Non-Abelian T-/U-duality

京都府立医科大学

酒谷 雄峰

Based on

YS, 1903.12175
YS, 1911.06320
Malek, YS, Thompson, 2007.08510
YS, 2009.04454
Musaev, YS, 2012.13263
Fernandez-Melgarejo, YS, 2104.00007

京大素粒子論研究室セミナー

話の流れ

通常の **T-duality**

Double Field Theory

Poisson-Lie T-duality

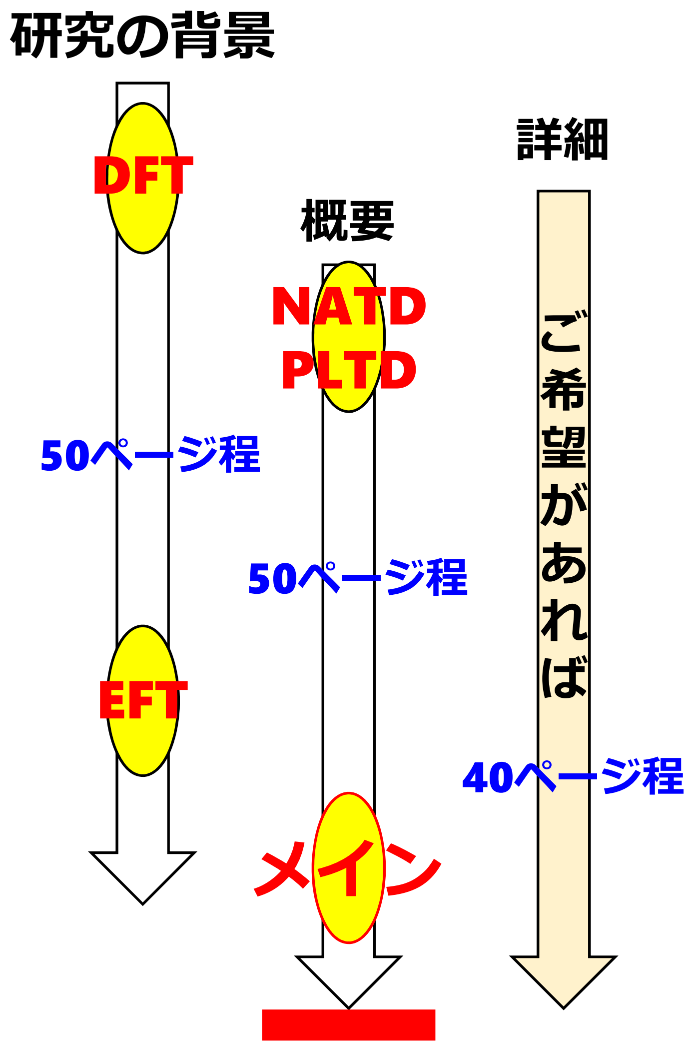
(Drinfel'd double)

通常の **U-duality**

Exceptional Field Theory

Nambu-Lie U-duality

(Exceptional Drinfel'd Algebra)



一周目

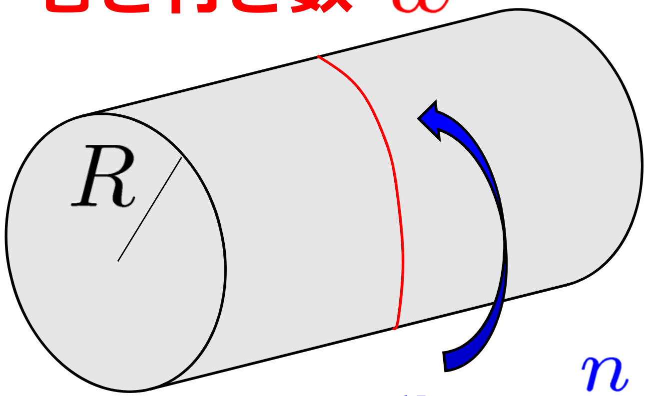
研究の背景

よく知られた T-duality

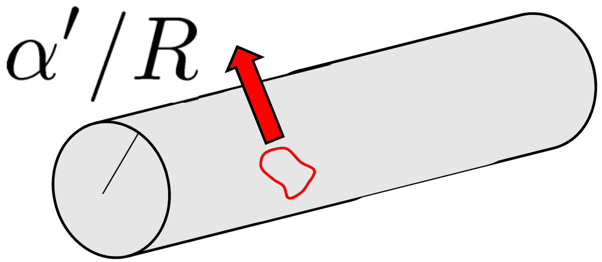
[Kikkawa,Yamasaki '84; ...]

S^{1}コンパクト化された時空上の
closed string 理論

巻き付き数 w



運動量 $\frac{n}{R}$



2乗質量

Left/right の数演算子

$$m^2 = n^2 \frac{1}{R^2} + w^2 \frac{R^2}{\alpha'^2} + \frac{2\left(N + \tilde{N} - 2\right)}{\alpha'}$$

対称性

$$n \leftrightarrow w\,, \qquad R \leftrightarrow \frac{\alpha'}{R}\,.$$

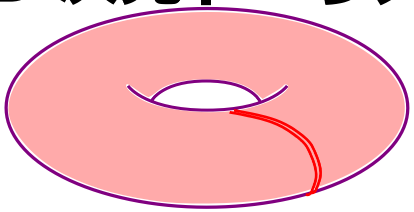
運動量　　巻き付き数

よく知られた T-duality

[Kikkawa,Yamasaki ‘84; …]

T^Dコンパクト化された時空上の
closed string 理論

***D* 次元トーラス**



$$x^m(\tau, \sigma + 2\pi) = x^m(\tau, \sigma) + 2\pi\sqrt{\alpha'}\, w^m\,,$$

$$g_{mn} = \delta_{mn}\,\frac{R_m^2}{\alpha'}\,, \qquad \underbrace{B_{mn}}_{\text{定数}}\,, \qquad p_m = \frac{n_m}{\sqrt{\alpha'}}\,.$$

n_m → p_m と書く

$$p_M = \begin{pmatrix} p_m \\ w^m \end{pmatrix}\,, \qquad \mathcal{H}^{MN} = \begin{pmatrix} g^{mn} & -(g^{-1}B)^m{}_n \\ (B\,g^{-1})_m{}^n & g_{mn} - (B\,g^{-1}B)_{mn} \end{pmatrix}$$

一般化運動量　　　一般化計量 (の逆行列)

$$({}^M) = ({}^m,\ {}_m)$$
$$({}_M) = ({}_m,\ {}^m)$$

$$m^2 = \frac{1}{\alpha'}\,p_M\,\mathcal{H}^{MN}\,p_N + \frac{2\,(N + \tilde{N} - 2)}{\alpha'}$$

よく知られた T-duality

$$p_M = \begin{pmatrix} p_m \\ w^m \end{pmatrix}$$

一般化運動量

$\xrightarrow{\text{条件}}$

Level-matching 条件

$$\boxed{N - \tilde{N} = p_m\, w^m} = \frac{1}{2}\, p_M\, \eta^{MN}\, p_N$$

$$\eta^{MN} \equiv \begin{pmatrix} 0 & \delta^m_n \\ \delta^n_m & 0 \end{pmatrix}$$

逆行列

$$\eta_{MN} \equiv \begin{pmatrix} 0 & \delta^n_m \\ \delta^m_n & 0 \end{pmatrix}$$

O(*D*,*D*) metric

$$\eta_{MN} = \mathrm{diag}(\overbrace{+1, \ldots, +1}^{D\text{個}}, \overbrace{-1, \ldots, -1}^{D\text{個}})$$

$$(C\,\eta\,C^{\mathrm{T}})_{MN} = \eta_{MN}$$

$$C_M{}^N \in \mathrm{O}(D, D)$$

以下, この計量で添え字の上げ下げをします.

$$(p^M = \eta^{MN}\, p_N)$$

O(*D*,*D*)変換

一般化運動量 $p_M = \begin{pmatrix} p_m \\ w^m \end{pmatrix}$ **混ぜ合わせるのが T-duality**

$$p_M \to p'_M = C_M{}^N\, p_N \qquad \left[\, C_M{}^N \in \mathrm{O}(D,D) \,\right]$$

Level-matching 条件は保たれる:

$$p'_M\, \eta^{MN}\, p'_N = p_M\, (C^{\mathrm{T}}\, \eta\, C)^{MN}\, p_N = p_M\, \eta^{MN}\, p_N = N - \tilde{N}\,.$$

2乗質量

$$\boxed{m^2 = \frac{1}{\alpha'}\, p_M\, \mathcal{H}^{MN}\, p_N + \frac{2\,(N + \tilde{N} - 2)}{\alpha'}}$$

同時に, 一般化計量も O(*D*,*D*) 変換すれば mass が不変.

$$\mathcal{H}_{MN} \to \mathcal{H}'_{MN} = C_M{}^P\, C_N{}^Q\, \mathcal{H}_{PQ}$$

$w \leftrightarrow n\,, \qquad R \leftrightarrow \frac{\alpha'}{R}$ **の一般化.**

O(*D*,*D*)変換

O(*D*,*D*)

$$(C\,\eta\,C^{\mathrm{T}})_{MN} = \eta_{MN}$$

$$\mathcal{H}_{MN} \to \mathcal{H}'_{MN} \equiv C_M{}^P\, C_N{}^Q\, \mathcal{H}_{PQ}\,.$$

1. GL(*D*)変換

$$C_M{}^N = \begin{pmatrix} \Lambda_m{}^n & 0 \\ 0 & (\Lambda^{-1})^m{}_n \end{pmatrix} \quad \begin{cases} g_{mn} \to \Lambda_m{}^p\, \Lambda_n{}^q\, g_{pq} \\ B_{mn} \to \Lambda_m{}^p\, \Lambda_n{}^q\, B_{pq} \end{cases}$$

2. B-shift

$$C_M{}^N = \begin{pmatrix} \delta_m^n & \omega_{mn} \\ 0 & \delta_n^m \end{pmatrix} \quad \begin{cases} g_{mn} \to g_{mn} \\ B_{mn} \to B_{mn} + \omega_{mn} \end{cases}$$

(factorized)
3. T-duality

$$C_M{}^N = \begin{pmatrix} \mathbf{1} - e_i & e_i \\ e_i & \mathbf{1} - e_i \end{pmatrix}$$

***i*-*i* 成分のみ 1**

$$R \leftrightarrow \frac{\alpha'}{R}$$

$$\begin{aligned} g'_{ab} &= g_{ab} - \frac{g_{ai}\, g_{bi} - B_{ai}\, B_{bi}}{g_{ii}}\,, \quad g'_{ai} = \frac{B_{ai}}{g_{ii}}\,, \quad g'_{ii} = \frac{1}{g_{ii}}\,, \\ B'_{ab} &= B_{ab} - \frac{B_{ad}\, g_{bi} - g_{ai}\, B_{bi}}{g_{ii}}\,, \quad B'_{ai} = \frac{g_{ai}}{g_{ii}}\,. \end{aligned}$$

Buscher 則

O(*D*,*D*)変換

$$\mathcal{H}_{MN} \to \mathcal{H}'_{MN} \equiv C_M{}^P\, C_N{}^Q\, \mathcal{H}_{PQ}\,.$$

β変換
(TsT変換)

$$C_M{}^N = \begin{pmatrix} \delta^n_m & 0 \\ \beta^{mn} & \delta^m_n \end{pmatrix} \qquad \left(E^{mn} \equiv [(g+B)^{-1}]^{mn} \right)$$

$$E^{mn} \to E^{mn} + \beta^{mn}$$

GL(*D*)

$$K^p{}_q \equiv \begin{pmatrix} \delta^p_m\, \delta^n_d & 0 \\ 0 & -\delta^m_q\, \delta^p_n \end{pmatrix},$$

B-shift

$$R^{p_1p_2} \equiv \begin{pmatrix} 0 & 2\,\delta^{p_1p_2}_{mn} \\ 0 & 0 \end{pmatrix},$$

β変換

$$R_{p_1p_2} \equiv \begin{pmatrix} 0 & 0 \\ 2\,\delta^{mn}_{p_1p_2} & 0 \end{pmatrix}.$$

O(*D*,*D*)代数

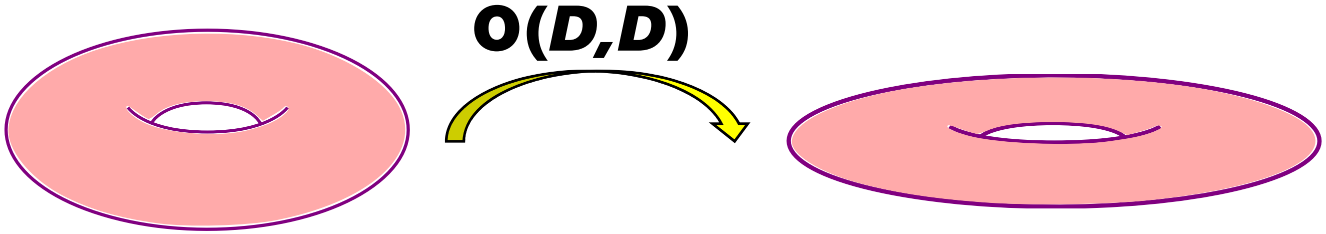
$$[K^m{}_n,\, K^p{}_q] = \delta^p_n\, K^m{}_q - \delta^m_q\, K^p{}_n\,,$$

$$[K^m{}_n,\, R^{pq}] = 2\,\delta^{pq}_{nr}\, R^{mr}\,, \qquad [K^m{}_n,\, R_{pq}] = -2\,\delta^{mr}_{pq}\, R_{nr}\,,$$

$$[R^{mn},\, R_{pq}] = -4\,\delta^{[m}_{[p}\, K^{n]}{}_{q]}\,, \qquad [R^{mn},\, R^{pq}] = [R_{mn},\, R_{pq}] = 0\,.$$

O(D,D) T-duality

質量スペクトルの不変性だけではない.



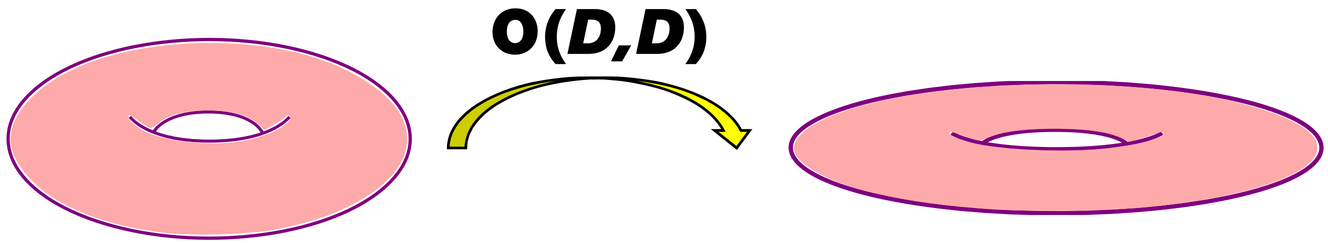
$$\mathcal{H}_{MN}$$

$$\mathcal{H}'_{MN} \equiv C_M{}^P \, C_N{}^Q \, \mathcal{H}_{PQ} \, .$$

これらのトーラス上で定義された string 理論が
互いに等価であることが知られている (e.g. 分配関数)

O(D,D) T-duality

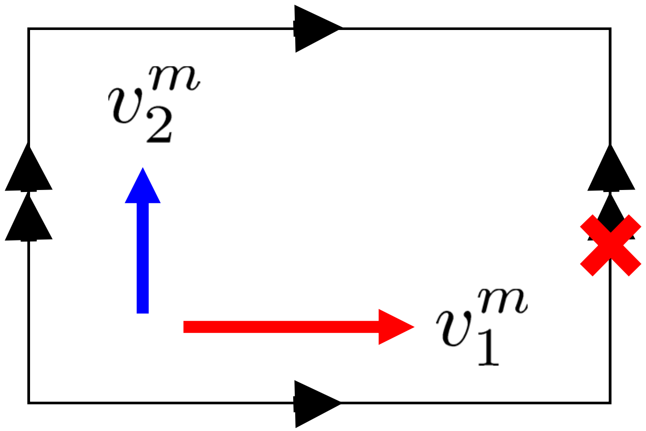
さらに, O(D,D)は SUGRA の運動方程式の対称性にもなっている.



$$\mathcal{H}_{MN} \qquad \mathcal{H}'_{MN} \equiv C_M{}^P\, C_N{}^Q\, \mathcal{H}_{PQ}\,.$$

SUGRA の解 ⇒ **SUGRA の解**

少なくとも, SUGRAのレベルでは
トーラスでコンパクト化しなくても
時空に並進対称性さえあれば O(D,D) 対称性はある.



T-duality で重要なのは
並進対称性の存在.

$$\left\{\begin{array}{l} \pounds_{v_a} g_{mn} = 0 \\ [v_a, v_b] = 0 \end{array}\right.$$

D 個の互いに交換するKillingベクトル場.

⇒ Abelian T-duality と呼ばれる.

並進対称性を持つ時空の例:

(勝手に作った Ricci flat 時空)

$$\left(R_{mn} = 0\right)$$

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + t^{\frac{4}{3}}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) + t^{-\frac{2}{3}}\,\mathrm{d}z^2$$

z方向にT-dual ⇓

x, y, z 方向に並進対称性

$$\begin{cases} \mathrm{d}s^2 = -\mathrm{d}t^2 + t^{\frac{4}{3}}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) + t^{+\frac{2}{3}}\,\mathrm{d}z^2\,, \\ \mathrm{e}^{-2\Phi} = t^{-\frac{2}{3}}\,. \end{cases}$$

Ricci曲率がゼロでない新たな SUGRA 解

$$R = \frac{4}{9t^2}\,.$$

T-duality はSUGRA解の生成手法として使える.

並進対称性を持つ時空の例:

(勝手に作った Ricci flat 時空) $\big(R_{mn}=0\big)$

$$\mathrm{d}s^2=-\mathrm{d}t^2+t^{\frac{4}{3}}\,(\mathrm{d}x^2+\mathrm{d}y^2)+t^{-\frac{2}{3}}\,\mathrm{d}z^2$$

z方向にT-dual ⇓

x, y, z 方向に並進対称性

$$\begin{cases}\mathrm{d}s^2=-\mathrm{d}t^2+t^{\frac{4}{3}}\,(\mathrm{d}x^2+\mathrm{d}y^2)+t^{+\frac{2}{3}}\,\mathrm{d}z^2\,,\\ \mathrm{e}^{-2\Phi}=t^{-\frac{2}{3}}\,.\end{cases}$$

⇓

x,y方向にもT-dualをとれば
また新たなSUGRA解が得られる.

[O(3,3) T-duality]

GL(*D*) **B-shift** **β変換**

$K^p{}_q, \qquad R^{pq}, \qquad R_{pq}$

ここまで

Abelian O(*D*,*D*) T-duality の復習

⇩

O(*D*,*D*)対称性を明白にするSUGRAの定式化

Double Field Theory のレビュー

(DFT) [Siegel '93; Hull, Zwiebach '09]

通常のSUGRA:　O(*D*,*D*)対称性は見えない

$$\mathcal{L} = \mathrm{e}^{2\Phi}\left[R + 4\,\nabla_m\Phi\,\nabla^m\Phi - \frac{1}{12}\,H_{mnp}\,H^{mnp}\right]$$

特別なO(*D*,*D*)

$$g'_{ab} = g_{ab} - \frac{g_{ai}\,g_{bi} - B_{ai}\,B_{bi}}{g_{ii}}\,, \quad g'_{ai} = \frac{B_{ai}}{g_{ii}}\,, \quad g'_{ii} = \frac{1}{g_{ii}}\,,$$

$$B'_{ab} = B_{ab} - \frac{B_{ad}\,g_{bi} - g_{ai}\,B_{bi}}{g_{ii}}\,, \quad B'_{ai} = \frac{g_{ai}}{g_{ii}}\,, \quad \mathrm{e}^{2\Phi'} = \frac{\mathrm{e}^{2\Phi'}}{g_{dd}}\,.$$

$$\mathcal{L}' = \mathrm{e}^{2\Phi'}\left[R' + 4\,\nabla'_m\Phi'\,\nabla'^m\Phi' - \frac{1}{12}\,H'_{mnp}\,H'^{mnp}\right]$$

***D*次元の並進対称性を仮定しない限り，**
SUGRA action は O(*D*,*D*)不変でない．

一方, DFT ではO(*D*,*D*)不変なactionを作る．

Double Field Theory

[Siegel '93;
Hull, Zwiebach '09]

通常, *D*次元のSUGRA

$$\mathcal{L} = \mathrm{e}^{2\Phi}\left[R + 4\,\nabla_m\Phi\,\nabla^m\Phi - \frac{1}{12}\,H_{mnp}\,H^{mnp}\right]$$

を記述するため*D*次元時空の座標 x^m を導入.

<u>String理論</u>

$$\begin{tikzcd} x^m \arrow[r, leftrightarrow, "\text{T-dual}"] \arrow[d, leftrightarrow, "\text{共役}"'] & \tilde{x}_m \arrow[d, leftrightarrow, "\text{共役}"'] \\ p_m \arrow[r, leftrightarrow, "\text{T-dual}"'] & w^m \end{tikzcd}$$

dual座標

**T-dual不変
ではない**

Double Field Theory

[Siegel '93;
Hull, Zwiebach '09]

通常, *D*次元のSUGRA

$$\mathcal{L} = \mathrm{e}^{2\Phi}\left[R + 4\,\nabla_m\Phi\,\nabla^m\Phi - \frac{1}{12}\,H_{mnp}\,H^{mnp}\right]$$

を記述するため*D*次元時空の座標 x^m を導入.

