Gravitational Positivity Bounds and the Standard Model

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mainly based on arXiv:2104.09682

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see also arXiv:2105.01436 w/Junsei Tokuda (Kobe U)

June 23rd 2021@京大

Outline

1. Positivity bounds on low-energy scattering amplitudes provide
   a criterion for a low-energy EFT to be UV completable in the standard manner

2. They provide a Swampland condition when applied to gravitational EFTs

3. Puzzles on positivity in gravitational QED [Alberte-de Rham-Jaitly-Tolley '20]
 - implies a cutoff scale $\Lambda \sim 10^8$ GeV (too low to believe???)
 - implies that massless QED $m_e \to 0$ is in the Swampland (sounds strange???)

4. Positivity in gravitational Standard Model [Aoki-Loc-TN-Tokuda '21]
 - the cutoff scale is improved up to $\Lambda \sim 10^{16}$ GeV
 - massless limit $m_e \to 0$ is allowed if we take $m_W \to 0$ simultaneously

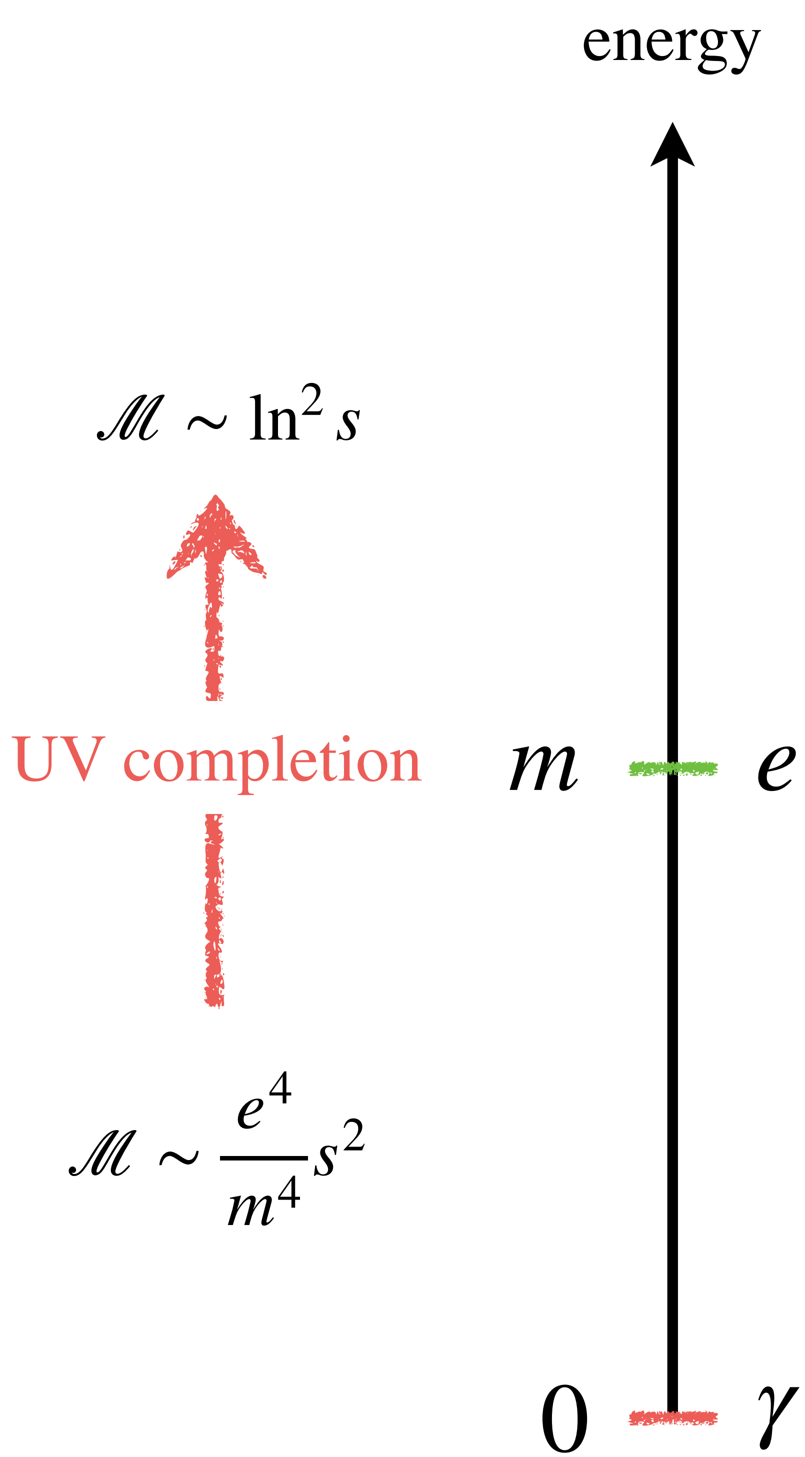
Introduction: EFT and UV completion

The EFT framework is useful for relating UV physics & IR physics

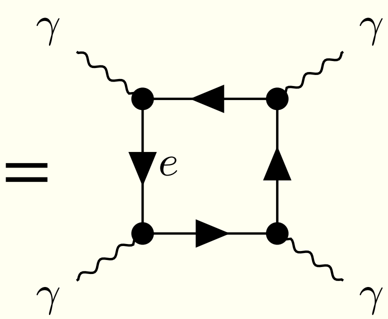
- UV completion of Fermi interactions predicted weak bosons
- UV completion of weak boson scattering predicted the Higgs boson
- The Swampland program is trying to clarify necessary conditions for a gravitational EFT to be UV completable.

In this introduction, I recap this idea and introduce positivity bounds using QED and the Euler-Heisenberg model as an example.

QED vs Euler-Heisenberg



QED: $\mathscr{L} = -\frac{1}{4} F_{\mu\nu}^2 - \bar{\psi}(\not{D} + m)\psi$

1-loop amplitude: $\mathscr{M} =$ 

Euler-Heisenberg :

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu}^2 + \alpha_1 (F_{\mu\nu} F^{\mu\nu})^2 + \alpha_2 (F_{\mu\nu} \widetilde{F}^{\mu\nu})^2$$

