Probing Hawking radiation through capacity of entanglement

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May 19, 2021

Based on 2102.02425 and 2105.08396

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Overview

1. Introduction: entropy of Hawking radiation
2. Review: capacity of entanglement
3. Capacity of Hawking radiation in toy models
4. Capacity formula in 2d dilaton gravity

Outline

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Hawking radiation from an evaporating BH

- Suppose the initial state of matter is pure

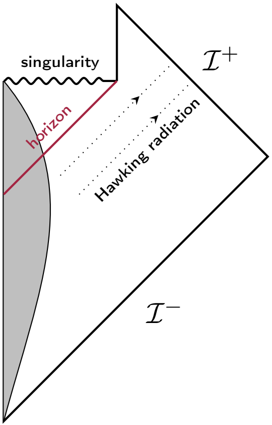
$$\rho_{\text{pure}} = |\Psi\rangle\langle\Psi|$$

but after gravitational collapse a black hole is formed

- BH starts to evaporate due to Hawking radiation
- After the evaporation of BH, the system is in a mixed state of thermal radiation:

$$\rho_{\text{pure}} \xrightarrow[\text{BH evaporation}]{} \rho_{\text{mixed}}$$

which appears to contradict with unitarity [Hawking 76]



Page curve for the radiation

To model an evaporating BH with radiation, suppose

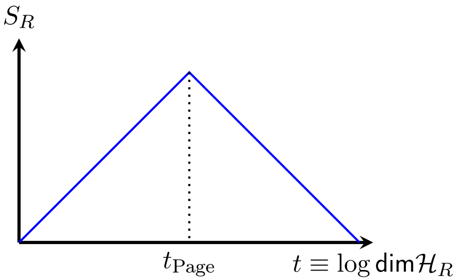
$$|\Psi\rangle \in \mathcal{H}_{\text{BH}} \otimes \mathcal{H}_R \,, \qquad \mathcal{H}_{\text{BH}} : \text{BH system} \,, \qquad \mathcal{H}_R : \text{radiation system}$$

- For a pure state $|\Psi\rangle$ Page showed [Page 93]
  when $\dim \mathcal{H}_R \ll \dim \mathcal{H}_{\text{BH}}$ the radiation system is almost maximally entangled :

$$S_R \approx \log \dim \mathcal{H}_R$$

- In the opposite limit, $\dim \mathcal{H}_R \gg \dim \mathcal{H}_{\text{BH}}$, from unitarity

$$S_R \approx \log(\dim \mathcal{H}_{\text{tot}} - \dim \mathcal{H}_R)$$



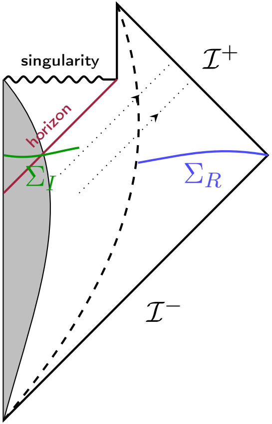
Island formula for the radiation entropy

- To reconcile with the Page curve, the entropy of radiation should be calculated by the island formula [Penington 19, Almheiri-Engelhardt-Marolf-Maxfield19, Almheiri-Mahajan-Maldacena-Zhao 19]:

$$S_R = \min_{\Sigma_I} \left\{ \text{ext}_{\Sigma_I} \left[\frac{\text{Area}(\partial\Sigma_I)}{4G_N} + S_{\text{mat}}(\Sigma_R \cup \Sigma_I) \right] \right\}$$

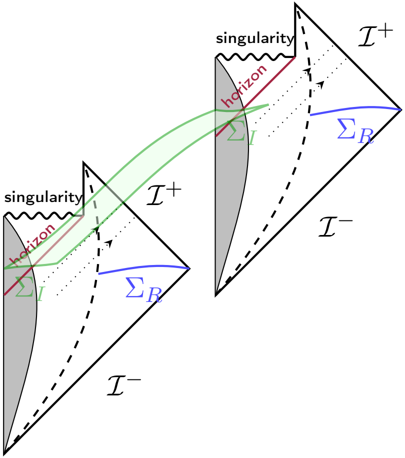
  - Σ_R: radiation region R
  - Σ_I: island region I