---

*2D*次元

一般化座標
(ダブル座標)

$$x^M = (x^m,\ \tilde{x}_m)$$

2*D*次元のダブル空間上の重力理論 = DFT

Double Field Theory

[Siegel '93; Hull, Zwiebach '09]

一般化計量 = ダブル空間上の計量

$\{g_{mn},\ B_{mn}\}$ ⇨ $\mathcal{H}_{MN} = \begin{pmatrix} (g - B\, g^{-1}\, B)_{mn} & B_{mp}\, g^{pn} \\ -g^{mp}\, B_{pn} & g^{mn} \end{pmatrix} \in \mathrm{O}(D, D)$

Φ ⇨ $\mathrm{e}^{-2d} = \mathrm{e}^{-2\Phi}\sqrt{|g|}$ T-duality 不変

R-R場 ⇨ O(*D*,*D*) spinor [Fukuma, Oota, Tanaka, '00]

ダブル空間上の"一般座標変換不変"な action

$$\mathcal{L}_{\mathrm{DFT}} = \mathrm{e}^{-2d}\,\mathcal{R} + \mathcal{L}_{\mathrm{RR}}\,.$$

一般化Ricci scalar

Double Field Theory

$$e^{-2d}\mathcal{R} \equiv e^{-2d}\Big(\mathcal{H}^{MN}\,\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} - 4\mathcal{H}^{MN}\,\partial_M d\,\partial_N d + 4\partial_M\mathcal{H}^{MN}\,\partial_N d$$
$$+ \frac{1}{8}\mathcal{H}^{MN}\,\partial_M\mathcal{H}^{PQ}\,\partial_N\mathcal{H}_{PQ} - \frac{1}{2}\mathcal{H}^{MN}\,\partial_M\mathcal{H}^{PQ}\,\partial_P\mathcal{H}_{NQ}\Big)\,.$$

[Hull, Hohm, Zwiebach '10]

(トーラスコンパクト化する前から)

O(*D*,*D*) 変換の下で不変

$$x^M \to (C^{-1})_N{}^M\,x^N$$

$$\Updownarrow$$

$$\mathcal{H}_{MN} \to C_M{}^P\,C_N{}^Q\,\mathcal{H}_{PQ}\,, \quad d \to d\,, \quad \partial_M \to C_M{}^N\,\partial_N\,.$$

$$\eta^{MN}\,\partial_M \otimes \partial_N = 0$$

ただし section condition を課す必要がある。

(理論の整合性より)

座標依存性の半分を消す!

Double Field Theory

$$\mathrm{e}^{-2d}\,\mathcal{R} \equiv \mathrm{e}^{-2d}\Big(\mathcal{H}^{MN}\,\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} - 4\mathcal{H}^{MN}\,\partial_M d\,\partial_N d + 4\partial_M\mathcal{H}^{MN}\,\partial_N d + \frac{1}{8}\,\mathcal{H}^{MN}\,\partial_M\mathcal{H}^{PQ}\,\partial_N\mathcal{H}_{PQ} - \frac{1}{2}\,\mathcal{H}^{MN}\,\partial_M\mathcal{H}^{PQ}\,\partial_P\mathcal{H}_{NQ}\Big)\,.$$

$$\downarrow \quad \partial_M = (\boxed{\partial_m}, \cancel{\tilde{\partial}^m})$$

$$\mathcal{L} = \mathrm{e}^{2\Phi}\left[R + 4\,\nabla_m\Phi\,\nabla^m\Phi - \frac{1}{12}\,H_{mnp}\,H^{mnp}\right] + \partial_m(\cdots)^m$$

通常のSUGRA = DFT で $\partial_M = (\partial_m, \cancel{\tilde{\partial}^m})$ としたもの.

$$\partial_M = (\boxed{\partial}\cancel{_m}, \cancel{\tilde{\partial}}\boxed{^m})$$

⇨ **dilaton が dual 座標に依存** **[坂本, YS, 吉田 '17]**

Generalized SUGRA

⇨ **R-R 1-form が dual 座標に依存**

massive type IIA SUGRA **[Hohm, Kwak '11]**

補足: Generalized SUGRA

(RR場無視)

[Arutyunov-Frolov-Hoare-Roiban-Tseytlin, '15;
Tseytlin, Wulff, '16]

E.O.M.

$$U_m \equiv I^n B_{nm}$$

$$g_{mn} \Rightarrow R_{mn} - \frac{1}{4} H_{mpq} H_n{}^{pq} + 2 \nabla_m \partial_n \Phi + \underline{\nabla_m U_n + \nabla_n U_m} = 0 \,,$$

$$\Phi \Rightarrow R + 4 \nabla^m \partial_m \Phi - 4 |\partial \Phi|^2 - \frac{1}{2} |H_3|^2 - \underline{4 \left(I^m I_m + U^m U_m + 2 U^m \partial_m \Phi - \nabla_m U^m \right)} = 0 \,,$$

$$B_{mn} \Rightarrow \frac{1}{2} \nabla^k H_{kmn} = \partial_k \Phi H^k{}_{mn} + \underline{U^k H_{kmn} + \nabla_m I_n - \nabla_n I_m} \,.$$

Killingベクトル場
(non-dynamical)

$\Downarrow$

$I^m = 0$ とするとSUGRAに帰着

うまい座標系で

$$I^m \partial_m = I^z \partial_z$$

定数

$$\begin{cases} \mathcal{H}_{MN} = \mathcal{H}_{MN}(x^i, \cancel{z}) \\ d = d_*(x^i, \cancel{z}) + I^z \tilde{z} \end{cases}$$

[坂本, YS, 吉田 '17]

DFTの E.O.M. $\ni \partial_M d$

Double Field Theory

$$\mathrm{e}^{-2d}\,\mathcal{R} \equiv \mathrm{e}^{-2d}\Big(\mathcal{H}^{MN}\,\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} - 4\mathcal{H}^{MN}\,\partial_M d\,\partial_N d + 4\partial_M\mathcal{H}^{MN}\,\partial_N d$$
$$+\frac{1}{8}\,\mathcal{H}^{MN}\,\partial_M\mathcal{H}^{PQ}\,\partial_N\mathcal{H}_{PQ} - \frac{1}{2}\,\mathcal{H}^{MN}\,\partial_M\mathcal{H}^{PQ}\,\partial_P\mathcal{H}_{NQ}\Big)\,.$$

O(*D*,*D*)対称性

$$\mathcal{H}_{MN} \to C_M{}^P\,C_N{}^Q\,\mathcal{H}_{PQ}\,,\qquad d \to d\,,\qquad x^M \to (C^{-1})_N{}^M\,x^N\,.$$

3次元空間のダブル $(x,\ y,\ z,\ \tilde{x},\ \tilde{y},\ \tilde{z})$

Buscher則

$$\mathcal{H}_{MN}(x,y)$$
$$d = z$$

DFTの解

(SUGRAの解)

z方向へ
T-duality

$z \leftrightarrow \tilde{z}$

$$\mathcal{H}'_{MN}(x,y)$$
$$d = \tilde{z}$$

DFTの解

時空に並進対称性がなくても
必ずDFTの解になる

しかし, この DFT の解は
dual 座標に依存しており,
通常の SUGRA では解釈が不明.

(実は gen. SUGRA の解)

3次元空間のダブル

$$(x,\ y,\ z,\ \tilde{x},\ \tilde{y},\ \tilde{z})$$

Buscher則

$$\mathcal{H}_{MN}(x,y) \qquad d = z$$

DFTの解

z方向へ
T-duality

$$z \leftrightarrow \tilde{z}$$

$$\mathcal{H}'_{MN}(x,y) \qquad d = \tilde{z}$$

DFTの解

時空に並進対称性がなくても
必ずDFTの解になる

一方, 時空に**並進対称性**がある場合

$$x^m = (x^\mu, \cancel{x^i})$$

D $\quad$ n

DFTの解 (SUGRAの解)	O(*n*,*n*)変換	DFTの解
$\mathcal{H}_{MN}(x^\mu)$ $d(x^\mu)$	$\Longrightarrow$ $x^i \leftrightarrow \tilde{x}_i$	$\mathcal{H}'_{MN}(x^\mu)$ $d(x^\mu)$

$\tilde{x}_i$ への依存性は出ない

⇩

SUGRAの解

O(*n*,*n*)対称性を得るには <u>*n* 次元の並進対称性</u>が必要

dual座標への依存性を生み出さない条件

ここまでのまとめ

***n*次元の並進対称性があれば**
string理論にはO(*n*,*n*)対称性がある.
Abelian T-duality

Double Field Theory
を使えば, SUGRA の
O(*n*,*n*)対称性が manifest!

Dual座標が見えても気にしなければ
並進対称性がなくてもO(*D*,*D*)対称性がある.

コメント

Non-Abelian T-duality や
Yang-Baxter変形 **(⇨ 後で説明)**

SUGRAの解

generalized SUGRAの解

非自明な変換

DFTの言葉で言うと:
Dualの座標が見えている解

$\tilde{x}_i$

O(*n*,*n*)変換

x^i

SUGRAの解′

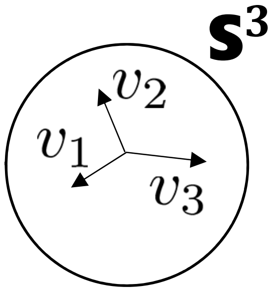
変な解ではない

Abelian T-duality

↓

Non-Abelian T-duality

(雰囲気だけ)



計量

$$dl^2 = d\theta^2 + \sin^2\theta\, d\phi^2 + (d\psi + \cos\theta\, d\phi)^2\,.$$

Killingベクトル場

$$\begin{cases} v_1 = \cos\psi\,\partial_\theta + \dfrac{\sin\psi}{\sin\theta}\,\partial_\phi - \dfrac{\sin\psi}{\tan\theta}\,\partial_\psi\,, \\ v_2 = -\sin\psi\,\partial_\theta + \dfrac{\cos\psi}{\sin\theta}\,\partial_\phi - \dfrac{\cos\psi}{\tan\theta}\,\partial_\psi\,, \\ v_3 = \partial_\psi\,. \end{cases}$$

SU(2) 代数: $[v_1,\, v_2] = v_3\,, \quad [v_2,\, v_3] = v_1\,, \quad [v_3,\, v_1] = v_2\,.$

~~3次元の並進対称性~~

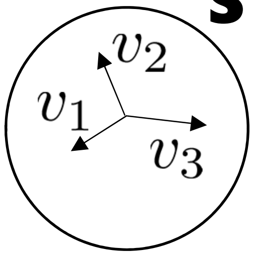
⇒ **~~O(3,3)対称性~~**

しかし, 交換しないKillingベクトルを利用した
Non-Abelian T-duality という対称性がある.

[de la Ossa, Quevedo '92]

Non-Abelian T-duality

S^3



$$\mathrm{d}l^2 = \frac{1}{4}\left[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 + (\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi)^2\right].$$

NATD **(詳細は二周目)**

3次元空間

$$\mathrm{d}l^2 = \frac{4\left(\delta_{ij} + 16\,u_i\,u_j\right)\mathrm{d}u^i\,\mathrm{d}u^j}{1 + 16\,u_k\,u^k}.$$

NATD がstring理論の対称性かは分からないが
少なくともSUGRAの対称性になっている.
(新たなSUGRA解の生成手法)

IIB SUGRA の解

[Sfetsos, Thompson '10]

AdS$_3$ × S^3 × T^4

$$\mathrm{d}s^2 = \frac{-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}z^2}{z^2} + \frac{1}{4}\left[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 + (\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi)^2\right] + \mathrm{d}s^2_{\mathrm{T}^4}\,,$$

$$G_3 = \frac{2\,\mathrm{d}t\wedge\mathrm{d}x\wedge\mathrm{d}z}{z^3} - \frac{\sin\theta}{4}\,\mathrm{d}\theta\wedge\mathrm{d}\phi\wedge\mathrm{d}\psi\,.$$

(R-R field strength)

⇓ **NATD**

$$\mathrm{d}s'^2 = \frac{-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}z^2}{z^2} + \frac{4\left(\delta_{ij} + 16\,u_i\,u_j\right)\mathrm{d}u^i\,\mathrm{d}u^j}{1 + 16\,u_k\,u^k} + \mathrm{d}s^2_{\mathrm{T}^4}\,,$$

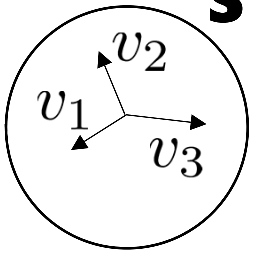
$$B'_2 = -\frac{8\,\epsilon_{ijk}\,u^i\,\mathrm{d}u^j\wedge\mathrm{d}u^k}{1 + 16\,u_k\,u^k}\,,\qquad \mathrm{e}^{-2\Phi'} = \frac{1 + 16\,u_k\,u^k}{64}\,,$$

$$G'_0 = \frac{1}{4}\,,\qquad G'_2 = \frac{2\,\epsilon_{ijk}\,u^i\,\mathrm{d}u^j\wedge\mathrm{d}u^k}{1 + 16\,u_l\,u^l}\,,$$

$$G'_4 = -\frac{2\,\mathrm{d}t\wedge\mathrm{d}x\wedge\mathrm{d}z\wedge u_i\,\mathrm{d}u^i}{z^3} - \frac{\mathrm{vol}(T^4)}{4}\,.$$

(massive) IIA SUGRA の解になる。

S^3



SU(2) isometry

NATD

$$\mathrm{d}l^2 = \frac{4\left(\delta_{ij} + 16\, u_i\, u_j\right) \mathrm{d}u^i\, \mathrm{d}u^j}{1 + 16\, u_k\, u^k}$$

Isometry??

NATD

Poisson-Lie
T-duality

[Klimcik, Severa '95]

一般化された"isometry"
があれば解の生成が実行できる
(Killingベクトルは必ずしもなくても良い)

Poisson-Lie T-duality

[Hassler, 1707.08624;
Demulder, Hassler, Thompson, 1810.11446;
YS, 1903.12175]

> **Poisson-Lie T-duality は**
> **Double Field Theory (DFT)**
> **の運動方程式の対称性.**

(⇨ 三周目)

PLTD

DFTの解 ⟹ **DFTの解**

(SUGRAの解)

(SUGRAの解/
generalized SUGRAの解/
massive type IIA SUGRAの解)

ここまでのまとめ

Killingベクトル場が**非可換な代数**をなす場合
NATD という対称性がある.

不完全 (一方通行)

Poisson-Lie T-duality へと拡張

DFT の対称性
(DFT の解を常に DFT の解へと移す)

Non-Abelian/Poisson-Lie T-duality
はよく理解されてきた.

String理論において, T-duality は
U-duality の部分群

M理論(11次元SUGRA)/ T^n

S^1コンパクト化

Type IIA SUGRA on T^D

(D=n-1)

O(D,D) T-duality

Type IIB SUGRA on T^D

今日考える理論

S-duality

M理論(11次元SUGRA)/ T^n

S^{1}コンパクト化

Type IIA SUGRA on T^D

$O(D,D)$
T-duality

Type IIB SUGRA on T^D

SL(2)
S-duality

Type IIB SUGRA

$$\mathsf{m}_{\alpha\beta} = \mathrm{e}^{\Phi} \begin{pmatrix} \mathrm{e}^{-2\,\Phi} + (\mathsf{C}_0)^2 & \mathsf{C}_0 \\ \mathsf{C}_0 & 1 \end{pmatrix}$$

dilaton/R-R 0-form

$$\mathcal{L}_{\mathrm{IIB}} = *\mathsf{R} + \frac{1}{4}\,\mathsf{F}_{1\alpha\beta} \wedge *\mathsf{F}_1^{\alpha\beta} - \frac{1}{2}\,\mathsf{m}_{\alpha\beta}\,\mathsf{F}_3^{\alpha} \wedge *\mathsf{F}_3^{\beta} - \frac{1}{4}\,\mathsf{F}_5 \wedge *\mathsf{F}_5 + \frac{1}{4}\,\epsilon_{\alpha\beta}\,\mathsf{A}_4 \wedge \mathsf{F}_3^{\alpha} \wedge \mathsf{F}_3^{\beta}\,.$$

Einstein-frame metric

g_{mn}

R-R 4-form

$$\mathsf{A}_4 \equiv \mathsf{C}_4 - \tfrac{1}{2}\,\mathsf{C}_2 \wedge \mathsf{B}_2$$

B場/R-R 2-form

$$(\mathsf{A}_2^{\alpha}) \equiv \begin{pmatrix} \mathsf{B}_2 \\ -\mathsf{C}_2 \end{pmatrix}$$

SL(2) S-duality

$$\mathsf{m}_{\alpha\beta} = \mathrm{e}^{\Phi} \begin{pmatrix} \mathrm{e}^{-2\Phi} + (\mathsf{C}_0)^2 & \mathsf{C}_0 \\ \mathsf{C}_0 & 1 \end{pmatrix}$$

SL(2) triplet

$$\mathcal{L}_{\mathrm{IIB}} = *\mathsf{R} + \frac{1}{4}\,\mathsf{F}_{1\alpha\beta} \wedge *\mathsf{F}_1^{\alpha\beta} - \frac{1}{2}\,\mathsf{m}_{\alpha\beta}\,\mathsf{F}_3^{\alpha} \wedge *\mathsf{F}_3^{\beta} - \frac{1}{4}\,\mathsf{F}_5 \wedge *\mathsf{F}_5 + \frac{1}{4}\,\epsilon_{\alpha\beta}\,\mathsf{A}_4 \wedge \mathsf{F}_3^{\alpha} \wedge \mathsf{F}_3^{\beta}\,.$$

SL(2) singlet

$$g_{mn}$$

SL(2) singlet

$$\mathsf{A}_4 \equiv \mathsf{C}_4 - \tfrac{1}{2}\,\mathsf{C}_2 \wedge \mathsf{B}_2$$

SL(2) doublet

$$(\mathsf{A}_2^{\alpha}) \equiv \begin{pmatrix} \mathsf{B}_2 \\ -\mathsf{C}_2 \end{pmatrix}$$

IIB 理論から出発...

Type **IIA** SUGRA on T^{D} $\Longleftrightarrow$ Type **IIB** SUGRA on T^{D}

$O(D,D)$ T-duality

SL(2) S-duality

解 → 解'

IIB S-duality⇨T-duality⇨ S-duality ... IIB

$\mathrm{E}_{n(n)}$ U-duality変換

(n=D+1)

$\mathrm{E}_{n(n)}$ 代数 $(n \le 8)$ (type IIBの言葉で)

O(D,D)

$K^{\mathsf{a}}{}_{\mathsf{b}}$

SL(2)

$R^{\alpha}{}_{\beta}$

$$R^{\mathsf{c}_1\mathsf{c}_2}_{\alpha} = \begin{pmatrix} R^{\mathsf{c}_1\mathsf{c}_2}_1 \\ R^{\mathsf{c}_1\mathsf{c}_2}_2 \end{pmatrix} \qquad R^{\alpha}_{\mathsf{c}_1\mathsf{c}_2} = \begin{pmatrix} R^1_{\mathsf{c}_1\mathsf{c}_2} \\ R^2_{\mathsf{c}_1\mathsf{c}_2} \end{pmatrix}$$

RR 4-form potential

$$R^{\mathsf{c}_1\cdots\mathsf{c}_4} \qquad R_{\mathsf{c}_1\cdots\mathsf{c}_4}$$

$$R^{\mathsf{c}_1\cdots\mathsf{c}_6}_{\alpha} = \begin{pmatrix} R^{\mathsf{c}_1\cdots\mathsf{c}_6}_1 \\ R^{\mathsf{c}_1\cdots\mathsf{c}_6}_2 \end{pmatrix} \qquad R^{\alpha}_{\mathsf{c}_1\cdots\mathsf{c}_6} = \begin{pmatrix} R^1_{\mathsf{c}_1\cdots\mathsf{c}_6} \\ R^2_{\mathsf{c}_1\cdots\mathsf{c}_6} \end{pmatrix}$$

$$R^{\mathsf{c}_1\cdots\mathsf{c}_7,\mathsf{c}} \qquad R_{\mathsf{c}_1\cdots\mathsf{c}_7,\mathsf{c}}$$