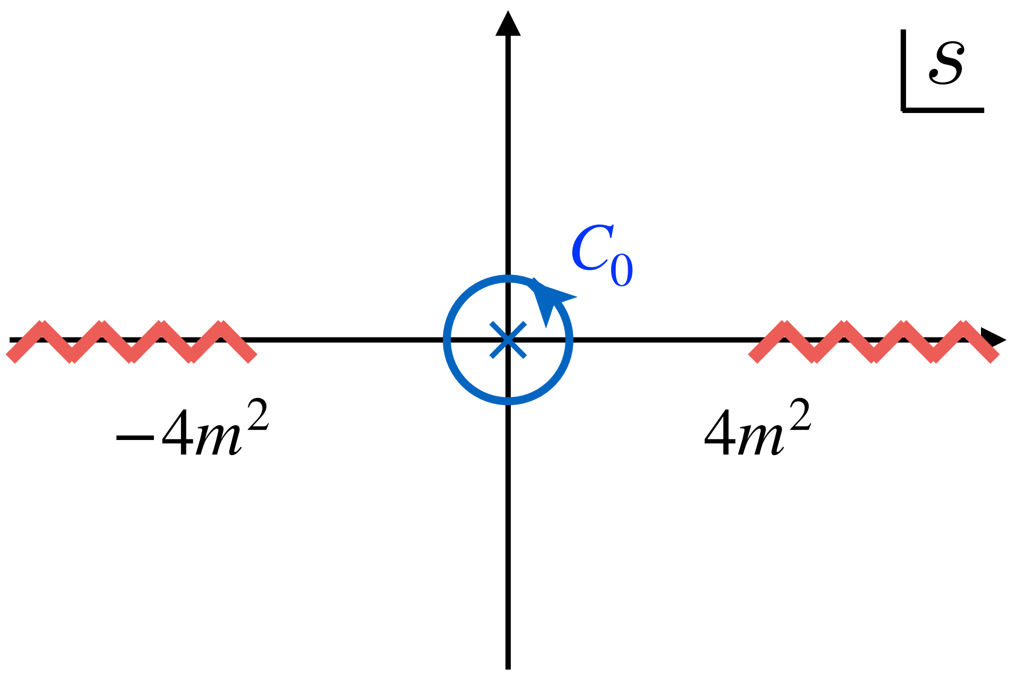
$$\alpha_1 = \frac{e^4}{1440\pi^2 m^4}, \quad \alpha_2 = \frac{7e^4}{5760\pi^2 m^4}.$$

Q. Why α_i are positive? → next slide

Why α_i are positive?

Consider an s-u crossing helicity sum of $\gamma\gamma \to \gamma\gamma$ scattering in the forward limit:

$$\mathscr{M} = \mathscr{M}_{++++} + \mathscr{M}_{----} + \mathscr{M}_{+-+-} + \mathscr{M}_{-+-+}$$



analytic structure of $\mathscr{M}(s, t=0)$

IR behavior: $\mathscr{M} = 32\left(\alpha_1 + \alpha_2\right) s^2 + \mathscr{O}\left(s^4\right)$

$$32\left(\alpha_1 + \alpha_2\right) = \oint_{C_0} \frac{ds}{2\pi i} \frac{\mathscr{M}(s, t=0)}{s^3}$$

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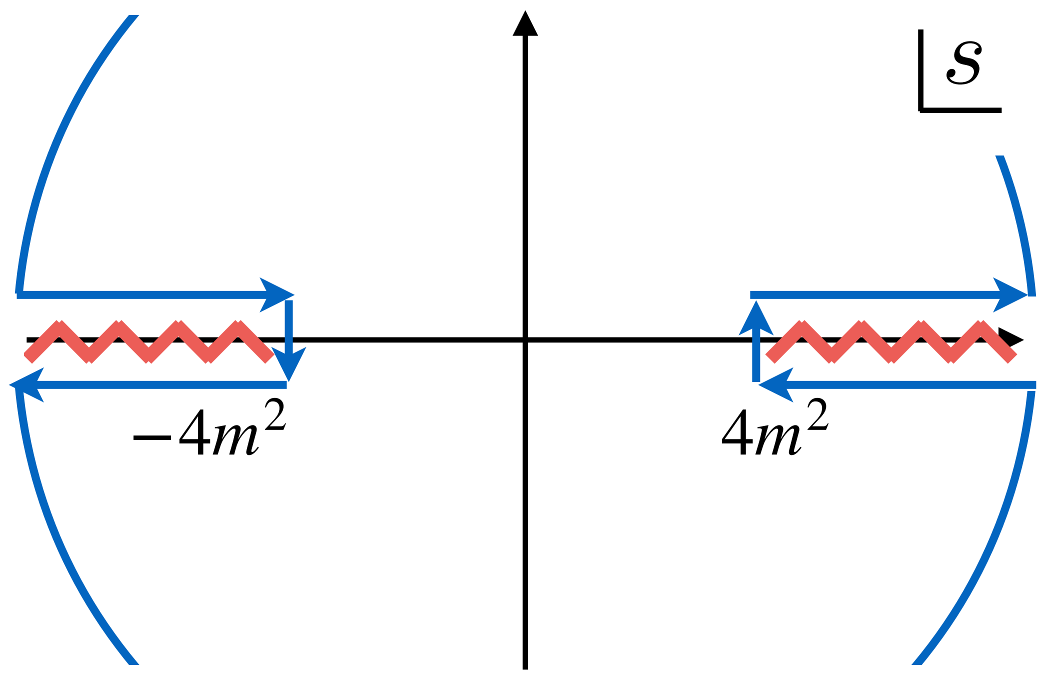

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$$32\left(\alpha_1 + \alpha_2\right) = \oint_{C_0} \frac{ds}{2\pi i} \frac{\mathcal{M}(s, t=0)}{s^3}$$

Deform the integration contour to rewrite it in the UV language:

$$32\left(\alpha_1 + \alpha_2\right) = \frac{2}{\pi} \int_{4m^2}^{\infty} ds \frac{\mathrm{Im}\mathcal{M}(s, t=0)}{s^3} + \oint_{C_\infty} \frac{ds}{2\pi i} \frac{\mathcal{M}(s, t=0)}{s^3}$$

※ used the s-u symmetry and Disc $\mathcal{M}(s, t=0) = 2i\, \mathrm{Im}\, \mathcal{M}(s, t=0)$

Why α_i are positive?

Consider an s-u crossing helicity sum of $\gamma\gamma \to \gamma\gamma$ scattering in the forward limit:

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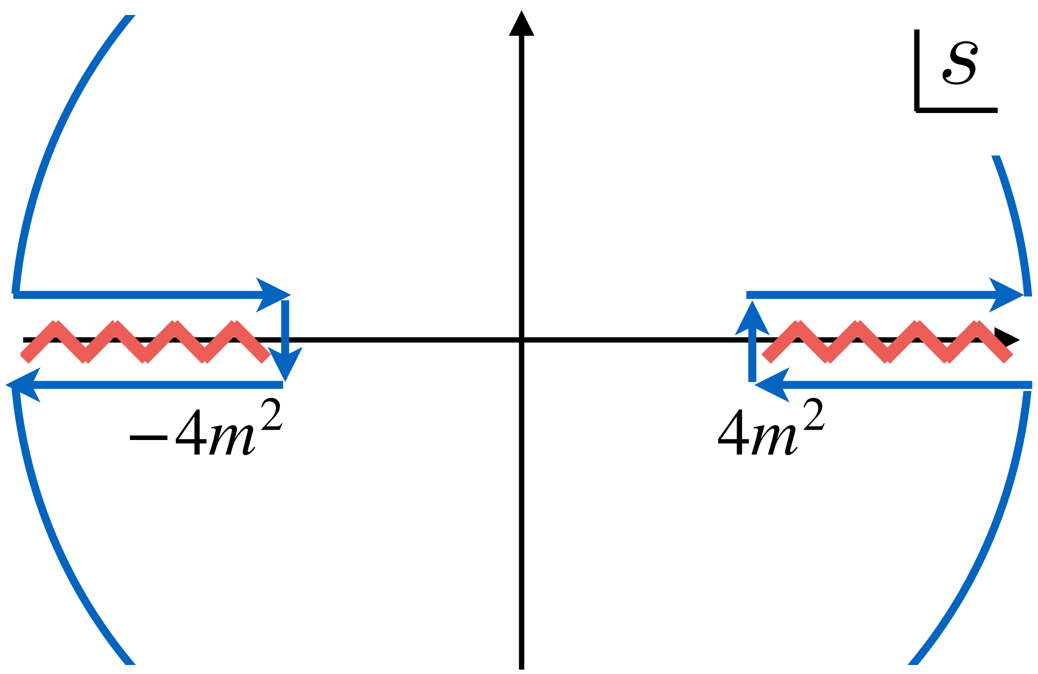

analytic structure of $\mathcal{M}(s, t=0)$

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$$32\left(\alpha_1 + \alpha_2\right) = \oint_{C_0} \frac{ds}{2\pi i} \frac{\mathcal{M}(s, t=0)}{s^3}$$

Positive because of unitarity!

