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Structure constants of operators on the Wilson loop from integrability at weak coupling

Naoki Kiryu (University of Tokyo, Komaba)

Based on arXiv:1706.02989 with Minkyoo Kim (University of Witwatersrand)

Seminar@Kyoto University

$\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence

$$\mathcal{N} = 4 \text{ SYM in 4d} \iff \text{Type IIB string on } \mathrm{AdS}_5 \times \mathrm{S}^5$$

Goal of this talk:

Proposal for computing 3-point functions of
open strings at finite coupling in the large N limit

Strategy:

3-point functions formula for close string called
hexagon method and weak coupling integrability analysis

1. Introduction

Integrability in $\mathcal{N} = 4$ SYM

2-point functions in $\mathcal{N} = 4$ SYM

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \frac{\delta_{ij}}{|x_1 - x_2|^{\Delta_i}}$$

$$\Delta_i = \Delta_i^{\text{tree}} + \gamma_i$$

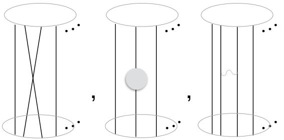
$\uparrow$
anomalous dimensions

Single trace operators:
e.g.

$$\mathcal{O}_i = \text{tr}(ZYZZY\cdots)$$
$$Z = \phi_1 + i\phi_2, \quad Y = \phi_3 + i\phi_4$$

1-loop anomalous dimension and spin chain Hamiltonian

['02 Minahan, Zarembo]

$\Gamma^{1-\text{loop}}\Big($  $\Big) \longleftrightarrow \mathcal{H}_{XXX} = \frac{\lambda}{2}\sum_{n=1}^{L}\sigma_n^i\sigma_{n+1}^i$

- Eigenfunctions

$$\text{BPS operator} : \text{tr}(Z\cdots Z) \longleftrightarrow \text{Vacuum} : |\uparrow\cdots\uparrow\rangle$$

$$\text{tr}(ZYZZY\cdots) \longleftrightarrow |\uparrow\downarrow\uparrow\uparrow\downarrow\cdots\rangle$$

- Eigenvalues

$$\gamma^{1-\text{loop}} \longleftrightarrow E_{\text{XXX}}$$

1-loop anomalous dimension can be efficiently computed using integrability.

One can also compute the anomalous dimension at finite coupling using integrability techniques.

['10 Beisert et al]

3-point functions

structure constants

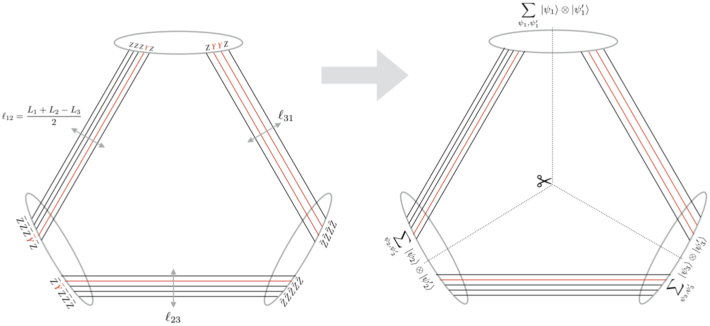
$$\langle\mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle=\frac{\overset{\substack{\text{structure constants}\\ \downarrow}}{C_{ijk}}}{|x_1-x_2|^{\Delta_i+\Delta_j-\Delta_k}|x_2-x_3|^{\Delta_j+\Delta_k-\Delta_i}|x_3-x_1|^{\Delta_k+\Delta_i-\Delta_j}}$$

Tree-level : Tailoring method ['10 Escobedo-Gromov-Sever-Vieira]

All-loop : Hexagon method ['15 Basso, Komatsu, Vieira]

C_{123} at tree-level

$$\mathcal{O}_i : \mathrm{tr}(Z{\color{red}Y}ZZ{\color{red}Y}\cdots) \xrightarrow{\text{map}} |\uparrow{\color{red}\downarrow}\uparrow\uparrow{\color{red}\downarrow}\cdots\rangle \xrightarrow{\text{cut}} \sum_{\psi_i,\psi_i'} |\psi_i\rangle \otimes |\psi_i'\rangle$$

