Seminar at Kyoto Theoretical Particle Physics Group, 21 April 2021

場の量子論によるバルクの再構築について
Bulk reconstruction of metrics by QFT data

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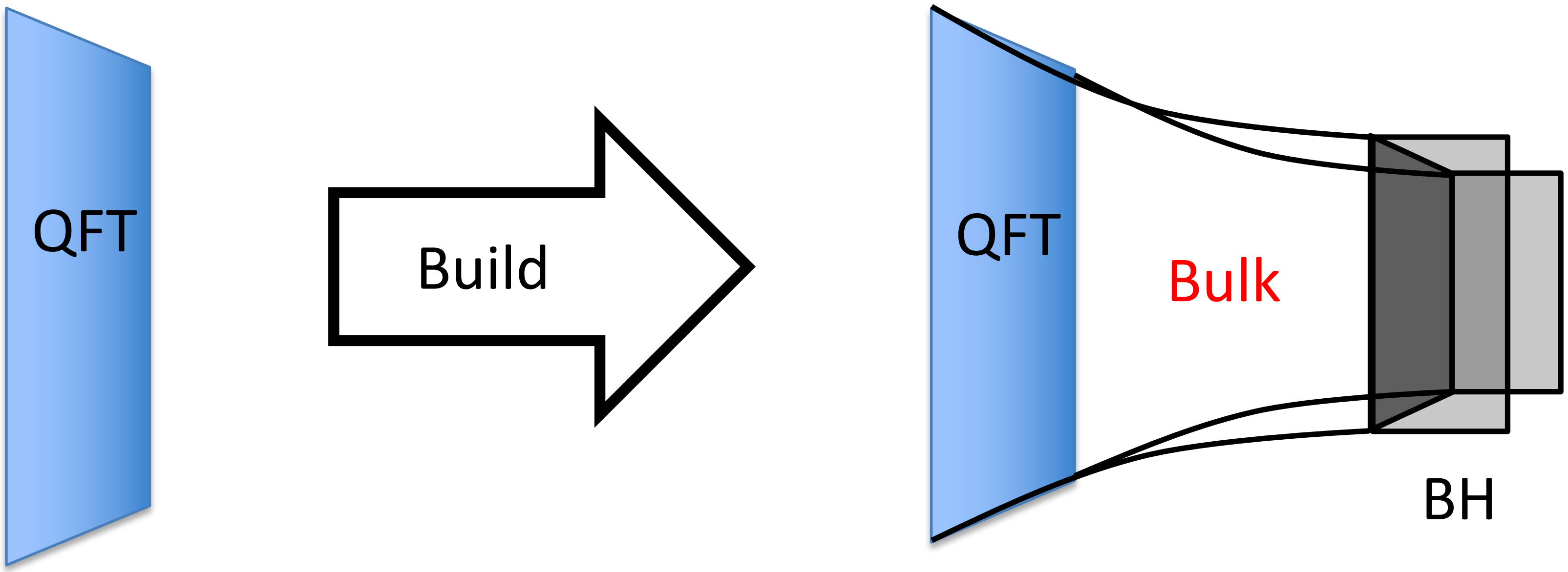
ArXiv:2008.10883
ArXiv:2103.13186 (w/ Ryota Watanabe)



$$ds^2 = -f(\eta)dt^2 + g(\eta)d\vec{x}^2 + d\eta^2$$

Surface dim.	AdS_{d+1}	CFT_d
1 + 1	Worldsheet	Wilson loop [Maldacena 1998]
(d-1) + 1	Worldvolume	Flavor sector [Karch, Katz 2002]
(d-1) + 0	RT surface	Entanglement [Ryu, Takayanagi 2006]
d + 0	Volume	Complexity [Stanford, Susskind 2014]
d + 0 (w/ boundary)	Island	Page curve

[Penington 2019] [Almheiri, Engelhardt, Marolf, Maxfield 2019]

Bulk Reconstruction of metrics

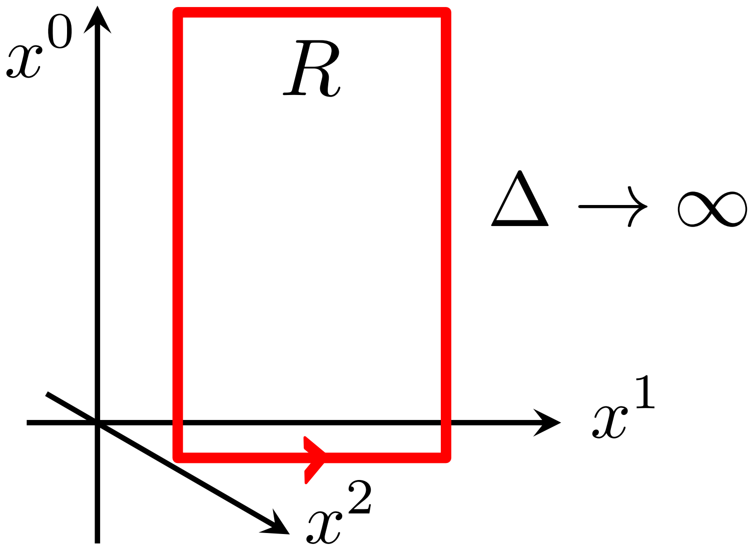
[KH 2008.10883][KH Watanabe 2103.13186]

1. Review: Wilson loops, AdS/CFT

Temporal / Spatial Wilson loops

$$W \equiv \mathrm{P}\,\mathrm{tr}\exp\left[i\int_{\square} A_\mu(x)dx^\mu\right]$$