$|\mathcal{M}(s, t=0)| < |s|^2$

Deform the integration contour to rewrite it in the UV language:

$$32\left(\alpha_1 + \alpha_2\right) = \frac{2}{\pi}\int_{4m^2}^{\infty} ds \frac{\mathrm{Im}\mathcal{M}(s, t=0)}{s^3} + \oint_{C_\infty} \frac{ds}{2\pi i} \frac{\mathcal{M}(s, t=0)}{s^3}$$

※ used the s-u symmetry and $\mathrm{Disc}\,\mathcal{M}(s, t=0) = 2i\,\mathrm{Im}\,\mathcal{M}(s, t=0)$

This implies that EFTs with $\alpha_1 + \alpha_2 < 0$ cannot be embedded into any unitary UV theory satisfying $|M(s, t=0)| < |s|^2 \ \ (|s| \to \infty)$.

In other words, $\alpha_1 + \alpha_2 > 0$ is required to have such a UV completion

※ generalization to other helicity sums shows $\alpha_1, \alpha_2 > 0$

(positivity bounds [Adams et al '06])

Froissart bound:

$|M(s, t=0)| < s \ln^2 s \ \ (|s| \to \infty)$ follows from locality etc in gapped theories

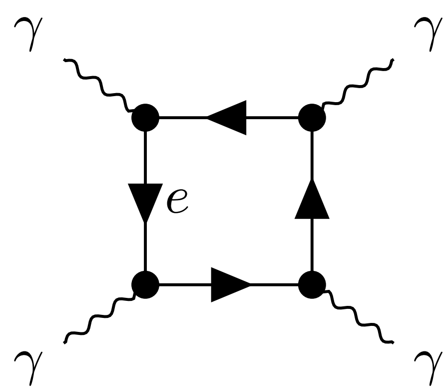
Contents

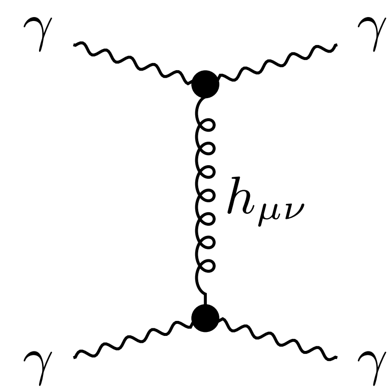
2. Gravitational Positivity Bounds

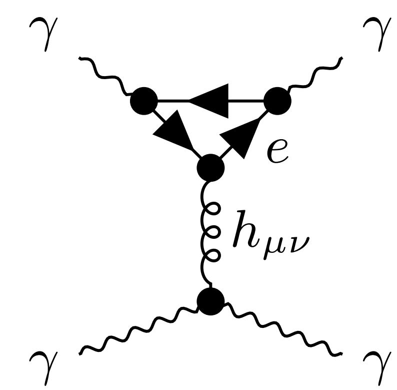
Gravitational QED as an EFT (1)

QED coupled to GR: $\mathscr{L} = \frac{M_{\rm Pl}^2}{2}R - \frac{1}{4}F_{\mu\nu}^2 - \bar{\psi}(\not{D}+m)\psi$

Consider 1-loop amplitudes:

$\mathscr{M}(s,t) =$ 

$+$ 

$+$ 

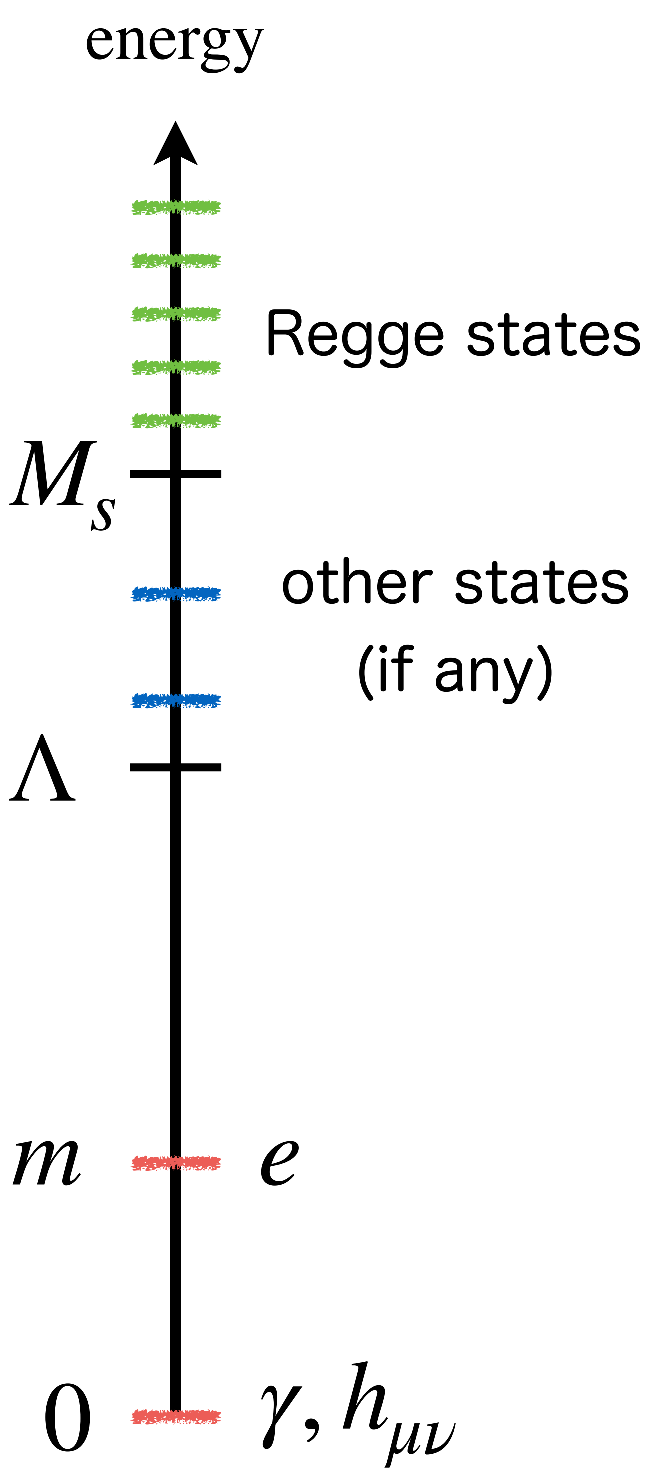
IR behavior: $\sim \frac{s^2}{m^4} + \mathcal{O}(t)$ $\qquad \sim \frac{s^2}{M_{\rm Pl}^2 t} + \mathcal{O}(t^0)$ $\qquad \sim \frac{s^2}{M_{\rm Pl}^2 m^2} + \mathcal{O}(t)$