- No island $\to$ linear growth at early time
- With island $\to$ saturation or decay at late time



Replica wormholes

- The island formula is a generalization of the Ryu-Takayanagi formula for entanglement entropy [Ryu-Takayanagi 06], which has a gravitational path integral derivation [Lewkowycz-Maldacena 13, ⋯]
- The island regions are accounted for by replica wormholes [Almheiri-Maldacena-Hartman-Shaghoulian-Tajdini 19, Penington-Shenker-Stanford-Yang 19]



Goal of this talk

- We will examine if Hawking radiation (or replica wormholes) can be captured by capacity of entanglement, a quantum information measure other than entanglement entropy
- Calculate the capacity for two toy models of radiating black holes:
  - End of the world (EOW) brane model [Penington-Shenker-Stanford-Yang 19]
  - Moving mirror model [Akal-Kusuki-Shiba-Takayanagi-Wei 20]
- Derive a formula for the capacity in 2d dilaton gravity
  - Apply to an eternal AdS_2 black hole coupled to a flat bath region at high temperature
- The capacity has a peak or discontinuity at the Page time, showing a good probe of the radiation

Outline

Entanglement entropy

Divide a system to A and $B = \bar{A}$: $\mathcal{H}_{\text{tot}} = \mathcal{H}_A \otimes \mathcal{H}_B$

Entanglement entropy

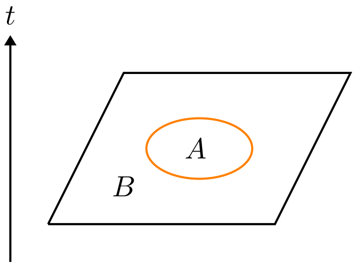
$$S_A = -\text{tr}_A \left[\rho_A \log \rho_A\right]$$

- The reduced density matrix

$$\rho_A \equiv \text{tr}_B[\rho_{\text{tot}}]$$

- For a pure ground state $|\Psi\rangle$

$$\rho_{\text{tot}} = |\Psi\rangle \langle\Psi|$$



Replica trick and Rényi entropy

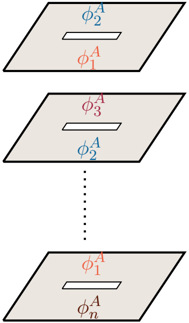
Entanglement entropy

$$S_A = \lim_{n \to 1} S_n$$

n^{th} Rényi entropy

$$S_n \equiv \frac{1}{1-n} \log \mathrm{tr}_A[\rho_A^n] = \frac{1}{1-n} \log Z(n)$$

$Z(n)$: partition function on the n-fold cover branched over A



Analogy to statistical mechanics

We regard $Z(n) \equiv \mathrm{tr}_A[\rho_A^n]$ as a thermal partition function at an inverse temperature $\beta \equiv n$:

Statistical mechanics	Rényi entropy
inverse temperature	$\beta = n$
Hamiltonian	$H_A = -\log \rho_A$
partition function	$Z(\beta) = \mathrm{tr}_A \left[e^{-\beta H_A} \right]$
free energy	$F(\beta) = -\beta^{-1} \log Z(\beta)$
energy	$E(\beta) = -\partial_\beta \log Z(\beta)$
thermal entropy	$\tilde{S}(\beta) = \beta^2 \partial_\beta F(\beta)$
heat capacity	$C(\beta) = -\beta \partial_\beta \tilde{S}(\beta)$

Capacity of entanglement

- The "thermal" entropy is *not* the Rényi entropy

$$S_n = -\frac{1}{n-1}\log Z(\beta) = \frac{n}{n-1}\,F(\beta) \neq \beta^2\,\partial_\beta F(\beta)$$

but a refined one (improved Rényi/modular entropy [Dong 16, Nakaguchi-TN 16]):

$$\tilde{S}_n \equiv \tilde{S}(\beta) = \beta^2\,\partial_\beta F(\beta) = n^2\,\partial_n\left(\frac{n-1}{n}\,S_n\right)$$

- The capacity of entanglement [Yao-Qi 10] is non-negative for a unitary theory:

$$C_n \equiv C(\beta) = n^2\langle\,(H_A - \langle\,H_A\,\rangle_n)^2\,\rangle_n \geq 0$$

where $\langle\,X\,\rangle_n \equiv \mathrm{tr}_A\left[X\,e^{-n\,H_A}\right]/Z(\beta)$