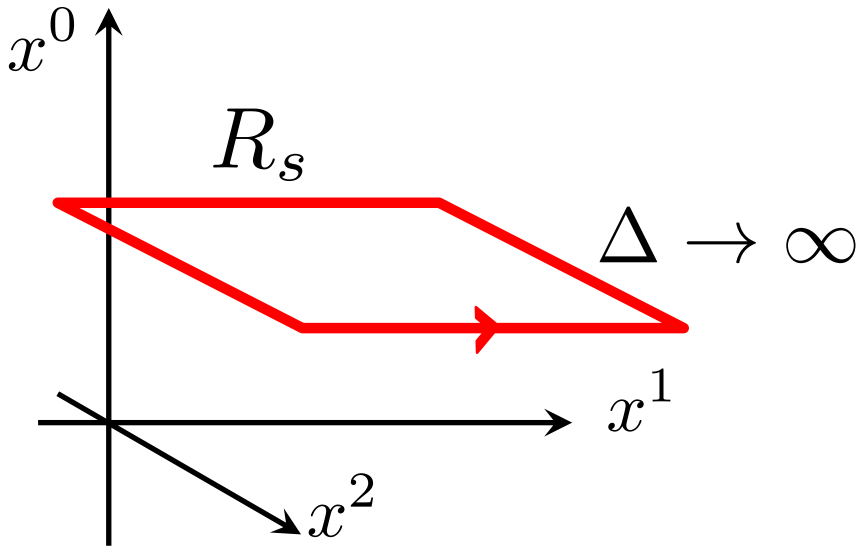
Temporal Wilson loop



$$\langle W\rangle \sim \exp[-E(R)\Delta]$$

Quark potential

Spatial Wilson loop

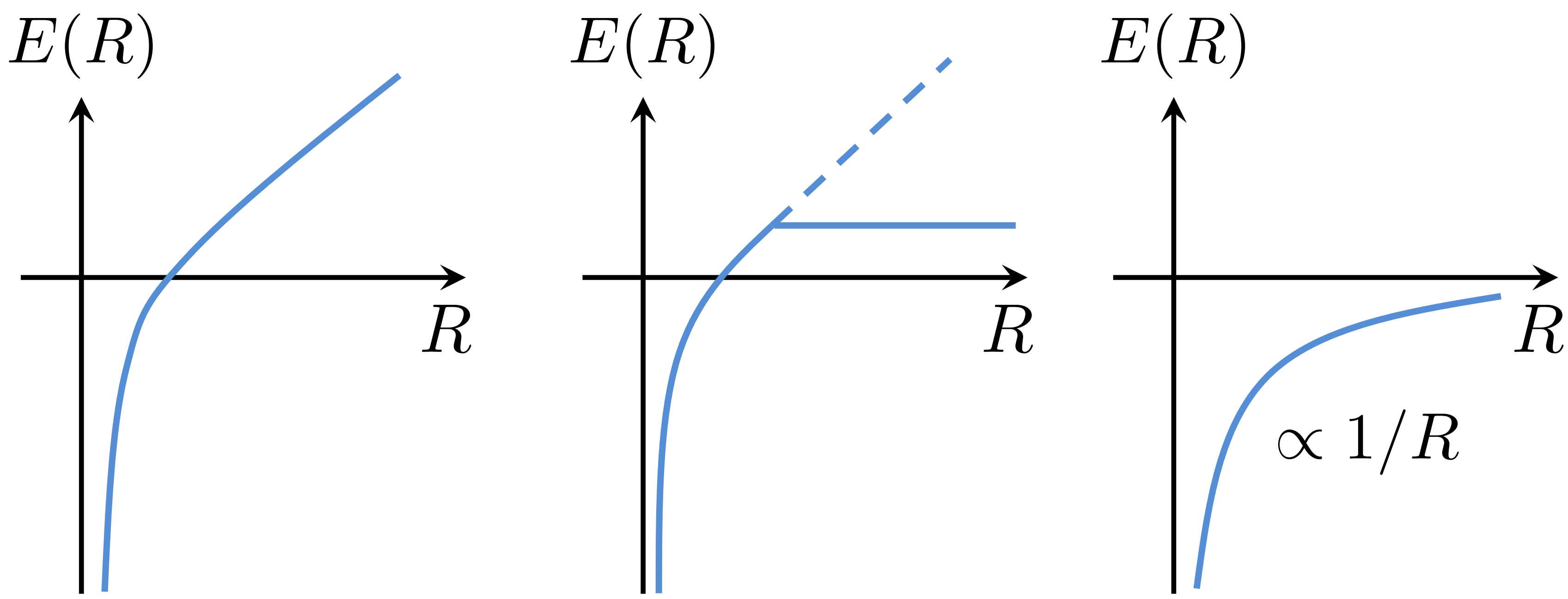


$$\langle W\rangle \sim \exp[-E_s(R_s)\Delta]$$

1. Review: Wilson loops, AdS/CFT

Phases of gauge theories

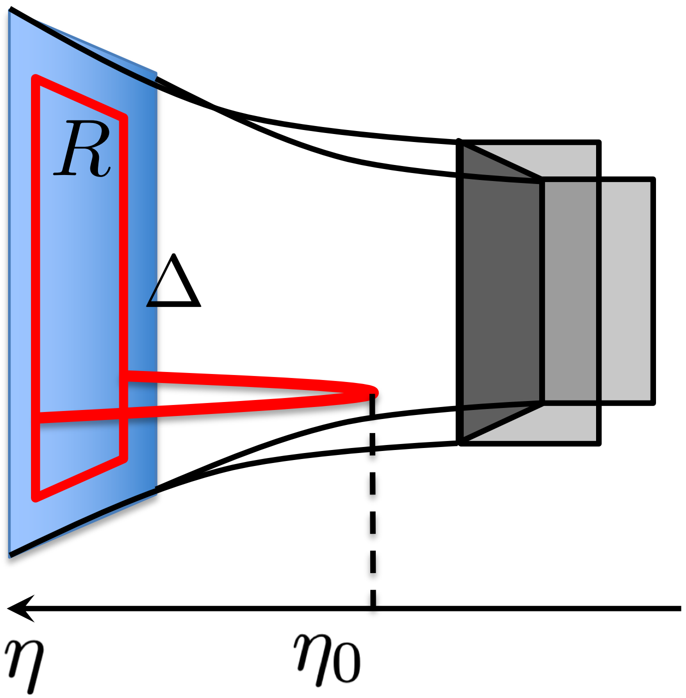
Quark potential



Confinement Deconfinement Conformal

1. Review: Wilson loops, AdS/CFT

AdS/CFT calculations of Wilson loop



Bulk metric in string frame

$$ds^2 = -f(\eta)dt^2 + g(\eta)d\vec{x}^2 + d\eta^2$$

$$f(\infty) = g(\infty) = \infty$$

[Maldacena 98] [Rey Yee 98]

Nambu-Goto string solution

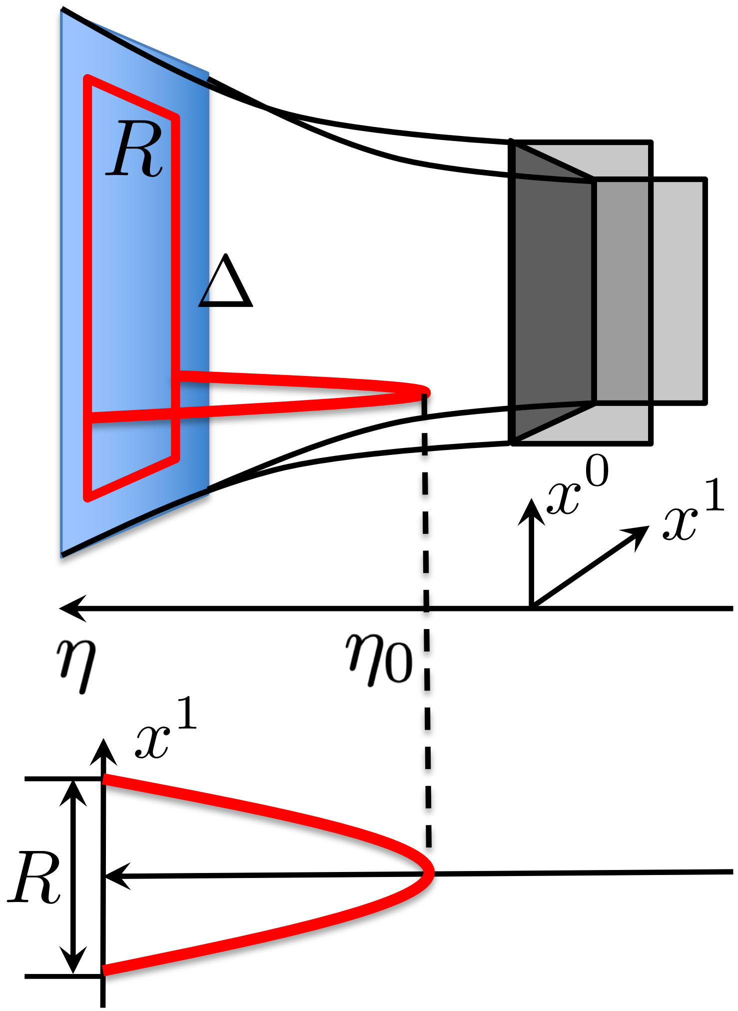
$$E(\eta_0) = \frac{1}{\pi\alpha'}\int_{\eta_0}^{\infty} d\eta\, \sqrt{f(\eta)}\sqrt{\frac{f(\eta)g(\eta)}{f(\eta)g(\eta) - f(\eta_0)g(\eta_0)}}$$