UV behavior: $\sim \ln^2 s$ $\qquad \sim \frac{s^2}{M_{\rm Pl}^2 t} + \mathcal{O}(t^0)$ $\qquad \sim \frac{s^2 \ln^2 s}{M_{\rm Pl}^2 m^2} + \mathcal{O}(t)$

Gravity dominates at UV and the amplitude grows as $\sim s^2$

Expectation: the UV behavior becomes mild in the UV complete theory

Gravitational QED as an EFT (2)



A UV criterion motivated by perturbative string theory:

$|\mathcal{M}(s,t)| < s^2$ $(|s| \to \infty, t < 0 : \text{fixed})$

cf. tree-level closed string exchange

$$\mathcal{M}(s,t)| \sim \frac{1}{M_{\rm Pl}^2 t} s^{2+\alpha' t/2} < s^2 \quad \text{for} \quad t < 0$$

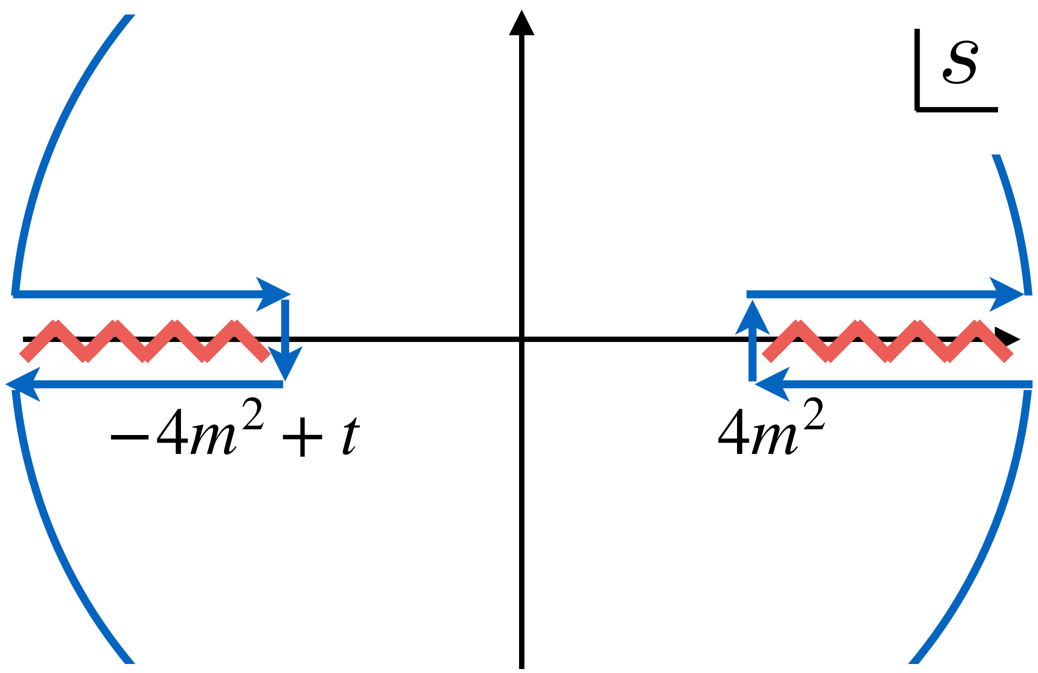
※ Higher spin Regge states make the UV behavior mild

UV completable?

Gravitational QED: $\mathscr{L} = \frac{M_{\rm Pl}^2}{2} R - \frac{1}{4} F_{\mu\nu}^2 - \bar{\psi}(\not{D} + m)\psi + \cdots$

cf. Froissart-Martin bound (gapped theories): $|M(s,t)| < s^2$ $(|s| \to \infty; 0 \leq t < 4m_{\rm ext}^2)$

Implications of analyticity



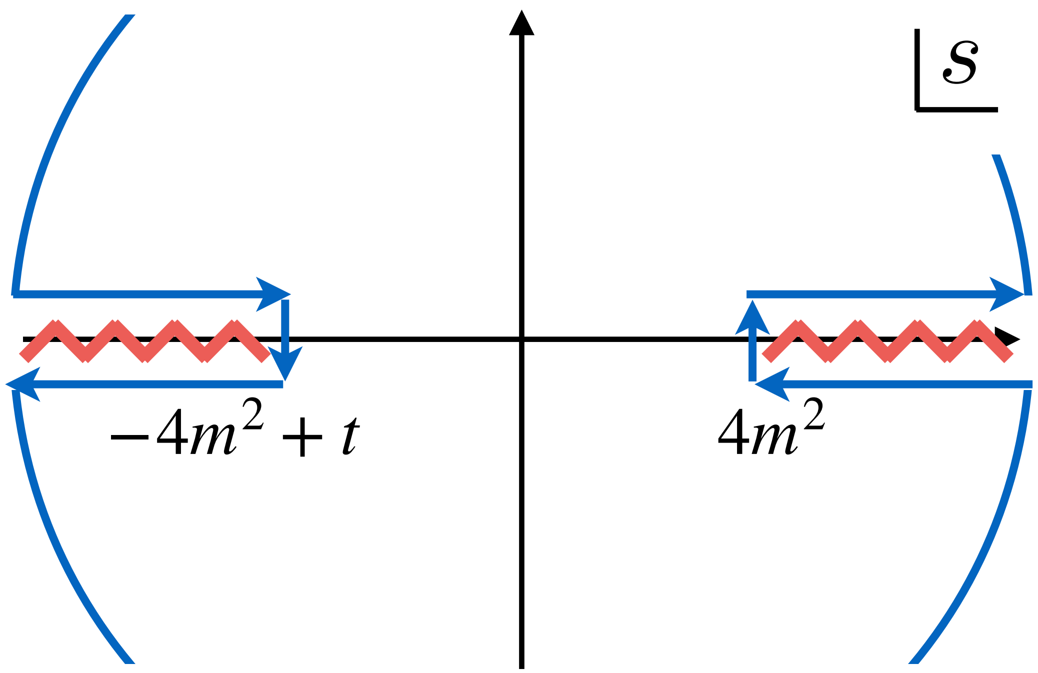
IR expansion of the $\gamma\gamma \to \gamma\gamma$ amplitude:

$$\mathcal{M}(s,t) = -\frac{4su}{M_{\rm Pl}^2 t} - \frac{4tu}{M_{\rm Pl}^2 s} - \frac{4ts}{M_{\rm Pl}^2 u} + \sum_{n=0}^{\infty} \frac{c_n(t)}{n!}\left(s + \frac{t}{2}\right)^n$$

Repeating the same argument as before, we find

$$c_2(t) - \frac{8}{M_{\rm Pl}^2 t} = \frac{4}{\pi}\int_{4m^2}^{\infty} ds \frac{{\rm Im}\mathcal{M}(s,t)}{(s+t/2)^3}$$