$E_{n(n)}$	SL(5)	SO(5,5)	$E_{6(6)}$	$E_{7(7)}$	$E_{8(8)}$
dim	24	45	78	133	248

**[Tumanov, West '14;
YS, Uehara '17]**

$$[K^{\mathsf{a}}{}_{\mathsf{b}},\, K^{\mathsf{c}}{}_{\mathsf{d}}] = \delta^{\mathsf{c}}_{\mathsf{b}}\, K^{\mathsf{a}}{}_{\mathsf{d}} - \delta^{\mathsf{a}}_{\mathsf{d}}\, K^{\mathsf{c}}{}_{\mathsf{b}}\,,$$

$$[K^{\mathsf{a}}{}_{\mathsf{b}},\, R^{\bar{\mathsf{c}}_2}_{\alpha}] = \delta^{\bar{\mathsf{c}}_2}_{\mathsf{bd}}\, R^{\mathsf{ad}}_{\alpha}\,, \qquad [K^{\mathsf{a}}{}_{\mathsf{b}},\, R^{\alpha}_{\bar{\mathsf{c}}_2}] = -\delta^{\mathsf{ad}}_{\bar{\mathsf{c}}_2}\, R^{\alpha}_{\mathsf{bd}}\,,$$

$$[K^{\mathsf{a}}{}_{\mathsf{b}},\, R^{\bar{\mathsf{c}}_4}] = \delta^{\bar{\mathsf{c}}_4}_{\mathsf{b}\bar{\mathsf{d}}_3}\, R^{\mathsf{a}\bar{\mathsf{d}}_3}\,, \qquad [K^{\mathsf{a}}{}_{\mathsf{b}},\, R_{\bar{\mathsf{c}}_4}] = -\delta^{\mathsf{a}\bar{\mathsf{d}}_3}_{\bar{\mathsf{c}}_4}\, R_{\mathsf{b}\bar{\mathsf{d}}_3}\,,$$

$$[K^{\mathsf{a}}{}_{\mathsf{b}},\, R^{\bar{\mathsf{c}}_6}_{\alpha}] = \delta^{\bar{\mathsf{c}}_6}_{\mathsf{b}\bar{\mathsf{d}}_5}\, R^{\mathsf{a}\bar{\mathsf{d}}_5}_{\alpha}\,, \qquad [K^{\mathsf{a}}{}_{\mathsf{b}},\, R^{\alpha}_{\bar{\mathsf{c}}_6}] = -\delta^{\mathsf{a}\bar{\mathsf{d}}_5}_{\bar{\mathsf{c}}_6}\, R^{\alpha}_{\mathsf{b}\bar{\mathsf{d}}_5}\,,$$

$$[K^{\mathsf{a}}{}_{\mathsf{b}},\, R^{\bar{\mathsf{c}}_7,\mathsf{c}'}] = \delta^{\bar{\mathsf{c}}_7}_{\mathsf{b}\bar{\mathsf{d}}_6}\, R^{\mathsf{a}\bar{\mathsf{d}}_6,\mathsf{c}'} + \delta^{\mathsf{c}'}_{\mathsf{b}}\, R^{\bar{\mathsf{c}}_7,\mathsf{a}}\,, \qquad [K^{\mathsf{a}}{}_{\mathsf{b}},\, R_{\bar{\mathsf{c}}_7,\mathsf{c}'}] = -\delta^{\mathsf{a}\bar{\mathsf{d}}_6}_{\bar{\mathsf{c}}_7}\, R_{\mathsf{b}\bar{\mathsf{d}}_6,\mathsf{c}'} - \delta^{\mathsf{a}}_{\mathsf{c}'}\, R_{\bar{\mathsf{c}}_7,\mathsf{b}}\,,$$

$$[R^{\alpha}{}_{\beta},\, R^{\gamma}{}_{\delta}] = \delta^{\gamma}_{\beta}\, R^{\alpha}{}_{\delta} - \delta^{\alpha}_{\delta}\, R^{\gamma}{}_{\beta}\,,$$

$$[R^{\alpha}{}_{\beta},\, R^{\bar{\mathsf{c}}_2}_{\gamma}] = -\big(\delta^{\alpha}_{\gamma}\,\delta^{\delta}_{\beta} - \tfrac{1}{2}\,\delta^{\alpha}_{\beta}\,\delta^{\delta}_{\gamma}\big)\, R^{\bar{\mathsf{c}}_2}_{\delta}\,, \qquad [R^{\alpha}{}_{\beta},\, R^{\gamma}_{\bar{\mathsf{c}}_2}] = \big(\delta^{\alpha}_{\delta}\,\delta^{\gamma}_{\beta} - \tfrac{1}{2}\,\delta^{\alpha}_{\beta}\,\delta^{\gamma}_{\delta}\big)\, R^{\delta}_{\bar{\mathsf{c}}_2}\,,$$

$$[R^{\alpha}{}_{\beta},\, R^{\bar{\mathsf{c}}_6}_{\gamma}] = -\big(\delta^{\alpha}_{\gamma}\,\delta^{\delta}_{\beta} - \tfrac{1}{2}\,\delta^{\alpha}_{\beta}\,\delta^{\delta}_{\gamma}\big)\, R^{\bar{\mathsf{c}}_6}_{\delta}\,, \qquad [R^{\alpha}{}_{\beta},\, R^{\gamma}_{\bar{\mathsf{c}}_6}] = \big(\delta^{\alpha}_{\delta}\,\delta^{\gamma}_{\beta} - \tfrac{1}{2}\,\delta^{\alpha}_{\beta}\,\delta^{\gamma}_{\delta}\big)\, R^{\delta}_{\bar{\mathsf{c}}_6}\,,$$

$$[R^{\bar{\mathsf{a}}_2}_{\alpha},\, R^{\bar{\mathsf{b}}_2}_{\beta}] = \epsilon_{\alpha\beta}\, R^{\bar{\mathsf{a}}_2\bar{\mathsf{b}}_2}\,, \quad [R^{\bar{\mathsf{a}}_2}_{\alpha},\, R^{\bar{\mathsf{b}}_4}] = -R^{\bar{\mathsf{a}}_2\bar{\mathsf{b}}_4}_{\alpha}\,, \quad [R^{\bar{\mathsf{a}}_2}_{\alpha},\, R^{\bar{\mathsf{b}}_6}_{\beta}] = \epsilon_{\alpha\beta}\,\delta^{\bar{\mathsf{b}}_6}_{\bar{\mathsf{c}}_5\mathsf{f}}\, R^{\bar{\mathsf{a}}_2\bar{\mathsf{c}}_4,\mathsf{f}}\,,$$

$$[R^{\bar{\mathsf{a}}_2}_{\alpha},\, R^{\beta}_{\bar{\mathsf{b}}_2}] = -\delta^{\beta}_{\alpha}\,\delta^{\bar{\mathsf{a}}_2}_{\mathsf{ce}}\,\delta^{\mathsf{de}}_{\bar{\mathsf{b}}_2}\, K^{\mathsf{c}}{}_{\mathsf{d}} + \tfrac{1}{4}\,\delta^{\beta}_{\alpha}\,\delta^{\bar{\mathsf{a}}_2}_{\bar{\mathsf{b}}_2}\, K + \delta^{\bar{\mathsf{a}}_2}_{\bar{\mathsf{b}}_2}\, R^{\beta}{}_{\alpha}\,,$$

$$[R^{\bar{\mathsf{a}}_2}_{\alpha},\, R_{\bar{\mathsf{b}}_4}] = -\epsilon_{\alpha\beta}\,\delta^{\bar{\mathsf{a}}_2\bar{\mathsf{c}}_2}_{\bar{\mathsf{b}}_4}\, R^{\beta}_{\bar{\mathsf{c}}_2}\,, \qquad [R^{\bar{\mathsf{a}}_2}_{\alpha},\, R^{\beta}_{\bar{\mathsf{b}}_6}] = \delta^{\beta}_{\alpha}\,\delta^{\bar{\mathsf{a}}_2\bar{\mathsf{c}}_4}_{\bar{\mathsf{b}}_6}\, R_{\bar{\mathsf{c}}_4}\,,$$

$$[R^{\bar{\mathsf{a}}_2}_{\alpha},\, R_{\bar{\mathsf{b}}_7,\mathsf{b}'}] = -\epsilon_{\alpha\beta}\,\delta^{\bar{\mathsf{a}}_2\bar{\mathsf{c}}_5}_{\bar{\mathsf{b}}_7}\, R^{\beta}_{\bar{\mathsf{c}}_5\mathsf{b}'}\,,$$

$$[R^{\bar{\mathsf{a}}_4},\, R^{\bar{\mathsf{b}}_4}] = -\delta^{\bar{\mathsf{b}}_4}_{\bar{\mathsf{c}}_3\mathsf{d}}\, R^{\bar{\mathsf{a}}_4\bar{\mathsf{c}}_3,\mathsf{d}}\,, \qquad [R^{\bar{\mathsf{a}}_4},\, R^{\alpha}_{\bar{\mathsf{b}}_2}] = \epsilon^{\alpha\beta}\,\delta^{\bar{\mathsf{a}}_4}_{\bar{\mathsf{b}}_2\bar{\mathsf{c}}_2}\, R^{\bar{\mathsf{c}}_2}_{\beta}\,,$$

$$[R^{\bar{\mathsf{a}}_4},\, R_{\bar{\mathsf{b}}_4}] = -\delta^{\bar{\mathsf{a}}_4}_{\mathsf{c}\bar{\mathsf{e}}_3}\,\delta^{\mathsf{d}\bar{\mathsf{e}}_3}_{\bar{\mathsf{b}}_4}\, K^{\mathsf{c}}{}_{\mathsf{d}} + \tfrac{2}{4}\,\delta^{\bar{\mathsf{a}}_4}_{\bar{\mathsf{b}}_4}\, K$$

$$[R^{\bar{\mathsf{a}}_4},\, R^{\alpha}_{\bar{\mathsf{b}}_6}] = -\delta^{\bar{\mathsf{a}}_4\bar{\mathsf{c}}_2}_{\bar{\mathsf{b}}_6}\, R^{\alpha}_{\bar{\mathsf{c}}_2}\,, \qquad [R^{\bar{\mathsf{a}}_4},\, R_{\bar{\mathsf{b}}_7,\mathsf{b}'}] = \delta^{\bar{\mathsf{a}}_4\bar{\mathsf{c}}_3}_{\bar{\mathsf{b}}_7}\, R_{\bar{\mathsf{c}}_3\mathsf{b}'}$$

$$[R^{\bar{\mathsf{a}}_6}_{\alpha},\, R^{\beta}_{\bar{\mathsf{b}}_2}] = -\delta^{\beta}_{\alpha}\,\delta^{\bar{\mathsf{a}}_6}_{\bar{\mathsf{b}}_2\bar{\mathsf{c}}_4}\, R^{\bar{\mathsf{c}}_4}\,, \qquad [R^{\bar{\mathsf{a}}_6}_{\alpha},\, R_{\bar{\mathsf{b}}_4}] = \delta^{\bar{\mathsf{a}}_6}_{\bar{\mathsf{b}}_4\bar{\mathsf{c}}_2}\, R^{\bar{\mathsf{c}}_2}_{\alpha}\,,$$

$$[R^{\bar{\mathsf{a}}_6}_{\alpha},\, R^{\beta}_{\bar{\mathsf{b}}_6}] = -\delta^{\beta}_{\alpha}\,\delta^{\bar{\mathsf{a}}_6}_{\mathsf{c}\bar{\mathsf{e}}_5}\,\delta^{\mathsf{d}\bar{\mathsf{e}}_5}_{\bar{\mathsf{b}}_6}\, K^{\mathsf{c}}{}_{\mathsf{d}} + \tfrac{3}{4}\,\delta^{\beta}_{\alpha}\,\delta^{\bar{\mathsf{a}}_6}_{\bar{\mathsf{b}}_6}\, K + \delta^{\bar{\mathsf{a}}_6}_{\bar{\mathsf{b}}_6}\, R^{\beta}{}_{\alpha}\,,$$

$$[R^{\bar{\mathsf{a}}_6}_{\alpha},\, R_{\bar{\mathsf{b}}_7,\mathsf{b}}] = \epsilon_{\alpha\beta}\,\delta^{\bar{\mathsf{a}}_6\mathsf{c}}_{\bar{\mathsf{b}}_7}\, R^{\beta}_{\mathsf{cb}}\,,$$

$$[R^{\bar{\mathsf{a}}_7,\mathsf{a}'},\, R^{\alpha}_{\bar{\mathsf{b}}_2}] = \epsilon^{\alpha\beta}\,\delta^{\bar{\mathsf{a}}_7}_{\bar{\mathsf{b}}_2\bar{\mathsf{c}}_5}\, R^{\bar{\mathsf{c}}_5\mathsf{a}'}_{\beta}\,, \qquad [R^{\bar{\mathsf{a}}_7,\mathsf{a}'},\, R_{\bar{\mathsf{b}}_4}] = -\delta^{\bar{\mathsf{a}}_7}_{\bar{\mathsf{b}}_4\bar{\mathsf{c}}_3}\, R^{\bar{\mathsf{c}}_3\mathsf{a}'}$$

$$[R^{\bar{\mathsf{a}}_7,\mathsf{a}},\, R^{\alpha}_{\bar{\mathsf{b}}_6}] = -\epsilon_{\alpha\beta}\,\delta^{\bar{\mathsf{a}}_7}_{\bar{\mathsf{b}}_6\mathsf{c}}\, R^{\mathsf{ca}}_{\beta}\,, \qquad [R^{\bar{\mathsf{a}}_7,\mathsf{a}},\, R_{\bar{\mathsf{b}}_7,\mathsf{b}}] = -\delta^{\bar{\mathsf{a}}_7}_{\bar{\mathsf{b}}_7}\, K^{\mathsf{a}}{}_{\mathsf{b}}\,,$$

$$[R^{\alpha}_{\bar{\mathsf{a}}_2},\, R^{\beta}_{\bar{\mathsf{b}}_2}] = \epsilon^{\alpha\beta}\, R_{\bar{\mathsf{a}}_2\bar{\mathsf{b}}_2}\,, \quad [R^{\alpha}_{\bar{\mathsf{a}}_2},\, R_{\bar{\mathsf{b}}_4}] = -R^{\alpha}_{\bar{\mathsf{a}}_2\bar{\mathsf{b}}_4}\,, \quad [R^{\alpha}_{\bar{\mathsf{a}}_2},\, R^{\beta}_{\bar{\mathsf{b}}_6}] = \epsilon^{\alpha\beta}\,\delta^{\bar{\mathsf{c}}_5\mathsf{d}}_{\bar{\mathsf{b}}_6}\, R_{\bar{\mathsf{a}}_2\bar{\mathsf{c}}_5,\mathsf{d}}\,,$$

$$[R_{\bar{\mathsf{a}}_4},\, R_{\bar{\mathsf{b}}_4}] = -\delta^{\bar{\mathsf{c}}_3\mathsf{f}}_{\bar{\mathsf{b}}_4}\, R_{\bar{\mathsf{a}}_4\bar{\mathsf{c}}_3,\mathsf{f}}\,.$$

M理論から出発...

M ⇨ S^1コンパクト化⇨T-duality⇨ ... ⇨ M

$E_{n(n)}$ U-duality変換

(n=D+1)

$\mathrm{E}_{n(n)}$ 代数 ($n \leq 8$) (M理論の言葉で)

M理論/T^n

トーラス上の座標変換 → GL(n)

$K^a{}_b$ GL(n)

$R^{a_1 a_2 a_3}$ $\qquad$ $R_{a_1 a_2 a_3}$

C$_3$のgauge変換

$R^{a_1 \cdots a_6}$ $\qquad$ $R_{a_1 \cdots a_6}$

C$_6$のgauge変換

$R^{a_1 \cdots a_8, a}$ $\qquad$ $R_{a_1 \cdots a_8, a}$

$E_{n(n)}$	SL(5)	SO(5,5)	$E_{6(6)}$	$E_{7(7)}$	$E_{8(8)}$
dim	24	45	78	133	248

$$[K^a{}_b,\, K^c{}_d] = \delta^c_b\, K^a{}_d - \delta^a_d\, K^c{}_b\,,$$
$$[K^a{}_b,\, R^{\bar{c}_3}] = \delta^{\bar{c}_3}_{b\bar{d}_2}\, R^{a\bar{d}_2}\,, \qquad [K^a{}_b,\, R_{\bar{c}_3}] = -\delta^{a\bar{d}_2}_{\bar{c}_3}\, R_{b\bar{d}_2}\,,$$
$$[K^a{}_b,\, R^{\bar{c}_6}] = \delta^{\bar{c}_6}_{b\bar{d}_5}\, R^{a\bar{d}_5}\,, \qquad [K^a{}_b,\, R_{\bar{c}_6}] = -\delta^{a\bar{d}_5}_{\bar{c}_6}\, R_{b\bar{d}_5}\,,$$
$$[K^a{}_b,\, R^{\bar{c}_8,c'}] = \delta^{\bar{c}_8}_{b\bar{d}_7}\, R^{a\bar{d}_7,c'} + \delta^{c'}_b\, R^{\bar{c}_8,a}\,, \qquad [K^a{}_b,\, R_{\bar{c}_8,c'}] = -\delta^{a\bar{d}_7}_{\bar{c}_8}\, R_{b\bar{d}_7,c'} - \delta^a_{c'}\, R_{\bar{c}_8,b}\,,$$
$$[R^{\bar{a}_3},\, R^{\bar{b}_3}] = R^{\bar{a}_3\bar{b}_3}\,, \quad [R^{\bar{a}_3},\, R^{\bar{b}_6}] = \delta^{\bar{b}_6}_{\bar{c}_5 c'}\, R^{\bar{a}_3\bar{c}_5,c'}\,,$$
$$[R^{\bar{a}_3},\, R_{\bar{b}_3}] = -\delta^{\bar{a}_3}_{\bar{e}_2 c}\, \delta^{\bar{e}_2 d}_{\bar{b}_3}\, K^c{}_d + \tfrac{1}{3}\, \delta^{\bar{a}_3}_{\bar{b}_3}\, K\,, \qquad [R^{\bar{a}_3},\, R_{\bar{b}_6}] = -\delta^{\bar{a}_3\bar{c}_3}_{\bar{b}_6}\, R_{\bar{c}_3}\,,$$
$$[R^{\bar{a}_3},\, R_{\bar{b}_8,b}] = -\delta^{\bar{a}_3\bar{c}_5}_{\bar{b}_8}\, R_{\bar{c}_5 b}\,, \qquad [R^{\bar{a}_6},\, R_{\bar{b}_3}] = \delta^{\bar{a}_6}_{\bar{b}_3\bar{c}_3}\, R^{\bar{c}_3}\,,$$
$$[R^{\bar{a}_6},\, R_{\bar{b}_6}] = -\delta^{\bar{a}_6}_{\bar{e}_5 c}\, \delta^{\bar{e}_5 d}_{\bar{b}_6}\, K^c{}_d + \tfrac{2}{3}\, \delta^{\bar{a}_6}_{\bar{b}_6}\, K\,, \qquad [R^{\bar{a}_6},\, R_{\bar{b}_8,b}] = -\delta^{\bar{a}_6\bar{c}_2}_{\bar{b}_8}\, R_{\bar{c}_2 b}\,,$$
$$[R^{\bar{a}_8,a},\, R_{\bar{b}_3}] = \delta^{\bar{a}_8}_{\bar{b}_3\bar{c}_5}\, R^{\bar{c}_5 a}\,, \qquad [R^{\bar{a}_8,a},\, R_{\bar{b}_6}] = \delta^{\bar{a}_8}_{\bar{b}_6\bar{c}_2}\, R^{\bar{c}_2 a}\,,$$
$$[R^{\bar{a}_8,a},\, R_{\bar{b}_8,b}] = -\delta^{\bar{a}_8}_{\bar{b}_8}\, K^a{}_b\,, \quad [R_{\bar{a}_3},\, R_{\bar{b}_3}] = R_{\bar{a}_3\bar{b}_3}\,, \quad [R_{\bar{a}_3},\, R_{\bar{b}_6}] = \delta^{\bar{c}_5 c'}_{\bar{b}_6}\, R_{\bar{a}_3\bar{c}_5,c'}\,.$$

[Berman, Godazgar, Perry, West '11]

再定義で type IIB の generator にできる. [YS, Uehara '17]

ここまでのまとめ

Abelian U-duality

T-duality と S-duality は
$\mathrm{E}_{n(n)}$ U-duality の部分群

M理論/T^{n}　　**Type IIB SUGRA/T^{D}**

(D=n-1)

Exceptional Field Theory

[Berman, Hohm, Godazgar, Perry, Samtleben, West,...]

(コンパクト化する前から)

$\mathrm{E}_{n(n)}$対称性を明白にした SUGRA

DFT $x^M = (x^m, \tilde{x}_m)$

正準共役

P F1

p_m w^m

$$\left\{ \begin{array}{l} \mathcal{H}_{MN} = \begin{pmatrix} (g - B\,g^{-1}\,B)_{mn} & B_{mp}\,g^{pn} \\ -g^{mp}\,B_{pn} & g^{mn} \end{pmatrix} \in \mathrm{O}(D,D) \\ \mathrm{e}^{-2d} \end{array} \right.$$

EFT

$$x^I = (x^i,\, y_{i_1 i_2},\, y_{i_1\cdots i_5},\, y_{i_1\cdots i_7,i},\, \cdots)$$

P M2 M5 KKM exotic branes

一般化計量

$$\mathcal{M}_{IJ} = \Big(\quad \cdots \quad \Big) \in E_{n(n)}$$

g_{ij} C_{ijk} $C_{i_1\cdots i_6}$

$g_{\mu\nu}$ $C_{\mu\nu\rho}$	g_{ij} C_{ijk}	$g_{\mu i}$
(11-n)次元	n次元	$C_{\mu ij}$ $C_{\mu\nu i}$
		A^I_μ $A^{\mathcal{I}}_{\mu\nu}$

Exceptional Space

$$x^I = (\overset{\textbf{P}}{x^i}, \overset{\textbf{M2}}{y_{i_1 i_2}}, \overset{\textbf{M5}}{y_{i_1 \cdots i_5}}, \overset{\textbf{KKM}}{y_{i_1 \cdots i_7, i}}, \cdots)$$

$\underline{n = 4}$	$4 + {}_4\mathrm{C}_2$	✖	✖	$= 10$
$\underline{n = 5}$	$5 + {}_5\mathrm{C}_2$	$+ \ {}_5\mathrm{C}_5$	✖	$= 16$
$\underline{n = 6}$	$6 + {}_6\mathrm{C}_2$	$+ \ {}_6\mathrm{C}_5$	✖	$= 27$
$\underline{n = 7}$	$7 + {}_7\mathrm{C}_2$	$+ \ {}_7\mathrm{C}_5$	$+ \quad 7$	$= 56$
$\underline{n = 8}$	$8 + {}_8\mathrm{C}_2$	$+ \ {}_8\mathrm{C}_5$	$+ \quad \cdots$	$= 248$

例外群 $E_{n(n)}$ のベクトル表現の次元を持つ空間

Exceptional Field Theory

[Hohm, Samtleben '13; ....]

$$\mathcal{L}_{\rm EFT} = \mathcal{L}_{\rm EH} + \mathcal{L}_{\rm scalar} + \mathcal{L}_{\rm pot} \ ,$$

$g_{\mu\nu}$ の **Ricci scalar**

$e \equiv \sqrt{-\det g_{\mu\nu}}\,.$

$\left(g_{\mu\nu} \equiv |\det g_{ij}|^{\frac{1}{d-2}}\,\mathrm{g}_{\mu\nu}\right)$

$$\begin{aligned}
\mathcal{L}_{\rm EH} &= eR\,, \\
\mathcal{L}_{\rm scalar} &= \frac{e}{4\alpha_n}\, g^{\mu\nu}\,\partial_\mu \mathcal{M}_{IJ}\,\partial_\nu \mathcal{M}^{IJ}\,, \\
\mathcal{L}_{\rm pot} &= \frac{e}{4\alpha_n}\,\mathcal{M}^{IJ}\,\partial_I \mathcal{M}^{KL}\,\partial_J \mathcal{M}_{KL} - \frac{e}{2}\,\mathcal{M}^{IJ}\,\partial_J \mathcal{M}^{KL}\,\partial_L \mathcal{M}_{IK} + e\,\partial_I \ln e\,\partial_J \mathcal{M}^{IJ} \\
&\quad + e\,\mathcal{M}^{IJ}\,\partial_I \ln e\,\partial_J \ln e + \frac{e}{4}\,\mathcal{M}^{MN}\,\partial_I g^{\mu\nu}\,\partial_J g_{\mu\nu}\,.
\end{aligned}$$

Exceptional Field Theory

$$x^I = (x^i, \cancel{y_{i_1 i_2}, y_{i_1\cdots i_5}, y_{i_1\cdots i_7,i}, \cdots})$$

[Berman, Godazgar, Perry, West, '12]

$$\Downarrow$$

$$\mathcal{L}_{\rm EFT} \to \sqrt{-g}\left(R - \frac{1}{2\cdot 4!}F_{m_1\cdots m_4}F^{m_1\cdots m_4} - \frac{1}{2\cdot 7!}F_{m_1\cdots m_7}F^{m_1\cdots m_7}\right)$$

EFT は 11D SUGRAを再現

Type IIB の言葉でも記述できる

exotic branes

$$x^I = (x^m,\ y^{\alpha}_m,\ y_{m_1m_2m_3},\ y^{\alpha}_{m_1\cdots m_5},\ y_{m_1\cdots m_6,n},\ y^{\alpha\beta}_{m_1\cdots m_7},\ \cdots).$$

D次元 P F1/D1 D3 D5/NS5 KKM 7-brane

$$\mathcal{M}_{IJ} = \left(\quad\cdots\quad\right) \in E_{n(n)}$$

g_{mn} A_2^{α} $A_4\cdots$

$\mathcal{L}_{\rm EFT} \to$ **Type IIB SUGRA**

[Blair, Malek, Park, '13; Lee, Rey, YS, '16]

Exceptional Field Theory

$\mathbf{E}_{n(n)}$ **U-duality共変な**
11D SUGRAとType IIB SUGRA
の両方を統一する枠組み.