$$R(\eta_0) = 2\int_{\eta_0}^{\infty} d\eta\, \frac{1}{\sqrt{g(\eta)}}\sqrt{\frac{f(\eta_0)g(\eta_0)}{f(\eta)g(\eta) - f(\eta_0)g(\eta_0)}}$$

Quark potential $E(R)$, $E_s(R_s)$

1. Review: Wilson loops, AdS/CFT

Details of the calculation



Bulk metric in string frame

$$ds^2 = -f(\eta)dt^2 + g(\eta)d\vec{x}^2 + d\eta^2$$

String worldsheet

$$x^M(\sigma^\alpha) \ (M = 0,1,2,3,\eta)\,(\alpha = 0,1)$$

(Static gauge $x^0 = \sigma^0,\ x^1 = \sigma^1$)

Nambu-Goto action

$$S = \frac{-1}{2\pi\alpha'}\int d\sigma^0 d\sigma^1 \sqrt{-\det \partial_\alpha x^M \partial_\beta x^N G_{MN}[x]}$$

$$= \frac{-\Delta}{2\pi\alpha'}\int d\sigma^1 \sqrt{f(\eta)\left(g(\eta) + \left(\frac{d\eta}{d\sigma^1}\right)^2\right)}$$

1. Review: Wilson loops, AdS/CFT

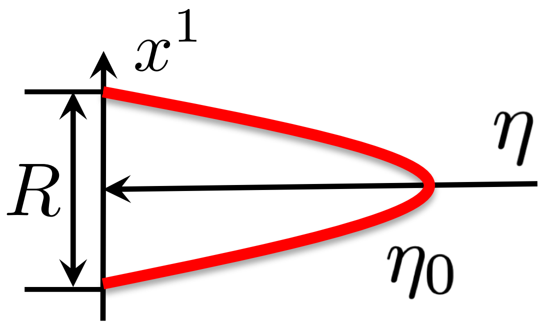
Details of the calculation (cnt'd)

