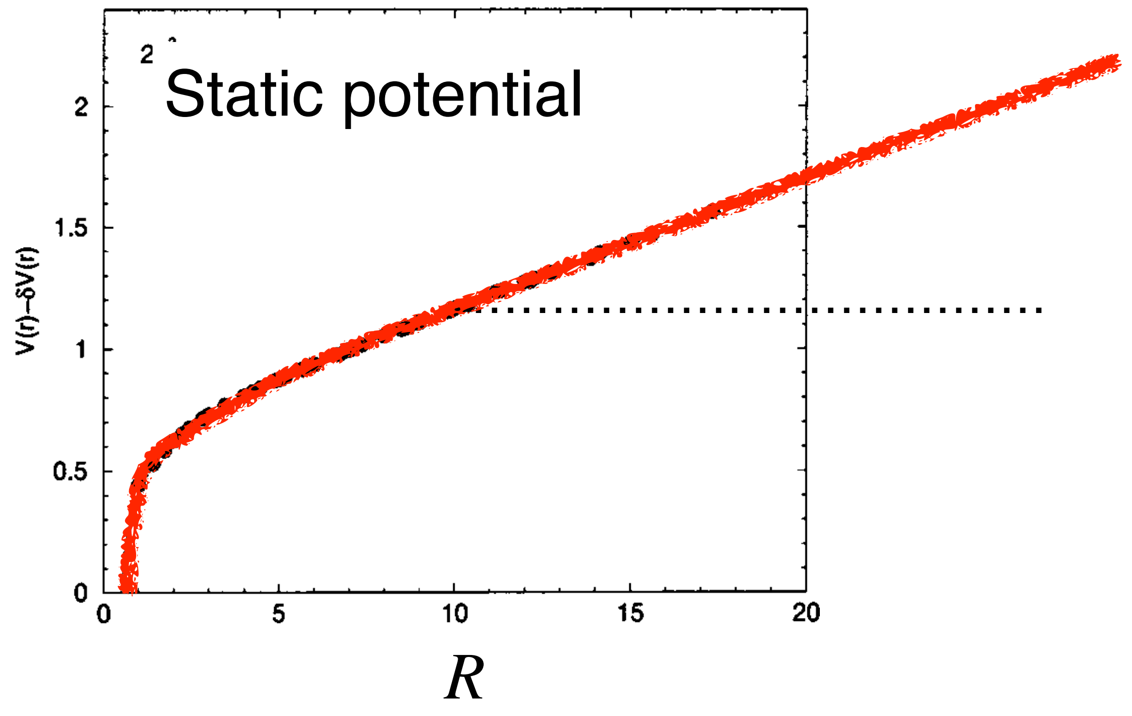


- Sub-volume method seems to trace an original branch even after the crossing point is passed.