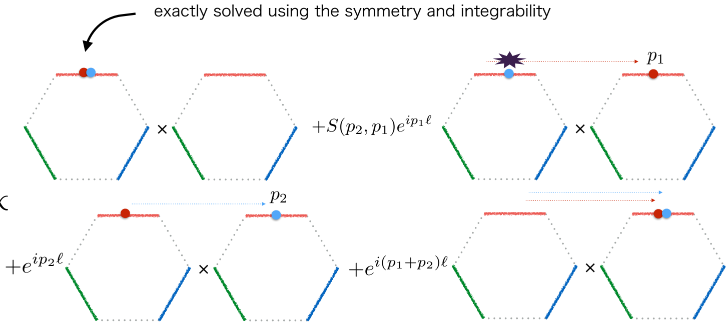


$$C_{123} \propto \sum_{\psi_i\psi_i'} \langle\psi_1|\psi_2'\rangle\langle\psi_2|\psi_3'\rangle\langle\psi_3|\psi_1'\rangle$$

Asymptotic C_{123} at all-loop

$\ell_{ij} >> 1$

$C_{123}^{\circ 2 \circ} \propto$



One can compute in principle the finite size corrections.

['15 Basso, Komatsu, Vieira], three-loop:['15 Eden, Sfondrini]['15 Basso, Goncalves, Komatsu, Vieira], four-loop:['15 Basso, Goncalves, Komatsu], multiple:['17 Basso, Goncalves, Komatsu]

C_{123} of closed strings

summation over the partitions of the magnon on the two hexagon form factors

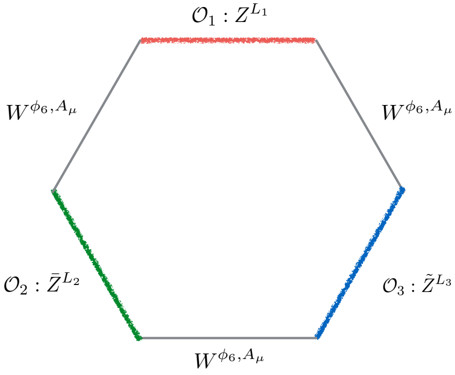
C_{123} of open strings using the hexagon method?

['17 Kim, *N.K*]

2. Set up

C_{123} of open strings from weak coupling analysis

C_{123} of operators on the Wilson loop



$$Z = \phi_1 + i\phi_2$$
$$Y = \phi_3 + i\phi_4$$
$$\tilde{Z} = Z + \bar{Z} + Y - \bar{Y} = 2(\phi_1 + i\phi_4)$$

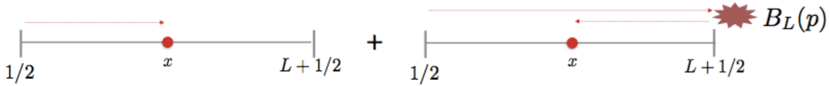
$$\langle W[\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)]\rangle, \quad W = \mathrm{tr}\left[P \exp\left(\oint d\tau (iA_\mu \dot{x}^\mu + \phi_6 |\dot{x}|)\right)\right]$$

Coordinate Bethe ansatz of open spin chain

['05 Okamura, Takayama, Yoshida]['06 Drukker, Kawamoto]

$$\mathcal{O}_i : \sum_{1\le x\le L_i} Z\cdots Z\underset{\substack{\uparrow\\ x}}{Y}Z\cdots Z \xrightarrow{\text{map}} \sum_{1\le x\le L_i} \psi^{(1)}(x)|\uparrow\cdots\uparrow\underset{\substack{\uparrow\\ x}}{\downarrow}\uparrow\cdots\uparrow\rangle$$

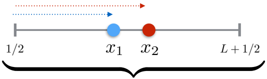
$$\psi^{(1)}(x) = e^{ip\left(x-\frac{1}{2}\right)} + e^{2ipL}B_L(p)e^{-ip\left(x-\frac{1}{2}\right)}$$

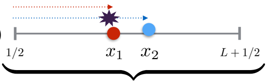


$B_L(p) = \ 1$: Neumann b.c. (Y-excitation)
$B_L(p) = -1$: Dirichlet b.c.