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Capacity of Hawking radiation in toy models of BH

[Kawabata-TN-Okuyama-Watanabe 21]

- We will examine if the capacity can probe the Hawking radiation, i.e., replica wormholes:

Capacity of entanglement $(n = 1)$

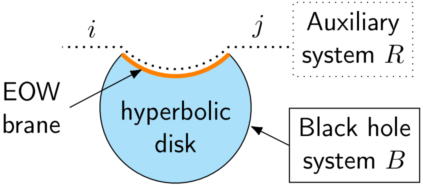
$$C \equiv C_{n=1} = -\partial_n \tilde{S}_n|_{n=1} \; (= -2\,\partial_n S_n|_{n=1})$$

- **Two toy models of radiating black holes**
  - End of the world (EOW) brane model [Penington-Shenker-Stanford-Yang 19]
  - Moving mirror model [Akal-Kusuki-Shiba-Takayanagi-Wei 20]

EOW brane model [Penington-Shenker-Stanford-Yang 19]

- A quantum mechanical model of a radiating black hole:

$$|\Psi\rangle = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} |\psi_i\rangle_B \, |i\rangle_R \qquad \langle\psi_j|\psi_i\rangle_B \, |i\rangle\langle j|_R =$$



  - B: BH system of dimension e^{S_0} (JT gravity + EOW brane)
  - R: auxiliary system of dimension k to measure Hawking radiation

- $^{\exists}$replica wormhole: $\langle\psi_i|\psi_k\rangle_B = \delta_{ij} + e^{-S_0/2}\, R_{ij}$ (R_{ij}: random variable)

$$\mathrm{tr}_R\,[\rho_R^n] = \frac{1}{(k\,e^{S_0})^n} \sum_{i_1,\cdots,i_n=1}^{k} \langle\psi_{i_1}|\psi_{i_2}\rangle_B \cdot \langle\psi_{i_2}|\psi_{i_3}\rangle_B \cdots \langle\psi_{i_n}|\psi_{i_1}\rangle_B$$

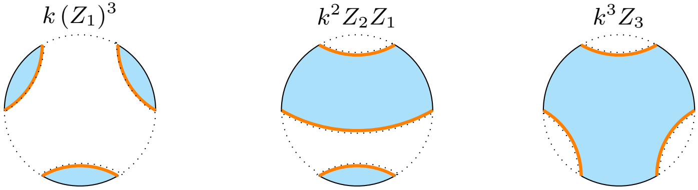
Planar approximation

- In the planar limit, $e^{S_0} \gg 1$ with $k\, e^{-S_0}$ fixed

$$\mathrm{tr}_R[\rho_R^n] \approx \frac{1}{k^{n-1}} \left[1 + \binom{n}{2} \cdot \frac{k\, Z_2}{(Z_1)^2} + \cdots + \frac{k^{n-1} Z_n}{(Z_1)^n} \right]$$

$Z_n (\propto e^{S_0})$: replica wormhole partition function of disk topology with n boundaries

Example $(n = 3)$:



fully disconnected | partly connected | fully connected

Entanglement entropy at early and late times

- $\dim \mathcal{H}_R = k \iff$ # of radiation particles $\approx \log k$
- $\log k$: time of BH evaporation
  - **Early time** $(\log k \ll S_0)$: fully disconnected solution dominates

$$\mathrm{tr}_R[\rho_R^n] \approx \frac{1}{k^{n-1}} \qquad \Rightarrow \qquad S_R \approx \log k$$

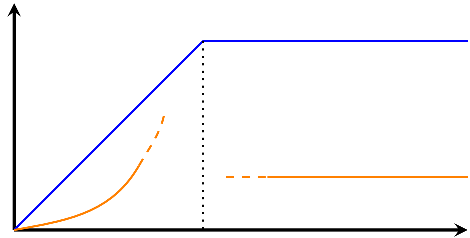
  - **Late time** $(\log k \gg S_0)$: fully connected solution dominates

$$\mathrm{tr}_R[\rho_R^n] \approx \frac{Z_n}{(Z_1)^n} \qquad \Rightarrow \qquad S_R \approx \lim_{n\to 1} (1-\partial_n) \log Z_n$$

