Missing final state puzzle in monopole-fermion scattering

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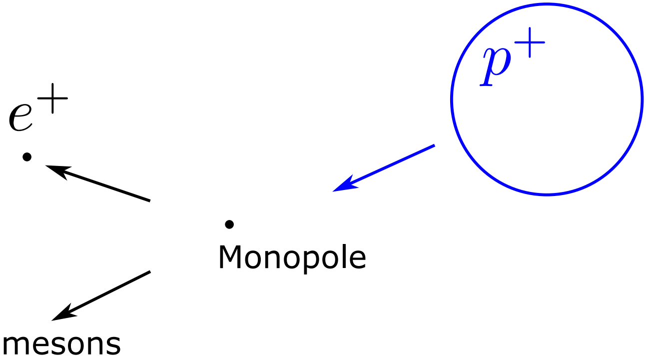
Rubakov-Callan effect



[V. A. Rubakov 1982, C. G. Callan 1982]

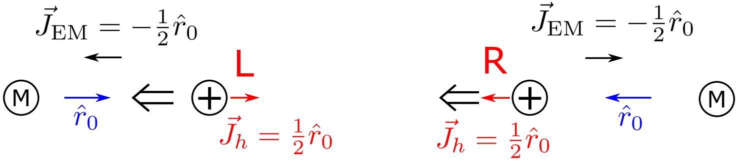
- When a proton collides with a GUT monopole, it decays into a positron and mesons.
- The effect has been used to set limits on the monopole flux in the Universe.

Helicity flip

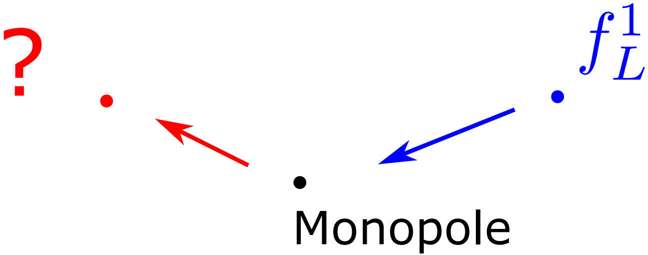
- When there are a magnetic monopole and a unit charge, **the electromagnetic field has the angular momentum with magnitude $1/2$**.

$$\vec{J}_{\rm EM} = \frac{1}{4\pi}\int d^3x\, \vec{x} \times (\vec{E} \times \vec{B}) = \frac{1}{2}\hat{r}_0$$

- Then the total angular momentum, the sum of $\vec{J}_{\rm EM}$ and the spin of the fermion, can be zero.
- When a charged fermion collides with a monopole, **its helicity has to flip**.



Puzzle: Two-flavor massless case



- The final state has to have the opposite helicity to the initial state and the same flavor charge.
- When there are two or more flavors of massless fermion, **the left and right handed particles have different flavor charge.**
- **There are no candidates of the final state, which are consistent with the helicity flip and the flavor charge conservation.**

Outline

Outline

Set up

- We consider an $SU(2)$ gauge theory with an adjoint Higgs and **4 flavors of Weyl fermions**, where $SU(2)$ is spontaneously broken down to $U(1)$.
- The global symmetry is $SU(4)$.

$$SU(2) \text{ doublets} \left\{ \overbrace{\left(\begin{pmatrix} a_1^+ \\ b_1^- \end{pmatrix}, \begin{pmatrix} a_2^+ \\ b_2^- \end{pmatrix}, \begin{pmatrix} a_3^+ \\ b_3^- \end{pmatrix}, \begin{pmatrix} a_4^+ \\ b_4^- \end{pmatrix} \right)}^{SU(4) \text{ quadruplets (fund. rep.)}} \right.$$

- The theory can be regarded as an approximation of $SU(5)$ GUT:

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} e_L^+ \\ d_L^3 \end{pmatrix}, \quad \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} \bar{d}_L^3 \\ e_L^- \end{pmatrix}, \quad \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} = \begin{pmatrix} u_L^1 \\ \bar{u}_L^2 \end{pmatrix}, \quad \begin{pmatrix} a_4 \\ b_4 \end{pmatrix} = \begin{pmatrix} \bar{u}_L^2 \\ \bar{u}_L^1 \end{pmatrix}.$$

