Weak renormalization group and critical phenomena

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Basic idea

- Functional Renormalization Group
  - Wilson RG, Non-perturbative RG, Exact RG···
- The method to evaluate the path integral

Path integral

$$Z = \int \mathcal{D}\phi \, e^{iS}$$

Functional integral

↔

FRG equation

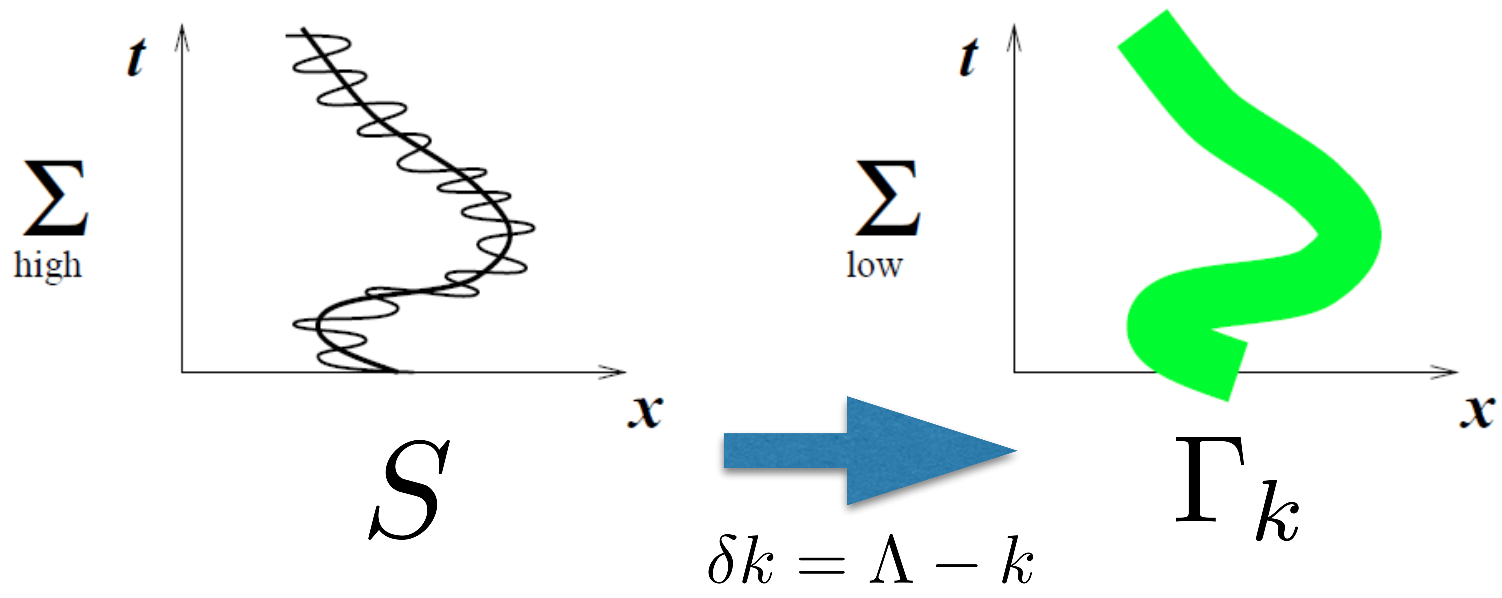
$$\frac{d\Gamma_k}{dk} = \beta\left[\frac{d^2\Gamma_k}{d\phi^2}; k\right]$$