Capacity and Page curve

- The asymptotic behavior of the capacity:

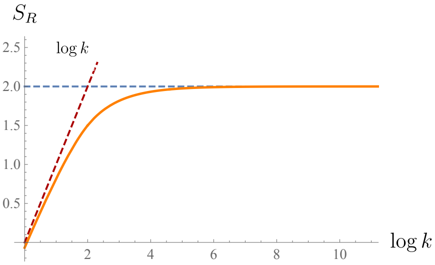
$$C_R \approx \begin{cases} k\,\frac{Z_2}{(Z_1)^2} \propto e^{\log k} & \textbf{(early time)} \\ \lim_{n\to 1} \partial_n^2 \log Z_n & \textbf{(late time)} \end{cases}$$

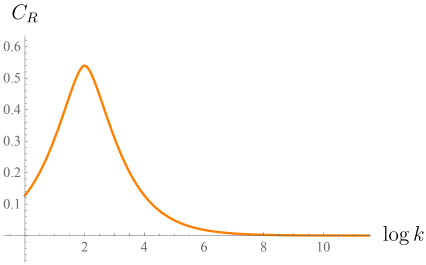


What happens for the capacity around the Page time?

Microcanonical ensemble

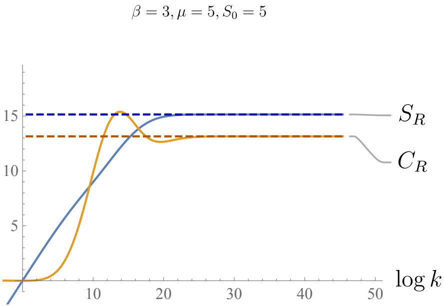
- Replica partition functions Z_n can be solved analytically in the microcanonical ensemble by fixing the energy of BH (in planar limit):
  - Entanglement entropy reproduces the Page curve for an eternal BH
  - The capacity shows a peak around the Page time and decays to zero at late time

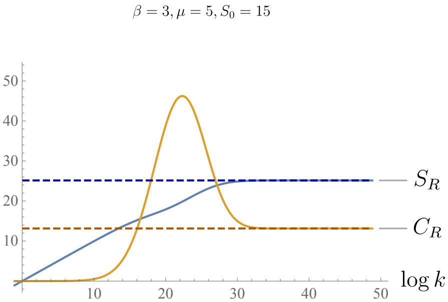




Canonical ensemble

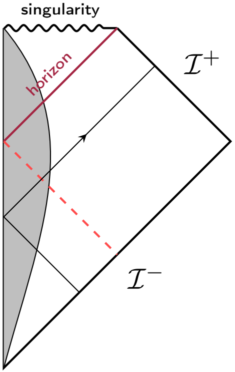
- Numerically calculate Z_n in the canonical ensemble by fixing the inverse temperature β of BH (in planar limit):
  - Entanglement entropy reproduces the similar Page curve as in the microcanonical ensemble
  - The capacity shows a peak around the Page time and approaches to a constant at late time

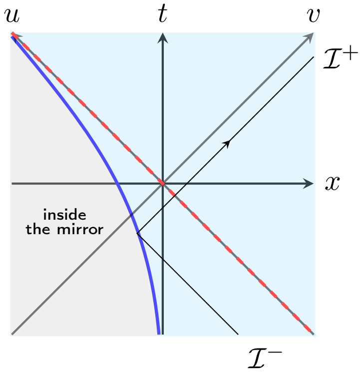




Moving mirror model of radiating BH [Davies-Fulling 76, Birrel-Davies 84, $\cdots$]