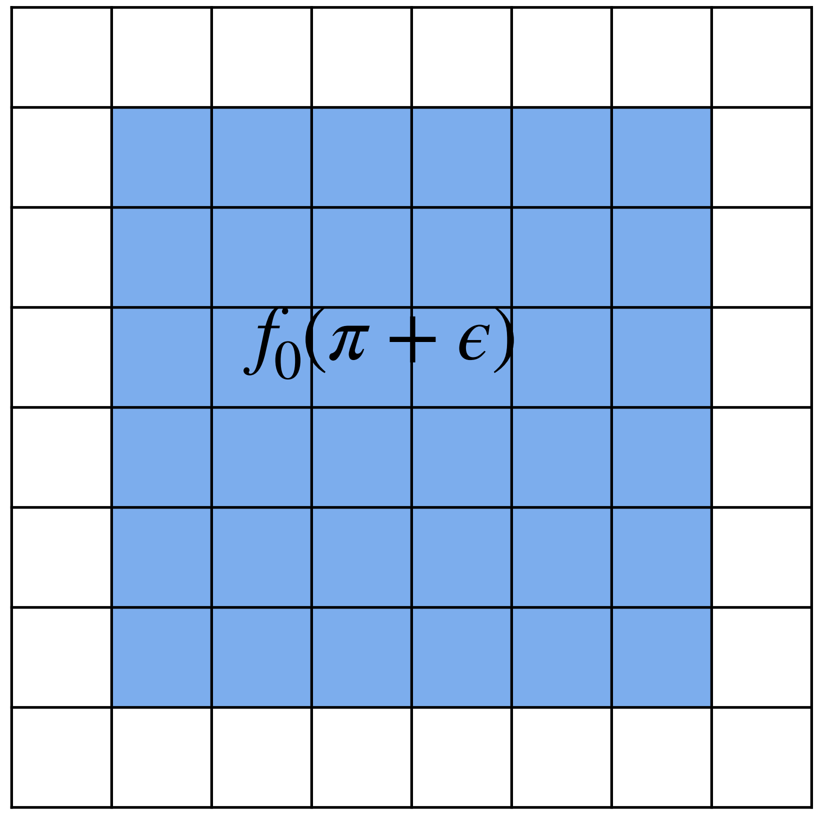
- Similar to the calculation of the static potential, where"string breaking" should happen but never occurs.

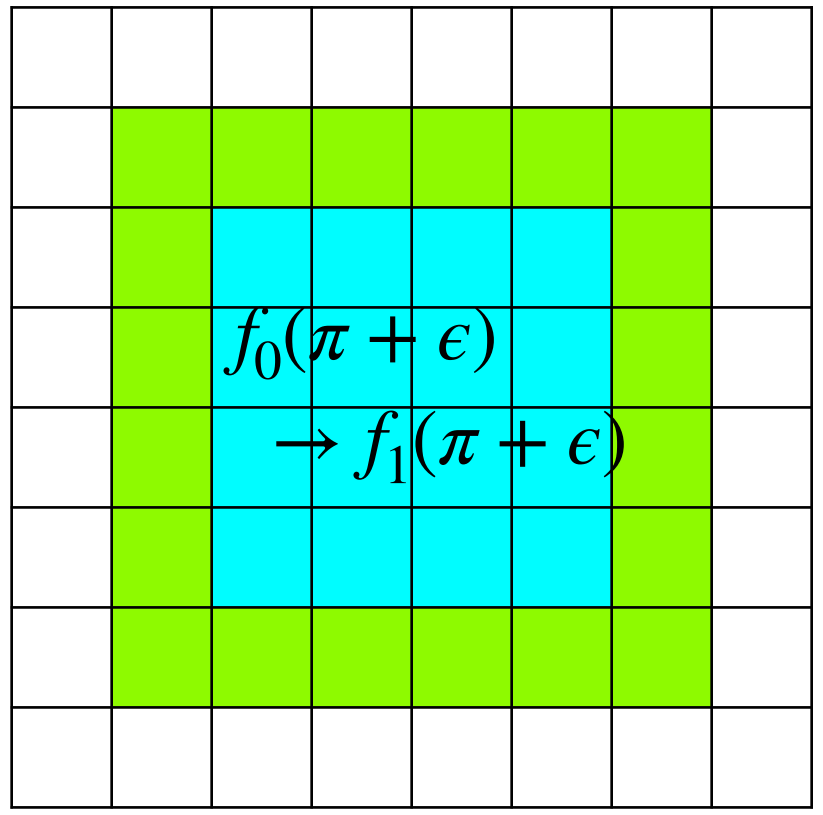


Chernodub (2010)



- In the present case, the domain to domain-wall transition should occur but does not in this method.