Implications of analyticity



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Repeating the same argument as before, we find

$$c_2(t) - \frac{8}{M_{\rm Pl}^2 t} = \frac{4}{\pi}\int_{4m^2}^{\infty} ds \frac{\mathrm{Im}\mathcal{M}(s,t)}{(s+t/2)^3} = \frac{4}{\pi}\int_{4m^2}^{\Lambda^2} ds \frac{\mathrm{Im}\mathcal{M}(s,t)}{(s+t/2)^3} + \frac{4}{\pi}\int_{\Lambda^2}^{\infty} ds \frac{\mathrm{Im}\mathcal{M}(s,t)}{(s+t/2)^3}$$

It is convenient to reformulate it as

$$B(\Lambda,t) := c_2(t) - \frac{4}{\pi}\int_{4m^2}^{\Lambda^2} ds \frac{\mathrm{Im}\mathcal{M}(s,t)}{(s+t/2)^3} = \frac{8}{M_{\rm Pl}^2 t} + \frac{4}{\pi}\int_{\Lambda^2}^{\infty} ds \frac{\mathrm{Im}\mathcal{M}(s,t)}{(s+t/2)^3}$$

※ red terms: calculable within the EFT, i.e., QED + GR

Gravitational positivity bounds [Tokuda-Aoki-Hirano '20]

In the previous slide, we derived

$$B(\Lambda, t) := c_2(t) - \frac{4}{\pi}\int_{4m^2}^{\Lambda^2} ds \frac{\mathrm{Im}\mathcal{M}(s,t)}{(s+t/2)^3} = \frac{8}{M_{\rm Pl}^2 t} + \frac{4}{\pi}\int_{\Lambda^2}^{\infty} ds \frac{\mathrm{Im}\mathcal{M}(s,t)}{(s+t/2)^3}$$

- $B(\Lambda, t)$ is finite in the forward limit $t \to 0$ and also calculable within the EFT
- each term of the r.h.s. diverges in the forward limit, but the sum has to be finite

Assume the following Regge behavior of the imaginary part:

$$\mathrm{Im}\mathcal{M}(s,t) = f(t)\left(\frac{s}{M_s^2}\right)^{2+\alpha' t + \alpha'' t^2 + \cdots} + \text{sub-leading terms.}$$

If one further makes the single scaling behavior $\left|f'/f\right|, |\alpha''/\alpha'| \lesssim \alpha' \sim M_s^{-2}$,

one can explicitly show that $B(\Lambda) := B(\Lambda, 0) > -\mathcal{O}(1)\cdot\dfrac{1}{M_{\rm Pl}^2 M_s^2}$.

※ It is important to clarify how generic this single scaling assumption is.

For related developments, see also Hamada-TN-Shiu '18, Herrero-Valea et al '20, Bellazzini et al'19, Alberte et al '20, Arkani-Hamed et al '20, Caron-Huot et al '21.

Summary of the section

standard assumptions of positivity + the single scaling assumption

$\rightarrow$ an approximate positivity $B(\Lambda) := c_2(0) - \dfrac{4}{\pi}\displaystyle\int_{4m^2}^{\Lambda^2} ds \frac{\mathrm{Im}\mathcal{M}(s,0)}{s^3} > -\mathcal{O}(1) \cdot \frac{1}{M_{\rm Pl}^2 M_s^2}$.

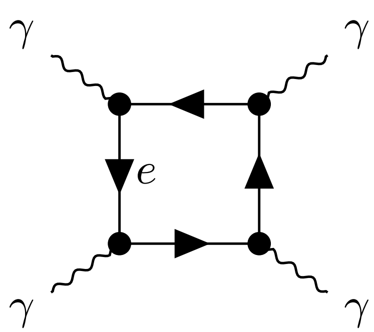
3. Positivity in Gravitational QED

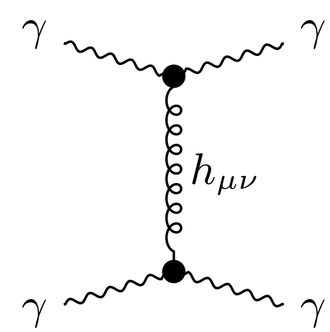
[Alberte-de Rham-Jaitly-Tolley ’20, see also Aoki-Loc-TN-Tokuda ’21]

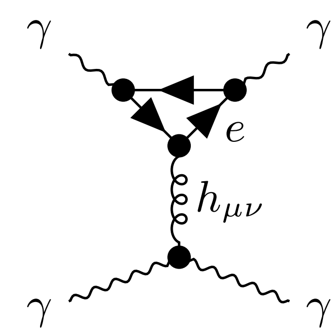
Decomposition of scattering amplitudes

gravitational positivity: $B(\Lambda) := c_2(0) - \frac{4}{\pi}\int_{4m^2}^{\Lambda^2} ds \frac{\mathrm{Im}\mathscr{M}(s,0)}{s^3} > -\mathcal{O}(1)\cdot\frac{1}{M_{\rm Pl}^2 M_s^2}$

- Decompose the $\gamma\gamma \to \gamma\gamma$ amplitude at IR as $\mathscr{M} = \mathscr{M}_{\rm QED} + \mathscr{M}_{\rm GR} + \mathscr{M}_{\rm UV}$

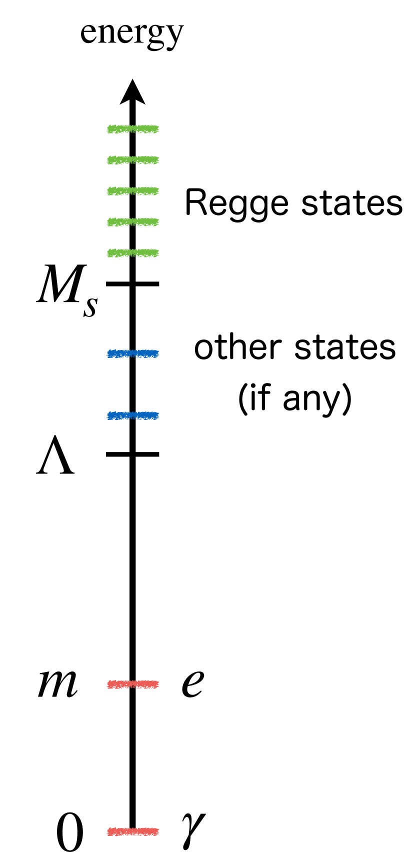
$\mathscr{M}_{\rm QED} =$ 

, $\mathscr{M}_{\rm GR} =$ 

$+$ 

$\mathscr{M}_{\rm UV}$: effects of UV dof $\lesssim \frac{1}{\Lambda^4}$

- We perform similar decompositions, e.g., as $B = B_{\rm QED} + B_{\rm GR} + B_{\rm UV}$



Evaluation of B's

gravitational positivity: $B(\Lambda) := c_2(0) - \frac{4}{\pi}\int_{4m^2}^{\Lambda^2} ds \frac{\mathrm{Im}\mathscr{M}(s,0)}{s^3} > -\mathcal{O}(1)\cdot\frac{1}{M_{\mathrm{Pl}}^2 M_s^2}$

- evaluation of B_{QED}

Technically, it is convenient to remind $|\mathscr{M}_{\mathrm{QED}}(s,0)| < s^2$,

so that $c_{2,\mathrm{QED}}(0) = \frac{4}{\pi}\int_{4m^2}^{\infty} ds \frac{\mathrm{Im}\mathscr{M}_{\mathrm{QED}}(s,0)}{s^3}$ (cf. positivity in non-gravitational QED)

This implies $B_{\mathrm{QED}}(\Lambda) = \frac{4}{\pi}\int_{\Lambda^2}^{\infty} ds \frac{\mathrm{Im}\mathscr{M}_{\mathrm{QED}}(s,0)}{s^3} = \frac{4e^4}{\pi^2\Lambda^4}\left(\ln\frac{\Lambda}{m} - \frac{1}{4}\right)$.