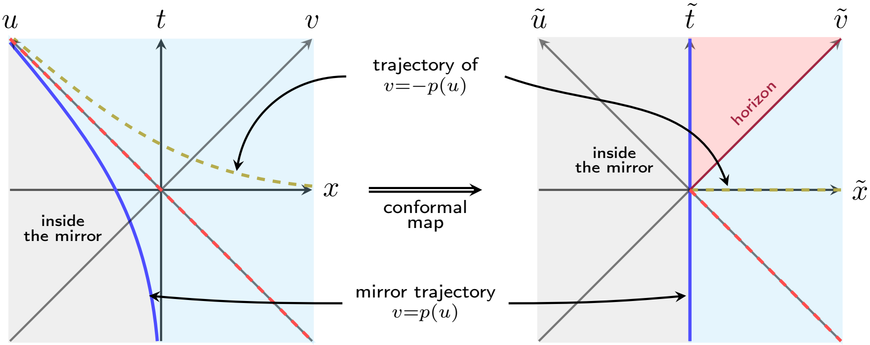
- CFT_2 on flat space with reflecting boundary condition at a moving mirror
- Known to have thermal energy flux (Hawking radiation) at null infinity





Conformal map to BCFT$_2$

- After a conformal map, the model becomes Boundary CFT$_2$ on the right half plane:



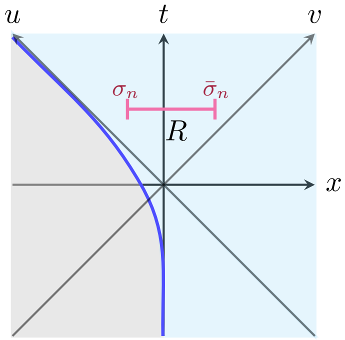
Measuring Hawking radiation

- Take an interval R at a fixed distance from the mirror and measure the radiation

- Replica partition functions can be calculated using twist operators:

$$\mathrm{tr}_R[\rho_R^n] \propto \langle\, \sigma_n(t_0, x_0)\, \bar{\sigma}_n(t_1, x_1)\, \rangle_{\text{BCFT}}$$

- Two-point functions in BCFT can be fixed by conformal block [McAvity-Osborn 93]



Holographic CFT

- Two-point functions greatly simplify in holographic CFT with large central charge [Takayanagi 11, Sully-Van Raamsdonk-Wakeham 20]:

$$\langle\, \tilde{\sigma}_n(\tilde{t}_0, \tilde{x}_0)\, \tilde{\bar{\sigma}}_n(\tilde{t}_1, \tilde{x}_1)\,\rangle_{\text{RHP}}$$

$$= \max \begin{cases} \langle\, \tilde{\sigma}_n(\tilde{t}_0, \tilde{x}_0)\, \tilde{\bar{\sigma}}_n(\tilde{t}_1, \tilde{x}_1)\,\rangle_{\mathbb{R}^{1,1}} & \textbf{(connected OPE channel)} \\ e^{2(1-n)S_{\text{bdy}}} \cdot \prod_{i\in\{0,1\}} \langle\, \tilde{\sigma}_n(\tilde{t}_i, \tilde{x}_i)\, \tilde{\bar{\sigma}}_n(\tilde{t}_i, \tilde{x}_i)\,\rangle_{\mathbb{R}^{1,1}}^{\frac{1}{2}} & \textbf{(disconnected OPE channel)} \end{cases}$$

- $S_{\text{bdy}} \equiv \log \langle 0|B\rangle$: boundary entropy for a boundary state $|B\rangle$

- Twist correlator in flat space:

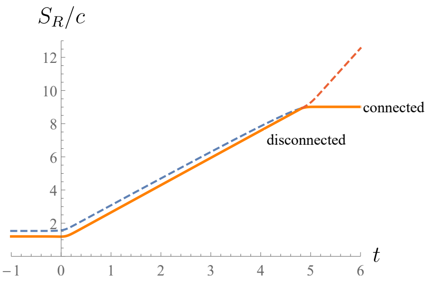
$$\langle\, \tilde{\sigma}_n(\tilde{t}, \tilde{x})\, \tilde{\bar{\sigma}}_n(\tilde{t}', \tilde{x}')\,\rangle_{\mathbb{R}^{1,1}} = \left|(\tilde{t}' - \tilde{t}')^2 - (\tilde{x} - \tilde{x}')^2\right|^{-\frac{c}{12}\left(n-\frac{1}{n}\right)}$$

Entanglement entropy and Page curve [Akal-Kusuki-Shiba-Takayanagi-Wei 20]

- Entanglement entropy can have two phases corresponding to the two OPE channels:

$$S_R = \min \left[S_R^{\text{con}}, S_R^{\text{dis}} \right]$$

- This model has a phase transition between the two phases and reproduces the Page curve for a non-evaporating BH