Coordinate Bethe Ansatz of open spin chain

$$f(p_1,p_2) \equiv e^{ip_1\left(x_1-\frac{1}{2}\right)+ip_2\left(x_2-\frac{1}{2}\right)} + S(p_2,p_1)\, e^{ip_2\left(x_1-\frac{1}{2}\right)+ip_1\left(x_2-\frac{1}{2}\right)}$$

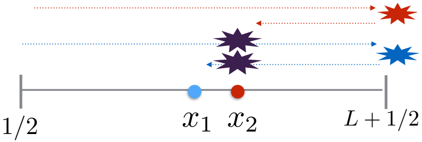




$$\begin{aligned}\psi^{(2)}(x_1,x_2) = {} & f(p_1,p_2) \leftarrow \text{closed cBA} \\ & + e^{2ip_2L}B_L(p_2)f(p_1,-p_2) \\ & + S(p_2,p_1)S(-p_1,p_2)e^{2ip_1L}B_L(p_1)f(-p_1,p_2) \\ & + S(p_2,p_1)S(-p_1,p_2)e^{2i(p_1+p_2)L}B_L(p_1)B_L(p_2)f(-p_1,-p_2)\end{aligned}$$

e.g.

$$S(p_2,p_1)S(-p_1,p_2)e^{2i(p_1+p_2)L} \times B_L(p_1)B_L(p_2)e^{-ip_1\left(x_1-\frac{1}{2}\right)-ip_2\left(x_2-\frac{1}{2}\right)} :$$



Coordinate Bethe Ansatz of open spin chain

$$f(p_1,p_2) = e^{ip_1\left(x_1-\frac{1}{2}\right)+ip_2\left(x_2-\frac{1}{2}\right)} + S(p_2,p_1)e^{ip_2\left(x_1-\frac{1}{2}\right)+ip_1\left(x_2-\frac{1}{2}\right)}$$

$$\begin{aligned}\psi^{(2)}(x_1,x_2) &= f(p_1,p_2)\\ &+ e^{2ip_2L}B_L(p_2)f(p_1,-p_2)\\ &+ S(p_2,p_1)S(-p_1,p_2)e^{2ip_1L}B_L(p_1)f(-p_1,p_2)\\ &+ S(p_2,p_1)S(-p_1,p_2)e^{2i(p_1+p_2)L}B_L(p_1)B_L(p_2)f(-p_1,-p_2)\end{aligned}$$

Multi-magnon:

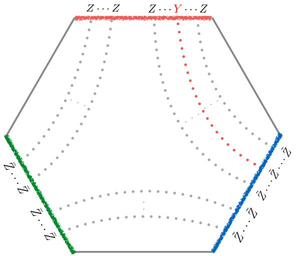
$$f(\hat{p}_1,\cdots,\hat{p}_M) \equiv \sum_{\sigma_1\neq\cdots\neq\sigma_M}^{M} \prod_{\substack{j<k\\ \sigma_k<\sigma_j}} S(\hat{p}_{\sigma_j},\hat{p}_{\sigma_k}) \prod_{m=1}^{M} e^{i\hat{p}_{\sigma_m}\left(x_m-\frac{1}{2}\right)}$$

$$\psi^{(M)} = \sum_{\mathsf{P}_+\cup\mathsf{P}_-=\{1,\ldots,M\}} \left[\prod_{k\in\mathsf{P}_-} \left(e^{2ip_k\ell_{13}}B_L(p_k)\right) \prod_{l<k} S(p_k,p_l)S(-p_k,p_l)\right] f(\hat{p}_1,\cdots,\hat{p}_M)$$

with $$\hat{p}_i = \begin{cases} p_i & i\in\mathsf{P}_+ \\ -p_i & i\in\mathsf{P}_- \end{cases}.$$