Section condition $\quad Y^{MN}_{PQ}\,\partial_M \otimes \partial_N = 0$

***n*個の座標を残す選択**

$$x^I = (x^i,\ \cancel{y_{i_1 i_2},\ y_{i_1\cdots i_5},\ y_{i_1\cdots i_7,i},\ \cdots})$$

$$x^I = (x^m,\ \cancel{\mathsf{y}^{\alpha}_m,\ \mathsf{y}_{m_1 m_2 m_3},\ \mathsf{y}^{\alpha}_{m_1\cdots m_5},\ \mathsf{y}_{m_1\cdots m_6,n},\ \mathsf{y}^{\alpha\beta}_{m_1\cdots m_7},\ \cdots})\,.$$

$\boldsymbol{D = (n-1)}$**個の座標を残す選択**

$E_{n(n)}$	SL(5)	SO(5,5)	$E_{6(6)}$	$E_{7(7)}$	$E_{8(8)}$
dim	10	16	27	56	248

次元の異なる2つの理論を
統一的に記述.

Abelian U-duality

⇨ **manifest!**

11D SUGRAの解 → **Type IIB SUGRAの解**

Non-Abelian T-duality があるなら
Non-Abelian U-duality もある?

今回のメイン

Non-Abelian U-duality

先行研究は皆無

(あったとしても PL T-duality + S-duality 程度)

Poisson-Lie T-duality の一般化

⇒ **Nambu-Lie U-duality を提案!**

[YS, 1911.06320;
Malek, Thompson, 1911.07833]

より一般的なSUGRA解の生成手法!

(まだ予想の段階)

非自明な具体例 [Musaev, YS, 2012.13263]

ここまでのまとめ

トーラスコンパクト化する前から
$\mathrm{E}_{n(n)}$ U-duality 対称性を明白にするSUGRA
= Exceptional Field Theory

11D SUGRA と type IIB SUGRA を統一

ここまでのまとめ

Exceptional space

M理論

P M2 M5 KKM

$$x^I = (x^i, y_{i_1 i_2}, y_{i_1\cdots i_5}, y_{i_1\cdots i_7,i}, \cdots)$$

$$x^I = (x^m, \mathrm{y}^{\alpha}_{m}, \mathrm{y}_{m_1 m_2 m_3}, \mathrm{y}^{\alpha}_{m_1\cdots m_5}, \mathrm{y}_{m_1\cdots m_6,n}, \mathrm{y}^{\alpha\beta}_{m_1\cdots m_7}, \cdots)\,.$$

P F1/D1 D3 D5/NS5 KKM 7-brane

Type IIB

Abelian U-duality を
Nambu-Lie U-duality へと一般化した.
予想: これは EFT の対称性

二周目

Non-Abelian T-duality (レビュー)
Poisson-Lie T-duality (レビュー)

⇩

Nambu-Lie U-duality

Non-Abelian T-duality

[de la Ossa, Quevedo '92]

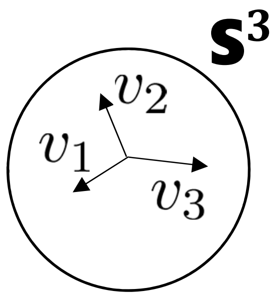
設定

話を簡単にするため,
*D*次元時空が*D*個のKillingベクトルを持つ場合に限定.
さらに, T-dual 変換前の時空では B場=0 とする.

*D*個のKillingベクトルがLie代数をなす:

$$[v_a,\, v_b] = f_{ab}{}^c\, v_c\,.$$

例: 群多様体



$f_{ab}{}^c$: 構造定数

v_a : 左不変ベクトル場

Non-Abelian T-duality

[de la Ossa, Quevedo '92,
Alvarez, Alvarez-Gaume, Lozano '94; ...]

Gaugeded action

簡単のため

$$S \equiv \frac{1}{4\pi\alpha'}\int_\Sigma g_{mn}\, Dx^m \wedge * Dx^n + \cancel{\frac{1}{2\pi\alpha'}\int_{\mathcal{B}} H_3}$$
$$+ \frac{1}{4\pi\alpha'}\int_\Sigma \left(2\, A^{\mathrm{a}} \wedge \mathrm{d}\tilde{x}_{\mathrm{a}} + f_{\mathrm{ab}}{}^{\mathrm{c}}\, \tilde{x}_{\mathrm{c}}\, A^{\mathrm{a}} \wedge A^{\mathrm{b}}\right)$$

補助場

$$\boxed{D_a x^m \equiv \partial_a x^m - A_a^{\mathrm{a}}\, v_{\mathrm{a}}^m}$$

補助場 $\tilde{x}_{\mathrm{a}}$ の運動方程式 ⇒ $F^{\mathrm{a}} = 0$ $\xrightarrow{A^{\mathrm{a}}\text{ 消去}}$ 通常の String 作用

$$\boxed{F^{\mathrm{a}} \equiv \mathrm{d}A^{\mathrm{a}} + \tfrac{1}{2} f_{\mathrm{bc}}{}^{\mathrm{a}}\, A^{\mathrm{b}} \wedge A^{\mathrm{c}}}$$

Non-Abelian T-duality

$$S \equiv \frac{1}{4\pi\alpha'}\int_\Sigma g_{mn}\, Dx^m \wedge * Dx^n + \frac{1}{4\pi\alpha'}\int_\Sigma \big(2\, A^{\mathrm{a}} \wedge \mathrm{d}\tilde{x}_{\mathrm{a}} + f_{\mathrm{ab}}{}^{\mathrm{c}}\, \tilde{x}_{\mathrm{c}}\, A^{\mathrm{a}} \wedge A^{\mathrm{b}}\big)$$

$$D_a x^m \equiv \partial_a x^m - A_a^{\mathrm{a}}\, v_{\mathrm{a}}^m\,, \qquad F^{\mathrm{a}} \equiv \mathrm{d}A^{\mathrm{a}} + \tfrac{1}{2}\, f_{\mathrm{bc}}{}^{\mathrm{a}}\, A^{\mathrm{b}} \wedge A^{\mathrm{c}}\,.$$

Gauge 対称性

$$\delta_\epsilon x^m(\sigma) = \epsilon^{\mathrm{a}}(\sigma)\, v_{\mathrm{a}}^m(x)\,, \qquad \delta_\epsilon \tilde{x}_{\mathrm{a}}(\sigma) = \cdots\,,$$

$$\delta_\epsilon A^{\mathrm{a}}(\sigma) = \mathrm{d}\epsilon^{\mathrm{a}}(\sigma) + f_{\mathrm{bc}}{}^{\mathrm{a}}\, A^{\mathrm{b}}(\sigma)\, \epsilon^{\mathrm{c}}(\sigma)\,.$$

定数

⇨ $x^m(\sigma) = c^m$ というgauge固定ができれば
作用から $x^m(\sigma)$ という変数が消せる.

Non-Abelian T-duality

$$S \equiv \frac{1}{4\pi\alpha'}\int_\Sigma g_{mn}\, Dx^m \wedge * Dx^n + \frac{1}{4\pi\alpha'}\int_\Sigma \left(2\, A^{\mathrm{a}} \wedge \mathrm{d}\tilde{x}_{\mathrm{a}} + f_{\mathrm{ab}}{}^{\mathrm{c}}\, \tilde{x}_{\mathrm{c}}\, A^{\mathrm{a}} \wedge A^{\mathrm{b}}\right)$$

$$D_a x^m \equiv \cancel{\partial_a x^m} - A_a^{\mathrm{a}}\, v_{\mathrm{a}}^m\,, \qquad F^{\mathrm{a}} \equiv \mathrm{d}A^{\mathrm{a}} + \tfrac{1}{2} f_{\mathrm{bc}}{}^{\mathrm{a}}\, A^{\mathrm{b}} \wedge A^{\mathrm{c}}\,.$$

$$x^m(\sigma) = c^m$$

A_a^{a} の運動方程式を使って A_a^{a} も消去

$\tilde{x}_{\mathrm{a}}$ だけを変数とする作用が得られる.

Dual model

補助場を座標とみなす　$x'^m = \tilde{x}_{\mathrm{a}}$

$$S = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\sqrt{-\gamma}\,(\gamma^{ab}-\varepsilon^{ab})\,E'_{mn}\,\partial_a x'^m\,\partial_b x'^n\,.$$

$$E'_{mn} = (\hat{g}_{\mathrm{ab}} + f_{\mathrm{ab}}{}^{\mathrm{c}}\,\tilde{x}_{\mathrm{c}})^{-1}$$

$$\hat{g}_{\mathrm{ab}} \equiv g_{mn}\,v^m_{\mathrm{a}}\,v^n_{\mathrm{b}}\big|_{x^m=c^m}$$

Dual 時空:　$g'_{mn} \equiv E'_{(mn)}\,,\quad B'_{mn} \equiv E'_{[mn]}$

上で定義された string 理論

簡単なまとめ: 伝統的なNATD

$$S \equiv \frac{1}{4\pi\alpha'} \int_\Sigma g_{mn}\, \mathrm{d}x^m \wedge * \mathrm{d}x^n$$

補助場 $\tilde{x}_{\mathrm{a}}$ を先に消去

Gauged action

NATD

x^m をゲージ固定

$$S = -\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \sqrt{-\gamma}\, (\gamma^{ab} - \varepsilon^{ab})\, E'_{mn}\, \partial_a x'^m\, \partial_b x'^n\,.$$

[de la Ossa, Quevedo '92,
Alvarez, Alvarez-Gaume, Lozano '94; ...]

Poisson-Lie T-duality

[Klimcik, Severa ’95]

$$E_{mn} \equiv g_{mn} + B_{mn}\,, \qquad E^{mn} \equiv (E^{-1})^{mn}\,.$$

NATD $\quad \pounds_{v_a} E_{mn} = 0\,, \qquad [v_a,\, v_b] = f_{ab}{}^{c}\, v_c\,.$

$f_a{}^{bc} = 0$

PLTD
$$\begin{cases} \pounds_{v_a} E_{mn} = -f_a{}^{bc}\, E_{mp}\, v_b^p\, v_c^q\, E_{qn}\,, \\ [v_a,\, v_b] = f_{ab}{}^{c}\, v_c\,. \end{cases}$$

なぜこんな補正を加えたのか?

(気持ちだけ説明)

String の運動方程式

$$v^m_{\mathrm{a}} = \delta^m_{\mathrm{a}}$$

$$[v_{\mathrm{a}},\, v_{\mathrm{b}}] = 0$$

Abelian のとき

$$\boxed{\mathcal{J}_{\mathrm{a}} \equiv v^m_{\mathrm{a}} \left(g_{mn} \, * \mathrm{d}x^n + B_{mn} \, \mathrm{d}x^n \right)}$$

$$\mathrm{d}\mathcal{J}_{\mathrm{a}} = \frac{1}{2} \left(\pounds_{v_{\mathrm{a}}} g_{pq} \, \mathrm{d}x^p \wedge *\mathrm{d}x^q + \pounds_{v_{\mathrm{a}}} B_{pq} \, \mathrm{d}x^p \wedge \mathrm{d}x^q \right) .$$

$$\Downarrow \qquad \pounds_{v_a} E_{mn} = 0$$

$$\mathrm{d}\mathcal{J}_{\mathrm{a}} = 0 \quad \Rightarrow \quad \mathcal{J}_{\mathrm{a}} = \mathrm{d}\tilde{x}_{\mathrm{a}}$$

$$\begin{cases} \mathcal{J}^{\mathrm{a}} = \mathrm{d}x^{\mathrm{a}} \\ \mathcal{J}_{\mathrm{a}} = \mathrm{d}\tilde{x}_{\mathrm{a}} \end{cases} \qquad \begin{cases} \mathrm{d}\mathcal{J}^{\mathrm{a}} = 0 & \textbf{(Bianch id.)} \\ \mathrm{d}\mathcal{J}_{\mathrm{a}} = 0 & \textbf{(E.O.M.)} \end{cases}$$

入れ替える対称性
が T-duality

String の運動方程式

$$\boxed{\mathcal{J}_{\mathrm{a}} \equiv v_{\mathrm{a}}^{m}\left(g_{mn} \, *\mathrm{d}x^{n} + B_{mn}\,\mathrm{d}x^{n}\right)}$$

$$\mathrm{d}\mathcal{J}_{\mathrm{a}} = \frac{1}{2}\left(\pounds_{v_{\mathrm{a}}} g_{pq}\,\mathrm{d}x^{p}\wedge *\mathrm{d}x^{q} + \pounds_{v_{\mathrm{a}}} B_{pq}\,\mathrm{d}x^{p}\wedge\mathrm{d}x^{q}\right).$$

$$\Downarrow \quad \pounds_{v_a} E_{mn} = -f_{a}{}^{bc}\,E_{mp}\,v_b^p\,v_c^q\,E_{qn}$$

$$\mathrm{d}\mathcal{J}_{\mathrm{a}} - \frac{1}{2} f_{\mathrm{a}}{}^{\mathrm{bc}}\,\mathcal{J}_{\mathrm{b}}\wedge\mathcal{J}_{\mathrm{c}} = 0 \quad \Longrightarrow \quad \mathcal{J}_{\mathrm{a}}\,\tilde{T}^{\mathrm{a}} = \mathrm{d}\tilde{g}\,\tilde{g}^{-1}$$

($\mathrm{d}\mathcal{J}_{\mathrm{a}} = 0$ **の non-Abelian 版**)

右不変1-form

$$\tilde{g} = \mathrm{e}^{\tilde{x}_a\,\tilde{T}^a}$$

$$[\tilde{T}^{\mathrm{a}},\,\tilde{T}^{\mathrm{b}}] = f_{\mathrm{c}}{}^{\mathrm{ab}}\,\tilde{T}^{\mathrm{c}}$$

$$\mathrm{d}\ell^{\mathrm{a}} + \frac{1}{2} f_{\mathrm{bc}}{}^{\mathrm{a}}\,\ell^{\mathrm{b}}\wedge\ell^{\mathrm{c}} = 0$$

$$g = \mathrm{e}^{x^a\,T_a}$$

$$[T_{\mathrm{a}},\,T_{\mathrm{b}}] = f_{\mathrm{ab}}{}^{\mathrm{c}}\,T_{\mathrm{c}}$$

$$\left\{\begin{array}{l} \ell = g^{-1}\,\mathrm{d}g \\ \mathcal{J} = \mathrm{d}\tilde{g}\,\tilde{g}^{-1} \end{array}\right.$$

入れ替える対称性
が Poisson-Lie T-duality

こんな条件を満たす時空を作れるか

$$\pounds_{v_a} E_{mn} = -f_a{}^{bc}\, E_{mp}\, v_b^p\, v_c^q\, E_{qn}\,,$$

$$[v_a,\, v_b] = f_{ab}{}^c\, v_c\,.$$

$$\pounds_{v_a} E^{mn} = f_a{}^{bc}\, v_b^m\, v_c^n$$ **微分方程式**

解: $$E^{mn}(x) = \hat{E}^{ab}\, e_a^m\, e_b^n + \pi^{mn}(x)$$

積分定数

Poisson-Lie structure

$$\pounds_{v_a} e_a^m = 0$$

$$\pounds_{v_a} \pi^{mn} = f_a{}^{bc}\, v_b^m\, v_c^n$$

$$3\,\pi^{q[m}\, \partial_q \pi^{np]} = 0 \longleftarrow$$ **Poisson tensor**

こんな条件を満たす時空を作れるか

$$\pounds_{v_a} E_{mn} = -f_a{}^{bc}\, E_{mp}\, v_b^p\, v_c^q\, E_{qn}\,,$$

$$[v_a,\, v_b] = f_{ab}{}^{c}\, v_c\,.$$

2D次元Lie代数

Drinfel'd double $T_A = (T_{\rm a},\, \tilde{T}^{\rm a})$

$$[T_{\rm a},\, T_{\rm b}] = f_{\rm ab}{}^{\rm c}\, T_{\rm c}\,, \qquad [\tilde{T}^{\rm a},\, \tilde{T}^{\rm b}] = f_{\rm c}{}^{\rm ab}\, \tilde{T}^{\rm c}\,,$$

$$[T_{\rm a},\, \tilde{T}^{\rm b}] = f_{\rm a}{}^{\rm bc}\, T_{\rm c} - f_{\rm ac}{}^{\rm b}\, \tilde{T}^{\rm c}\,.$$

$$g = {\rm e}^{x^{\rm a} T_{\rm a}}$$

左/右不変ベクトル場

$$\begin{cases} \ell_m^a\, {\rm d}x^m\, T_a = g^{-1}\, {\rm d}g \\ r_m^a\, {\rm d}x^m\, T_a = {\rm d}g\, g^{-1} \end{cases} \Rightarrow \begin{matrix} \ell_m^a\, v_b^m = \delta_b^a \\ r_m^a\, e_b^m = \delta_b^a \end{matrix} \qquad \begin{cases} [v_a,\, v_b] = f_{ab}{}^{c}\, v_c \\ [e_a,\, e_b] = -f_{ab}{}^{c}\, e_c \\ [v_a,\, e_b] = 0 \end{cases}$$

こんな条件を満たす時空を作れるか

$$\pounds_{v_a} E_{mn} = -f_a{}^{bc}\, E_{mp}\, v_b^p\, v_c^q\, E_{qn}\,,$$

$$[v_a,\, v_b] = f_{ab}{}^c\, v_c\,.$$

Drinfel'd double $\quad T_A = (T_{\rm a},\, \tilde{T}^{\rm a})$

$$[T_{\rm a},\, T_{\rm b}] = f_{\rm ab}{}^{\rm c}\, T_{\rm c}\,, \qquad [\tilde{T}^{\rm a},\, \tilde{T}^{\rm b}] = f_{\rm c}{}^{\rm ab}\, \tilde{T}^{\rm c}\,,$$

$$[T_{\rm a},\, \tilde{T}^{\rm b}] = f_{\rm a}{}^{\rm bc}\, T_{\rm c} - f_{\rm ac}{}^{\rm b}\, \tilde{T}^{\rm c}\,.$$

$$g = {\rm e}^{x^{\rm a} T_{\rm a}}$$

$$\pi^{mn}(x) = \pi^{ab}(x)\, e_a^m(x)\, e_b^n(x)$$

$$g^{-1}(x) \begin{pmatrix} T_{\rm a} \\ \tilde{T}^{\rm a} \end{pmatrix} g(x) = \begin{pmatrix} \delta_a^c & 0 \\ -\pi^{ac}(x) & \delta_c^a \end{pmatrix} \begin{pmatrix} a_c{}^b(x) & 0 \\ 0 & (a^{-1})_c{}^b(x) \end{pmatrix} \begin{pmatrix} T_{\rm b} \\ \tilde{T}^{\rm b} \end{pmatrix}$$

話を逆にすると,

Drinfel'd double $T_A = (T_{\rm a},\, \tilde{T}^{\rm a})$

$$[T_{\rm a},\, T_{\rm b}] = f_{\rm ab}{}^{\rm c}\, T_{\rm c}\,, \qquad [\tilde{T}^{\rm a},\, \tilde{T}^{\rm b}] = f_{\rm c}{}^{\rm ab}\, \tilde{T}^{\rm c}\,,$$

$$[T_{\rm a},\, \tilde{T}^{\rm b}] = f_{\rm a}{}^{\rm bc}\, T_{\rm c} - f_{\rm ac}{}^{\rm b}\, \tilde{T}^{\rm c}\,.$$

$$E^{mn}(x) = \hat{E}^{ab}\, e_a^m\, e_b^n + \pi^{mn}(x)$$

定数

逆行列

$$g_{mn} \equiv E_{(mn)}\,, \quad B_{mn} \equiv E_{[mn]}\,.$$

$$\begin{cases} \pounds_{v_a} E_{mn} = -f_a{}^{bc}\, E_{mp}\, v_b^p\, v_c^q\, E_{qn}\,, \\ [v_a,\, v_b] = f_{ab}{}^c\, v_c\,. \end{cases}$$

Poisson-Lie T-duality が実行できる

Poisson-Lie T-duality とは?