$$\Gamma_{k=\Lambda} = S$$

Functional differential equation

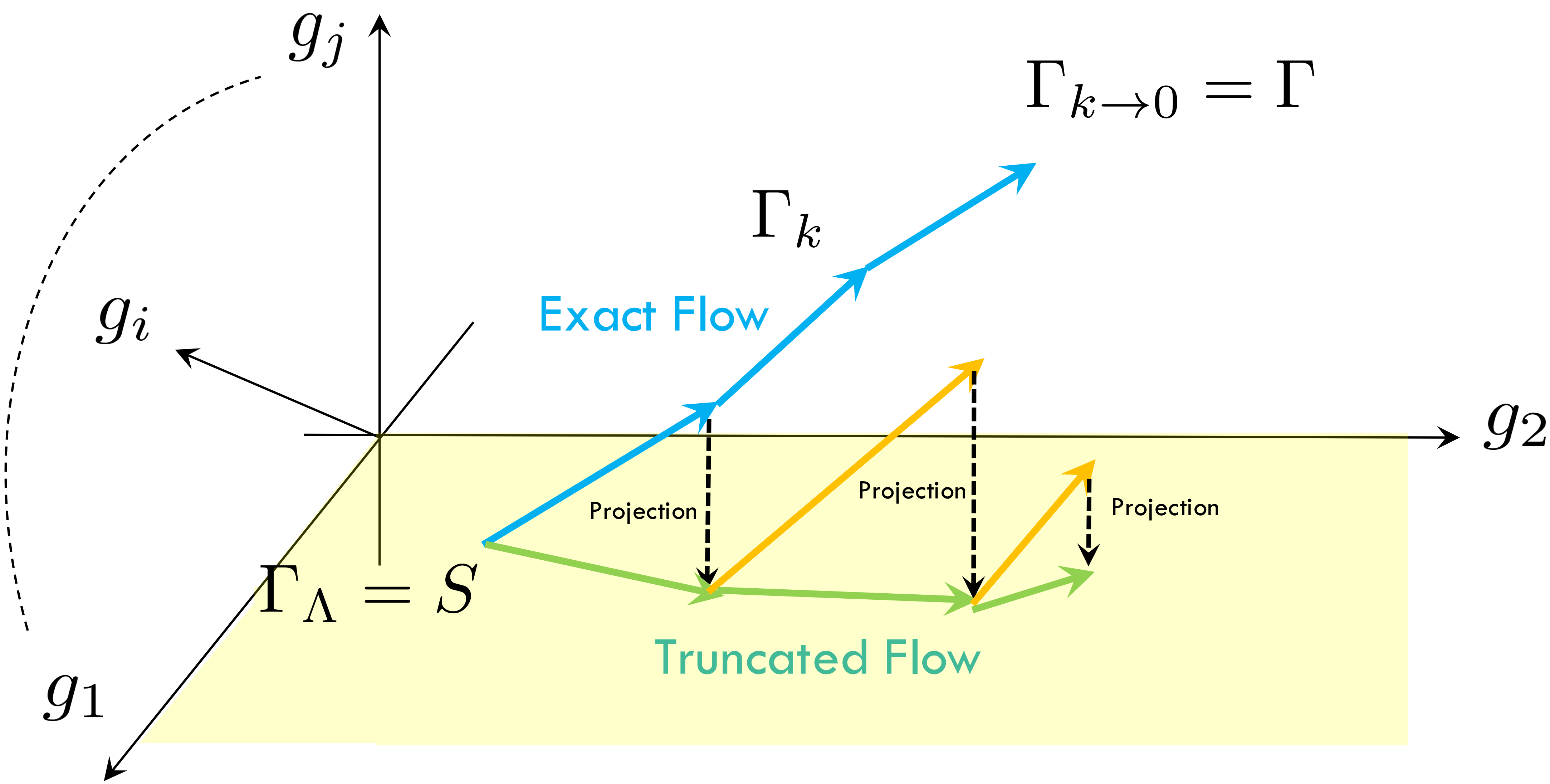
RG transformation

- Coarse-graining: Summing up quantum fluctuations



- Rescaling
  - Normalize the energy unit and the kinetic term.

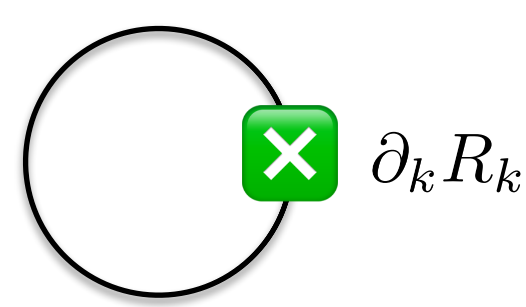
RG flow



Wetterich equation

1-loop structure!

$$\partial_k \Gamma_k = \frac{1}{2}\mathrm{STr}\left[\frac{\partial_k R_k}{\frac{\delta^2 \Gamma_k}{\delta \phi^2} + R_k}\right] =$$



Advantages of FRG

- The FRG is useful to evaluate the fixed point structure and critical exponents.

$$\Gamma_k = \int \mathrm{d}^d x \, [g_1 \mathcal{O}_1 + g_2 \mathcal{O}_2 + \cdots]$$

$$k \frac{\partial g_i}{\partial k} = \beta_i(g_i) = 0 \quad \longrightarrow \quad g_i^*$$

$$g_i = g_i^* + C_i \left(\frac{k}{k_0} \right)^{-\theta_i}$$

Advantages of FRG

- Systematically improve the approximation
    - The mean-field approximation and Schwinger-Dyson equation have difficulty to improve the approximation.
- No sign problem
    - Lattice Monte Carlo simulation suffers from the sign problem.

Plan

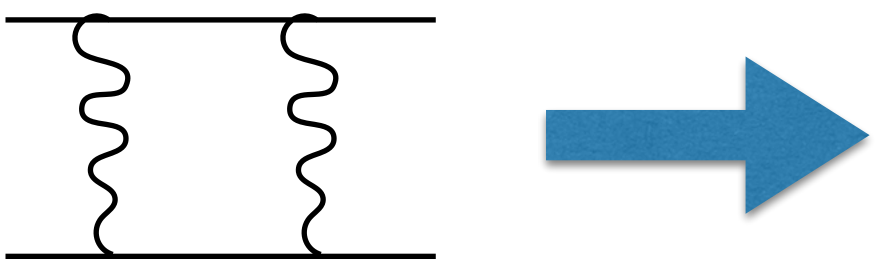
1. Four-Fermi coupling diverges.