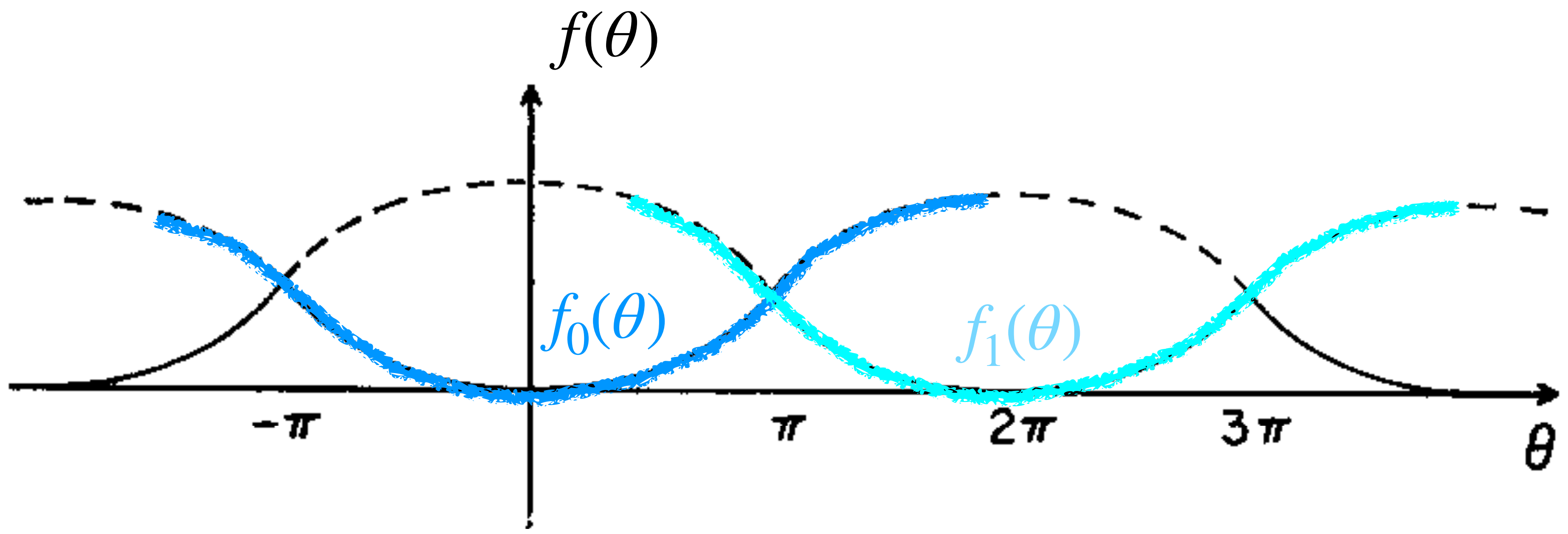




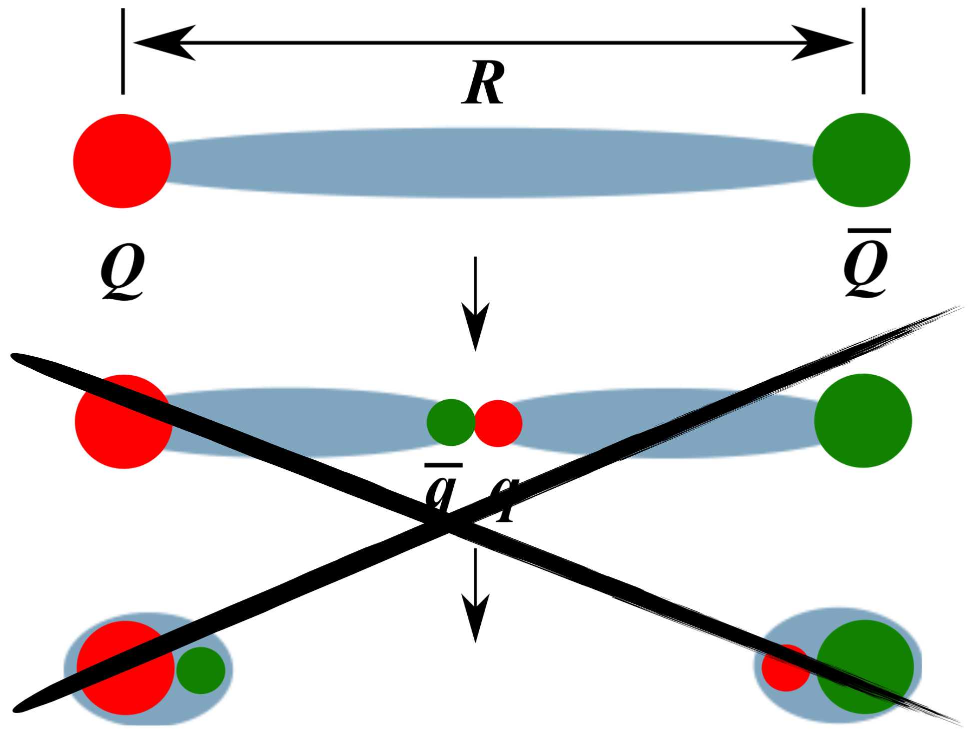
4d SU(N) YM has an topological object called a bag or a domain-wall [Luscher (1978)].