The low energy effective theory

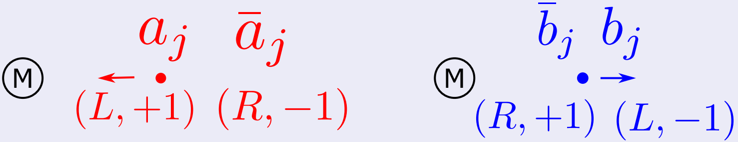
- We approximate the theory as the gauge theory of the unbroken $U(1)$.

$$
\begin{aligned}
&a : \text{The } U(1) \text{ gauge field,}\\
&f = da,\\
&a_j : \text{The left-handed Weyl fermions with charge } +1,\\
&b_j : \text{The left-handed Weyl fermions with charge } -1.
\end{aligned}
$$

- The 't Hooft-Polyakov monopole is approximated by the (background) Dirac monopole.
- X, Y bosons, GUT Higgs bosons are considered to be infinitely heavy.

The s-wave scattering

- In a monopole background, there are **s-wave states** of fermions (the total angular momentum is zero) due to the contribution from the EM field.
- According to the analysis of the Dirac equation, the higher partial waves cannot reach the monopole core. **[R. Jackiw & C. Rebbi, 1976]**
  When the energy of the incoming particle is sufficiently small, we can neglect the effect of the higher partial waves.
- In our setup, for s-wave fermions, only a_j and $\bar{a}_j$ can be incoming particles, and only b_j and $\bar{b}_j$ can be outgoing particles.

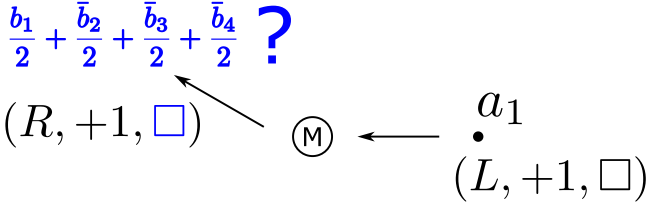


The missing final state puzzle

- The helicity, the $U(1)$ charge and the representation of $SU(4)$ are

$$a_j : (L, +1, \Box), \quad b_j : (L, -1, \Box), \quad \bar{a}_j : (R, -1, \overline{\Box}), \quad \bar{b}_j : (R, +1, \overline{\Box})$$

- If the initial state is a_1, the quantum number of the final state has to be $(R, +1, \Box)$. However, there are no particles with this quantum number.
- When we consider the multi-particle state, it has to consist of **b_j and $\bar{b}_j$** because the s-wave states of a_j and $\bar{a}_j$ cannot be out-going particles and the higher partial waves cannot reach the core of the monopole.
- The only possible answer is **"the semiton state"**.



$b_j/2$: "Semiton", the state with b_j **number** $1/2$.

What is the "semiton state"

$$|\frac{b_1}{2}, \frac{\bar{b}_2}{2}, \frac{\bar{b}_3}{2}, \frac{\bar{b}_4}{2}, M\rangle \ ?$$

- The state has **half fermion numbers**. (The state is the eigenstate of $\int d^3x \bar{b}_1 \sigma^0 b_1$ with the eigenvalue $1/2$.)
- The state is **orthogonal to any multi-particle state of $a_j, b_j, \bar{a}_j, \bar{b}_j$.**
- By projecting the fields to the s-wave component (**the s-wave approximation**), it was found that the state is actually the final state.
  [C. G. Callan 1984, J. Polchinski 1984, J. M. Maldacena & A. W. W. Ludwig 1997]

The problem is what an appropriate **interpretation** of the state is. A non-particle state? A multi-particle state?
Our claim is that it is **a new single-particle state that can exist only in the monopole background and only when the fermions are massless**.

Interpretation of the final state

Probabilistic interpretation [C. G. Callan (1984), V. A. Rubakov (1988)]:

- In the $SU(5)$ GUT,

$$a_1 = e_L^+, \quad a_2 = \overline{d}_L^3, \quad a_3 = u_L^1, \quad a_4 = u_L^2,$$
$$b_1 = d_L^3, \quad b_2 = e_L^-, \quad b_3 = \overline{u}_L^2, \quad b_4 = \overline{u}_L^1.$$

- "The semiton state" is

$$\frac{1}{2}e_R^+ + \frac{1}{2}u_R^1 + \frac{1}{2}u_R^2 + \frac{1}{2}d_L^3$$

- The state is interpreted as

$$\frac{1}{\sqrt{2}}\left|e_R^+, \mathrm{M}\right\rangle + \frac{1}{\sqrt{2}}\left|u_R^1 u_R^2 d_L^3, \mathrm{M}\right\rangle \quad ?$$

  This is problematic because $e_R^+ = \bar{b}_2$ is in the antifundamental representation of $SU(4)$, i.e., the flavor charge is not conserved.