※ Notice in particular that $\lim_{\Lambda\to\infty} B_{\mathrm{QED}}(\Lambda) = 0$.

- A straightforward computation shows $B_{\mathrm{GR}}(\Lambda) = -\frac{11e^2}{90\pi^2 m^2 M_{\mathrm{Pl}}^2}$

This gives a negative contribution that survives even in the limit $\Lambda \to \infty$.

Cutoff scale of gravitational QED

gravitational positivity: $B(\Lambda) := c_2(0) - \frac{4}{\pi}\int_{4m^2}^{\Lambda^2} ds \frac{\mathrm{Im}\mathscr{M}(s,0)}{s^3} > -\mathcal{O}(1)\cdot\frac{1}{M_{\mathrm{Pl}}^2 M_s^2}$

Now the gravitational positivity bound reads

$$\frac{4e^4}{\pi^2\Lambda^4}\left(\ln\frac{\Lambda}{m} - \frac{1}{4}\right) - \frac{11e^2}{90\pi^2 m^2 M_{\mathrm{Pl}}^2} + \frac{\alpha_{\mathrm{UV}}}{\Lambda^4} > -\mathcal{O}(1)\cdot\frac{1}{M_{\mathrm{Pl}}^2 M_s^2} \qquad (|\alpha_{\mathrm{UV}}| \lesssim 1)$$

Since $m \ll \Lambda \lesssim M_s$, we find $\frac{64\alpha^2}{\Lambda^4}\left(\ln\frac{\Lambda}{m} - \frac{1}{4}\right) + \frac{\alpha_{\mathrm{UV}}}{\Lambda^4} > \frac{22\alpha}{45\pi m^2 M_{\mathrm{Pl}}^2}$,

which gives an upper bound on the cutoff scale:

$$\Lambda \lesssim \min\left[\sqrt{emM_{\mathrm{Pl}}}, |\alpha_{\mathrm{UV}}|^{-1/4}\sqrt{mM_{\mathrm{Pl}}/e}\right] \sim 10^8\ \mathrm{GeV}.$$

↑

for QED parameters in our real world

Summary so far

- standard assumptions of positivity + the single scaling assumption implies

an approximate positivity bound $B(\Lambda) := c_2(0) - \frac{4}{\pi}\int_{4m^2}^{\Lambda^2} ds \frac{\mathrm{Im}\mathscr{M}(s,0)}{s^3} > -\mathcal{O}(1)\cdot\frac{1}{M_{\mathrm{Pl}}^2 M_s^2}.$

- when applied to gravitational QED, this implies a cutoff $\Lambda \lesssim \sqrt{mM_{\mathrm{Pl}}/e} \sim 10^8$ GeV.

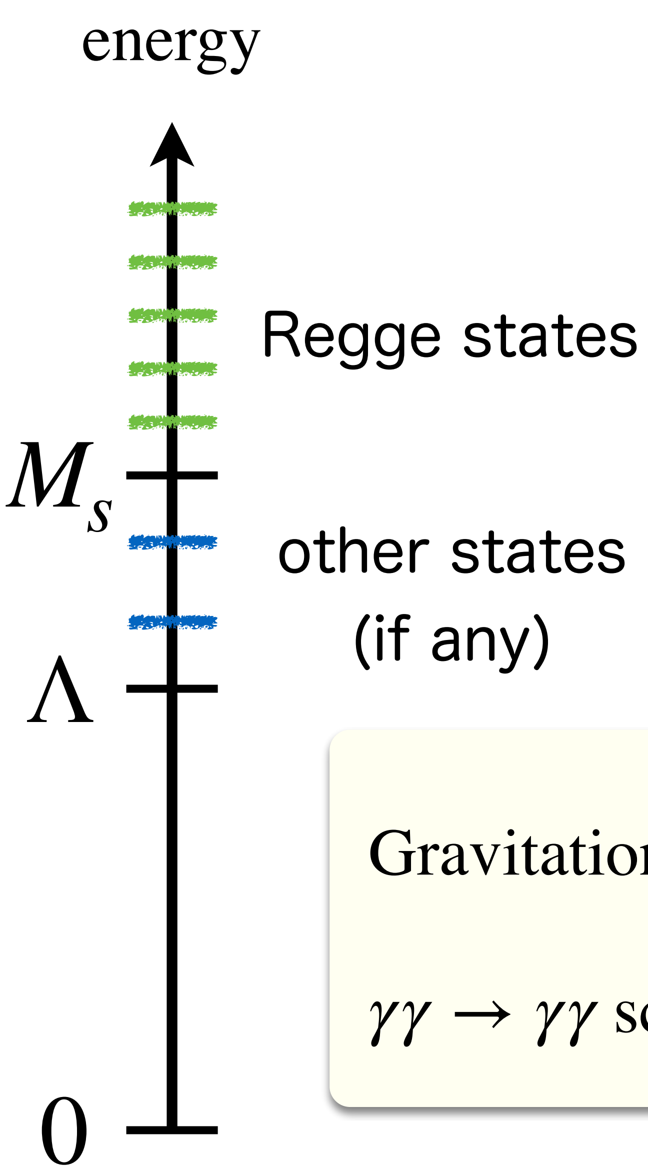
too small to believe the bound??? massless limit is not allowed???

→ we extended the analysis to the Standard Model

4. Positivity in Gravitational Standard Model

[Aoki-Loc-TN-Tokuda ’21]

Gravitational Standard Model



Gravitational Standard Model: $\mathscr{L} = \mathscr{L}_{\rm SM} + \frac{M_{\rm Pl}^2}{2} R + \cdots$

$\gamma\gamma \to \gamma\gamma$ scattering: $\mathscr{M} = \mathscr{M}_{\rm QED} + \mathscr{M}_{\rm weak} + \mathscr{M}_{\rm QCD} + \mathscr{M}_{\rm GR} + \mathscr{M}_{\rm UV}$

What to do is the same as the QED case except for

(A) there exist charged spin 1 particles (W bosons)