Interpretation

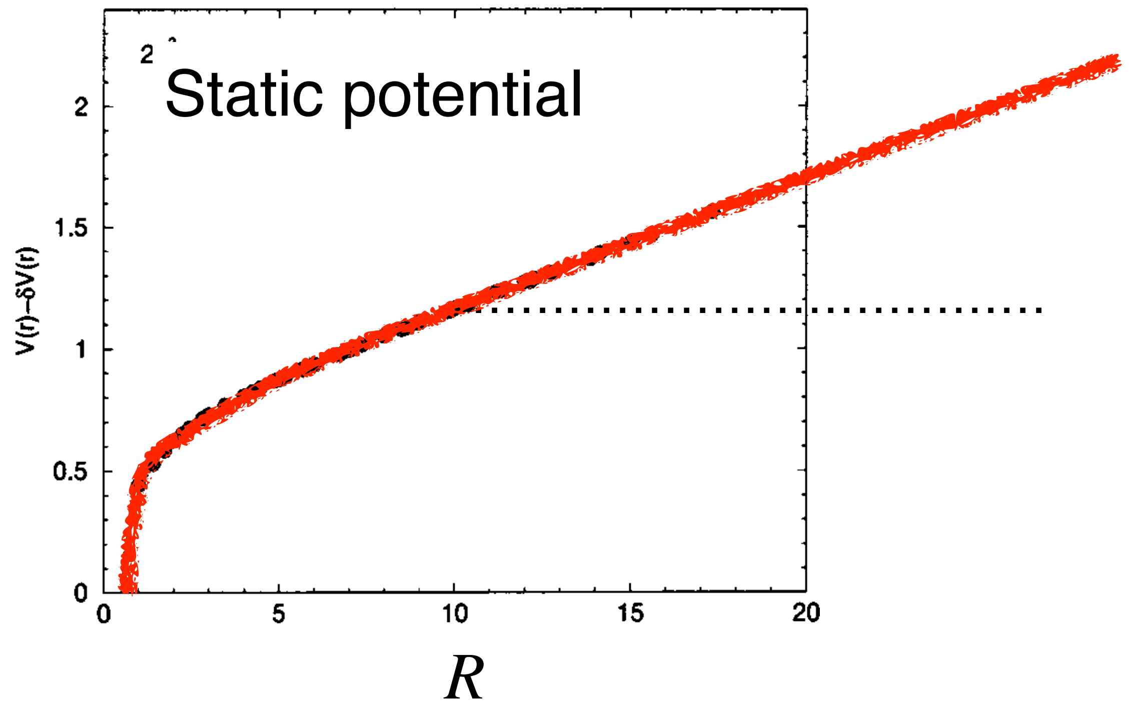
- Sub-volume method seems to trace an original branch even after the crossing point is passed.