2. Functional renormalization group with weak solution (Weak renormalization group)

3. Chiral phase diagram

D χ SB

- Effective interaction



four-Fermi structure

$$(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma^5\psi)^2$$

- Effective model describing the D χ SB:

  - Nambu-Jona-Lasinio (NJL) model

$$\mathcal{L}_{\mathrm{NJL}} = \bar{\psi}i\partial\!\!\!/\psi + \frac{G}{2}\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma^5\psi)^2\right]$$

    - Invariant under

$$\psi \to e^{i\gamma^5\theta}\psi \qquad \bar{\psi} \to \bar{\psi}e^{i\gamma^5\theta}$$

NJL model in FRG

- Initial action at k=Λ: Simplified NJL model

$$S = \int d^4x \left[\bar{\psi} \partial\!\!\!/ \psi - \frac{G_0}{2} (\bar{\psi}\psi)^2 \right]$$

- Invariant under $\psi \to \gamma^5 \psi, \ \bar{\psi} \to -\bar{\psi}\gamma^5$

- Effective action (with LPA)

$$\sigma = \bar{\psi}\psi$$

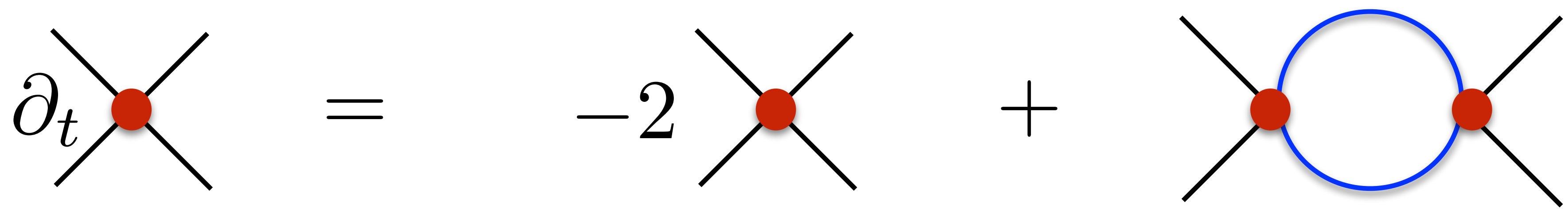
$$\Gamma_k = \int d^4x \left[\bar{\psi} \partial\!\!\!/ \psi - V(\sigma; k) \right]$$

$$V(\sigma; k) = \frac{G_k}{2}\sigma^2 + \frac{G_{8,k}}{4}\sigma^4 + \cdots$$

Four-Fermi coupling diverges at critical scale

$$\tilde{G} = \frac{G_k k^2}{2\pi^2}$$

- RG equation of G

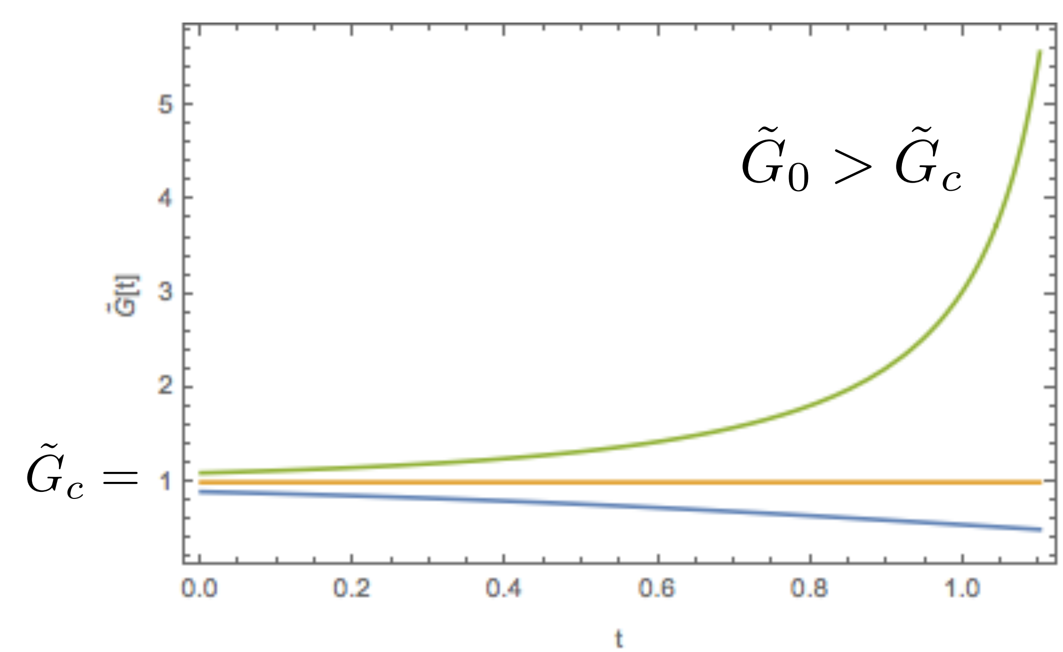


$$\partial_t \tilde{G} = -2\tilde{G} + 2\tilde{G}^2 \quad \xrightarrow{\text{Fixed Point}} \quad \tilde{G}_c = 1$$

- Solution:

$$\tilde{G}(t) = \frac{\tilde{G}_c \tilde{G}_0}{\tilde{G}_0 - (\tilde{G}_0 - \tilde{G}_c)e^{2t}}$$

It diverges at $t_c = \frac{1}{2}\log\left(\frac{\tilde{G}_0}{\tilde{G}_0 - \tilde{G}_c}\right)$



The divergence is signal of 2nd-order phase transition.