Drinfel'd double

$$[T_{\rm a},\, T_{\rm b}] = f_{\rm ab}{}^{\rm c}\, T_{\rm c}\,, \qquad [\tilde{T}^{\rm a},\, \tilde{T}^{\rm b}] = f_{\rm c}{}^{\rm ab}\, \tilde{T}^{\rm c}\,,$$

$$[T_{\rm a},\, \tilde{T}^{\rm b}] = f_{\rm a}{}^{\rm bc}\, T_{\rm c} - f_{\rm ac}{}^{\rm b}\, \tilde{T}^{\rm c}\,.$$

2*D*次元Lie代数

$$T_A = (T_{\rm a},\, \tilde{T}^{\rm a})$$

O(D,D) 変換

PL T-duality

$$T'_A = (T'_{\rm a},\, \tilde{T}'^{\rm a}) = C_A{}^B\, T_B$$

$$[T'_{\rm a},\, T'_{\rm b}] = f'_{\rm ab}{}^{\rm c}\, T'_{\rm c}\,, \qquad [\tilde{T}'^{\rm a},\, \tilde{T}'^{\rm b}] = f'_{\rm c}{}^{\rm ab}\, \tilde{T}'^{\rm c}\,,$$

$$[T'_{\rm a},\, \tilde{T}'^{\rm b}] = f'_{\rm a}{}^{\rm bc}\, T'_{\rm c} - f'_{\rm ac}{}^{\rm b}\, \tilde{T}'^{\rm c}\,.$$

2*D*次元Lie代数

Drinfel'd double

Poisson-Lie T-duality

O(D,D)変換

$$T_A = (T_{\rm a},\, \tilde{T}^{\rm a}) \qquad T'_A = C_A{}^B\, T_B \qquad T'_A = (T'_{\rm a},\, \tilde{T}'^{\rm a})$$

$$f_{ab}{}^c, \quad f_a{}^{bc} \qquad\qquad f'_{ab}{}^c, \quad f'_a{}^{bc}$$

$$E^{mn}(x) = \hat{E}^{ab}\, e_a^m\, e_b^n + \pi^{mn} \qquad\qquad E'^{mn}(x) = \hat{E}'^{ab}\, e'^m_a\, e'^n_b + \pi'^{mn}$$

定数 **定数**

$$\hat{E}^{ab} \to \hat{E}'^{ab}$$

Abelian O(*D*,*D*)

$$C_A{}^B$$

$$\left\{\begin{aligned} g_{mn} &\equiv E_{(mn)}\,, \\ B_{mn} &\equiv E_{[mn]}\,. \end{aligned}\right. \qquad \left\{\begin{aligned} g'_{mn} &\equiv E'_{(mn)}\,, \\ B'_{mn} &\equiv E'_{[mn]}\,. \end{aligned}\right.$$

PL T-duality

具体例: Abelian T-duality

O(D,D)変換

$$T_A = (T_{\rm a}, \tilde{T}^{\rm a}) \qquad T'_A = C_A{}^B\, T_B \qquad T'_A = (T'_{\rm a}, \tilde{T}'^{\rm a})$$

$$\cancel{f_{ab}{}^c, \; f_a{}^{bc}} \qquad \cancel{f'_{ab}{}^c, \; f'_a{}^{bc}}$$

$$\delta_a^m \qquad \delta_a^m$$

$$E^{mn}(x) = \hat{E}^{ab}\, e_a^m\, e_b^n + \cancel{\pi^{mn}} \qquad E'^{mn}(x) = \hat{E}'^{ab}\, e'^m_a\, e'^n_b + \cancel{\pi'^{mn}}$$

定数 **定数**

$\hat{E}^{ab} \to \hat{E}'^{ab}$

Abelian O(*D*,*D*)

$C_A{}^B$

$$\begin{cases} g_{mn} \equiv E_{(mn)}\,, \\ B_{mn} \equiv E_{[mn]}\,. \end{cases} \qquad \begin{cases} g'_{mn} \equiv E'_{(mn)}\,, \\ B'_{mn} \equiv E'_{[mn]}\,. \end{cases}$$

具体例: Non-Abelian T-duality

赤 $\leftrightarrow$ 青

$$T_A = (T_{\rm a},\, \tilde{T}^{\rm a}) \qquad T'_A = C_A{}^B\, T_B \qquad T'_A = (T'_{\rm a},\, \tilde{T}'^{\rm a})$$

$$f_{ab}{}^c,\quad \cancel{f_a{}^{bc}} \qquad\qquad \cancel{f'_{ab}{}^c},\quad f'_a{}^{bc}$$

$$\delta_a^m$$

$$E^{mn}(x) = \hat{E}^{ab}\, e_a^m\, e_b^n + \cancel{\pi^{mn}} \qquad\qquad E'^{mn}(x) = \hat{E}'^{ab}\, e'^m_a\, e'^n_b + \underline{\pi'^{mn}}$$

定数 　　　　定数

$$f'_a{}^{mn}\, x'^a$$

$$\boxed{\begin{gathered}\hat{E}^{ab} \to \hat{E}'^{ab} \\ \textbf{Abelian O}(D,D) \\ C_A{}^B\end{gathered}}$$

$$\begin{cases} g_{mn} \equiv E_{(mn)}\,, \\ B_{mn} \equiv E_{[mn]}\,. \end{cases} \qquad\qquad \begin{cases} g'_{mn} \equiv E'_{(mn)}\,, \\ B'_{mn} \equiv E'_{[mn]}\,. \end{cases}$$

具体例: Yang-Baxter 変形

$$T_A = (T_{\rm a},\, \tilde{T}^{\rm a}) \qquad T'_A = C_A{}^B\, T_B \qquad T'_A = (T'_{\rm a},\, \tilde{T}'^{\rm a})$$

$$f_{ab}{}^c, \quad \cancel{f_a{}^{bc}} \qquad\qquad f_{ab}{}^c, \quad f'_a{}^{bc}$$

$$C_A{}^B = \begin{pmatrix} \delta_a^b & 0 \\ r^{ab} & \delta_b^a \end{pmatrix} \qquad \begin{cases} T'_a = T_a\,, \\ T'^a = T^a + r^{ab}\, T_b\,. \end{cases}$$

$$[T_{\rm a},\, T_{\rm b}] = f_{\rm ab}{}^{\rm c}\, T_{\rm c}\,, \quad [\tilde{T}^{\rm a},\, \tilde{T}^{\rm b}] = 0\,,$$
$$[T_{\rm a},\, \tilde{T}^{\rm b}] = -f_{\rm ac}{}^{\rm b}\, \tilde{T}^{\rm c}\,.$$

$$f'_a{}^{bc} = 2\, r^{e[b}\, f_{da}{}^{c]}$$

$$[T'_{\rm a},\, T'_{\rm b}] = f_{\rm ab}{}^{\rm c}\, T'_{\rm c}\,,$$
$$[T'_{\rm a},\, \tilde{T}'^{\rm b}] = f'_{\rm a}{}^{\rm bc}\, T'_{\rm c} - f_{\rm ac}{}^{\rm b}\, \tilde{T}'^{\rm c}\,,$$
$$[\tilde{T}'^{\rm a},\, \tilde{T}'^{\rm b}] = f'_{\rm c}{}^{\rm ab}\, \tilde{T}'^{\rm c} + 3\, r^{d_1[a}\, r^{|d_2|b}\, f_{d_1 d_2}{}^{c]}\, T'_{\rm c}\,,$$

具体例: Yang-Baxter 変形

$$T_A = (T_{\mathrm{a}},\, \tilde{T}^{\mathrm{a}}) \xrightarrow{\;T'_A = C_A{}^B\, T_B\;} T'_A = (T'_{\mathrm{a}},\, \tilde{T}'^{\mathrm{a}})$$

$$f_{ab}{}^{c}, \quad \cancel{f_a{}^{bc}}$$

$$f_{ab}{}^{c}, \quad f'_a{}^{bc} = 2\, r^{e[b}\, f_{da}{}^{c]}$$

$$C_A{}^B = \begin{pmatrix} \delta_a^b & 0 \\ r^{ab} & \delta_b^a \end{pmatrix} \qquad \begin{cases} T'_a = T_a\,, \\ T'^a = T^a + r^{ab}\, T_b\,. \end{cases}$$

$$[T'_{\mathrm{a}},\, T'_{\mathrm{b}}] = f_{\mathrm{ab}}{}^{\mathrm{c}}\, T'_{\mathrm{c}}\,,$$

$$[T'_{\mathrm{a}},\, \tilde{T}'^{\mathrm{b}}] = f'_{\mathrm{a}}{}^{\mathrm{bc}}\, T'_{\mathrm{c}} - f_{\mathrm{ac}}{}^{\mathrm{b}}\, \tilde{T}'^{\mathrm{c}}\,,$$

$$[\tilde{T}'^{\mathrm{a}},\, \tilde{T}'^{\mathrm{b}}] = f'_{\mathrm{c}}{}^{\mathrm{ab}}\, \tilde{T}'^{\mathrm{c}} + \underbrace{\boxed{3\, r^{d_1[a}\, r^{|d_2|b}\, f_{d_1 d_2}{}^{c]}}}_{=0}\, T'_{\mathrm{c}}\,,$$

Classical Yang-Baxter Equations

具体例: Yang-Baxter 変形

$$T_A = (T_{\mathrm{a}},\ \tilde{T}^{\mathrm{a}}) \qquad T'_A = (T'_{\mathrm{a}},\ \tilde{T}'^{\mathrm{a}})$$

$$T'_A = C_A{}^B\, T_B$$

$$f_{ab}{}^c, \quad \cancel{f_a{}^{bc}} \qquad\qquad f_{ab}{}^c, \quad f'_a{}^{bc} = 2\, r^{e[b}\, f_{da}{}^{c]}$$

$$C_A{}^B = \begin{pmatrix} \delta_a^b & 0 \\ r^{ab} & \delta_b^a \end{pmatrix}$$

classical r-matrix

$$E^{mn}(x) = \hat{E}^{ab}\, e_a^m\, e_b^n \qquad\qquad E'^{mn} = E^{mn} + r^{ab}\, v_a^m\, v_b^n$$

定数

$$\begin{cases} g_{mn} \equiv E_{(mn)}\,, \\ B_{mn} \equiv E_{[mn]}\,. \end{cases} \qquad\qquad \begin{cases} g'_{mn} \equiv E'_{(mn)}\,, \\ B'_{mn} \equiv E'_{[mn]}\,. \end{cases}$$

YB 変形

SUGRA解の生成手法 **[吉田さん達]**

Poisson-Lie T-plurality

Drinfel'd double

2D次元のLie群

$({f'}_{ab}{}^{c},\ {f'}_{a}{}^{bc})$

YB変形

2つの代数への分解が複数通り存在

$({f'}_{ab}{}^{c},\ 0)$

NATD

$(0,\ {f'}_{ab}{}^{c})$

$(f_{ab}{}^{c},\ f_{a}{}^{bc})$ **PLTD** $(f_{a}{}^{bc},\ f_{ab}{}^{c})$

Poisson-Lie T-plurality と呼ぶ。 **[Unge '02]**

(単に PL T-duality とも呼ぶ)

Poisson-Lie T-plurality

Drinfel'd double

2D次元のLie群

$$E'''_{mn}$$

$$E''_{mn}$$

$$\left(f_{ab}{}^{c},\ f_{a}{}^{bc}\right)$$

$$\Downarrow$$

$$E_{mn} = (g+B)_{mn}$$

$$\left(f_{a}{}^{bc},\ f_{ab}{}^{c}\right)$$

$$\Downarrow$$

$$E'_{mn}$$

1つのSUGRA解から
一連のSUGRA解を
生成できる.

コメント: Jacobi-Lie T-plurality

[Fernandez-Melgarejo, YS, 2104.00007]

Drinfel'd double

Jacobi-Lie bialgebra

DD$^+$ $\quad \left[\,\mathrm{O}(D,D)\times\mathbb{R}^{+}\,\right]$

1対1に対応

$$T_{\rm a}\circ T_{\rm b}=f_{\rm ab}{}^{\rm c}\,T_{\rm c}\,,\qquad \tilde{T}^{\rm a}\circ\tilde{T}^{\rm b}=f_{\rm c}{}^{\rm ab}\,\tilde{T}^{\rm c}\,,$$

$$T_{\rm a}\circ\tilde{T}^{\rm b}=f_{\rm a}{}^{\rm bc}\,T_{\rm c}-f_{\rm ac}{}^{\rm b}\,\tilde{T}^{\rm c}+2\,Z_a\,\tilde{T}^{\rm b}\,,$$

$$\tilde{T}^{\rm a}\circ T_{\rm b}=-f_{\rm b}{}^{\rm ac}\,T_{\rm c}+\left(f_{\rm bc}{}^{\rm a}+2\,\delta^a_b\,Z_c-2\,\delta^a_c\,Z_b\right)\tilde{T}^{\rm c}\,.$$

$$\begin{cases} T_{\rm a}\circ T_{\rm b}=-T_{\rm b}\circ T_{\rm a}=[T_{\rm a},\,T_{\rm b}] \\ T_{\rm a}\circ\tilde{T}^{\rm b}\neq-\tilde{T}^{\rm b}\circ T_{\rm a} \end{cases}$$

Lie代数ではない
⇨ Leibniz 代数

~~Jacobi~~ Leibniz 恒等式を満たす

$$T_A\circ(T_B\circ T_C)=(T_A\circ T_B)\circ T_C+T_B\circ(T_A\circ T_C)\,.$$

コメント: Jacobi-Lie T-plurality

[Fernandez-Melgarejo, YS, 2104.00007]

$$T_{\mathrm{a}} \circ T_{\mathrm{b}} = f_{\mathrm{ab}}{}^{\mathrm{c}}\, T_{\mathrm{c}}\,, \qquad \tilde{T}^{\mathrm{a}} \circ \tilde{T}^{\mathrm{b}} = f_{\mathrm{c}}{}^{\mathrm{ab}}\, \tilde{T}^{\mathrm{c}}\,,$$

$$T_{\mathrm{a}} \circ \tilde{T}^{\mathrm{b}} = f_{\mathrm{a}}{}^{\mathrm{bc}}\, T_{\mathrm{c}} - f_{\mathrm{ac}}{}^{\mathrm{b}}\, \tilde{T}^{\mathrm{c}} + 2\, Z_a\, \tilde{T}^{\mathrm{b}}\,,$$

$$\tilde{T}^{\mathrm{a}} \circ T_{\mathrm{b}} = -f_{\mathrm{b}}{}^{\mathrm{ac}}\, T_{\mathrm{c}} + \big(f_{\mathrm{bc}}{}^{\mathrm{a}} + 2\,\delta^a_b\, Z_c - 2\,\delta^a_c\, Z_b\big)\, \tilde{T}^{\mathrm{c}}\,.$$

e_a^m $\quad \pi^{mn}$ $\quad \Delta$

Jacobi-Lie
structure

$$E^{mn}(x) = \mathrm{e}^{2\Delta}\big[\underbrace{\hat{E}^{ab}}_{\text{定数}}\, e_a^m\, e_b^n + \pi^{mn}(x)\big]$$

$$g_{mn} \equiv E_{(mn)}\,, \qquad B_{mn} \equiv E_{[mn]}\,.$$

コメント: Jacobi-Lie T-plurality

[Fernandez-Melgarejo, YS, 2104.00007]

O(*D*,*D*)変換

DD$^+$ $\longrightarrow$ **DD^{+}'**

$$T'_A = C_A{}^B\, T_B$$

$$g_{mn}\,,\quad B_{mn} \qquad\qquad g'_{mn}\,,\quad B'_{mn}$$

Jacobi-Lie T-plurality

DFT (RR場込み) の対称性

CYBEの解

例: <u>Yang-Baxter変形</u>

$$E'^{mn} = E^{mn} + r^{ab}\, v_a^m\, v_b^n$$

Δ

Non-Abelian T-duality

$$\Downarrow$$

Non-Abelian U-duality

Non-Abelian U-duality

方針1: [YS, 1903.12175]
伝統的な NATD のときと同様,
membrane 理論をゲージ化.

2+1次元の
gauged action は作れる [Duff, Lu, '90; Hull, Spence, '91] を参考に

String 座標 $x^i(\sigma)$

補助場 $\tilde{x}_a(\sigma) = x'^i(\sigma)$

membrane 座標 $x^i(\sigma)$

補助場 $y_{ab}(\sigma)$

数が合わない,
ゲージ場が消去できない…

dual action をどう作るのかよくわからない.

Non-Abelian U-duality

<u>方針2:</u> [YS, 1911.06320]

Drinfel'd double の一般化を探るのが自然?

O(*D*,*D*): $T_A = (T_{\mathrm{a}},\ \tilde{T}^{\mathrm{a}})$ **(Drinfel'd double)**

F1

$\mathbf{E}_{n(n)}$: $T_A = (T_{\mathrm{a}},\ T^{a_1 a_2},\ T^{a_1\cdots a_5},\ \cdots)$

M2 M5

ベクトル表現

ダブルではなく赤い generator を沢山導入.

Non-Abelian U-duality

$\mathbf{E}_{n(n)}$:
$$T_A = (T_{\mathrm{a}},\, T^{a_1 a_2},\, T^{a_1\cdots a_5},\, \cdots)$$

$$[T_{\mathrm{a}},\, T_{\mathrm{b}}] = f_{\mathrm{ab}}{}^{\mathrm{c}}\, T_{\mathrm{c}}\,,$$
$$[T_{\mathrm{a}},\, T^{\mathrm{b}_1\mathrm{b}_2}] = \cdots ??$$
$$[T_{\mathrm{a}},\, T^{\mathrm{b}_1\cdots\mathrm{b}_5}] = \cdots ??$$

ある方針から代数を決定.

(⇨ 三周目)

具体例: SL(5) EDA

$\mathrm{E}_{4(4)}$ = **SL(5)**

$$T_A = (T_{\mathrm{a}},\, T^{a_1 a_2})$$

**[YS, 1911.06320;
Malek, Thompson, 1911.07833]**

$$T_a \circ T_b = f_{ab}{}^{c}\, T_c\,,$$

$$T_a \circ T^{b_1 b_2} = f_a{}^{b_1 b_2 c}\, T_c + 2\, f_{ac}{}^{[b_1}\, T^{b_2]c} + 3\, Z_a\, T^{b_1 b_2}\,,$$

$$T^{a_1 a_2} \circ T_b = -f_b{}^{a_1 a_2 c}\, T_c + 3\, f_{[c_1 c_2}{}^{[a_1}\, \delta^{a_2]}_{b]}\, T^{c_1 c_2} - 9\, Z_c\, \delta^{[c}_b\, T^{a_1 a_2]}\,,$$

$$T^{a_1 a_2} \circ T^{b_1 b_2} = -2\, f_c{}^{a_1 a_2 [b_1}\, T^{b_2]c}\,.$$

Exceptional Drinfel'd Algebra (EDA)

Lie代数ではなくLeibniz代数

“ $\mathcal{E}_n$ 代数”

EDA

$$T_a \circ T_b = f_{ab}{}^{c}\, T_c\,,$$
$$T_a \circ T^{b_1 b_2} = f_a{}^{b_1 b_2 c}\, T_c + 2\, f_{ac}{}^{[b_1}\, T^{b_2]c} + 3\, Z_a\, T^{b_1 b_2}\,,$$
$$T^{a_1 a_2} \circ T_b = -f_b{}^{a_1 a_2 c}\, T_c + 3\, f_{[c_1 c_2}{}^{[a_1}\, \delta^{a_2]}_{b]}\, T^{c_1 c_2} - 9\, Z_c\, \delta^{[c}_{b}\, T^{a_1 a_2]}\,,$$
$$T^{a_1 a_2} \circ T^{b_1 b_2} = -2\, f_c{}^{a_1 a_2 [b_1}\, T^{b_2]c}\,.$$

Leibniz恒等式

$$T_A \circ (T_B \circ T_C) = (T_A \circ T_B) \circ T_C + T_B \circ (T_A \circ T_C)\,.$$

(Jacobi恒等式)

$$0 = 3\, f_{[ab}{}^{e}\, f_{c]e}{}^{d}\,,$$

(コサイクル条件)

$$0 = 6\, f_{[a|d}{}^{[c_1|}\, f_{|b]}{}^{d|c_2 c_3]} - f_{ab}{}^{d}\, f_d{}^{c_1 c_2 c_3} - 6\, Z_{[a}\, f_{b]}{}^{c_1 c_2 c_3}\,,$$
$$0 = f_c{}^{d a_1 a_2}\, f_d{}^{b_1 b_2 b_3} - 3\, f_d{}^{a_1 a_2 [b_1}\, f_c{}^{b_2 b_3]d}\,,$$
$$0 = f_{c_1 c_2}{}^{a}\, f_b{}^{d c_1 c_2} - 6\, Z_c\, f_b{}^{adc}\,,$$
$$0 = f_{ab}{}^{c}\, Z_c\,.$$

(Fundamental identity)

(c.f. Lie 3-algebra で出てきた条件)

Drinfel'd doubleは
Poisson-Lie structure を与えた.