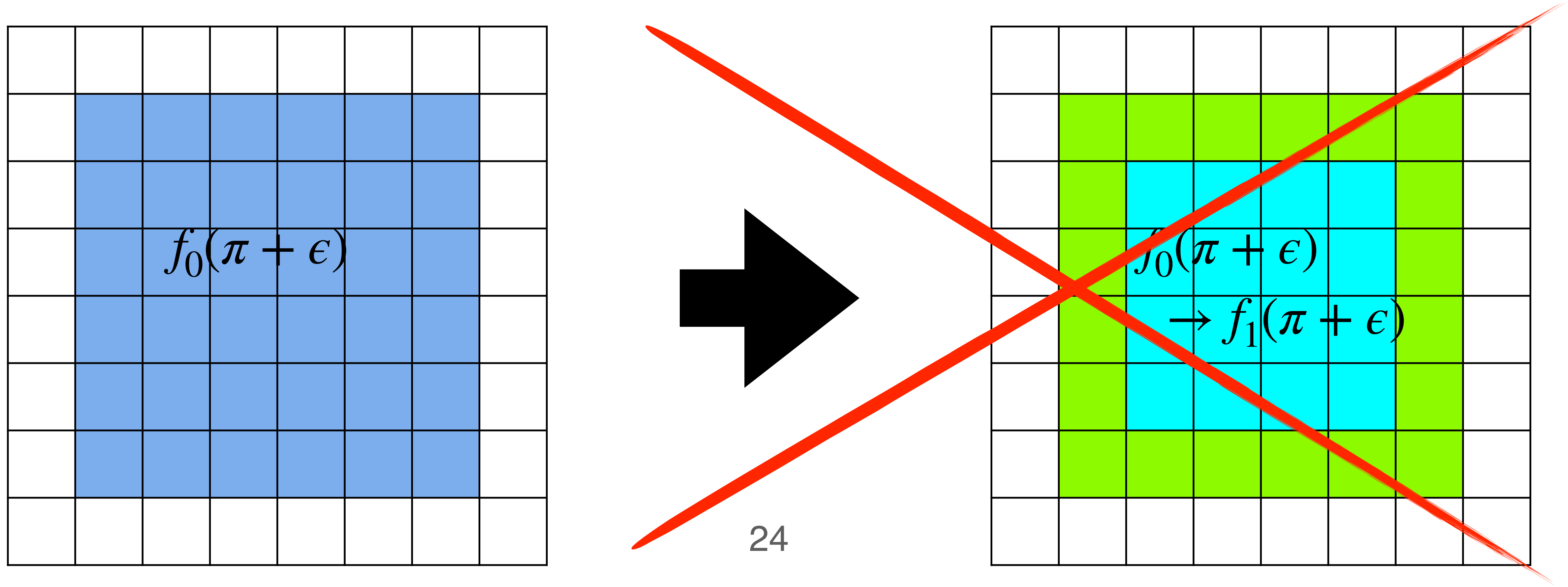
- Similar to the calculation of the static potential, where“string breaking” should happen but never occurs.



Chernodub (2010)



- In the present case, the domain to domain-wall transition should occur but does not in this method.