Outline

S-wave approximation and bosonization

- The final state was obtained as the soliton in the bosonized theory in the s-wave approximation, where we only consider the spherically symmetric fields.

$$S_{4d} = \int d^4x \left[-\frac{1}{4g^2} f \star f + \sum_{j=1}^{4} \left(i\bar{a}_j \, \bar{\sigma}^\mu D_\mu a_j + i\bar{b}_j \, \bar{\sigma}^\mu D_\mu b_j \right) \right],$$

$$\xrightarrow{\text{S-wave approximation, bosonization, soving Gauss's law}}$$

$$S_{2d} = \int_0^\infty dr \int_{-\infty}^{\infty} dt \left[\frac{1}{8\pi} \sum_{i=1}^{4} ((\partial_t \phi_i)^2 - (\partial_r \phi_i)^2) - \frac{g^4}{32\pi^3} \frac{1}{r^2} (\sum_{i=1}^{4} \phi_i)^2 \right],$$

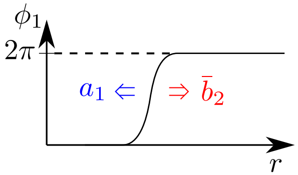
- The correspondence between the fermion and boson fields is roughly

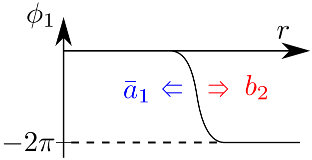
$$a_1 b_2 \leftrightarrow e^{i\phi_1}, \quad a_2 b_1 \leftrightarrow e^{i\phi_2}, \quad a_3 b_4 \leftrightarrow e^{i\phi_3}, \quad a_4 b_3 \leftrightarrow e^{i\phi_4}.$$

Fermions are kinks

- The fermions correspond to the kink solitons.

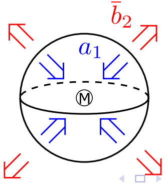
$$\text{Kinks} = \text{Fermions with the charge } +1$$
$$\text{Anti-kinks} = \text{Fermions with the charge } -1$$
$$\text{Incoming (anti-)kinks} = a_j \ (\bar{a}_j)$$
$$\text{Outgoing (anti-)kinks} = \bar{b}_j \ (b_j)$$





- The kink corresponds to the s-wave state of the fermion in 4d.

$$\phi_1 : 0 \to 2\pi$$



Currents and boundary conditions

- The currents of $U(1)$ and the maximal torus of $SU(4)$ is ($\alpha, \beta = t, r$)

$$
\begin{aligned}
4\pi r^2 J^\alpha &= \frac{1}{2\pi} \sum_j \varepsilon_{\alpha\beta} \partial_\beta \phi_j, \\
4\pi r^2 J^\alpha_{(1,-1,0,0)} &= \frac{1}{2\pi} \varepsilon_{\alpha\beta} \partial_\beta (\phi_1 - \phi_2), \\
4\pi r^2 J^\alpha_{(0,0,1,-1)} &= \frac{1}{2\pi} \varepsilon_{\alpha\beta} \partial_\beta (\phi_3 - \phi_4), \\
4\pi r^2 J^\alpha_{(1,1,-1,-1)} &= \frac{1}{2\pi} \partial_\alpha (\phi_1 + \phi_2 - \phi_3 - \phi_4).
\end{aligned}
$$

  $J^\alpha_{(\ldots)}$ is the current of $U(1)$ generated by $\mathrm{diag}(\ldots)$.
- The boundary condition should be imposed so that any charge does not flow into the infinitesimal region around the monopole for the conservation:

$$4\pi r^2 J^r = 0, \quad 4\pi r^2 J^r_{(\ldots)} = 0 \text{ at } r = 0,$$

  which implies

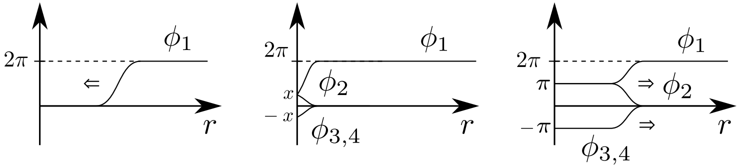
$$
\begin{aligned}
&\partial_t(\phi_1 + \phi_2 + \phi_3 + \phi_4) = \partial_t(\phi_1 - \phi_2) = \partial_t(\phi_3 - \phi_4) = 0, \\
&\partial_r(\phi_1 + \phi_2 - \phi_3 - \phi_4) = 0.
\end{aligned}
$$