(B) hadrons may contribute if some of s, t, u is below the QCD scale

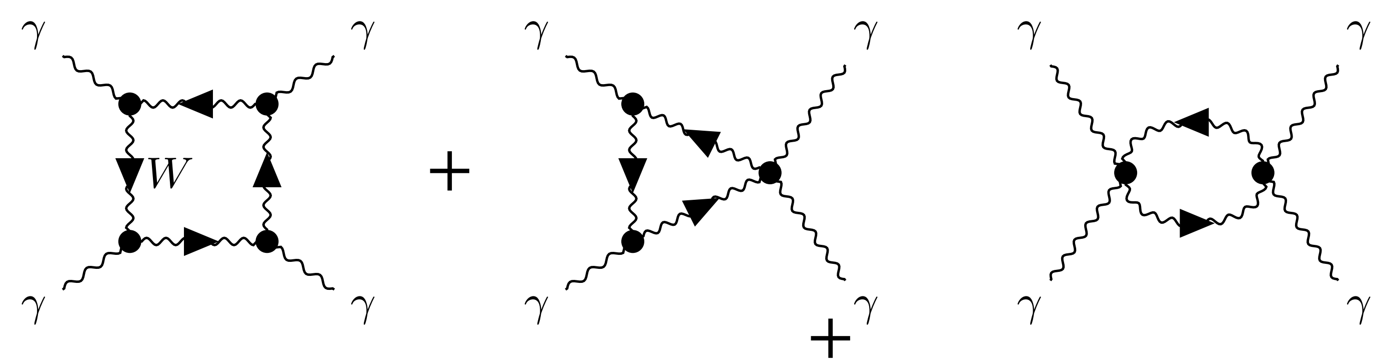
Weak sector analysis

gravitational positivity: $B(\Lambda) := c_2(0) - \frac{4}{\pi}\int_{4m^2}^{\Lambda^2} ds \frac{\mathrm{Im}\mathscr{M}(s,0)}{s^3} > -\mathcal{O}(1)\cdot\frac{1}{M_{\mathrm{Pl}}^2 M_s^2}$

- just like the QED case, we have $B_{\mathrm{weak}}(\Lambda) = \frac{4}{\pi}\int_{\Lambda^2}^{\infty} ds \frac{\mathrm{Im}\mathscr{M}_{\mathrm{weak}}(s,0)}{s^3}$.

- due to the spin 1 nature, W boson contributions grow faster than the QED case

$\mathscr{M}_{\mathrm{weak}} \simeq$



$$\simeq \frac{2e^4}{\pi^2 m_W^2} s \ln\frac{m_W^2}{-s} + (s \leftrightarrow -s) \qquad \text{cf. } \mathscr{M}_{\mathrm{QED}} \sim \ln^2 s$$

- we then find $B_{\mathrm{weak}}(\Lambda) = \frac{8e^4}{\pi^2 m_W^2 \Lambda^2} > B_{\mathrm{QED}}(\Lambda) = \frac{4e^4}{\pi^2\Lambda^4}\left(\ln\frac{\Lambda}{m} - \frac{1}{4}\right)$

- on the other hand, weak boson loops are sub-dominant in B_{GR}

QCD sector analysis

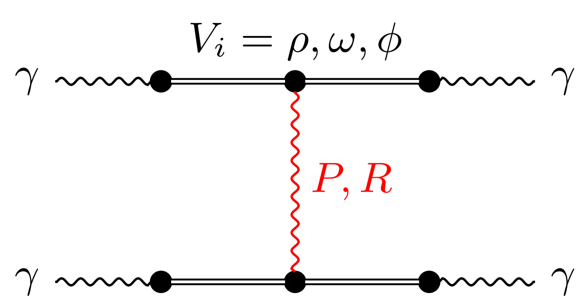
gravitational positivity: $B(\Lambda) := c_2(0) - \frac{4}{\pi}\int_{4m^2}^{\Lambda^2} ds \frac{\mathrm{Im}\mathcal{M}(s,0)}{s^3} > -\mathcal{O}(1)\cdot\frac{1}{M_{\mathrm{Pl}}^2 M_s^2}$

- again, we have $B_{\mathrm{QCD}}(\Lambda) = \frac{4}{\pi}\int_{\Lambda^2}^{\infty} ds \frac{\mathrm{Im}\mathcal{M}_{\mathrm{QCD}}(s,0)}{s^3}$.

- while the amplitude on the r.h.s. is high-energy, the momentum transfer is small

 $\rightarrow$ t-channel exchange of hadrons is relevant

$\mathrm{Im}\mathcal{M}_{\mathrm{QCD}} \simeq \mathrm{Im}$



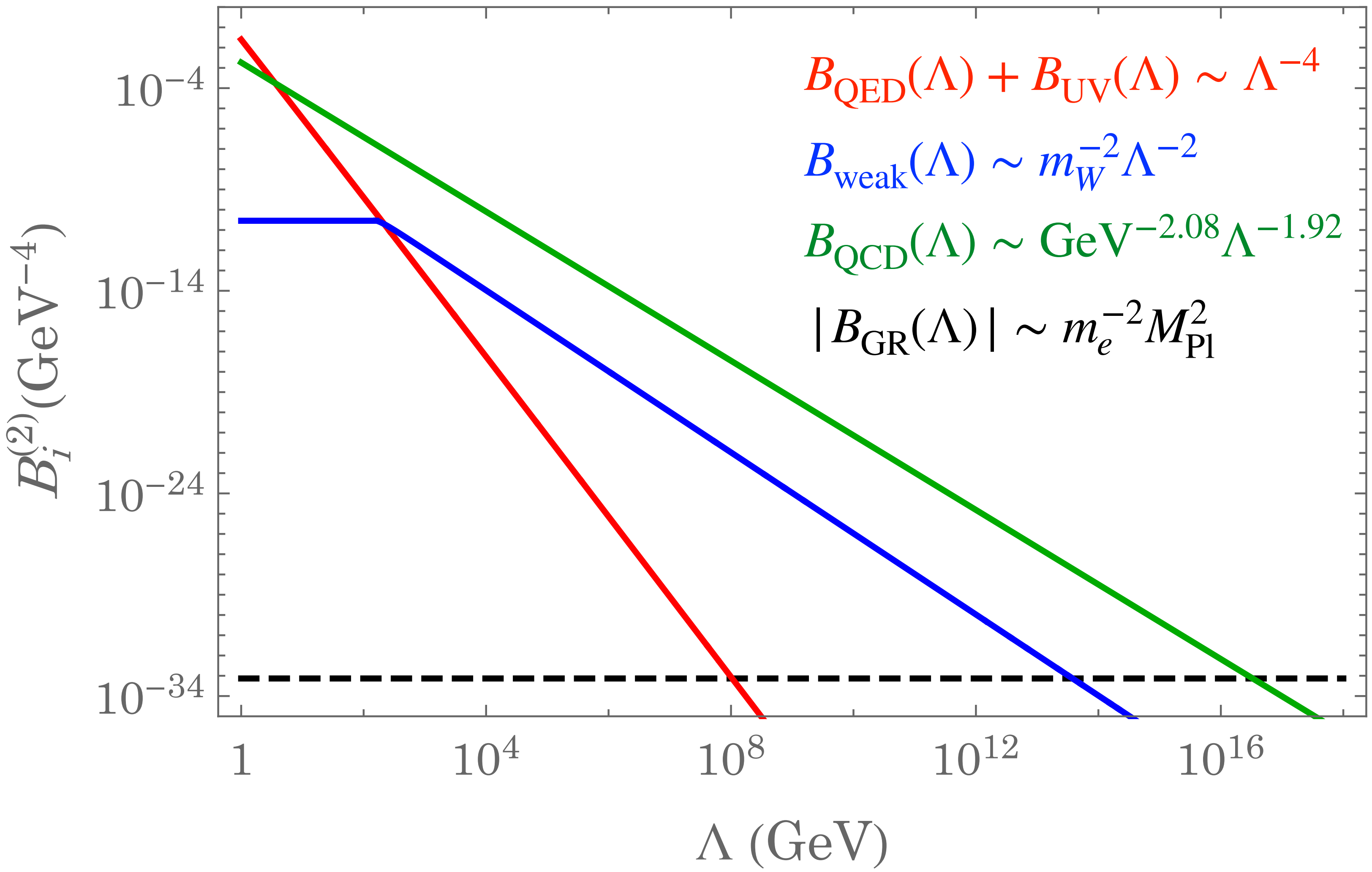
(P: Pomeroon, R: Reggeon)