- Path integral(partition function)

$$Z = e^{W} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, e^{-S+\int d^4x \, m_0 \bar{\psi}\psi}$$

- Susceptibility

$$\chi := \left.\frac{\partial^2 W}{\partial m_0^2}\right|_{m_0=0} \sim \langle(\bar{\psi}\psi)^2\rangle \sim G$$

- 2nd order phase transition
  ⇔Divergence of susceptibility (2nd order derivative of W)

The divergence is signal of 2nd-order phase transition.

- The divergence is physical: signal of symmetry breaking (2nd order).
- However…
  - Once the RG equation diverges, we cannot follow the RG flow after the divergence.
  - The physical values, e.g., chiral condensate, should be evaluated at infrared scale $\Lambda \to 0$ ($t \to \infty$)

Legendre effective potential

- Legendre effective potential

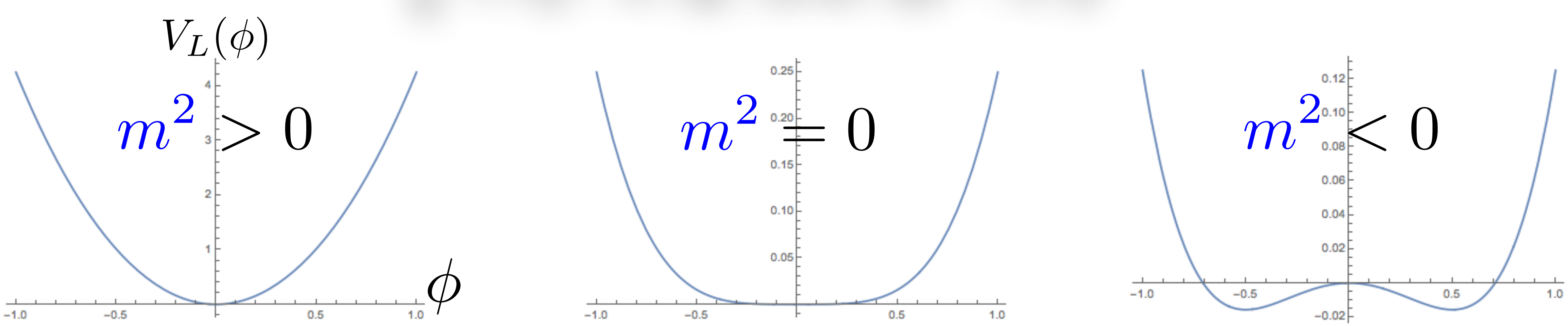
$$V_L(\phi) = -W(m_0) + m_0\phi \qquad \phi = \frac{\partial W}{\partial m_0} = \langle\bar{\psi}\psi\rangle$$

$$V_L(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 + \cdots$$

- Well-known relation:

$$1 = \frac{\partial^2 W}{\partial m_0^2} \cdot \frac{\partial V_L}{\partial \phi^2}\bigg|_{m_0=0,\phi=0} = \chi m^2$$

$$\chi \sim G \to \infty \iff m^2 \to 0$$



$t < t_c$ $\qquad$ $t = t_c$ $\qquad$ $t > t_c$

Summary so far

- The four-Fermi coupling constant is the (chiral) susceptibility.
- Divergence of G ⇔ 2nd order phase transition
- It is difficult to go to the broken phase.

Plan

1. Four-Fermi coupling diverges.
2. Functional renormalization group with weak solution (Weak renormalisation group)
3. Chiral phase diagram

How to access to broken phase ?

- Bosonization (auxiliary field method) Phys.Rev. D61 (2000) 045008
  - Inserting the Gauss integral: $1 = \mathcal{N} \int \mathcal{D}\phi \, e^{-\int d^4x \, \frac{y^2}{2G} \left(\phi - \frac{G}{y} \bar{\psi}\psi\right)^2}$

$$\frac{y^2}{2G}\phi^2 = \frac{m^2}{2}\phi^2$$

- Dynamical Bosonization Phys.Rev. D94 (2016) no.3, 034016
  - Scale-depend field: $\phi \longrightarrow \phi_k$
- External field method Prog.Theor.Phys. 121 (2009) 875-884
  - Introduce $m_0 \bar{\psi}\psi$

$$\langle \bar{\psi}\psi \rangle = \lim_{m_0 \to +0} \lim_{t \to \infty} \frac{\partial V\left(\psi, \bar{\psi}; t\right)}{\partial m_0}$$

Weak solution method

- We introduce neither an auxiliary field nor an external field.
- Mathematically define the solution with divergences!