Scattering process

- Let the initial state be a_1. In $g \to 0$ limit, the solution is

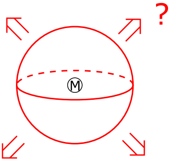
$$\phi_1 = f(t+r) - \frac{1}{2}f(t-r), \quad \phi_2 = \frac{1}{2}f(t-r), \quad \phi_{3,4} = -\frac{1}{2}f(t-r)$$
$$f(\infty) = 2\pi, \quad f(-\infty) = 0.$$

- The final state corresponds to



- The final state seems to be $b_1/2 + \bar{b}_2/2 + \bar{b}_3/2 + \bar{b}_4/2$, which is hardly interpreted.

$$\begin{cases} \phi_1 : \pi \to 2\pi \\ \phi_2 : \pi \to 0 \\ \phi_3 : -\pi \to 0 \\ \phi_4 : -\pi \to 0 \end{cases}$$



Massive case: No puzzle

In the massive case, this puzzle disappears.

- Let us introduce the mass term,

$$m_1(a_1 b_2 + a_2 b_1 + \text{h.c.}) + m_2(a_3 b_4 + a_4 b_3 + \text{h.c.}),$$

- The global symmetry reduces to $U(1) \times U(1)$, which is generated by

$$H_1 = \text{diag}(1, -1, 0, 0), \quad H_2 = \text{diag}(0, 0, 1, -1),$$

  and the charge correspond to

$$H_3 = \text{diag}(1, 1, -1, -1)$$

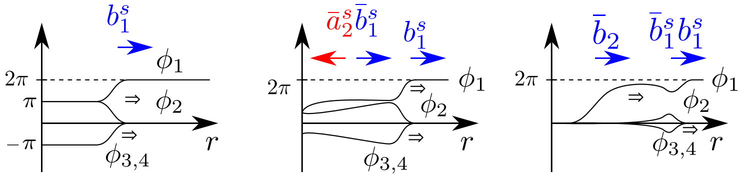
  is not conserved.

- There is a candidate of the final state

$$a_1 : (L, +1, Q_{(1,-1,0,0)} = 1, Q_{(0,0,1,-1)} = 0)$$
$$\to \bar{b}_2 : (R, +1, Q_{(1,-1,0,0)} = 1, Q_{(0,0,1,-1)} = 0).$$

Numerical result in massive case

The numerical simulation was done by [S. Dawson & A. N. Schellekens (1983)].



- Due to the mass term

$$\mu(m_1 \cos\phi_1 + m_1 \cos\phi_2 + m_2 \cos\phi_3 + m_2 \cos\phi_4)$$

  $\phi_1 = \phi_2 = -\phi_3 = -\phi_4 = \pi$ is no longer the vacuum configuration. Thus the potential energy increases as the semiton state goes out.

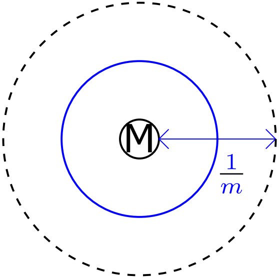
- At some point, **the following pair of semiton states** are produced:

$$\bar{a}_2^s := \frac{1}{2}a_1 + \frac{1}{2}\bar{a}_2 + \frac{1}{2}a_3 + \frac{1}{2}a_4 \text{ and } b_1^s := \frac{1}{2}\bar{b}_1 + \frac{1}{2}b_2 + \frac{1}{2}b_3 + \frac{1}{2}b_4$$

- The state $\frac{1}{2}a_1 + \frac{1}{2}\bar{a}_2 + \frac{1}{2}a_3 + \frac{1}{2}a_4$ collides with the monopole, and then $\bar{b}_2$ appears.

What happens when we take the massless limit?

- In the scattering process, when the semitons reach $r \sim 1/m$, the values of ϕ_i near the core start to change.
- This means that, in the region where $r \ll 1/m$, the theory can be regarded as the massless theory.
- In the massless limit, every point is near the monopole.



Outline

The fermion condensate

- **Fact**: The following fermionic operators have **nonzero expectation values**:

$$\boldsymbol{\langle (a_{i_1} b_{i_2})(a_{i_3} b_{i_4}) \rangle = \frac{1}{r^6} c_3 \varepsilon_{i_1 i_2 i_3 i_4}}.$$

- To reproduce the helicity flip, operators that mixes a_j and b_j have to have non-zero expectation value.
- The operator should be $SU(4)$ invariant to maintain the $SU(4)$ symmetry without NG bosons with the $SU(4)$ charges.