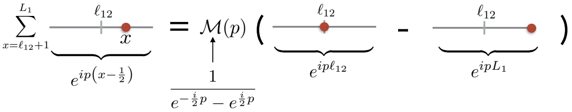
3. 3-point functions

tree-level analysis



$$\tilde{Z} = Z + \bar{Z} + Y - \bar{Y}$$

$$C_{123}^{1\circ\circ} \propto \sum_{x=\ell_{12}+1}^{L_1} \psi^{(1)}(x) = \sum_{x=\ell_{12}+1}^{L_1} \left(e^{ip\left(x-\frac{1}{2}\right)} + e^{2ipL_1} e^{-ip\left(x-\frac{1}{2}\right)}\right)$$

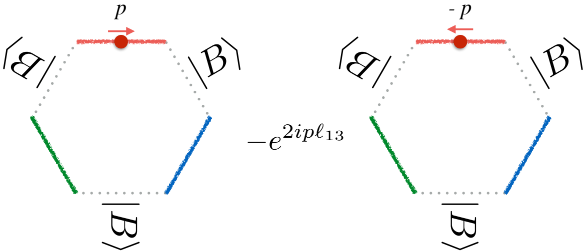


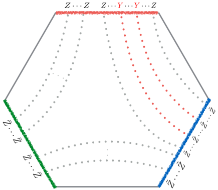
$$C_{123}^{1\circ\circ} \propto \mathcal{M}(p)(e^{ip\ell_{12}} - e^{ipL_1}) + \mathcal{M}(-p)e^{2ipL_1}(e^{-ip\ell_{12}} - e^{-ipL_1})$$
$$= \mathcal{M}(p)(e^{ip\ell_{12}} - e^{2ipL_1} e^{-ip\ell_{12}})$$

$$C_{123}^{1\circ\circ} \propto \mathcal{M}_{\ell_{12}}(p)(1 - e^{2ip\ell_{13}}), \quad \mathcal{M}_{\ell_{12}}(p) \equiv \mathcal{M}(p) e^{ip\ell_{12}}$$
$$= \mathcal{M}_{\ell_{12}}(p)(h^{\text{tree}}(p) - e^{2ip\ell_{13}} h^{\text{tree}}(-p))$$

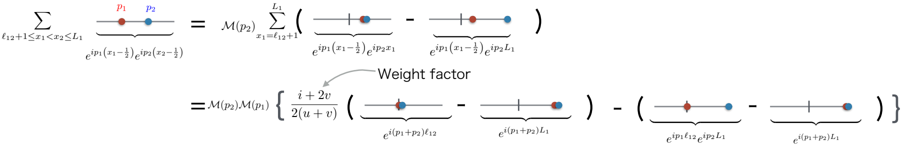
$$h^{\text{tree}}(p) = 1 : \text{1-magnon tree-level hexagon form factor}$$

inspired by hexagon method, as if





$$C_{123}^{2\circ\circ} \propto \sum_{\ell_{12}+1 \le x_1 < x_2 \le L_1} \psi^{(2)}(x_1, x_2)$$



$$C_{123}^{2\circ\circ} \propto \mathcal{M}_{\ell_{12}}(p_1)\mathcal{M}_{\ell_{12}}(p_2)\Big[\frac{u-v}{1+u-v} - S(p_2,p_1)S(-p_2,p_1)e^{2ip_1\ell_{13}}\frac{-u-v}{1-u-v}$$
$$- e^{2ip_2\ell_{13}}\frac{u+v}{1+u+v} + S(p_2,p_1)S(-p_2,p_1)e^{2i(p_1+p_2)\ell_{13}}\frac{-u+v}{1-u+v}\Big]$$

$$cf \;\; \frac{i+2v}{2(u+v)} + S(p_2,p_1)\frac{i+2u}{2(v+u)} = \frac{u-v}{1+u-v} \equiv h^{\text{tree}}(u,v)$$