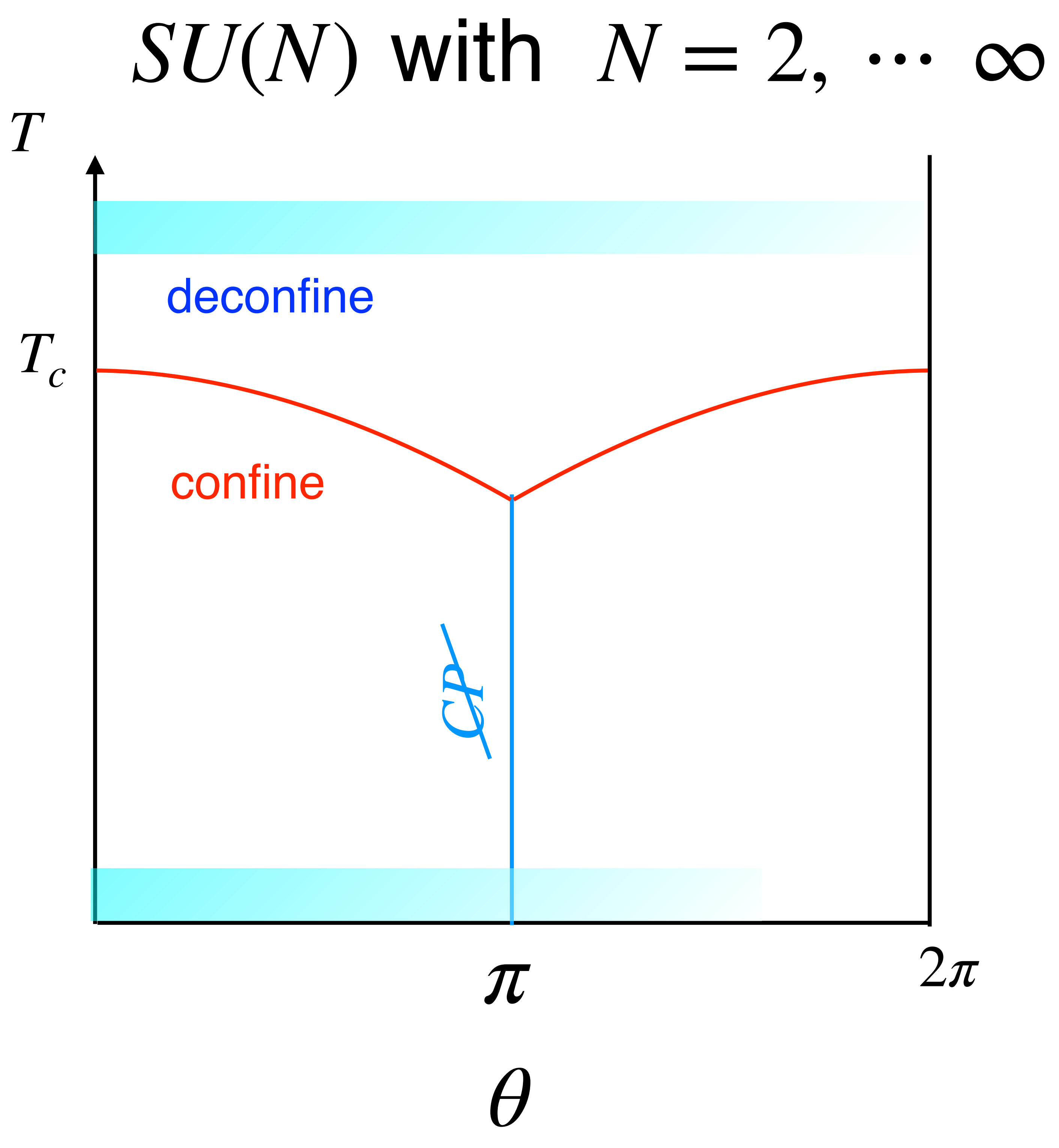
4d SU(N) YM has an topological object called a bag or a domain-wall [Luscher (1978)].

Summary and conclusion

- We have developed a sub-volume method, which enables us to calculate $f(\theta)$ up to $\theta \sim 3\pi/2$ in SU(2) Yang-Mills theory.
- Combining with the theory requirement $f(\pi - \theta) = f(\pi + \theta)$, our result provides with the evidence for spontaneous CPV at $\theta = \pi$ and at $T = 0$ and the existence of a bag-like object.
  ⇒ N=2 belongs to large N class (not like CP^1 model).
- The same method roughly reproduces the DIGA result, $f(\theta) \sim \chi(1 - \cos\theta)$, above T_c , which makes the above result more confident.