RG equation is PDE

- RG equation as partial differential equation (PDE)

$$\partial_t V(\sigma;t) = -F(M;t) \xrightarrow{\partial_\sigma} \partial_t M + \partial_\sigma F(M;t) = 0$$

Beta function: $F(M;t) = -\frac{k^3}{\pi^2}\sqrt{k^2+M^2}$ $\quad k = \Lambda e^{-t}$

Mass function: $M = \partial_\sigma V$

- Initial condition $V(\sigma;t=0) = \frac{G_0}{2}\sigma^2$

$$M(\sigma;t=0) = G_0\sigma$$

Solving RG equation

$$\partial_t M + \partial_\sigma F(M;t) = 0$$

with $M(\sigma; t=0) = G_0 \sigma$

- Due to the divergence of G, the derivatives with respect to t and σ cannot be defined.

Weak solution

- Introduce the test function $\varphi(\sigma;t)$

  Smooth and satisfying $\varphi(\pm\infty;t)=\varphi(\sigma;\infty)=0$

$$\int_0^\infty dt\int_{-\infty}^\infty d\sigma\left[\frac{\partial M}{\partial t}+\frac{\partial F(M;t)}{\partial\sigma}\right]\varphi(\sigma;t)=0$$

↓ integration by parts

$$\int_{-\infty}^\infty d\sigma\left[(M\varphi)|_{t=0}^{t=\infty}-\int_0^\infty dt\,M\frac{\partial\varphi}{\partial t}\right]+\int_0^\infty dt\left[(\varphi F(M;t))|_{\sigma=-\infty}^{\sigma=\infty}-\int_{-\infty}^\infty d\sigma\,M\frac{\partial\varphi}{\partial\sigma}\right]=0$$

Weak RG equation

$$\int_0^\infty dt\int_{-\infty}^\infty d\sigma\left[M\frac{\partial\varphi}{\partial t}+F(M;t)\frac{\partial\varphi}{\partial\sigma}\right]+\int_{-\infty}^\infty d\sigma\,M(\sigma;0)\,\varphi(\sigma;0)=0$$

Its solution is called “weak solution”.

Characteristics

- RG equation of NJL model as PDE

$$\partial_t M(\sigma;t) + \partial_\sigma F(M;t) = 0$$

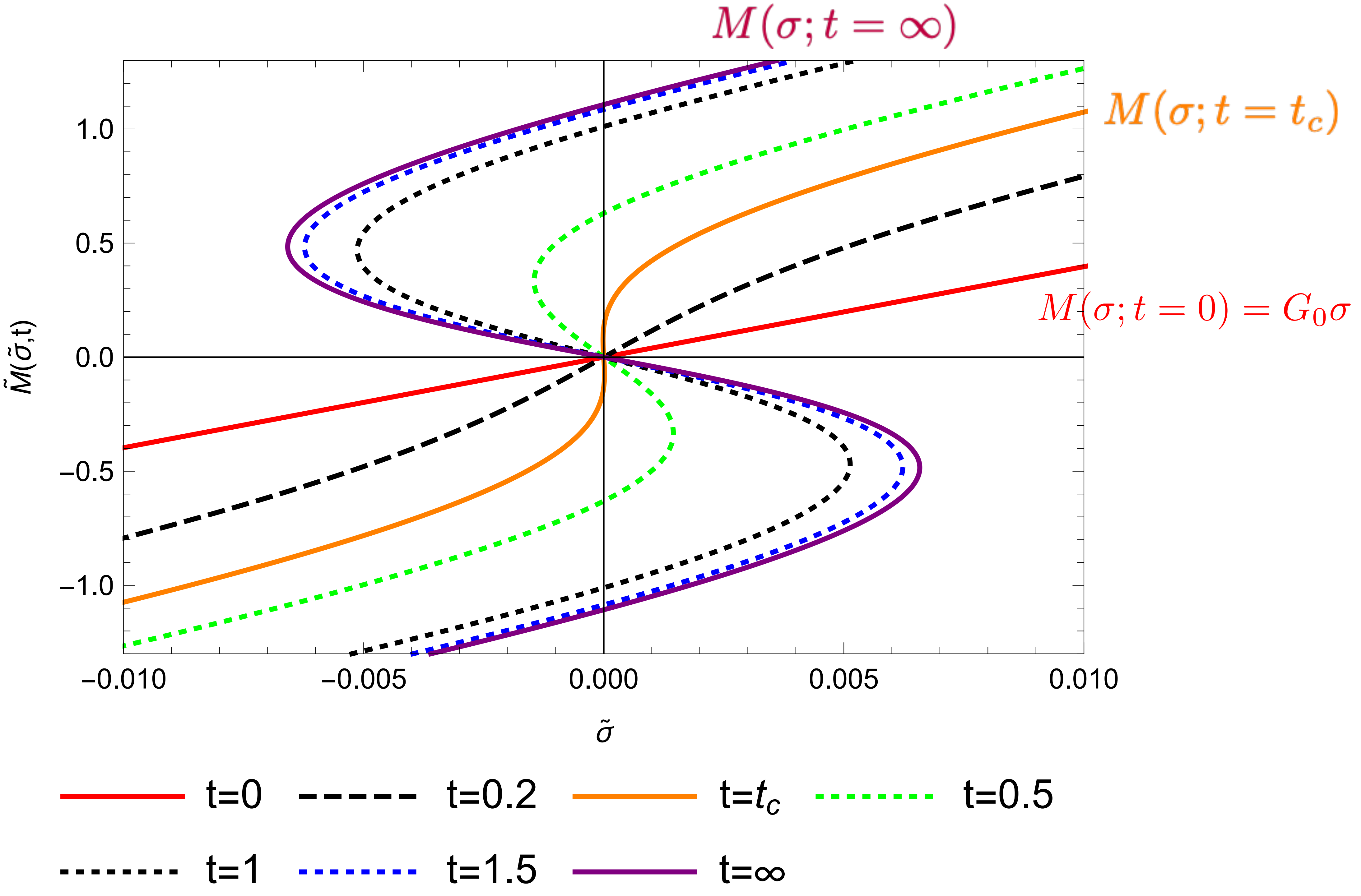
- Characteristic equation (coupled ODE)