Capacity in the moving mirror model

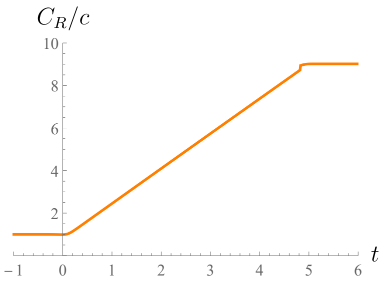
- The capacity takes a universal form in each phase:

$$C_R = \begin{cases} S_R^{\text{con}} & \text{(connected channel)} \\ S_R^{\text{dis}} - 2\,S_{\text{bdy}} & \text{(disconnected channel)} \end{cases}$$

- $^\exists$ discontinuity at the Page time:

$$C^{\text{con}} - C^{\text{dis}}\Big|_{t_{\text{Page}}} = 2\,S_{\text{bdy}}$$

- The capacity captures a phase transition between the two phases



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2d dilaton gravity + large-c CFT

- A general dilaton gravity in two dimensions ($4G_N = 1$):

$$I_{\text{grav}} = I_{\text{EH}} + I_{\text{dil}}$$

  - Einstein term (topological):

$$I_{\text{EH}} = -\frac{S_0}{4\pi}\int_{\Sigma_2}\mathcal{R} - \frac{S_0}{2\pi}\int_{\partial\Sigma_2}\mathcal{K} = -S_0\,\chi[\Sigma_2]$$

  - Dilaton term (constraint):

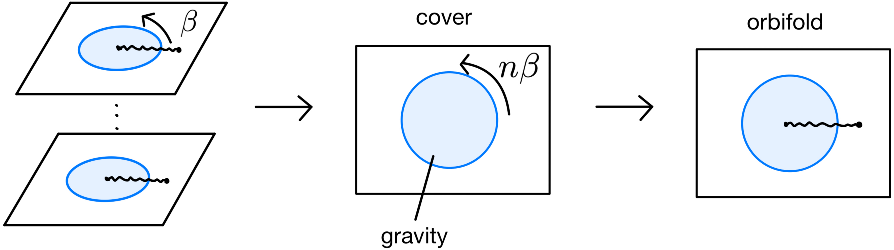
$$I_{\text{dil}} = -\frac{1}{4\pi}\int_{\Sigma_2}\Phi\,\left[\mathcal{R} + U(\Phi)(\nabla\Phi)^2 + V(\Phi)\right] - \frac{\Phi_b}{2\pi}\int_{\partial\Sigma_2}\mathcal{K}$$

- For JT gravity on AdS_2:

$$\Phi = \phi\,, \quad U = 0\,, \quad V = 2$$

Replica calculation in gravitational path integral

- Two descriptions of the replica geometry [Lewkowycz-Maldacena 13]:
  - n-fold cover $\widetilde{\mathcal{M}}_n$: no singularity in gravity region
  - orbifold $\mathcal{M}_n \equiv \widetilde{\mathcal{M}}_n/\mathbb{Z}_n$: conical singularities at $\mathbb{Z}_n$ fixed points in gravity region



Replica partition function

- The on-shell actions related by [Dong 16, Nakaguchi-TN 16]

$$\frac{1}{n} I_{\text{grav}}[\widetilde{\mathcal{M}}_n] = I_{\text{grav}}[\mathcal{M}_n] + \left(1 - \frac{1}{n}\right) \underbrace{\mathcal{A}^{(n)}}_{\text{localized on singularities}}$$

  - The "area" term:

$$\mathcal{A}^{(n)} = \sum_i \left[S_0 + \Phi^{(n)}(w_i)\right] \qquad w_i \text{ : conical singularities in gravity region}$$

- In the semiclassical limit, $1 \ll c \ll 1/G_N$:

$$-\frac{1}{n} \log \operatorname{Tr} \rho^n = \frac{1}{n} I_{\text{grav}}[\widetilde{\mathcal{M}}_n] - \frac{1}{n} \log Z_{\text{CFT}}[\widetilde{\mathcal{M}}_n]$$
$$= I_{\text{grav}}[\mathcal{M}_n] + \left(1 - \frac{1}{n}\right) \mathcal{A}^{(n)} - \frac{1}{n} \log Z_{\text{CFT}}[\widetilde{\mathcal{M}}_n]$$