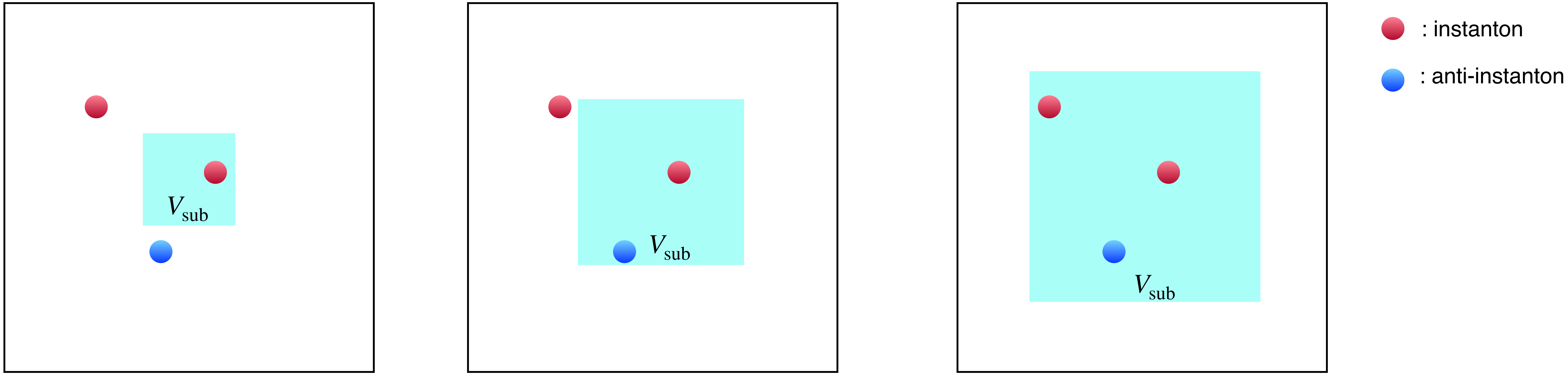
Future studies

- exploring the location of $T_c(\theta)$
- Suppose that $T_c(\theta)$ depends on θ. Then, does it mean that the β-function depend on θ ?
- Also interesting to apply the sub-volume method to the finite density system.



Intuitive understanding of periodic behavior of $f(\theta)$

$$f(\theta) = -\lim_{V_{\rm sub}\to\infty} \frac{1}{V_{\rm sub}} \ln\langle e^{-i\theta Q_{\rm sub}}\rangle = -\lim_{V_{\rm sub}\to\infty} \frac{1}{V_{\rm sub}} \ln\langle\, cos(\theta Q_{\rm sub})\,\rangle$$



$Q_{\rm sub} = +1$ $\qquad$ $Q_{\rm sub} = 0$ $\qquad$ $Q_{\rm sub} = +1$

In this case, $Q_{\rm sub}$ is almost always integer if $\rho^4_{\rm instanton} \ll V_{\rm sub}$.

$\Rightarrow f(\theta)\Big|_{\theta\approx 2\pi} \sim 0 \ \Rightarrow\ 2\pi$-periodicity can be expected.

θ-vacuum

- The vacuum can have an integer winding number, labeled by $|n\rangle$.
- But, this label is changed by gauge transformation, e.g. $U_{(1)}|n\rangle \to |n+1\rangle$.
- Define $|\theta\rangle = \sum_{n-\infty}^{+\infty} e^{in\theta}|n\rangle \iff U_{(1)}|\theta\rangle = e^{-i\theta}|\theta\rangle$
- $\langle\theta_+|\theta_-\rangle_J = \sum_{m,n} e^{in\theta}e^{-im\theta}\langle m_+|n_-\rangle_J = \sum_Q e^{i\theta Q}\sum_m \langle m_+|m_- + Q\rangle_J$

$$= \sum_Q \int_{\in Q} \mathscr{D}A\ e^{-S_g+i\theta Q+\int J\cdot A}\delta\left(Q - \frac{g^2}{32\pi^2}\int d^4x G\tilde{G}\right)$$

$$= \int \mathscr{D}A\ e^{-S_g+i\theta Q+\int J\cdot A}$$