The “effective” theory of the phases of the condensates

Let us consider the effective theory of the operators' phases.

- It is convenient to express them using **the phases of the fermion fields.** Let α_j be the phase of a_j and β_j be that of b_j.
- There are four independent variables, e.g.,

$$\theta_A = \sum_j (\alpha_j + \beta_j), \quad \theta_{1j} = \alpha_1 - \beta_1 - \alpha_j + \beta_j \quad \text{for } j = 2, 3, 4.$$

- To reproduce the chiral anomaly, the effective Lagrangian contains

$$\theta_A f \wedge f/(8\pi^2) + \theta_{12} f \wedge F_{12}/(4\pi^2) + \cdots ,$$

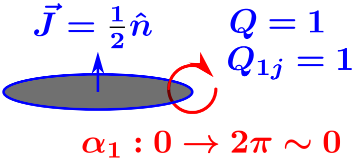
F_{12} : $U(1) \in SU(4)$ field strength corresponding to $\mathrm{diag}(1, -1, 0, 0)$

$$\Rightarrow \quad j^\mu = \varepsilon^{\mu\nu\rho\sigma} \frac{1}{4\pi^2} \partial_\nu \theta_A f_{\rho\sigma}, \quad j^\mu_{12} = \varepsilon^{\mu\nu\rho\sigma} \frac{1}{4\pi^2} \partial_\nu \theta_{12} f_{\rho\sigma} \cdots$$

(cf. **Axions**)

Strings of the phase is fermions

- Let us consider **a string configuration** of the phase, around which the phase winds. (c.f. **axion strings**)
- Due to the topological coupling to the gauge field, there have to be **edge modes** on the string. **[C. G. Callan & J. A. Harvey 1985]**
- There is an edge excitation that has **the unit charge and the same flavour charges as the fermion**.
- The edge mode gives the object **spin $1/2$**.$\Rightarrow$ **The object can be regarded as a fermion.** (c.f. $N_f = 1$ QCD **[Z. Komargodski 2018]** η' strings = baryons)
- We will see that the object is suitable to the final state of the scattering

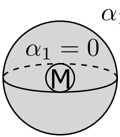


= Fermion

Monopole bags and strings

- **A monopole bag** has the electric and flavor charges:

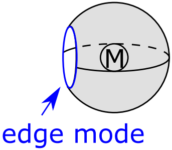
$$Q = \int d^3x \; j^0 = \frac{1}{4\pi^2} \int_0^\infty dr \partial_r \theta_A \int_{S_r^2} \vec{B} \cdot d\vec{S} = 1.$$



Witten effect:
$Q = +1$
$Q_{1j} = +1$

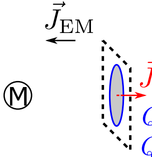
$\Longleftrightarrow$ $a_1 : Q = +1,$
$Q_{1j} = +1$

- Because $\boldsymbol{\alpha_1 \sim \alpha_1 + 2\pi}$, the wall of α_1 can have **a boundary**, which is **a string of** α_1.
- The edge modes contribute to the charge so that the total charge is an integer.



Witten effect + edge contribution:
$Q = +1$
$Q_{1j} = +1$

$\Longrightarrow$



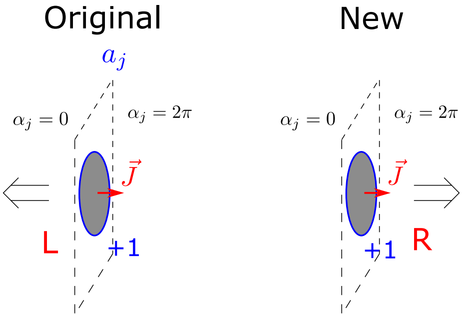
edge contribution
$Q = +1$
$Q_{1j} = +1$

= fermion

The final state of the scattering

- The objects can be regarded as **fermions**.
- Some of them have **opposite helicity** to the original fermions.