Nambu-Goto action $S = \frac{-\Delta}{2\pi\alpha'} \int dx^1 \sqrt{f(\eta)\left(g(\eta) + \left(\frac{d\eta}{dx^1}\right)^2\right)}$

Conserved quantity

$$\text{const.} = \frac{\delta \mathcal{L}}{\delta \eta'}\eta' - \mathcal{L} \quad \Rightarrow \quad \frac{d\eta}{dx^1} = \sqrt{g(\eta)}\sqrt{\frac{f(\eta)g(\eta)}{f(\eta_0)g(\eta_0)} - 1}$$

$$\Rightarrow \begin{cases} R(\eta_0) = 2\int_{\eta_0}^{\infty} d\eta \, \frac{1}{\sqrt{g(\eta)}}\sqrt{\frac{f(\eta_0)g(\eta_0)}{f(\eta)g(\eta) - f(\eta_0)g(\eta_0)}} \\ E(\eta_0) = \frac{1}{\pi\alpha'}\int_{\eta_0}^{\infty} d\eta \, \sqrt{f(\eta)}\sqrt{\frac{f(\eta)g(\eta)}{f(\eta)g(\eta) - f(\eta_0)g(\eta_0)}} \end{cases}$$



$\Rightarrow$ $E(R)$ is obtained.

Bulk Reconstruction of metrics

[KH 2008.10883][KH Watanabe 2103.13186]

2. The formulas

Solving inverse problem

Bulk metric in string frame

$$ds^2 = -f(\eta)dt^2 + g(\eta)d\vec{x}^2 + d\eta^2$$

$$f(\infty) = g(\infty) = \infty$$

[Maldacena 98] [Rey Yee 98]

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$$E(\eta_0) = \frac{1}{\pi\alpha'}\int_{\eta_0}^{\infty} d\eta\,\sqrt{f(\eta)}\sqrt{\frac{f(\eta)g(\eta)}{f(\eta)g(\eta)-f(\eta_0)g(\eta_0)}}$$

$$R(\eta_0) = 2\int_{\eta_0}^{\infty} d\eta\,\frac{1}{\sqrt{g(\eta)}}\sqrt{\frac{f(\eta_0)g(\eta_0)}{f(\eta)g(\eta)-f(\eta_0)g(\eta_0)}}$$

Quark potential $E(R)$, $E_s(R_s)$

2. The formulas

Bulk reconstruction at T = 0

$$E(R) \quad \Rightarrow \quad ds^2 = f(\eta)(-dt^2 + d\vec{x}^2) + d\eta^2$$

Given a quark potential $E(R)$, solve

$$f_0 = 2\pi\alpha' \frac{dE(R)}{dR}$$

to get R as a function of f_0. Then substitute it to the following differential equation

$$\frac{d\eta(f)}{df} = \frac{1}{\pi}\sqrt{f}\frac{d}{df}\int_{\infty}^{f} df_0 \frac{R(f_0)}{\sqrt{f_0^2 - f^2}}\,.$$

Integrate this to find $\eta = \eta(f)$. Finally, invert it to find a bulk metric $f(\eta)$.

Cf. Formulas for entanglement entropy [Bilson 08]

2. The formulas

Bulk reconstruction at T > 0

$E(R)\,,\,E_s(R_s)$ ⇨ $ds^2 = -f(\eta)dt^2 + g(\eta)d\vec{x}^2 + d\eta^2$

Given a potential $E_s(R_s)$ for a spatial Wilson loop and $E(R)$ for a temporal Wilson loop, solve

$$g_0 = 2\pi\alpha' \frac{dE_s(R_s)}{dR_s}\,, \quad h_0 = 2\pi\alpha' \frac{dE(R)}{dR}\,,$$

to get $R_s(g_0)$ and $R(h_0)$. First, substitute $R_s(g_0)$ to the differential equation

$$\frac{d\eta(g)}{dg} = \frac{1}{\pi}\sqrt{g}\frac{d}{dg}\int_\infty^g dg_0 \frac{R_s(g_0)}{\sqrt{g_0^2 - g^2}}\,.$$

Integrate it to find $\eta = \eta(g)$. Invert it to find a bulk metric component $g(\eta)$. Then substitute the explicit $g(\eta)$ and also $R(h_0)$ to the differential equation

$$\frac{d\eta(h)}{dh} = \frac{1}{\pi}\sqrt{g(\eta(h))}\frac{d}{dh}\int_\infty^h dh_0 \frac{R(h_0)}{\sqrt{h_0^2 - h^2}}\,.$$

Solve this to find $\eta(h)$, which is inverted to $h(\eta)$. Then obtain another component of the bulk metric as $f(\eta) = h(\eta)^2/g(\eta)$.

2. The formulas

Solving inverse problem

Bulk metric in string frame

$$ds^2 = -f(\eta)dt^2 + g(\eta)d\vec{x}^2 + d\eta^2$$

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$$E(\eta_0) = \frac{1}{\pi\alpha'}\int_{\eta_0}^{\infty} d\eta \sqrt{f(\eta)}\sqrt{\frac{f(\eta)g(\eta)}{f(\eta)g(\eta) - f(\eta_0)g(\eta_0)}}$$

$$R(\eta_0) = 2\int_{\eta_0}^{\infty} d\eta \frac{1}{\sqrt{g(\eta)}}\sqrt{\frac{f(\eta_0)g(\eta_0)}{f(\eta)g(\eta) - f(\eta_0)g(\eta_0)}}$$

Trick 1

Trick 2

Quark potential $E(R)$, $E_s(R_s)$

2. The formulas

Essential parts of the proof

Trick 1

Q. η_0 is the variable which was eliminated.
How can one recover it?!

A. Notice $f(\eta_0)$ is equal to dE/dR

Cf. Differential entropy [Myers Rao Sugishita 14]
[Balasubramanian Chowdhury Czech de Boer Heller 13]