Result

By mathematical induction, we proved

$$\left(\frac{C_{123}^{M\circ\circ}}{C_{123}^{\circ\circ\circ}}\right)^2 = \frac{(e^{i(p_1+\cdots+p_M)\ell_{12}}\mathcal{K}^{(M)})^2}{\det(\partial_{u_i}\phi_j)\prod_{i<j} S(p_j,p_i)e^{i(p_1+\cdots+p_M)L_1}}$$

$$\uparrow$$

$$\text{Gaudin norm: } e^{i\phi_j} \equiv e^{ip_jL_1}\prod_{k\neq j} S(p_k,p_j)S(-p_k,p_j)$$

$$\mathcal{K}_{\text{tree}}^{(M)} = \sum_{\mathsf{P}_+\cup\mathsf{P}_-=\{1,\ldots,M\}}\left[\prod_{k\in\mathsf{P}_-}(-e^{2ip_k\ell_{13}})\prod_{l<k} S(p_k,p_l)S(-p_k,p_l)\right]\prod_{i<j} h_{YY}^{\text{tree}}(\hat{p}_i,\hat{p}_j)$$

with

$$\hat{p}_i = \begin{cases} p_i & i\in\mathsf{P}_+ \\ -p_i & i\in\mathsf{P}_- \end{cases}.$$

Proposal

Recall the 2-magnon case:

$$C_{123}^{2\circ\circ} \propto h^{\text{tree}}(u,v) - S(p_2,p_1)S(-p_2,p_1)e^{2ip_1\ell_{13}}h^{\text{tree}}(-u,v)$$
$$- e^{2ip_2\ell_{13}}h^{\text{tree}}(u,-v) + S(p_2,p_1)S(-p_2,p_1)e^{2i(p_1+p_2)\ell_{13}}h^{\text{tree}}(-u,-v)$$

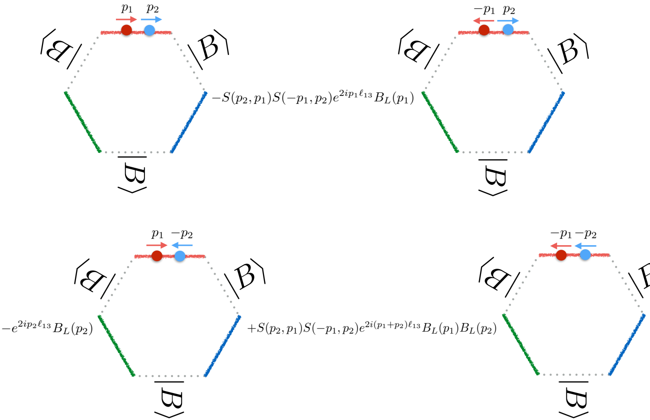
Inspired by the closed case: replacing by finite coupling factors

- hexagon form factor
- S-matrix factor

$$C_{123}^{2\circ\circ} \propto h(u,v) - S^{\text{finite}}(p_2,p_1)S^{\text{finite}}(-p_2,p_1)e^{2ip_1\ell_{13}}B_L(p_1)h(-u,v)$$
$$- e^{2ip_2\ell_{13}}B_L(p_2)h(u,-v) + S^{\text{finite}}(p_2,p_1)S^{\text{finite}}(-p_2,p_1)e^{2i(p_1+p_2)\ell_{13}}B_L(p_1)B_L(p_2)h(-u,-v)$$

Proposal

$$C_{123}^{2\circ\circ} \propto h(u,v) - S^{\text{finite}}(p_2,p_1)S^{\text{finite}}(-p_2,p_1)e^{2ip_1\ell_{13}}B_L(p_1)h(-u,v)$$
$$- e^{2ip_2\ell_{13}}B_L(p_2)h(u,-v) + S^{\text{finite}}(p_2,p_1)S^{\text{finite}}(-p_2,p_1)e^{2i(p_1+p_2)\ell_{13}}B_L(p_1)B_L(p_2)h(-u,-v)$$