Island and capacity formulas in 2d dilaton gravity

- The island formula [Almheiri-Maldacena-Hartman-Shaghoulian-Tajdini 19, Penington-Shenker-Stanford-Yang 19]:

$$S \equiv \frac{1}{1-n} \log \operatorname{Tr} \rho^n \Big|_{n=1} = \sum_{i \in \partial I} \left[S_0 + \Phi(w_i) \right] + S_{\text{CFT}}$$

- The capacity formula [Kawabata-TN-Okuyama-Watanabe 21]:

$$C \equiv -2\partial_n \left(\frac{1}{1-n} \log \operatorname{Tr} \rho^n \right) \Bigg|_{n=1} = -\sum_{i \in \partial I} \partial_n \Phi^{(n)}(w_i) \big|_{n=1} + C_{\text{CFT}}$$

Implication of the formula: discontinuity at Page time

- The capacity should be calculated
  - with the values w_i fixed by the QES condition (which does not depend on $\widetilde{\mathcal{M}}_{n\neq 1}$!):

$$\partial_{w_i}[S_0 + \Phi(w_i) + S_{\mathsf{CFT}}] = 0$$

  - on the dominant saddle with least entropy (dictated by the island formula)

- When there are two competing saddle solutions (with and without island)
  - the entropy is continuous at the Page time: $\Delta S \equiv S_{\text{no-island}} - S_{\text{island}}|_{\text{Page}} = 0$
  - the capacity typically shows a discontinuity (we have no rigorous proof):

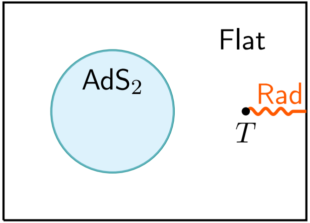
$$\begin{aligned}\Delta C &\equiv C_{\text{no-island}} - C_{\text{island}}|_{\text{Page}} \\ &= \sum_{i\in\partial I}\left[S_0 + \Phi(w_i) + \partial_n \Phi^{(n)}(w_i)|_{n=1}\right] \neq 0\end{aligned}$$

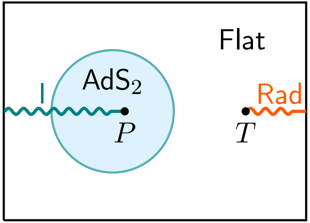
Example: eternal AdS_2 BH at high temperature

- Technical difficulties in calculating the capacity:
  - For $\partial_n \Phi^{(n)}(w_i)\big|_{n=1}$ solve the dilaton EOM on $\tilde{\mathcal{M}}_{n\neq 1}$
  - For C_{CFT} solve the conformal welding problem [Almheiri-Maldacena-Hartman-Shaghoulian-Tajdini 19]

  $\Rightarrow$ Both problems are hard to solve in general!

- We can circumvent these problems for the semi-infinite radiation region in high temperature limit



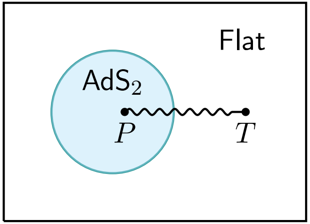


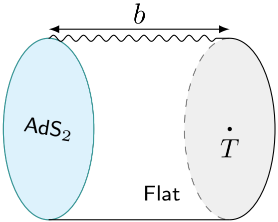
Comparison with thermodynamic quantities

- The island phase always favored (no phase transition)
- S and C coincide with thermodynamic entropy and capacity:

$$S_{\text{island}} = S_{\text{th}}(\beta) \ , \qquad C_{\text{island}} = C_{\text{th}}(\beta)$$

- We speculate this coincidence is universal in the island/replica wormhole phase





Summary and future direction

- The capacity of entanglement can be a good probe of Hawking radiation
  - EOW model: sensitive to the dominant replica wormhole saddle, dependent on the choice of ensembles
  - Moving mirror model: discontinuous at the Page time (in holographic CFT)
- The capacity formula implies
  - a discontinuity at the Page time
  - coincidence with thermal capacity in the island phase