Trick 2

Q. Metric is inside the integral.
How can one extract it?!

A. Use an integral inversion formula for Volterra eq.

$$F(x) = \int_x^a dt\, y(t) \frac{x}{\sqrt{t^2 - x^2}} \qquad \Rightarrow \qquad y(t) = \frac{-2}{\pi} \frac{d}{dt} \int_t^a dx\, \frac{F(x)}{\sqrt{x^2 - t^2}}$$

Bulk Reconstruction of metrics

[KH 2008.10883][KH Watanabe 2103.13186]

3. Examples of reconstruction

Rebuilding AdS and near-horizon D-branes

CFT at T=0 : $E(R) = -\frac{c}{2\pi\alpha'}\frac{1}{R}$

$\Rightarrow$ AdS spacetime w/ AdS radius $L = \frac{2\Gamma(5/4)\sqrt{c}}{\Gamma(3/4)\sqrt{\pi}}$

$$ds^2 = Ae^{2\eta/L}(-dt^2 + d\vec{x}^2) + d\eta^2 + (\text{internal space})$$

Power-law potential at T=0 : $E(R) = -\frac{c}{2\pi\alpha'}\frac{1}{R^{n-1}}$

$\Rightarrow$ Near-horizon geometry of D*p*-branes

$$ds^2 = \eta^{\frac{2(7-p)}{p-3}}\left(-dt^2 + \sum_{i=1}^{p}(dx^i)^2\right) + d\eta^2 + (\text{internal space})$$

$n = (7-p)/(5-p)$

3. Examples of reconstruction

Confinement $\Rightarrow$ Bulk IR bottom

Theorem *Assume the linear confinement: at large R, the quark potential is given by*

$$\frac{dE(R)}{dR} = \sigma + \frac{c}{R^n} + (\textit{higher in } 1/R)\,.$$

Here $\sigma(>0)$ is the confining string tension. The second term (with $c>0$ and $n>0$) is the leading correction.
Then the bulk metric function $f(\eta)\,(=g(\eta)\,)$ has an IR bottom: $f(\eta)$ approaches a minimum $f = 2\pi\alpha'\sigma$ at which the gradient $df/d\eta$ vanishes. The location of the IR bottom in the η coordinate is

- *at a finite value of η, when $n>2$ (or when the correction vanishes faster than the power-law).*
- *at $\eta = -\infty$, when $2 \geq n > 0$.*

Cf: Bulk IR bottom $\Rightarrow$ confinement

[Kinar Schreiber Sonnenschein 98]

Bulk Reconstruction of metrics

[KH 2008.10883][KH Watanabe 2103.13186]

4. Inside horizons

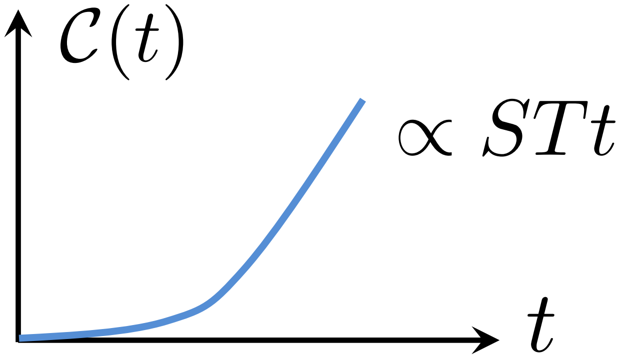
"CV conjecture" : Complexity = Volume

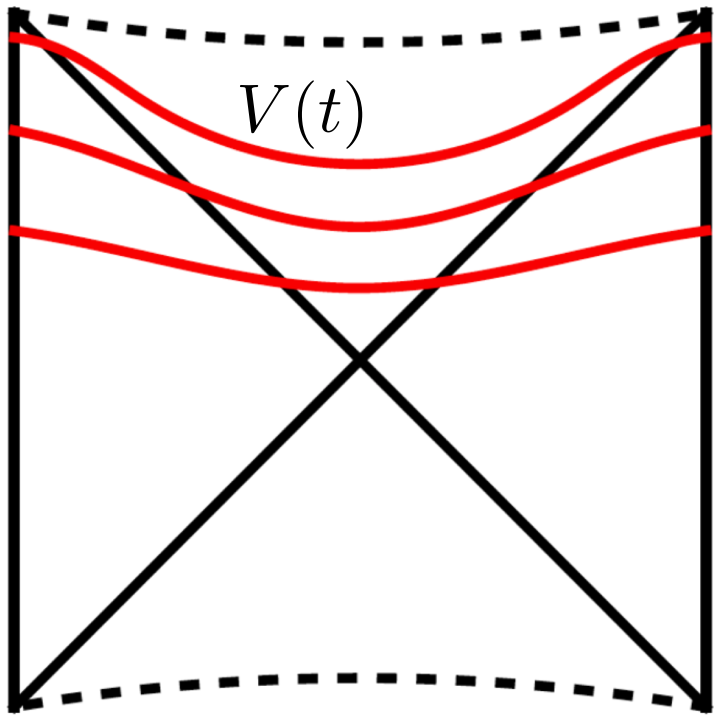
[Stanford, Susskind 2014]

Complexity $\mathcal{C}$

= Min. Num. of quantum gates to construct $|\mathrm{TFD}(t)\rangle$ from $|\mathrm{TFD}(t=0)\rangle$

$$|\mathrm{TFD}\rangle \equiv \frac{1}{\sqrt{Z}} \sum_n e^{-\beta E_n/2} |n\rangle_L \otimes |n\rangle_R$$





Eternal
AdS Schwarzschild
black hole

4. Inside horizons

Inside the horizons by complexity

[KH Watanabe 2103.13186]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma^2_{k,d-1}$$

Assume that

- the time derivative of the complexity $\dot{\mathcal{C}}(t)$ in the dual QFT is given.
- $\dot{\mathcal{C}}(t)$ is monotonic in time.
- the metric $f(r)$ outside the horizon ($r \geq r_\mathrm{h}$) is known.