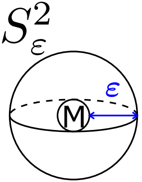
  $\Rightarrow$ **New fermions!**
- **The final state of the monopole-fermion scattering is identified with the new fermions.**



The boundary condition at the core of the monopole

- In the process of the scattering, **the boundary condition** at the core of the monopole plays an important role.
- The boundary condition is determined so that **the monopole does not have the electric and flavor charges**:

$$\int_{S_\varepsilon^2} \vec{j} \cdot d\vec{S} = 0, \quad \int_{S_\varepsilon^2} \vec{j}_{1k} \cdot d\vec{S} = 0,$$



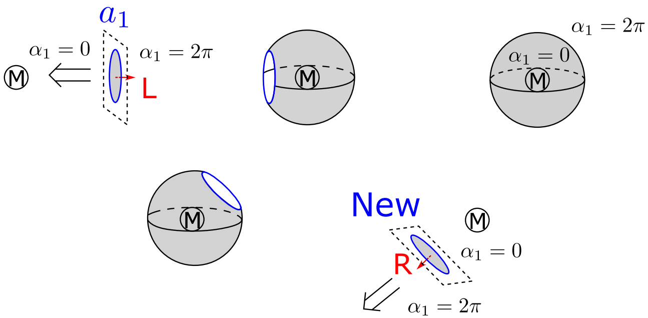
which implies that

$$\partial_t \theta_A|_{r=0} = 0, \quad \partial_t \theta_{1k}|_{r=0} = 0, \quad \forall k,$$

i.e., **the phases of the condensations can not change at the core of the monopole**.

A soliton picture of the scattering

- As the soliton approaches the monopole, the charge moves from the edge to the bulk due to the Witten effect.
- When the wall reaches the monopole, the wall bounces back due to the boundary condition, and the string goes throw the monopole. Then the string can shrink and disappear because it has no charge.
- The wall can break by creating the string.
- The final state corresponds to the new particle with $(R, +1, \square)$.



(cf. Scattering of an axion string and a monopole **[I. Kogan 1993]**)

Summary

- When a charged fermion collides with a monopole, the helicity of the s-wave component of the fermion has to flip.
- Puzzle: If there are two flavors of massless Dirac fermions, **any fermions in the action cannot be the final state of the monopole-fermion scattering, which should be consistent with the flavor charge conservation and the helicity flip**.
- We solve this puzzle by identifying the final state as **a new fermion, which can be described as a soliton of the fermion condensates**.

Backup

The edge state

- By substituting the 2π jump of α_j into the effective Lagrangian $\sum_j (\alpha_j + \beta_j) f \wedge f/(8\pi^2)$, we obtain the Chern-Simons theory as the theory on the wall:

$$\frac{1}{4\pi} \int a \wedge f$$

- When the wall has the boundary, the CS theory is not gauge invariant.
  $\Rightarrow$ There has to be a chiral edge mode:

$$\frac{1}{4\pi} \int_{\mathbb{R} \times D^2} a \wedge f + \frac{1}{4\pi} \int_{\partial D^2} \left(D_x \phi (D_t \phi + v D_x \phi)\, dx dt - \phi f \right),$$
$$D\phi := d\phi - a, \quad a \to a + d\lambda, \quad \phi \to \phi + \lambda.$$

- ϕ is a 2π-periodic scalar, thus we can define the winding number of ϕ,

$$\frac{1}{2\pi} \int_{\partial D^2} d\phi.$$

- The pancake with the exited edge state with this winding number ± 1 can be considered as a fermion.

The charge of the edge state

- The $U(1)$ charge is

$$Q = \frac{1}{2\pi}\int_{\partial D^2}(d\phi - a) + \frac{1}{2\pi}\int_{D^2} f.$$

  In the gauge where the Dirac string does not penetrate the pancake, the charge is given as a winding number of ϕ around the edge.

- The quantum eigenstate of $\int_{\partial D^2} d\phi/2\pi$ with the eigenvalue $+1$ (-1) in the edge theory has the charge ± 1.
- Classically, the state corresponds to the solution of the eq. of motion of the edge theory. If we neglect the gauge fields, it is

$$\phi = \pm 2\pi(x - vt)/L.$$

- By introducing the background fields of the maximal torus of $SU(4)$, we can confirm that the edge state also have the flavor charge corresponding to the fundamental representation.

The spin of the pancake

- The spin of the object is given as the generator of the translation along the edge:

$$J^z = \frac{L}{2\pi} P_x = \frac{L}{8\pi^2} \int_0^L dx\, (\partial_x \phi)^2,$$

where we neglect the gauge fields.

- By substituting the solution $\phi = \pm 2\pi(x - vt)/L$, we obtain

$$J^z = \frac{1}{2}.$$

- The direction of the spin depends only on the orientation of the wall, and does not depend on the charge.