$g^{-1}\,T_A\,g = \ldots$

$$f_a{}^{bc} \quad \Rightarrow \quad \pi^{mn}(x)$$

bi-vector

$$\begin{cases} 3\,\pi^{q[m}\,\partial_q \pi^{np]} = 0 \\ \pounds_{v_a}\pi^{mn} = f_a{}^{bc}\,v_b^m\,v_c^n \end{cases}$$

Poisson tensor

同様に,Exceptional Drinfel'd Algebra (EDA)は
以下の tri-vector field を与える! [YS, 1911.06320]

$$\mathrm{e}^{-x^c\,T_c\circ}\,T_A = \begin{pmatrix} \delta_a^c & 0 \\ -\frac{\pi^{a_1 a_2 c}}{\sqrt{2!}} & \delta^{a_1 a_2}_{c_1 c_2} \end{pmatrix} \begin{pmatrix} a_c{}^b & 0 \\ 0 & (a^{-1})_{[b_1}{}^{c_1}\,(a^{-1})_{b_2]}{}^{c_2} \end{pmatrix} T_B$$

Nambu-Poisson tensor

$$f_a{}^{bcd} \quad \Rightarrow \quad \pi^{ijk}(x)$$

tri-vector

$$\begin{cases} 3\,\pi^{l[j_1 j_2}\,\nabla_l \pi^{k]i_1 i_2} = \pi^{i_1 i_2 l}\,\nabla_l \pi^{j_1 j_2 k} \\ \text{(Weitzenböck接続: } \Gamma_i{}^j{}_k \equiv v_a^j\,\partial_i \ell_k^a \text{)} \\ \pounds_{v_a}\pi^{ijk} = f_a{}^{bcd}\,v_b^i\,v_c^j\,v_d^k \end{cases}$$

Nambu-Lie structure

$$f_{ab}{}^{c} \qquad f_{a}{}^{bcd} \qquad Z_a$$

$$e_a^i(x) \qquad \pi^{ijk}(x) \qquad \Delta(x)$$

$$\Downarrow$$

$$g_{ij}(x), \qquad C_{ijk}(x)$$

複雑ですが, SUGRA場の構成方法は
Poisson-Lie T-duality の場合と同様.
(DFT, EFTを知っていれば素直 ⇨ 三周目)

Non-Abelian U-duality
(Nambu-Lie U-duality)

EDA

$$T_a \circ T_b = f_{ab}{}^{c}\, T_c\,,$$
$$T_a \circ T^{b_1 b_2} = f_a{}^{b_1 b_2 c}\, T_c + 2\, f_{ac}{}^{[b_1}\, T^{b_2]c} + 3\, Z_a\, T^{b_1 b_2}\,,$$
$$T^{a_1 a_2} \circ T_b = -f_b{}^{a_1 a_2 c}\, T_c + 3\, f_{[c_1 c_2}{}^{[a_1}\, \delta^{a_2]}_{b]}\, T^{c_1 c_2} - 9\, Z_c\, \delta^{[c}_b\, T^{a_1 a_2]}\,,$$
$$T^{a_1 a_2} \circ T^{b_1 b_2} = -2\, f_c{}^{a_1 a_2 [b_1}\, T^{b_2]c}\,.$$

Nambu-Lie structure

SUGRA の場

$$T_A = (T_{\mathrm{a}},\, T^{a_1 a_2},\, T^{a_1 \cdots a_5},\, \cdots)$$

$$T'_A = C_A{}^{B}\, T_B \quad (C_A{}^{B} \in E_{n(n)})$$

$$T'_A = (T'_{\mathrm{a}},\, T'^{a_1 a_2},\, T'^{a_1 \cdots a_5},\, \cdots)$$

SUGRA の場'

EDA'

Non-Abelian U-duality
(Nambu-Lie U-duality)

予想

Nambu-Lie U-duality は
SUGRA解をSUGRA解に移す
(EFTの対称性)

[Musaev, YS, 2012.13263]

非自明な具体例を数多く見つけた

SUGRA の場

N-L U-duality

SUGRA の場'

$E_{4(4)}$ **EDA** **(10次元Leibniz代数)** **[YS, 1911.06320; Malek, Thompson, 1911.07833]**

$E_{6(6)}$ **EDA** **(27次元Leibniz代数)** $T_A = (T_{\mathrm{a}},\, T^{a_1 a_2},\, T^{a_1 \cdots a_5})$

[Malek, YS, Thompson, 2007.08510]

embedding tensor formalism

$E_{8(8)}$ **EDA** **(248次元Leibniz代数)** **[YS, 2009.04454]**

$$f_a{}^{a_1 a_2 a_3} \;\Downarrow\; \pi^{i_1 i_2 i_3} \qquad f_a{}^{a_1 \cdots a_6} \;\Downarrow\; \pi^{i_1 \cdots i_6} \qquad f_a{}^{a_1 \cdots a_8, a} \;\Downarrow\; \pi^{i_1 \cdots i_8, i}$$

Type IIB 理論の言葉で書いたEDAも作った.

$$T_A = (T_{\mathrm{a}},\, T_\alpha^a,\, T^{a_1 a_2 a_3},\, T_\alpha^{a_1 \cdots a_5},\, \cdots)$$

F1/D1 D3 D5/NS5

$E_{7(7)}$ EDA (M理論)

$$
\begin{aligned}
T_a \circ T_b &= f_{ab}{}^c\, T_c\,,\\
T_a \circ T^{b_1 b_2} &= f_a{}^{b_1 b_2 c}\, T_c + 2\, f_{ac}{}^{[b_1}\, T^{b_2]c} + 3\, Z_a\, T^{b_1 b_2}\,,\\
T_a \circ T^{b_1\cdots b_5} &= -f_a{}^{b_1\cdots b_5 c}\, T_c - 10\, f_a{}^{[b_1 b_2 b_3}\, T^{b_4 b_5]} - 5\, f_{ac}{}^{[b_1}\, T^{b_2\cdots b_5]c} + 6\, Z_a\, T^{b_1\cdots b_5}\,,\\
T_a \circ T^{b_1\cdots b_7,b'} &= 7\, f_a{}^{[b_1\cdots b_6}\, T^{b_7]b'} - 21\, f_a{}^{b'[b_1 b_2}\, T^{b_3\cdots b_7]}\\
&\quad - 7\, f_{ac}{}^{[b_1}\, T^{b_2\cdots b_7]c,b'} - f_{ac}{}^{b'}\, T^{b_1\cdots b_7,c} + 9\, Z_a\, T^{b_1\cdots b_7,b'}\,,\\
T^{a_1 a_2} \circ T_b &= -f_b{}^{a_1 a_2 c}\, T_c + 3\, f_{[c_1 c_2}{}^{[a_1}\, \delta^{a_2]}_{b]}\, T^{c_1 c_2} - 9\, Z_c\, \delta^{[c}_b\, T^{a_1 a_2]}\,,\\
T^{a_1 a_2} \circ T^{b_1 b_2} &= -2\, f_c{}^{a_1 a_2 [b_1}\, T^{b_2]c} - f_{c_1 c_2}{}^{[a_1}\, T^{a_2] b_1 b_2 c_1 c_2} + 3\, Z_c\, T^{a_1 a_2 b_1 b_2 c}\,,\\
T^{a_1 a_2} \circ T^{b_1\cdots b_5} &= 5\, f_c{}^{a_1 a_2 [b_1}\, T^{b_2\cdots b_5]c} - 3\, \delta^{a_1 a_2}_{de}\, f_{c_1 c_2}{}^d\, T^{b_1\cdots b_5 [c_1 c_2, e]} + 9\, Z_c\, T^{b_1\cdots b_5 [a_1 a_2, c]}\,,\\
T^{a_1 a_2} \circ T^{b_1\cdots b_7,b'} &= 7\, f_c{}^{a_1 a_2 [b_1}\, T^{b_2\cdots b_7]c,b'} + f_c{}^{a_1 a_2 b'}\, T^{b_1\cdots b_7,c}\,,\\
T^{a_1\cdots a_5} \circ T_b &= f_b{}^{a_1\cdots a_5 c}\, T_c + 10\, f_b{}^{[a_1 a_2 a_3}\, T^{a_4 a_5]} + 20\, f_c{}^{[a_1 a_2 a_3}\, \delta^{a_4}_b\, T^{a_5]c}\\
&\quad + 5\, f_{bc}{}^{[a_1}\, T^{a_2\cdots a_5]c} + 10\, f_{c_1 c_2}{}^{[a_1}\, \delta^{a_2}_b\, T^{a_3 a_4 a_5] c_1 c_2} - 36\, Z_c\, \delta^{[c}_b\, T^{a_1\cdots a_5]}\,,\\
T^{a_1\cdots a_5} \circ T^{b_1 b_2} &= 2\, f_c{}^{a_1\cdots a_5 [b_1}\, T^{b_2]c} - 10\, f_c{}^{[a_1 a_2 a_3}\, T^{a_4 a_5] b_1 b_2 c}\\
&\quad + 5\, f_{c_1 c_2}{}^{[a_1}\, T^{a_2\cdots a_5] c_1 c_2 [b_1, b_2]} - 12\, Z_c\, T^{c a_1\cdots a_5 [b_1, b_2]}\,,\\
T^{a_1\cdots a_5} \circ T^{b_1\cdots b_5} &= -5\, f_c{}^{a_1\cdots a_5 [b_1}\, T^{b_2\cdots b_5]c} - 30\, f_c{}^{[a_1 a_2 a_3}\, \delta^{a_4 a_5]c}_{d_1 d_2 e}\, T^{b_1\cdots b_5 d_1 d_2, e}\,,\\
T^{a_1\cdots a_5} \circ T^{b_1\cdots b_7,b'} &= -7\, f_c{}^{a_1\cdots a_5 [b_1}\, T^{b_2\cdots b_7]c,b'} - f_c{}^{a_1\cdots a_5 b'}\, T^{b_1\cdots b_7,c}\,,\\
T^{a_1\cdots a_7,a'} \circ T_b &= -21\, f_c{}^{[a_1\cdots a_6}\, \delta^{a_7]a' c}_{b d_1 d_2}\, T^{d_1 d_2} - 126\, f_c{}^{a'[a_1 a_2}\, \delta^{a_3\cdots a_7]c}_{b d_1\cdots d_5}\, T^{d_1\cdots d_5}\,,\\
T^{a_1\cdots a_7,a'} \circ T^{b_1 b_2} &= 7\, f_c{}^{[a_1\cdots a_6}\, T^{a_7]a' c b_1 b_2} - 42\, f_c{}^{a'[a_1 a_2}\, T^{a_3\cdots a_7]c[b_1, b_2]}\,,\\
T^{a_1\cdots a_7,a'} \circ T^{b_1\cdots b_5} &= 21\, f_c{}^{[a_1\cdots a_6}\, \delta^{a_7]a' c}_{d_1 d_2 e}\, T^{b_1\cdots b_5 d_1 d_2, e}\,,\\
T^{a_1\cdots a_7,a'} \circ T^{b_1\cdots b_7,b'} &= 0\,.
\end{aligned}
$$

$E_{6(6)}$ EDA (Type IIB)

$$
\begin{aligned}
T_{a} \circ T_{b} &= f_{ab}{}^{c}\, T_{c}\,, \\
T_{a} \circ T^{b}_{\beta} &= f_{a\beta}{}^{cb}\, T_{c} + f_{a\beta}{}^{\gamma}\, T^{b}_{\gamma} - f_{ac}{}^{b}\, T^{c}_{\beta} + 2\, Z_{a}\, T^{b}_{\beta}\,, \\
T_{a} \circ T^{b_1 b_2 b_3} &= f_{a}{}^{c b_1 b_2 b_3}\, T_{c} + 3\, \epsilon^{\gamma\delta}\, f_{a\gamma}{}^{[b_1 b_2}\, T^{b_3]}_{\delta} - 3\, f_{ac}{}^{[b_1}\, T^{b_2 b_3]c} + 4\, Z_{a}\, T^{b_1 b_2 b_3}\,, \\
T_{a} \circ T^{b_1 \cdots b_5}_{\beta} &= 5\, f_{a}{}^{[b_1 \cdots b_4}\, T^{b_5]}_{\beta} - 10\, f_{a\beta}{}^{[b_1 b_2}\, T^{b_3 b_4 b_5]} \\
&\quad + f_{a\beta}{}^{\gamma}\, T^{b_1 \cdots b_5}_{\gamma} - 5\, f_{ac}{}^{[b_1}\, T^{b_2 \cdots b_5]c}_{\beta} + 6\, Z_{a}\, T^{b_1 \cdots b_5}_{\beta}\,, \\
T^{a}_{\alpha} \circ T_{b} &= f_{b\alpha}{}^{ac}\, T_{c} + 2\, \delta^{a}_{[b}\, f_{c]\alpha}{}^{\gamma}\, T^{c}_{\gamma} + f_{bc}{}^{a}\, T^{c}_{\alpha} + 4\, Z_{c}\, \delta^{[a}_{b}\, T^{c]}_{\alpha}\,, \\
T^{a}_{\alpha} \circ T^{b}_{\beta} &= -f_{c\alpha}{}^{ab}\, T^{c}_{\beta} - f_{c\alpha}{}^{\gamma}\, \epsilon_{\gamma\beta}\, T^{cab} + \tfrac{1}{2}\, \epsilon_{\alpha\beta}\, f_{c_1 c_2}{}^{a}\, T^{c_1 c_2 b} - 2\, \epsilon_{\alpha\beta}\, Z_{c}\, T^{abc}\,, \\
T^{a}_{\alpha} \circ T^{b_1 b_2 b_3} &= -3\, f_{c\alpha}{}^{a[b_1}\, T^{b_2 b_3]c} - f_{c\alpha}{}^{\gamma}\, T^{acb_1 b_2 b_3}_{\gamma} - \tfrac{1}{2}\, f_{c_1 c_2}{}^{a}\, T^{c_1 c_2 b_1 b_2 b_3}_{\alpha} \\
&\quad + 2\, Z_{c}\, T^{ab_1 b_2 b_3 c}_{\alpha}\,, \\
T^{a}_{\alpha} \circ T^{b_1 \cdots b_5}_{\beta} &= -5\, f_{c\alpha}{}^{a[b_1}\, T^{b_2 \cdots b_5]c}_{\beta}\,, \\
T^{a_1 a_2 a_3} \circ T_{b} &= -f_{b}{}^{c a_1 a_2 a_3}\, T_{c} - 6\, \epsilon^{\gamma\delta}\, f_{[b|\gamma}{}^{[a_1 a_2}\, \delta^{a_3]}_{|c]}\, T^{c}_{\delta} \\
&\quad + 3\, f_{bc}{}^{[a_1}\, T^{a_2 a_3]c} + 3\, f_{c_1 c_2}{}^{[a_1}\, \delta^{a_2}_{b}\, T^{a_3]c_1 c_2} + 16\, Z_{c}\, \delta^{[a_1}_{b}\, T^{a_2 a_3 c]}\,, \\
T^{a_1 a_2 a_3} \circ T^{b}_{\beta} &= -f_{c}{}^{a_1 a_2 a_3 b}\, T^{c}_{\beta} + 3\, f_{c\beta}{}^{[a_1 a_2}\, T^{a_3]bc} + \tfrac{3}{2}\, f_{c_1 c_2}{}^{[a_1}\, T^{a_2 a_3]bc_1 c_2}_{\beta} \\
&\quad - 4\, Z_{c}\, T^{a_1 a_2 a_3 bc}_{\beta}\,, \\
T^{a_1 a_2 a_3} \circ T^{b_1 b_2 b_3} &= -3\, f_{c}{}^{a_1 a_2 a_3 [b_1}\, T^{b_2 b_3]c} + 3\, \epsilon^{\gamma\delta}\, f_{c\gamma}{}^{[a_1 a_2}\, T^{a_3]b_1 b_2 b_3 c}_{\delta}\,, \\
T^{a_1 a_2 a_3} \circ T^{b_1 \cdots b_5}_{\beta} &= -5\, f_{c}{}^{a_1 a_2 a_3 [b_1}\, T^{b_2 \cdots b_5]c}_{\beta}\,, \\
T^{a_1 \cdots a_5}_{\alpha} \circ T_{b} &= -10\, f_{[b}{}^{[a_1 \cdots a_4}\, \delta^{a_5]}_{c]}\, T^{c}_{\alpha} - 30\, f_{c\alpha}{}^{[a_1 a_2}\, \delta^{a_3}_{b}\, T^{a_4 a_5]c} \\
&\quad + 10\, f_{b\alpha}{}^{[a_1 a_2}\, T^{a_3 a_4 a_5]} + 5\, f_{c\alpha}{}^{\gamma}\, \delta^{[a_1}_{b}\, T^{a_2 \cdots a_5]c}_{\gamma} - f_{b\alpha}{}^{\gamma}\, T^{a_1 \cdots a_5}_{\gamma} \\
&\quad + 5\, f_{bc}{}^{[a_1}\, T^{a_2 \cdots a_5]c}_{\alpha} + 10\, f_{c_1 c_2}{}^{[a_1}\, \delta^{a_2}_{b}\, T^{a_3 a_4 a_5]c_1 c_2}_{\alpha} + 36\, Z_{c}\, \delta^{[a_1}_{b}\, T^{a_2 \cdots a_5 c]}_{\alpha}\,, \\
T^{a_1 \cdots a_5}_{\alpha} \circ T^{b}_{\beta} &= -5\, \epsilon_{\alpha\beta}\, f_{c}{}^{[a_1 \cdots a_4}\, T^{a_5]bc} + 10\, f_{c\alpha}{}^{[a_1 a_2}\, T^{a_3 a_4 a_5]bc}_{\beta}\,, \\
T^{a_1 \cdots a_5}_{\alpha} \circ T^{b_1 b_2 b_3} &= 5\, f_{c}{}^{[a_1 \cdots a_4}\, T^{a_5]b_1 b_2 b_3 c}_{\alpha}\,, \\
T^{a_1 \cdots a_5}_{\alpha} \circ T^{b_1 \cdots b_5}_{\beta} &= 0\,.
\end{aligned}
$$

M理論の解 ⇨ type IIBの解

$R^{\alpha}{}_{\beta}$

$f_{a\alpha}{}^{\beta}$

$f_{23}{}^{1} = 1$

$\mathrm{E}_{3(3)}$ **EDA**

$f_{22}{}^{1} = 1$

$$g_{ij} = r_i^a\, r_j^b\, \hat{g}_{ab}$$

$$\hat{g}_{MN} = \begin{pmatrix} 0 & 0 & 1 & 0 & & 0 \\ 0 & 1 & 0 & 0 & & 0 \\ 1 & 0 & 2\,y & 0 & & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ & & & & \ddots & \\ 0 & 0 & 0 & 0 & & 1 \end{pmatrix}$$

NLUD ⇨

$$g_{mn} = \sqrt{2\,y} \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix},$$

$$C_0 = -\frac{1}{2\,y}\,, \qquad \mathrm{e}^{\Phi} = 2\,y\,.$$

11D flat Minkowski

Type IIB* SUGRAの解

Non-Abelian U-duality を使って
新たなSUGRA解を沢山生成できる!

[Musaev, YS, 2012.13263]

一般化されたCYBE

$\mathrm{E}_{4(4)}$ **EDA**
(M理論)

$$f_a{}^{bcd} = 0$$

$$
\begin{aligned}
T_a \circ T_b &= f_{ab}{}^{c}\, T_c\,, \\
T_a \circ T^{b_1 b_2} &= \cancel{f_a{}^{b_1 b_2 c}}\, T_c + 2\, f_{ac}{}^{[b_1}\, T^{b_2]c} + 3\, Z_a\, T^{b_1 b_2}\,, \\
T^{a_1 a_2} \circ T_b &= -\cancel{f_b{}^{a_1 a_2 c}}\, T_c + 3\, f_{[c_1 c_2}{}^{[a_1}\, \delta^{a_2]}_{b]}\, T^{c_1 c_2} - 9\, Z_c\, \delta^{[c}_{b}\, T^{a_1 a_2]}\,, \\
T^{a_1 a_2} \circ T^{b_1 b_2} &= -2\, \cancel{f_c{}^{a_1 a_2 [b_1}}\, T^{b_2]c}\,.
\end{aligned}
$$

$$C_A{}^B = \begin{pmatrix} \delta^b_a & 0 \\ \frac{r^{a_1 a_2 b}}{\sqrt{2}} & \delta^{a_1 a_2}_{b_1 b_2} \end{pmatrix} \in E_{4(4)}$$

r-matrix
のM理論版

$$r^{abc}$$

$\mathrm{E}_{4(4)}$ **EDA**
(M理論)

$$f_a{}^{b_1 b_2 b_3} = 3\, f_{ac}{}^{[b_1|}\, r^{c|b_2 b_3]} - 3\, Z_a\, r^{b_1 b_2 b_3}$$

r-matrix の従うべき式…

一般化されたCYBE

$$3\, f_{cd}{}^{[b_1}\, r^{|c|b_2 b_3]}\, r^{a_1 a_2 d} = 0\,, \qquad f_{cd}{}^{a}\, r^{bcd} = 0\,.$$

generalized CYBE

EDAに特有の条件

これを満たす “classical r-matrix” を使って一般化されたYang-Baxter変形ができる.

Yang-Baxter 変形の例

$\mathrm{E}_{6(6)}$ EDA

$$
f_{24}{}^{1} = 1\,, \qquad f_{34}{}^{2} = 1\,, \qquad f_{a}{}^{bcd} = 0\,, \qquad Z_a = 0\,.
$$

$$
g_{ij} = r_i^a\, r_j^b\, \hat{g}_{ab}
$$

$$
\hat{g}_{MN} = \begin{array}{c} \begin{matrix} x & y & z & w & & \end{matrix} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & & 0 \\ 0 & 1 & -2\,w & 0 & & 0 \\ 1 & -2\,w & 2\,w^2 & 0 & & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ & & & & \ddots & \\ 0 & 0 & 0 & 0 & & 1 \end{pmatrix} \end{array}
$$

11次元 SUGRA の解

Yang-Baxter 変形の例

$$f_{24}{}^{1} = 1\,,\quad f_{34}{}^{2} = 1\,,\quad f_{a}{}^{bcd} = 0\,,\quad Z_a = 0\,.$$

$\mathrm{E}_{6(6)}$ **EDA**

$$r^{123} = \frac{1}{\sqrt{2}}\,,\quad r^{456} = -\sqrt{-2}\,,\quad r^{1\cdots 6} = \frac{1}{2}\,.$$

[Malek, YS, Thompson, 2007.08510]

$$f_{24}{}^{1} = 1\,,\quad f_{34}{}^{2} = 1\,,$$

$$f_{2}{}^{156} = \sqrt{2}\,,\quad f_{3}{}^{256} = \sqrt{2}\,,\quad Z_a = 0\,.$$

Yang-Baxter 変形の例

[Musaev, YS, 2012.13263]

$$f_{24}{}^{1} = 1\,, \quad f_{34}{}^{2} = 1\,,$$

$$f_{2}{}^{156} = \sqrt{2}\,, \qquad f_{3}{}^{256} = \sqrt{2}\,, \quad Z_a = 0\,.$$

計量, Nambu-Lie structure を計算し,
11D SUGRAの場に翻訳

$$\hat{g}_{MN} = -\frac{1}{2(1+z^2)^{2/3}} \begin{pmatrix} 0 & 0 & 4(1+z^2) & 0 & 0 & 0 \\ & 4 & -4(2w+yz) & -2z & 0 & 0 \\ & & 4w[4yz+2w(1-z^2)]-4y^2 & 2(2wz-y) & 0 & 0 \\ & & & 1+2z^2 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix} \oplus (1+z^2)^{1/3}\,\mathbf{1}_5\,,$$

$$C_3 = -\sqrt{2}\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z - \frac{1}{\sqrt{2}\,(1+z^2)}\left[z\,\mathrm{d}y + (y-2wz)\,\mathrm{d}z + \tfrac{\mathrm{d}w}{2}\right]\wedge\mathrm{d}u\wedge\mathrm{d}v\,,$$

$$C_6 = -\frac{1}{4\,(1+z^2)}\,\mathrm{d}x\wedge\cdots\wedge\mathrm{d}u\,.$$

11次元 SUGRA の解

まとめ

Poisson-Lie T-duality

Drinfel'd double の generator の再定義

↓

Poisson-Lie structure ⇒ SUGRA解

Nambu-Lie U-duality

Exceptional Drinfel'd Algebra
の generator の再定義

↓

Nambu-Lie structure ⇒ SUGRA解

展望

$$\mathcal{M}_{IJ}(x) = E_I{}^A\, E_J{}^B\, \hat{\mathcal{M}}_{AB} \qquad \mathcal{M}'_{IJ}(x) = E'_I{}^A\, E'_J{}^B\, \hat{\mathcal{M}}'_{AB}$$

NLUD

EFT の解を必ず EFT の解に移すのかは未確認.