Then, solve

$$\frac{d\mathcal{C}}{dt_\mathrm{R}}(t_\mathrm{R}) = \frac{2\Omega_{k,d-1}}{G_\mathrm{N} L}\sqrt{-F_\mathrm{min}}$$

to get t_R as a function of F_min. In addition, outside of the horizon, invert $F = F(r) \equiv f(r) r^{2(d-1)}$ to find $r = r(F)$. Substitute them into

$$Q(F_\mathrm{min}) \equiv \frac{t_\mathrm{R}(F_\mathrm{min})}{\sqrt{-F_\mathrm{min}}} + \int_\epsilon^\infty dF \, \frac{dr}{dF} \frac{r(F)^{2(d-1)}}{F} \frac{1}{\sqrt{F - F_\mathrm{min}}},$$

with a positive infinitesimal parameter ϵ. For $F \leq -\epsilon$, calculate

$$y(F) \equiv \frac{1}{\pi} \frac{d}{dF} \int_F^{-\epsilon} dU \, \frac{Q(U)}{\sqrt{U - F}},$$

and integrate the differential equation

$$\frac{dr}{dF} \frac{r(F)^{2(d-1)}}{F} = y(F)$$

to find $r = r(F)$. Finally, invert $r(F)$ to find $F(r)$, and the metric inside the black hole is given as $f(r) = F(r) r^{-2(d-1)}$.

4. Inside horizons

Reconstructing BTZ black hole

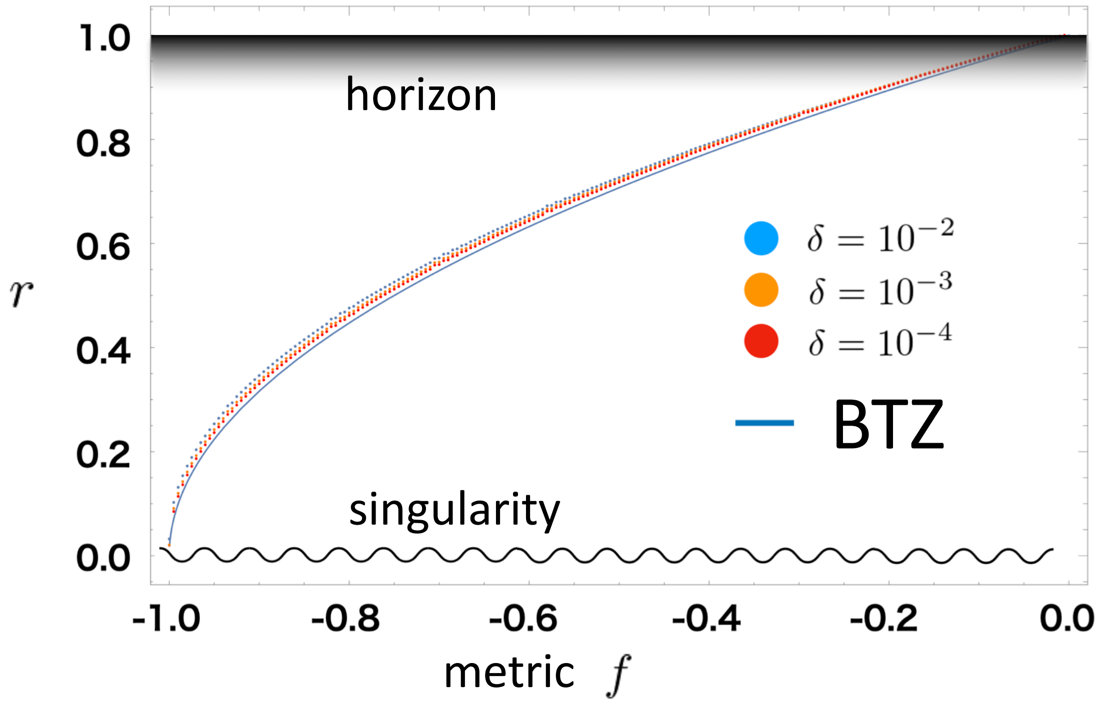
[KH Watanabe 2103.13186]

QFT data input:

Hartman-Maldacena entanglement entropy

$$S_{\mathrm{HM}} = \frac{A}{4G_{\mathrm{N}}}, \quad \frac{dA}{dt_{\mathrm{R}}} = 2\frac{r_{\mathrm{h}}}{L} \tanh \frac{r_{\mathrm{h}} t_{\mathrm{R}}}{L^2}$$

Result of the formula :