- employing the Vector Meson Dominance (VDM) model,

$$\mathrm{Im}\mathcal{M}_{\mathrm{QCD}} \simeq \frac{25e^4}{16\pi^2}\left(\frac{s}{\mathrm{GeV}^2}\right)^{1.08}$$

(See our paper for model-(in)sensitivity)

Cutoff scale of gravitational SM

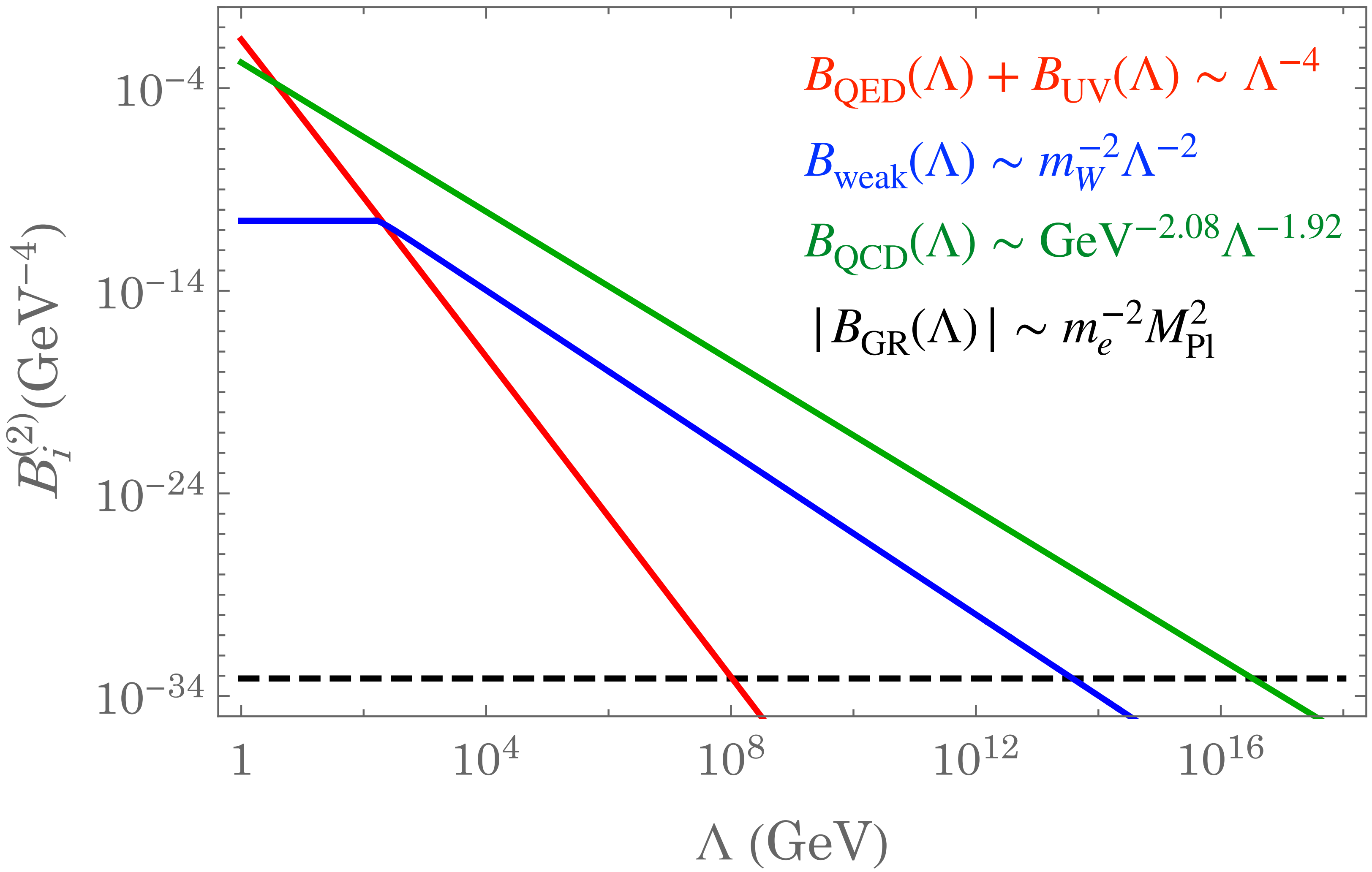


gravitational positivity:

$$B_{\rm QED}(\Lambda) + B_{\rm UV}(\Lambda) + B_{\rm weak}(\Lambda) + B_{\rm QCD}(\Lambda) > -B_{\rm GR}(\Lambda) - \mathcal{O}(1)\cdot\frac{1}{M_{\rm Pl}^2 M_s^2}$$

$\rightarrow$ this defines the cutoff of the gravitational SM $\Lambda \simeq 3\times 10^{16}$ GeV.

A remark on EW theory w/o QCD



gravitational positivity implies:

$$B_{\rm weak}(\Lambda) > -B_{\rm GR}(\Lambda) \quad \rightleftarrows \quad \frac{m_W}{M_{\rm Pl}} < \sqrt{\frac{720}{11}}\, e\, \frac{m_e}{\Lambda}$$

- Possible explanation for the hierarchy between the EW scale and the Planck scale??
- Massless limit $m_e \to 0$ is allowed if we take the limit $m_W \to 0$ simultaneously

Summary and prospects

Summary

1. Gravitational positivity bounds

standard assumptions of positivity + the single scaling assumption implies

an approximate positivity bound $B(\Lambda) := c_2(0) - \frac{4}{\pi}\int_{4m^2}^{\Lambda^2} ds \frac{\mathrm{Im}\mathcal{M}(s,0)}{s^3} > -\mathcal{O}(1)\cdot\frac{1}{M_{\mathrm{Pl}}^2 M_s^2}$.

2. Puzzles on positivity in gravitational QED [Alberte-de Rham-Jaitly-Tolley '20, …]

- implies a cutoff scale $\Lambda \sim 10^8$ GeV (too low to believe???)
- implies that massless QED $m_e \to 0$ is in the Swampland (sounds strange???)

3. Positivity in gravitational Standard Model [Aoki-Loc-TN-Tokuda '21]

- the cutoff scale is improved up to $\Lambda \sim 10^{16}$ GeV
- when applied to EW theory w/o QCD, we find $\frac{m_W}{M_{\mathrm{Pl}}} < \sqrt{\frac{720}{11}}\, e\, \frac{m_e}{\Lambda}$.

※ massless limit $m_e \to 0$ is allowed if we take $m_W \to 0$ simultaneously

Future directions

- How generic the single scaling assumption is? → detailed study of string amplitudes
- connections to other principles such as energy conditions, entropy bounds?
- phenomenological applications

  e.g., bounds on scalar potentials [TN-Tokuda '21], dark matters, neutrinos, …
- possible implications for Higgs mechanism in string theory (brane recombination)?

Thank you!