EDA は Leibniz代数

$$T_A \circ T_B = X_{AB}{}^C\, T_C\,.$$

$$(T_A \circ T_B \neq -T_B \circ T_A)$$

Lie群のように多様体の構造を持つのか?

EDAから
Nambu-Lie structure
を系統的に作れる.

$$\{f, g, h\} = \pi^{ijk}\, \partial_i f\, \partial_j g\, \partial_k h$$

南部括弧

応用?

展望

今回, maximal SUGRA のみ扱った

Duality 群 = $E_{n(n)}$

⇨ **Exceptional Drinfel'd Algebra**

Half-maximal SUGRA

Duality 群 = $R^{+} \times O(D,D+n)$ (5次元以上)

$= SL(2)\times O(6,6+n)$ (4次元 N=4)

Half-maximal Extended Drinfel'd Algebra

[YS, arXiv:2105.xxxxx]

Non-Abelian duality

Heterotic SUGRA/ T^D ⇔ **Type I SUGRA/ T^D**

…

三周目

PLTD は DFT の対称性

[Hassler, 1707.08624;
Demulder, Hassler, Thompson, 1810.11446;
YS, 1903.12175]

⇩

どうやって EDA を決定したのか
(発見法的に説明)

Nambu-Lie structure から
SUGRAの場をどう決定するのか

Poisson-Lie T-duality
は
DFT の対称性

[Hassler, 1707.08624;
Demulder, Hassler, Thompson, 1810.11446;
YS, 1903.12175]

Lie 微分

一般相対論では,
無限小の座標変換はLie 微分で生成される:

$$\pounds_v w_m = v^n \, \partial_n w_m + \partial_m v^n \, w_n \, .$$

一般座標変換不変な理論:

$$\mathcal{L}_{\mathrm{GR}} = \sqrt{-g} \, R \, .$$

Einstein gravity

一般化 Lie 微分

ダブル空間の場合,
無限小の座標変換は 一般化 Lie 微分で生成される.

$$\hat{\pounds}_V W_M = \underline{V^N \partial_N W_M + (\partial_M V^P} - \partial^P V_M) W_P$$

ここが違う

$$\pounds_v w_m = v^n \partial_n w_m + \partial_m v^n w_n$$

導出

string 理論の Hamiltonian形式: [Siegel '93]

Closed SFT のゲージ対称性: [Hull, Zwiebach 0904.4664]

一般化 Lie 微分の意味

$$\delta_V \mathcal{H}_{MN} = \hat{\pounds}_V \mathcal{H}_{MN} = V^P\,\partial_P \mathcal{H}_{MN} + (\partial_M V^P - \partial^P V_M)\,\mathcal{H}_{PN} + (\partial_N V^P - \partial^P V_N)\,\mathcal{H}_{MP}\,.$$

$$\tilde{\partial}^m = 0$$

$$(\mathcal{H}_{MN}) = \begin{pmatrix} (g - B\,g^{-1}\,B)_{mn} & B_{mp}\,g^{pn} \\ -g^{mp}\,B_{pn} & g^{mn} \end{pmatrix}$$

$$V^M = (v^m,\ \tilde{v}_m)$$

$$\begin{cases} \delta_V g_{mn} &= \pounds_v g_{mn} \\ \delta_V B_{mn} &= \pounds_v B_{mn} + (\partial_m \tilde{v}_n - \partial_n \tilde{v}_m)\,. \end{cases}$$

一般化 Lie微分は

通常のLie微分(座標変換)とB場のゲージ変換の組み合わせ.

Einstein 重力

$$g_{mn}(x) = e^a_m(x)\, e^b_n(x)\, \hat{g}_{ab}$$

定数

$$\pounds_{e_a} e^m_b \equiv -f_{ab}{}^c(x)\, e^m_c$$

geometric flux

$$\sqrt{-g}\, R = \sqrt{-g}\left(2\, D^a f_{ba}{}^b - f_{ab}{}^a\, f_c{}^{bc} - \frac{1}{2}\, f_{abc}\, f^{cba} - \frac{1}{4}\, f_{abc}\, f^{abc}\right)$$

$$D_a \equiv e^m_a\, \partial_m$$

曲率を flux とその微分で表せる.

Double Field Theory

Generalized vielbein

定数

$$\mathcal{H}_{MN}(x) = E_M{}^A(x)\, E_N{}^B(x)\, \hat{\mathcal{H}}_{AB}$$

2種類のフラックスを定義:

一般化 flux

$$\hat{\pounds}_{E_A} E_B{}^M = -\mathcal{F}_{AB}{}^C(x)\, E_C{}^M\,,$$

$$\mathcal{F}_A \equiv -\mathcal{D}^B E_A{}^M\, E_{MB} + 2\,\mathcal{D}_A d\,.$$

$$\mathcal{D}_A \equiv E_A{}^M\, \partial_M$$

dilaton flux

DFT の Flux Formulation

[Geissbuhler, Marques, Nunez, Penas, arXiv:1304.1472]

Dilaton の E.O.M.　　**一般化計量の E.O.M.**

$$\mathcal{R} = 0\,, \qquad \mathcal{G}^{AB} = 0\,.$$

$$\mathcal{R} \equiv -2\,\bar{P}^{AB}\left(2\,\mathcal{D}_A\mathcal{F}_B - \mathcal{F}_A\,\mathcal{F}_B\right) - \frac{1}{3}\,\bar{P}^{ABCDEF}\,\mathcal{F}_{ABC}\,\mathcal{F}_{DEF}\,,$$

$$\mathcal{G}^{AB} \equiv -4\,\bar{P}^{C[A}\,\mathcal{D}^{B]}\mathcal{F}_C + 2\,(\mathcal{F}_C - \mathcal{D}_C)\,\check{\mathcal{F}}^{C[AB]} - 2\,\check{\mathcal{F}}^{CD[A}\,\mathcal{F}_{CD}{}^{B]}\,.$$

フラックスとその微分
と projection だけ

$$\check{\mathcal{F}}^{ABC} \equiv \bar{P}^{ABCDEF}\,\mathcal{F}_{DEF}$$

定数

$$P_{AB} \equiv \frac{1}{2}\left(\eta_{AB} + \hat{\mathcal{H}}_{AB}\right)\,, \qquad \bar{P}_{AB} \equiv \frac{1}{2}\left(\eta_{AB} - \hat{\mathcal{H}}_{AB}\right)\,,$$

$$\begin{aligned}\bar{P}^{ABCDEF} \equiv\ & \bar{P}^{AD}\,\bar{P}^{BE}\,\bar{P}^{CF} + P^{AD}\,\bar{P}^{BE}\,\bar{P}^{CF} \\ & + \bar{P}^{AD}\,P^{BE}\,\bar{P}^{CF} + \bar{P}^{AD}\,\bar{P}^{BE}\,P^{CF}\end{aligned}$$

Poisson-Lie T-duality の設定

$$E^{mn}(x) = \hat{E}^{ab}\, e_a^m\, e_b^n + \pi^{mn}(x)$$

定数

Poisson-Lie structure

右不変ベクトル場

$$\mathcal{H}_{MN}(x) = E_M{}^A(x)\, E_N{}^B(x)\, \hat{\mathcal{H}}_{AB}$$

逆行列

$$(E_A{}^M) = \begin{pmatrix} \delta_a^b & 0 \\ -\pi^{ab} & \delta_b^a \end{pmatrix} \begin{pmatrix} e_b^m & 0 \\ 0 & r_m^b \end{pmatrix}$$

$$\begin{pmatrix} (\hat{g} - \hat{B}\, \hat{g}^{-1}\, \hat{B})_{ab} & \hat{B}_{ac}\, \hat{g}^{cb} \\ -\hat{g}^{ac}\, \hat{B}_{cb} & \hat{g}^{ab} \end{pmatrix}$$

定数

Poisson-Lie symmetric な時空

Poisson-Lie symmetricな時空

Frame場 $$(E_A{}^M) = \begin{pmatrix} \delta_a^b & 0 \\ -\pi^{ab} & \delta_b^a \end{pmatrix} \begin{pmatrix} e_b^m & 0 \\ 0 & r_m^b \end{pmatrix}$$

⇓

$$\hat{\pounds}_{E_A} E_B{}^M = -\mathcal{F}_{AB}{}^C(x)\, E_C{}^M$$ **を計算**

$$\boxed{\mathcal{F}_{AB}{}^C(x) = F_{AB}{}^C \quad \text{定数になる!}}$$

しかも, Drinfel'd double の構造定数

[Hassler, 1707.08624;
Demulder, Hassler, Thompson, 1810.11446]

→ **次頁**

Drinfel'd double の構造定数

$$[T_{\mathrm{a}}, T_{\mathrm{b}}] = f_{\mathrm{ab}}{}^{\mathrm{c}}\, T_{\mathrm{c}}\,, \qquad [\tilde{T}^{\mathrm{a}}, \tilde{T}^{\mathrm{b}}] = f_{\mathrm{c}}{}^{\mathrm{ab}}\, \tilde{T}^{\mathrm{c}}\,,$$

$$[T_{\mathrm{a}}, \tilde{T}^{\mathrm{b}}] = f_{\mathrm{a}}{}^{\mathrm{bc}}\, T_{\mathrm{c}} - f_{\mathrm{ac}}{}^{\mathrm{b}}\, \tilde{T}^{\mathrm{c}}\,.$$

まとめる

$$T_A = (T_{\mathrm{a}}, \tilde{T}^{\mathrm{a}})$$

$$[T_A, T_B] = F_{AB}{}^{C}\, T_C$$

$$F_{ab}{}^{c} = f_{ab}{}^{c}\,, \quad F_{abc} = 0\,, \quad F_{a}{}^{bc} = f_{a}{}^{bc}\,,$$

$$F_{a}{}^{b}{}_{c} = -f_{ac}{}^{b}\,, \quad F^{ab}{}_{c} = f_{c}{}^{ab}\,, \quad F^{abc} = 0\,.$$

dilaton flux

$$\mathcal{F}_A \equiv -\mathcal{D}^B E_A{}^M\, E_{MB} + 2\,\mathcal{D}_A d\,.$$

Dilatonを適切に選べば　$\mathcal{F}_A = 0$ **とできる.**

$$\mathcal{R} \equiv -2\,\bar{P}^{AB}\left(2\,\cancel{\mathcal{D}_A\mathcal{F}_B - \mathcal{F}_A}\,\mathcal{F}_B\right) - \frac{1}{3}\,\bar{P}^{ABCDEF}\,\mathcal{F}_{ABC}\,\mathcal{F}_{DEF}\,,$$

$$\mathcal{G}^{AB} \equiv -4\,\cancel{\bar{P}^{C[A}\mathcal{D}^{B]}}\mathcal{F}_C + \cancel{2\,(\mathcal{F}_C} - \mathcal{D}_C)\,\check{\mathcal{F}}^{C[AB]} - 2\,\check{\mathcal{F}}^{CD[A}\,\mathcal{F}_{CD}{}^{B]}\,.$$

$\mathcal{F}_A \neq 0$ **の場合** ⇒ **[YS, 1903.12175]**

DFTの運動方程式

$$\mathcal{R} = 0\,, \qquad \mathcal{G}^{AB} = 0\,.$$

$$\mathcal{R} \equiv -2\,\bar{P}^{AB}\,\cancel{\left(2\,\mathcal{D}_A\mathcal{F}_B - \mathcal{F}_A\,\mathcal{F}_B\right)} - \frac{1}{3}\,\bar{P}^{ABCDEF}\,\mathcal{F}_{ABC}\,\mathcal{F}_{DEF}\,,$$

$$\mathcal{G}^{AB} \equiv -4\,\cancel{\bar{P}^{C[A}\,\mathcal{D}^{B]}\mathcal{F}_C + 2\,(\mathcal{F}_C - \mathcal{D}_C)}\,\check{\mathcal{F}}^{C[AB]} - 2\,\check{\mathcal{F}}^{CD[A}\,\mathcal{F}_{CD}{}^{B]}\,.$$

<u>**代数方程式**</u>

$$\bar{P}^{ABCDEF}\,F_{ABC}\,F_{DEF} = 0\,,$$

$$\check{F}^{CD[A}\,F_{CD}{}^{B]} = 0\,.$$

DFTの運動方程式

<u>もう少し具体的に書くと</u>

$$\frac{1}{12}\,F_{ABC}\,F_{DEF}\left(3\,\hat{\mathcal{H}}^{AD}\,\eta^{BE}\,\eta^{CF}-\hat{\mathcal{H}}^{AD}\,\hat{\mathcal{H}}^{BE}\,\hat{\mathcal{H}}^{CF}\right)=0\,,$$

$$\frac{1}{2}\left(\eta^{CE}\,\eta^{DF}-\hat{\mathcal{H}}^{CE}\,\hat{\mathcal{H}}^{DF}\right)\hat{\mathcal{H}}^{G[A}\,F_{CD}{}^{B]}\,F_{EFG}=0\,.$$

$\Uparrow$

これを満たす定数行列 $\hat{\mathcal{H}}_{AB}$ が見つかれば

PL symmetric な時空は DFT の解.

$$\mathcal{H}_{MN}(x)=E_M{}^A(x)\,E_N{}^B(x)\,\hat{\mathcal{H}}_{AB}$$

PL T-plurality

$$\begin{cases} \hat{\mathcal{H}}'_{AB} = C_A{}^C\, C_B{}^D\, \hat{\mathcal{H}}_{CD}\,. \\ T'_A = C_A{}^B\, T_B\,. \end{cases}$$

$$[T_A,\, T_B] = F_{AB}{}^C\, T_C$$

$$F'_{ABC} = C_A{}^E\, C_B{}^F\, C_C{}^G\, F_{EFG}\,.$$

DFT の運動方程式はこの変換の下で manifest に covariant.

$$\frac{1}{12}\, F_{ABC}\, F_{DEF}\, \big(3\, \hat{\mathcal{H}}^{AD}\, \eta^{BE}\, \eta^{CF} - \hat{\mathcal{H}}^{AD}\, \hat{\mathcal{H}}^{BE}\, \hat{\mathcal{H}}^{CF}\big) = 0\,,$$

$$\frac{1}{2}\, \big(\eta^{CE}\, \eta^{DF} - \hat{\mathcal{H}}^{CE}\, \hat{\mathcal{H}}^{DF}\big)\, \hat{\mathcal{H}}^{G[A}\, F_{CD}{}^{B]}\, F_{EFG} = 0\,.$$

DFT の解は DFT の別の解へと map される.

Dilatonはどう作るか?

$$\mathcal{F}_A \equiv -\mathcal{D}^B E_A{}^M E_{MB} + 2\,\mathcal{D}_A d = 0\,.$$

$$\Longrightarrow \quad \mathrm{e}^{-2\,d(x)} = \mathrm{e}^{-2\,\hat{d}}\,\lvert\det(\ell^{\mathrm{a}}_m)\rvert\,.$$

[Unge '02]

定数　　左不変1-form

※ ただし, dual algebra の構造定数のトレースがゼロでない場合 $(f_{\mathrm{b}}{}^{\mathrm{ba}} \neq 0)$

$$\mathcal{F}_A = 0 \quad \Longrightarrow \quad d(x) \ \to\ d(x) + \frac{1}{2}\, f_{\mathrm{b}}{}^{\mathrm{ba}}\, v^m_{\mathrm{a}}\, \tilde{x}_m\,.$$

[Demulder, Hassler, Thompson, 1810.11446]

この場合, 時空は generalized SUGRA の解になる.

ここまでのまとめ

Einstein重力と同様,
DFT の運動方程式は flux とその微分の方程式.

Poisson-Lie symmetric な時空では
flux が定数になり, DFT の e.o.m. は代数方程式

$$\frac{1}{12}\,F_{ABC}\,F_{DEF}\,\bigl(3\,\hat{\mathcal{H}}^{AD}\,\eta^{BE}\,\eta^{CF}-\hat{\mathcal{H}}^{AD}\,\hat{\mathcal{H}}^{BE}\,\hat{\mathcal{H}}^{CF}\bigr)=0\,,$$
$$\frac{1}{2}\,\bigl(\eta^{CE}\,\eta^{DF}-\hat{\mathcal{H}}^{CE}\,\hat{\mathcal{H}}^{DF}\bigr)\,\hat{\mathcal{H}}^{G[A}\,F_{CD}{}^{B]}\,F_{EFG}=0\,.$$

O(*D*,*D*) 対称性

$$\begin{cases} \hat{\mathcal{H}}'_{AB}=C_A{}^C\,C_B{}^D\,\hat{\mathcal{H}}_{CD}\,. \\ F'_{ABC}=C_A{}^E\,C_B{}^F\,C_C{}^G\,F_{EFG}\,. \end{cases}$$

ここまでのまとめ

$$\{T_A,\, \hat{\mathcal{H}}_{AB}\} \xrightarrow{\textbf{O}(D,D)} \{T'_A,\, \hat{\mathcal{H}}'_{AB}\}$$

$$\mathcal{H}_{MN}(x) = E_M{}^A(x)\, E_N{}^B(x)\, \hat{\mathcal{H}}_{AB}$$

DFTの解

$$\mathcal{H}'_{MN}(x) = E'_M{}^A(x)\, E'_N{}^B(x)\, \hat{\mathcal{H}}'_{AB}$$

DFTの解'

どうやって EDA を決定したのか
(発見法的に説明)

⇩

もっと形式的な説明

[YS, 2009.04454]

大事な関係式

$$\boxed{\hat{\pounds}_{E_A} {E_B}^M = -{F_{AB}}^C\, {E_C}^M}$$

Drinfel'd double のLie代数

$$[T_A, T_B] = {F_{AB}}^C\, T_C$$

と同じ代数を満たす
frame 場を構築できるというのがポイント.

$$({E_A}^M) = \begin{pmatrix} \delta^b_a & 0 \\ -\pi^{ab} & \delta^a_b \end{pmatrix} \begin{pmatrix} e^m_b & 0 \\ 0 & r^b_m \end{pmatrix}$$

U-duality 版の拡張

$$\boxed{\hat{\pounds}_{E_A} E_B{}^M = -X_{AB}{}^C\, E_C{}^M}$$

EFTの一般化Lie微分　　**定数**

となる frame 場を探せば
Exceptional Drinfel'd Algebra が見つかるはず!

$$T_A \circ T_B = X_{AB}{}^C\, T_C$$

U-duality 版の拡張

$$\boxed{\hat{\pounds}_{E_A} E_B{}^M = -X_{AB}{}^C \, E_C{}^M}$$

$$[\hat{\pounds}_{E_A}, \hat{\pounds}_{E_B}] E_C{}^M = \hat{\pounds}_{\hat{\pounds}_{E_A} E_B} E_C{}^M$$

$$\Updownarrow$$

$$X_{AD}{}^E \, X_{BC}{}^D - X_{BD}{}^E \, X_{AC}{}^D = X_{AB}{}^D \, X_{DC}{}^E$$

$$\Updownarrow$$

$$T_A \circ (T_B \circ T_C) = (T_A \circ T_B) \circ T_C + T_B \circ (T_A \circ T_C) \,.$$

Leibniz恒等式 $\longrightarrow$ **Leibniz 代数**

一般化 Lie 微分

DFTの場合

$$\hat{\pounds}_V W^M \equiv \underline{V^N \partial_N W^M - W^N \partial_N V^M} + \eta^{MN} \eta_{PQ} \partial_N V^P W^Q \, .$$

EFTの場合

$$\hat{\pounds}_V W^I \equiv \underline{V^J \partial_J W^I - W^J \partial_J V^I} + \boxed{Y^{IJ}_{KL}} \partial_J V^K W^L \, .$$

一般化 Lie 微分

以下, *n*≦4 の場合に制限

$$\hat{\pounds}_V W^I \equiv V^J \partial_J W^I - W^J \partial_J V^I + Y^{IJ}_{KL}\, \partial_J V^K\, W^L \,.$$

$$x^I = (x^i,\, \cancel{y_{i_1 i_2}})$$

$$\hat{\pounds}_V W^I = \begin{pmatrix} \pounds_v w^i \\ \pounds_v w_{i_1 i_2} - (\iota_w \mathrm{d} v_2)_{i_1 i_2} \end{pmatrix} .$$

$$V^I = \begin{pmatrix} v^i \\ v_{i_1 i_2} \end{pmatrix}$$

diffeo. parameter

C_3のgauge変換

$$\delta C_3 = \pounds_v C_3 + \mathrm{d} v_2$$

Frame 場

$$
{E_A}^M = \begin{pmatrix} \delta_a^b & 0 \\ -\pi^{ab} & \delta_b^a \end{pmatrix} \begin{pmatrix} e_b^m & 0 \\ 0 & r^b{}_m \end{pmatrix}
$$