Bulk Reconstruction of metrics

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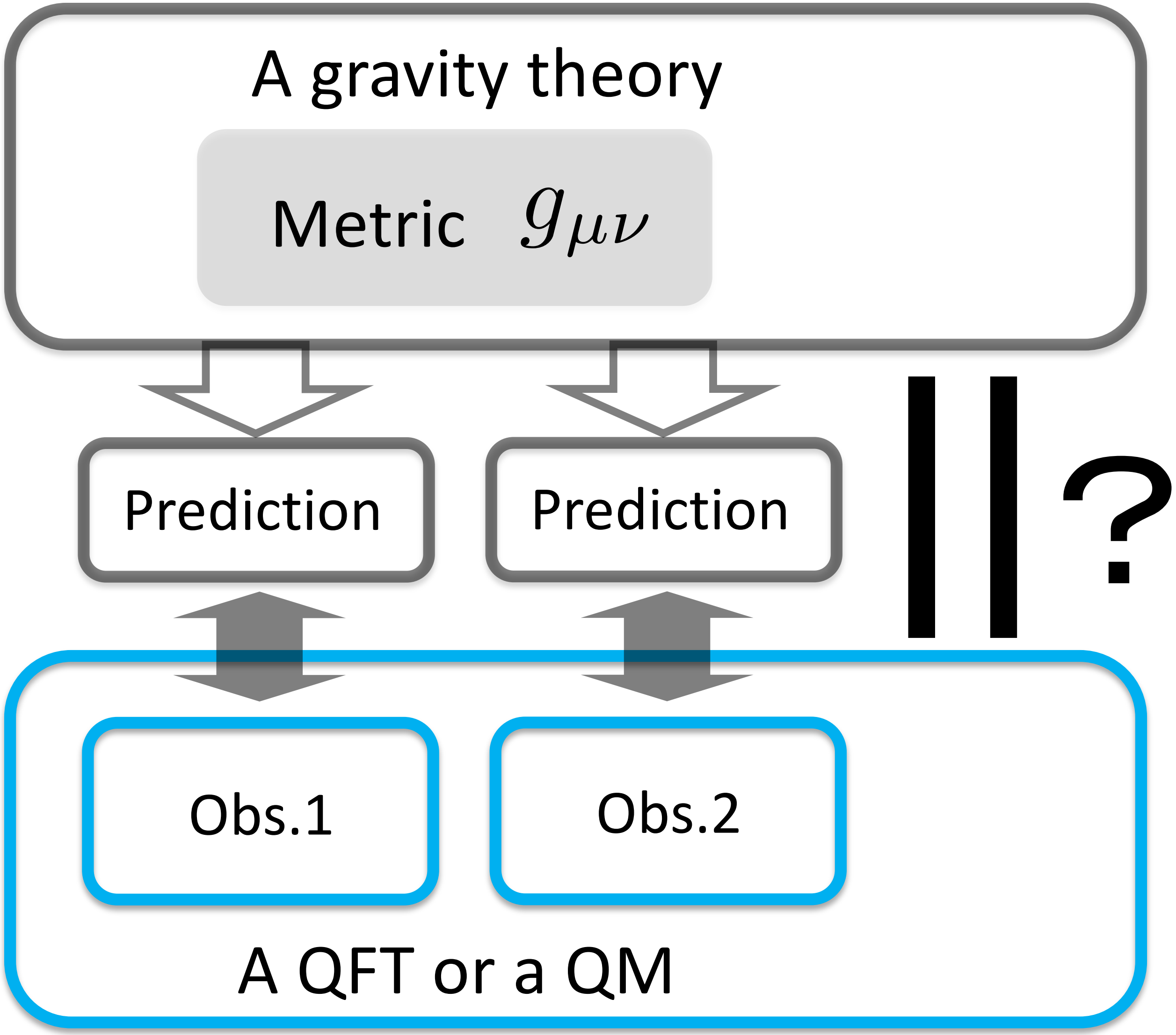
5. Uses and issues

Comparison of methods

Reconstruction method	No use of Einstein eq	Lattice input
Holographic renormalization [deHaro Solodukhin Skenderis 00]		✓
Entanglement, Complexity [Hammersley 07] [Bilson 08]… [KH Watanabe 21]	✓	
Correlators [Hammersley 06] [Hubeny Liu Rangamani 06]	✓	
AdS/DL [KH Tanaka Tomiya Sugishita 18]	✓	✓
Wilson loop [KH 20]	✓	✓

5. Uses and issues

Falsification test of gravity dual



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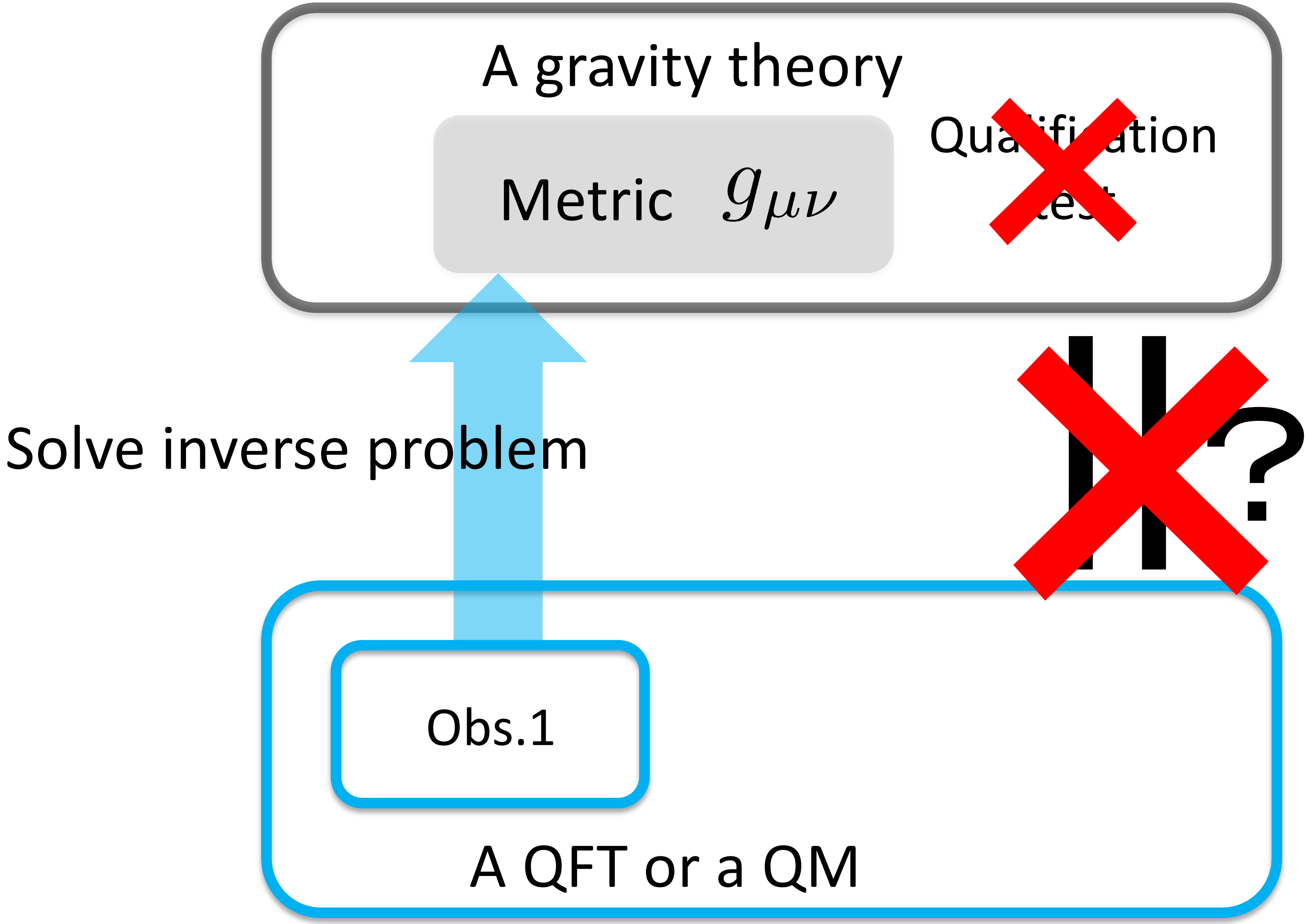
Falsification test of gravity dual