Proposal

Asymptotic 3-point functions of the open strings:

$$\left(\frac{C_{123}^{M\circ\circ}}{C_{123}^{\circ\circ\circ}}\right)^2 = \frac{(e^{i(p_1+\cdots+p_M)\ell_{12}}\mathcal{K}^{(M)})^2}{\det(\partial_{u_i}\phi_j)\prod_{i<j}S(p_j,p_i)e^{i(p_1+\cdots+p_M)L_1}}$$

$$\uparrow$$

Gaudin norm: $e^{i\phi_j} \equiv e^{ip_jL_1}\prod_{k\neq j}S(p_k,p_j)S(-p_k,p_j)$

$$\mathcal{K}^{(M)}_{\text{finite}} = \sum_{\mathsf{P}_+\cup\mathsf{P}_-=\{1,\ldots,M\}}\left[\prod_{k\in\mathsf{P}_-}(-e^{2ip_k\ell_{13}}B_L(p_k))\prod_{l<k}S(p_k,p_l)S(-p_k,p_l)\right]\prod_{i<j}h^{\text{finite}}_{YY}(\hat{p}_i,\hat{p}_j)$$

Proposal

Asymptotic 3-point functions of the open strings:

$$\left(\frac{C_{123}^{M\circ\circ}}{C_{123}^{\circ\circ\circ}}\right)^2 = \frac{(e^{i(p_1+\cdots+p_M)\ell_{12}}\mathcal{K}^{(M)})^2}{\det(\partial_{u_i}\phi_j)\prod_{i<j}S(p_j,p_i)e^{i(p_1+\cdots+p_M)L_1}}$$

$\uparrow$

Gaudin norm: $e^{i\phi_j} \equiv e^{ip_jL_1}\prod_{k\neq j}S(p_k,p_j)S(-p_k,p_j)$

$$\mathcal{K}_{\text{finite}}^{(M)} = \sum_{\mathsf{P}_+\cup\mathsf{P}_-=\{1,\ldots,M\}}\left[\prod_{k\in\mathsf{P}_-}(-e^{2ip_k\ell_{13}}B_L(p_k))\prod_{l<k}S(p_k,p_l)S(-p_k,p_l)\right]\prod_{i<j}h_{YY}^{\text{finite}}(\hat{p}_i,\hat{p}_j)$$

Summation over all possible ways of changing the signs of magnon momenta in the hexagon form factors with appropriate factors

c.f.

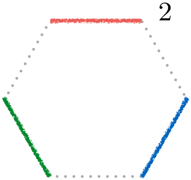
$$C_{123}^{2\circ\circ} \propto h(u,v) - S^{\text{finite}}(p_2,p_1)S^{\text{finite}}(-p_2,p_1)e^{2ip_1\ell_{13}}B_L(p_1)h(-u,v)$$
$$- e^{2ip_2\ell_{13}}B_L(p_2)h(u,-v) + S^{\text{finite}}(p_2,p_1)S^{\text{finite}}(-p_2,p_1)e^{2i(p_1+p_2)\ell_{13}}B_L(p_1)B_L(p_2)h(-u,-v)$$

Summary

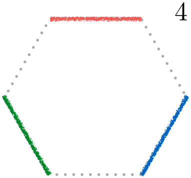
3−pt

4−pt

closed strings:

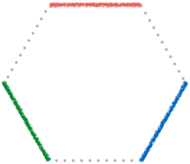


['15 Basso, Komatsu, Vieira]



['15 Basso, Coronado, Komatsu, Lam, Vieira, Zhong]

open strings:



$\times |B\rangle^3$

['17 Kim, N.K]

in progress

Outlook

1. Higher rank sectors or higher loop: $B_L(p) \neq 1$?
2. Other integrable "strings"?
3. Spin chain length 0 limit?[to appear Kim, N.K, Komatsu and Nishimura]
4. Higher point functions?[work in rogress N.K, Komatsu]

fin.