GL(*D*)

R_{ab} **β変換**

R_{abc}

GL(*n*)

$$
\begin{aligned}
{E_A}^I &\equiv \begin{pmatrix} \delta_a^b & 0 \\ -\frac{\pi^{a_1 a_2 b}}{\sqrt{2!}} & \delta^{a_1 a_2}_{b_1 b_2} \end{pmatrix} \begin{pmatrix} e_b^i & 0 \\ 0 & r^{[b_1}_{[i_1}\, r^{b_2]}_{i_2]} \end{pmatrix} \\
&= \begin{pmatrix} e_a^i & 0 \\ -\frac{\pi^{a_1 a_2 b}\, e_b^i}{\sqrt{2!}} & r^{[a_1}_{[i_1}\, r^{a_2]}_{i_2]} \end{pmatrix} .
\end{aligned}
$$

どうやって代数を作るか?

$$E_A{}^I = \begin{pmatrix} e_a^i & 0 \\ -\frac{\pi^{a_1 a_2 b}\, e_b^i}{\sqrt{2!}} & r^{[a_1}_{[i_1}\, r^{a_2]}_{i_2]} \end{pmatrix}$$

$$\begin{cases} E_a{}^I = \begin{pmatrix} e_a^i \\ 0 \end{pmatrix} \\ E^{a_1 a_2\, I} = \begin{pmatrix} -\pi^{a_1 a_2 b}\, e_b^i \\ 2\, r^{[a_1}_{[i_1}\, r^{a_2]}_{i_2]} \end{pmatrix} \end{cases}$$

$$\Downarrow$$

$$\hat{\pounds}_V W^I = \begin{pmatrix} \pounds_v w^i \\ \pounds_v w_{i_1 i_2} - (\iota_w \mathrm{d} v_2)_{i_1 i_2} \end{pmatrix}.$$

一般化Lie微分の代数を計算!

$$\boxed{\hat{\pounds}_{E_A} E_B{}^I = -\boldsymbol{X}_{AB}{}^C\, E_C{}^I}$$

結果

一般化 flux

$$\hat{\pounds}_{E_A} E_B{}^I = -\boldsymbol{X}_{AB}{}^C\, E_C{}^I\,,$$

定数ではない

$\pi^{abc}(x)$ をうまく選んで定数に。

$$\begin{aligned}
\boldsymbol{X}_{ab}{}^c &= f_{ab}{}^c\,,\\
\boldsymbol{X}_{abc_1c_2} &= 0\,,\\
\boldsymbol{X}_a{}^{b_1b_2c} &= D_a\pi^{b_1b_2c} - 3\,f_{ad}{}^{[c}\,\pi^{b_1b_2]d}\,,\\
\boldsymbol{X}_a{}^{b_1b_2}{}_{c_1c_2} &= 4\,f_{ad}{}^e\,\delta^{b_1b_2}_{ef}\,\delta^{fd}_{c_1c_2}\,,\\
\boldsymbol{X}^{a_1a_2}{}_b{}^c &= -\big(D_b\pi^{a_1a_2c} - 3\,f_{bd}{}^{[c}\,\pi^{a_1a_2]d} - f_{d_1d_2}{}^{[a_1}\,\delta^{a_2]}_b\,\pi^{d_1d_2c}\big)\,,\\
\boldsymbol{X}^{a_1a_2}{}_{bc_1c_2} &= 6\,f_{[bc_1}{}^{[a_1}\,\delta^{a_2]}_{c_2]}\,,\\
\boldsymbol{X}^{a_1a_2b_1b_2c} &= -\pi^{a_1a_2d}\,D_d\pi^{b_1b_2c} + 3\,\pi^{[b_1b_2|d}\,D_d\pi^{a_1a_2|c]}\\
&\quad + 2\,f_{gh}{}^{[a_1}\,\pi^{a_2]cg}\,\pi^{b_1b_2h} - 3\,f_{gh}{}^{[b_1}\,\pi^{b_2c]g}\,\pi^{a_1a_2h} - f_{gh}{}^{[a_1}\,\pi^{a_2]b_1b_2}\,\pi^{ghc}\,,\\
\boldsymbol{X}^{a_1a_2b_1b_2}{}_{c_1c_2} &= -\big(4\,D_d\pi^{a_1a_2[b_1}\,\delta^{b_2]d}_{c_1c_2} + 4\,\pi^{a_1a_2d}\,f_{de}{}^{[b_1}\,\delta^{b_2]e}_{c_1c_2} - 4\,\pi^{b_1b_2d}\,f_{de}{}^{[a_1}\,\delta^{a_2]e}_{c_1c_2}\\
&\quad + 2\,f_{c_1c_2}{}^{[a_1}\,\pi^{a_2]b_1b_2}\big)\,.
\end{aligned}$$

$$(D_a \equiv e_a^i\,\partial_i)$$

PL T-duality の場合

$$g^{-1}\,T_A\,g \equiv M_A{}^B(x)\,T_B\,,$$

$$(M_A{}^B) = \begin{pmatrix} \delta_a^c & 0 \\ -\pi^{ac} & \delta_c^a \end{pmatrix} \begin{pmatrix} a_c{}^b & 0 \\ 0 & (a^{-1})_b{}^c \end{pmatrix}.$$

⇨ 単位元 $g = e$ では, $M_A{}^B(x=0) = \delta_A^B$

$$\pi^{ab}(x=0) = 0, \qquad a_a{}^b(x=0) = \delta_a^b\,.$$

必ず $\pi^{ab} = 0$ となる場所が存在する.

⇨ $\pi^{abc} = 0$ となる場所があるだろう.

結果

$$\hat{\pounds}_{E_A} E_B{}^I = -\boldsymbol{X}_{AB}{}^C\, E_C{}^I\,,$$

$$\boldsymbol{X}_{ab}{}^c = f_{ab}{}^c\,,\qquad f_a{}^{b_1b_2c}\qquad (f_a{}^{bcd} = f_a{}^{[bcd]})$$

$$\boldsymbol{X}_{abc_1c_2} = 0\,,$$

$$\boldsymbol{X}_a{}^{b_1b_2c} = \boxed{D_a\pi^{b_1b_2c}} - 3\, f_{ad}{}^{[c}\,\cancel{\pi^{b_1b_2]d}}\,,$$

$$\boldsymbol{X}_a{}^{b_1b_2}{}_{c_1c_2} = 4\, f_{ad}{}^e\, \delta^{b_1b_2}_{ef}\, \delta^{fd}_{c_1c_2}\,,$$

$$\boldsymbol{X}^{a_1a_2}{}_b{}^c = -\bigl(\boxed{D_b\pi^{a_1a_2c}} - 3\, f_{bd}{}^{[c}\,\cancel{\pi^{a_1a_2]d} - f_{d_1d_2}{}^{[a_1}\,\delta_b^{a_2]}\,\pi^{d_1d_2c}}\bigr)\,,$$

$$\boldsymbol{X}^{a_1a_2}{}_{bc_1c_2} = 6\, f_{[bc_1}{}^{[a_1}\,\delta^{a_2]}_{c_2]}\,,$$

$$\boldsymbol{X}^{a_1a_2b_1b_2c} = -\pi^{a_1a_2d}\,\cancel{D_d\pi^{b_1b_2c} + 3\,\pi^{[b_1b_2|d}\,D_d\pi^{a_1a_2|c]}}$$
$$\cancel{+ 2\, f_{gh}{}^{[a_1}\,\pi^{a_2]cg}\,\pi^{b_1b_2h} - 3\, f_{gh}{}^{[b_1}\,\pi^{b_2c]g}\,\pi^{a_1a_2h} - f_{gh}}{}^{[a_1}\,\pi^{a_2]b_1b_2}\,\pi^{ghc}\,,$$

$$\boldsymbol{X}^{a_1a_2b_1b_2}{}_{c_1c_2} = -\bigl(4\,\boxed{D_d\pi^{a_1a_2[b_1}}\,\delta^{b_2]d}_{c_1c_2}\,\cancel{+ 4\,\pi^{a_1a_2d}\, f_{de}{}^{[b_1}\,\delta^{b_2]e}_{c_1c_2} - 4\,\pi^{b_1b_2d}\, f_{de}{}^{[a_1}\,\delta^{a_2]e}_{c_1c_2}}$$
$$+ 2\, f_{c_1c_2}\,\cancel{{}^{[a_1}\,\pi^{a_2]b_1b_2}}\bigr)\,.$$

EDA

$$T_A \circ T_B = X_{AB}{}^C\, T_C \,.$$

$$T_a \circ T_b = f_{ab}{}^c\, T_c \,,$$

$$T_a \circ T^{b_1 b_2} = f_a{}^{b_1 b_2 c}\, T_c + 2\, f_{ac}{}^{[b_1}\, T^{b_2]c} \,,$$

$$T^{a_1 a_2} \circ T_b = -f_b{}^{a_1 a_2 c}\, T_c + 3\, f_{[c_1 c_2}{}^{[a_1}\, \delta_{b]}^{a_2]}\, T^{c_1 c_2} \,,$$

$$T^{a_1 a_2} \circ T^{b_1 b_2} = -2\, f_d{}^{a_1 a_2 [b_1}\, T^{b_2]d} \,.$$

$T_A \circ T_B \neq -T_B \circ T_A$ **Lie代数ではない**

逆に, 代数からNambu-Lie structure を作る

$$\pi^{ijk}$$

Group element を作る:

$$g^{-1}(x) \equiv \mathrm{e}^{h(x)} = \mathrm{e}^{-x^a T_a} \quad .$$

“Adjoint action” を計算:

$$\begin{aligned} g^{-1}(x) \circ T_A &\equiv 1 + h \circ T_A + \frac{1}{2!} h \circ (h \circ T_A) + \frac{1}{3!} h \circ (h \circ (h \circ T_A)) + \cdots \\ &\equiv M_A{}^B(x)\, T_B \, . \end{aligned}$$

$$\boxed{M_A{}^B(x) = \begin{pmatrix} \delta_a^c & 0 \\ -\frac{\pi^{a_1 a_2 c}}{\sqrt{2!}} & \delta^{a_1 a_2}_{c_1 c_2} \end{pmatrix} \begin{pmatrix} a_c{}^b & 0 \\ 0 & (a^{-1})_{[b_1}{}^{c_1} (a^{-1})_{b_2]}{}^{c_2} \end{pmatrix}}$$

$$\Rightarrow \qquad \pi^{abc}(x=0) = 0$$

いくつかの恒等式

<u>Leibniz恒等式:</u> $$g \circ (T_A \circ T_B) = (g \circ T_A) \circ (g \circ T_B)$$

$$
\begin{aligned}
&(a^{-1})_a{}^e\,(a^{-1})_b{}^f\,a_g{}^c\,f_{ef}{}^g = f_{ab}{}^c\,,\\
&a_a{}^e\,(a^{-1})_{f_1}{}^{b_1}\,(a^{-1})_{f_2}{}^{b_2}\,(a^{-1})_{f_3}{}^{b_3}\,f_e{}^{f_1f_2f_3} = f_a{}^{b_1b_2b_3} + 3\,f_{ac}{}^{[b_1}\,\pi^{b_2b_3]c}\,,\\
&3\left(f_{e[c}{}^{a_1}\,\delta^{[a_2}_{d]}\,\pi^{b_1b_2]e} - f_{e[c}{}^{a_2}\,\delta^{[a_1}_{d]}\,\pi^{b_1b_2]e}\right) + f_{cd}{}^{[a_1}\,\pi^{a_2]b_1b_2} = 0\,,\\
&f_d{}^{b_1b_2c}\,\pi^{a_1a_2d} - 3\,f_d{}^{a_1a_2[b_1}\,\pi^{b_2c]d} = 3\,f_{de}{}^{[c}\,\pi^{b_1b_2]d}\,\pi^{a_1a_2e} - 4\,f_{de}{}^{[a_1}\,\pi^{a_2]d[b_1}\,\pi^{b_2]ec}\,,\\
&f_{ab}{}^c\,\pi^{abd} = 0\,.
\end{aligned}
$$

いくつかの恒等式

微分恒等式:

$$M_A{}^D\, D_c(M^{-1})_D{}^B = X_{cA}{}^B$$

$$D_c a_a{}^b = a_a{}^d\, a_c{}^e\, f_{de}{}^b\,,$$

$$D_c \pi^{a_1 a_2 a_3} = (a^{-1})_{b_1}{}^{a_1}\, (a^{-1})_{b_2}{}^{a_2}\, (a^{-1})_{b_3}{}^{a_3}\, a_c{}^d\, f_d{}^{b_1 b_2 b_3}\,.$$

$$\pi^{i_1 i_2 i_3} = \pi^{a_1 a_2 a_3}\, e^{i_1}_{a_1}\, e^{i_2}_{a_2}\, e^{i_3}_{a_3}$$

Nambu-Lie structure

$$\pounds_{v_a} \pi^{ijk} = f_a{}^{bcd}\, v^i_b\, v^j_c\, v^k_d$$

$$3\, \pi^{l[j_1 j_2}\, \nabla_l \pi^{k] i_1 i_2} = \pi^{i_1 i_2 l}\, \nabla_l \pi^{j_1 j_2 k}$$

(Weitzenböck接続: $\Gamma_i{}^j{}_k \equiv v^j_a\, \partial_i \ell^a_k$ **)**

構造定数

$$\boldsymbol{X}_{ab}{}^{c} = f_{ab}{}^{c}\,,$$

$$\boldsymbol{X}_{abc_1c_2} = 0\,,$$

$$\boldsymbol{X}_{a}{}^{b_1b_2c} = D_a\pi^{b_1b_2c} - 3\,f_{ad}{}^{[c}\,\pi^{b_1b_2]d}\,,$$

$$\boldsymbol{X}_{a}{}^{b_1b_2}{}_{c_1c_2} = 4\,f_{ad}{}^{e}\,\delta^{b_1b_2}_{ef}\,\delta^{fd}_{c_1c_2}\,,$$

$$\boldsymbol{X}^{a_1a_2}{}_{b}{}^{c} = -\big(D_b\pi^{a_1a_2c} - 3\,f_{bd}{}^{[c}\,\pi^{a_1a_2]d} - f_{d_1d_2}{}^{[a_1}\,\delta^{a_2]}_{b}\,\pi^{d_1d_2c}\big)\,,$$

$$\boldsymbol{X}^{a_1a_2}{}_{bc_1c_2} = 6\,f_{[bc_1}{}^{[a_1}\,\delta^{a_2]}_{c_2]}\,,$$

$$\begin{aligned}\boldsymbol{X}^{a_1a_2b_1b_2c} &= -\pi^{a_1a_2d}\,D_d\pi^{b_1b_2c} + 3\,\pi^{[b_1b_2|d}\,D_d\pi^{a_1a_2|c]} \\ &\quad + 2\,f_{gh}{}^{[a_1}\,\pi^{a_2]cg}\,\pi^{b_1b_2h} - 3\,f_{gh}{}^{[b_1}\,\pi^{b_2c]g}\,\pi^{a_1a_2h} - f_{gh}{}^{[a_1}\,\pi^{a_2]b_1b_2}\,\pi^{ghc}\,,\end{aligned}$$

$$\begin{aligned}\boldsymbol{X}^{a_1a_2b_1b_2}{}_{c_1c_2} &= -\big(4\,D_d\pi^{a_1a_2[b_1}\,\delta^{b_2]d}_{c_1c_2} + 4\,\pi^{a_1a_2d}\,f_{de}{}^{[b_1}\,\delta^{b_2]e}_{c_1c_2} - 4\,\pi^{b_1b_2d}\,f_{de}{}^{[a_1}\,\delta^{a_2]e}_{c_1c_2} \\ &\quad + 2\,f_{c_1c_2}{}^{[a_1}\,\pi^{a_2]b_1b_2}\big)\,.\end{aligned}$$

構造定数 $X_{AB}{}^{C}$ になることが確認できた!

$$\boldsymbol{X}_{AB}{}^{C} = X_{AB}{}^{C}$$

FluxをEDAの構造定数とするframe場の作り方は分った.

$$\hat{\pounds}_{E_A} E_B{}^I = -\boldsymbol{X}_{AB}{}^C E_C{}^I$$

予想

EFT のe.o.m.もfluxとその微分でかけるはず.

もしそうなら

$$\begin{cases} \hat{\mathcal{M}}'_{AB} = C_A{}^C C_B{}^D \hat{\mathcal{M}}_{CD} \\ T'_A = C_A{}^B T_B \end{cases}$$

は EFT の対称性.

$$\mathcal{M}_{IJ}(x) = E_I{}^A(x) E_J{}^B(x) \hat{\mathcal{M}}_{AB}$$

Nambu-Lie structure から
SUGRAの場をどう決定するのか

DFTの場合

B変換

$$\mathcal{H}_{MN} = \begin{pmatrix} (g - B\, g^{-1}\, B)_{mn} & B_{mp}\, g^{pn} \\ -g^{mp}\, B_{pn} & g^{mn} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & B \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ -B & \mathbf{1} \end{pmatrix}$$

$$= E_M{}^A\, E_N{}^B\, \hat{\mathcal{H}}_{AB}$$

$$\hat{\mathcal{H}}_{AB} = \begin{pmatrix} (\hat{g} - \hat{B}\, \hat{g}^{-1}\, \hat{B})_{ab} & \hat{B}_{ac}\, \hat{g}^{cb} \\ -\hat{g}^{ac}\, \hat{B}_{cb} & \hat{g}^{ab} \end{pmatrix}$$

$$E_M{}^A = \begin{pmatrix} r_m^a & 0 \\ 0 & e_a^m \end{pmatrix} \begin{pmatrix} \delta_a^b & 0 \\ \pi^{ab} & \delta_b^a \end{pmatrix}$$

$$\Rightarrow \quad E^{mn}(x) = \hat{E}^{ab}\, e_a^m\, e_b^n + \pi^{mn}(x)$$

EFTの場合

$$\mathcal{H}_{MN} = \begin{pmatrix} \mathbf{1} & B \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ -B & \mathbf{1} \end{pmatrix}$$

$$\mathrm{e}^{\frac{1}{2!}\, B_{mn}\, R^{mn}} \in \mathrm{O}(D, D)$$

$$\mathrm{e}^{\frac{1}{3!}\, C_{i_1 i_2 i_3}\, R^{i_1 i_2 i_3}}\ \mathrm{e}^{\frac{1}{6!}\, C_{i_1\cdots i_6}\, R^{i_1\cdots i_6}}\ \underbrace{\cdots}_{\mathrm{E}_8\ \textbf{EFT}} \in E_{n(n)}$$

C_3 C_6 $C_{8,1}$

EFTの場合

$$\mathcal{M}_{IJ} = (L^{\mathrm{T}}\,\hat{\mathcal{M}}\,L)_{IJ}\,.$$

$$g^{i_1\cdots i_p,\,j_1\cdots j_p} \equiv g^{i_1k_1}\cdots g^{i_pk_p}\,\delta^{[j_1}_{[k_1}\cdots\delta^{j_p]}_{k_p]}$$

$$\hat{\mathcal{M}} \equiv |g|^{\frac{1}{9-n}} \begin{pmatrix} g_{ij} & 0 & 0 & 0 \\ 0 & g^{i_1i_2,\,j_1j_2} & 0 & 0 \\ 0 & 0 & g^{i_1\cdots i_5,\,j_1\cdots j_5} & 0 \\ 0 & 0 & 0 & g^{i_1\cdots i_7,\,j_1\cdots j_7}\,g^{ij} \end{pmatrix};$$

$$L = \begin{pmatrix} \delta^i_j & 0 & 0 & 0 \\ -\frac{C_{i_1i_2j}}{\sqrt{2!}} & \delta^{j_1j_2}_{i_1i_2} & 0 & 0 \\ \frac{C_{i_1\cdots i_5j} - 5C_{[i_1i_2i_3}C_{i_4i_5]j}}{\sqrt{5!}} & \frac{20\delta^{j_1j_2}_{[i_1i_2}C_{i_3i_4i_5]}}{\sqrt{2!\,5!}} & \delta^{j_1\cdots j_5}_{i_1\cdots i_5} & 0 \\ \frac{21C_{i[i_1i_2}(C_{i_3\cdots i_7]j} - \frac{5}{3}C_{i_3i_4i_5}C_{i_6i_7]j})}{\sqrt{7!}} & -\frac{42\delta^{j_1j_2}_{[i_1i_2}(C_{|i|i_3\cdots i_7]} - 5C_{|i|i_3i_4}C_{i_5i_6i_7]})}{\sqrt{2!\,7!}} & \frac{7!\delta^{j_1\cdots j_5}_{[i_1\cdots i_5}C_{i_6i_7]i}}{2!\sqrt{5!\,7!}} & \delta^{j_1\cdots j_7}_{i_1\cdots i_7}\,\delta^j_i \end{pmatrix}.$$

$(C_3)^3 + C_3C_6 + C_{8,1}$ $\qquad$ $C_6 + (C_3)^2$ $\qquad$ C_3

EFTの場合

$C_3,\, C_6 \quad g_{ij}, \quad C_3,\, C_6$

上三角 **下三角**

$$\mathcal{M}_{IJ} = (L^{\mathrm{T}}\,\hat{\mathcal{M}}\,L)_{IJ}$$

$$= E_M{}^A\,\hat{\mathcal{M}}_{AB}\,(E^{\mathrm{T}})^B{}_N$$

下三角 **上三角**

$r_i^a,\, \pi^{ijk},\, \pi^{i_1\cdots i_6} \qquad r_i^a,\, \pi^{ijk},\, \pi^{i_1\cdots i_6}$

$$\begin{cases} g_{ij} = \cdots \\ C_{ijk} = \cdots \\ \end{cases}$$

と表すのは難しいが
具体例で解くことはできる.

おわり