5. Uses and issues

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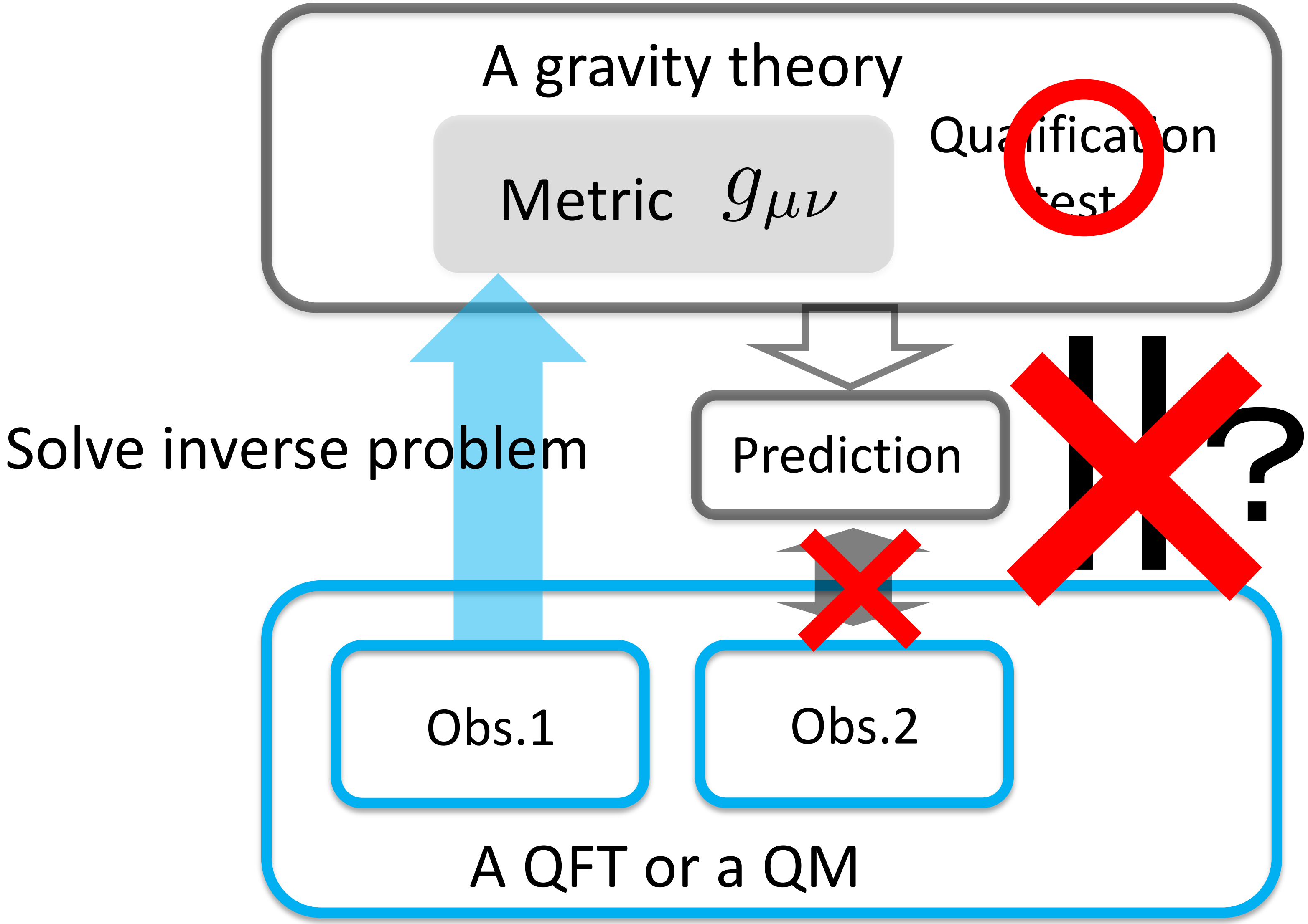

Bulk Reconstruction of metrics

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