$$\frac{d\sigma(s)}{ds} = \frac{\partial F}{\partial M}, \qquad \frac{dt(s)}{ds} = 1, \qquad \frac{dM(\sigma;s)}{ds} = 0$$

↓

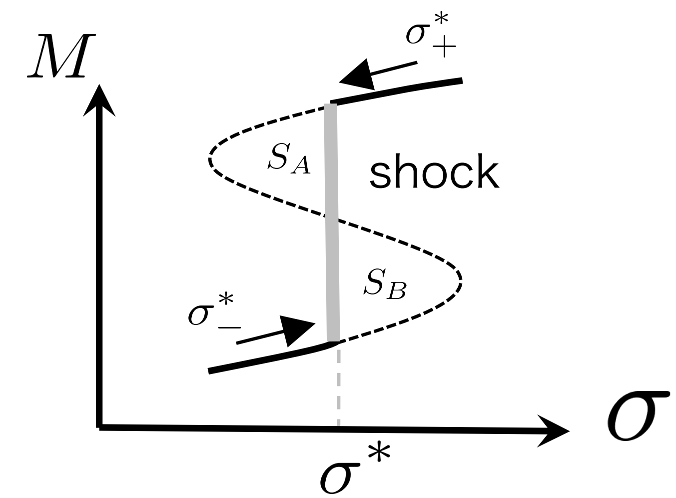
$$\frac{d\sigma(t)}{dt} = \frac{\partial F}{\partial M}, \qquad \frac{dM(\sigma;t)}{dt} = 0$$

Mass function



How to uniquely determine the solution?

- After the critical scale, the mass function becomes multi-valued function.



- To uniquely determine,

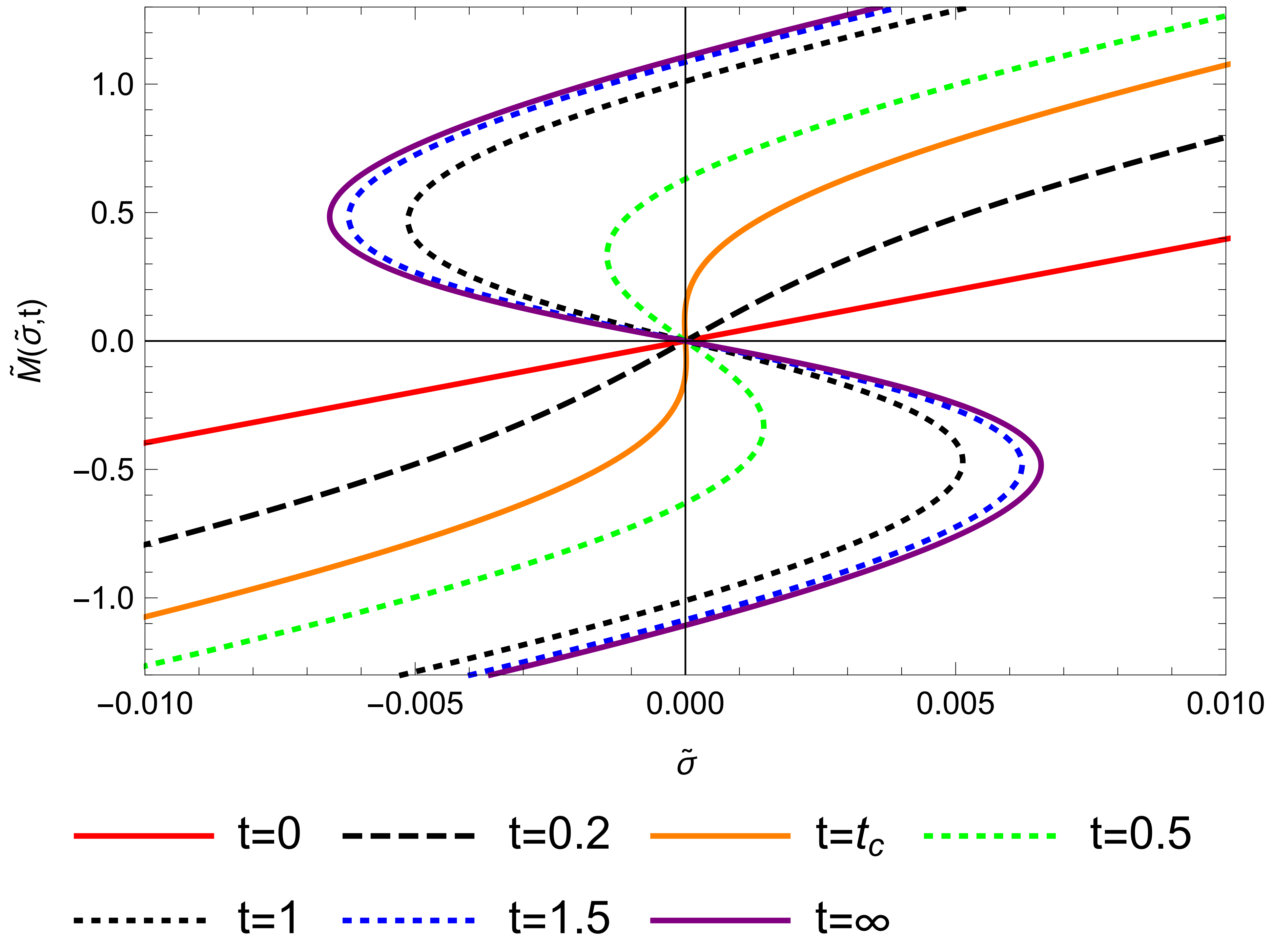
Rankine-Hugoniot condition:

$$\frac{d\sigma^*}{dt} = \frac{F\Big(M\big(\sigma^*_+(t)\big);t\Big) - F\Big(M\big(\sigma^*_-(t)\big);t\Big)}{M\big(\sigma^*_+(t);t\big) - M\big(\sigma^*_-(t);t\big)}$$

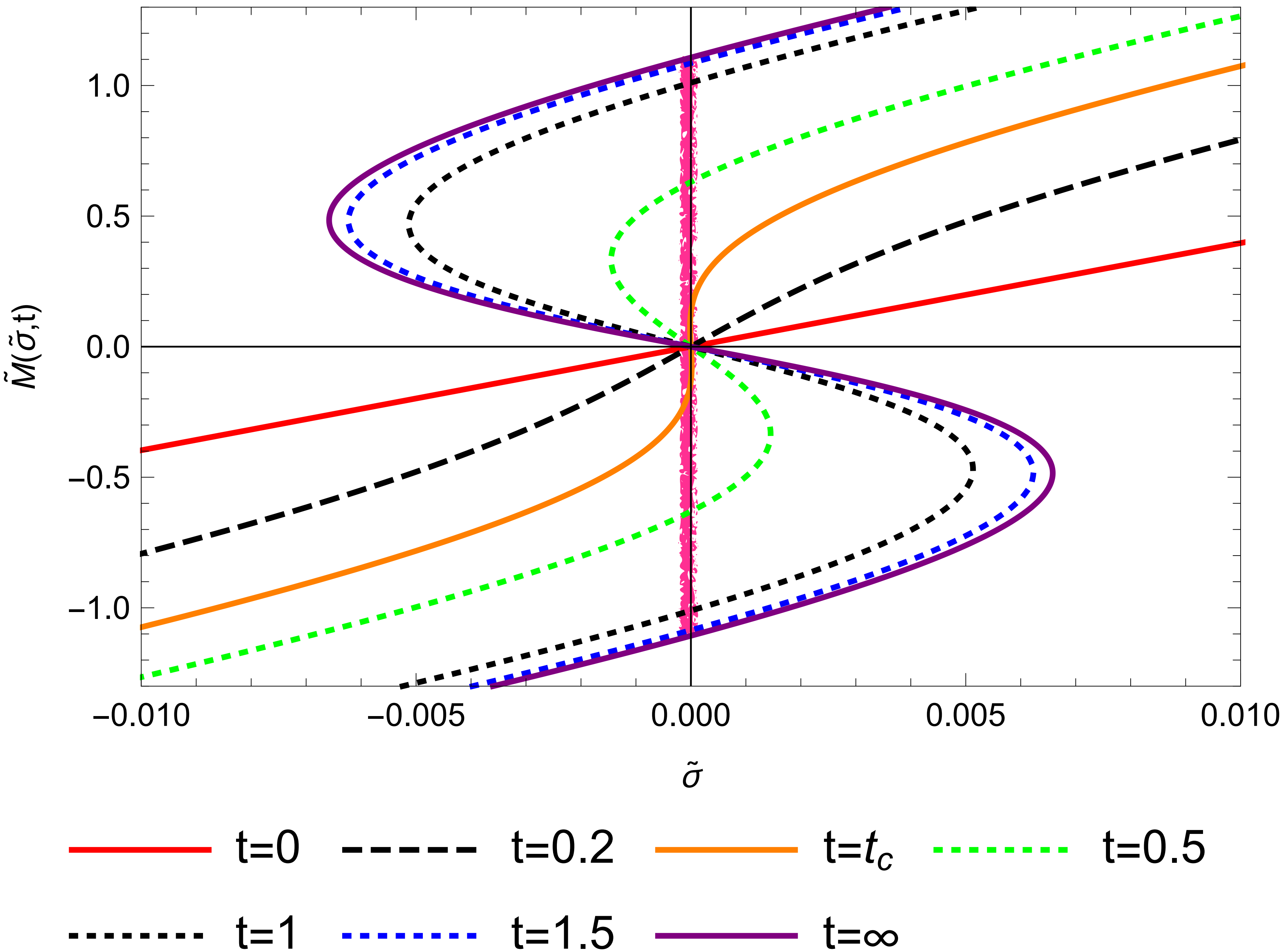
- Geometrically equal area law

$$S_A(t) - S_B(t) = \mathrm{constant} = 0$$

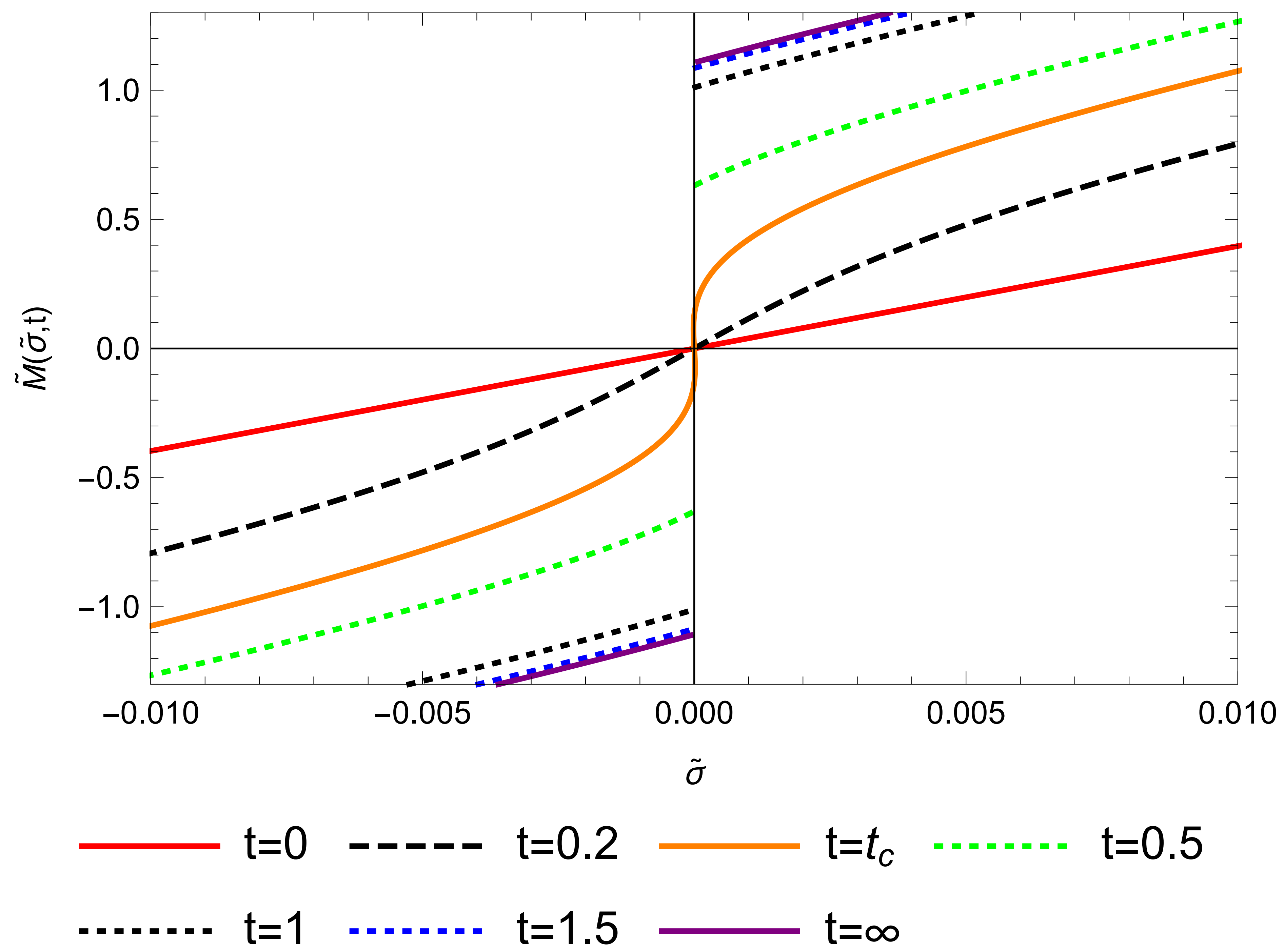
Mass function



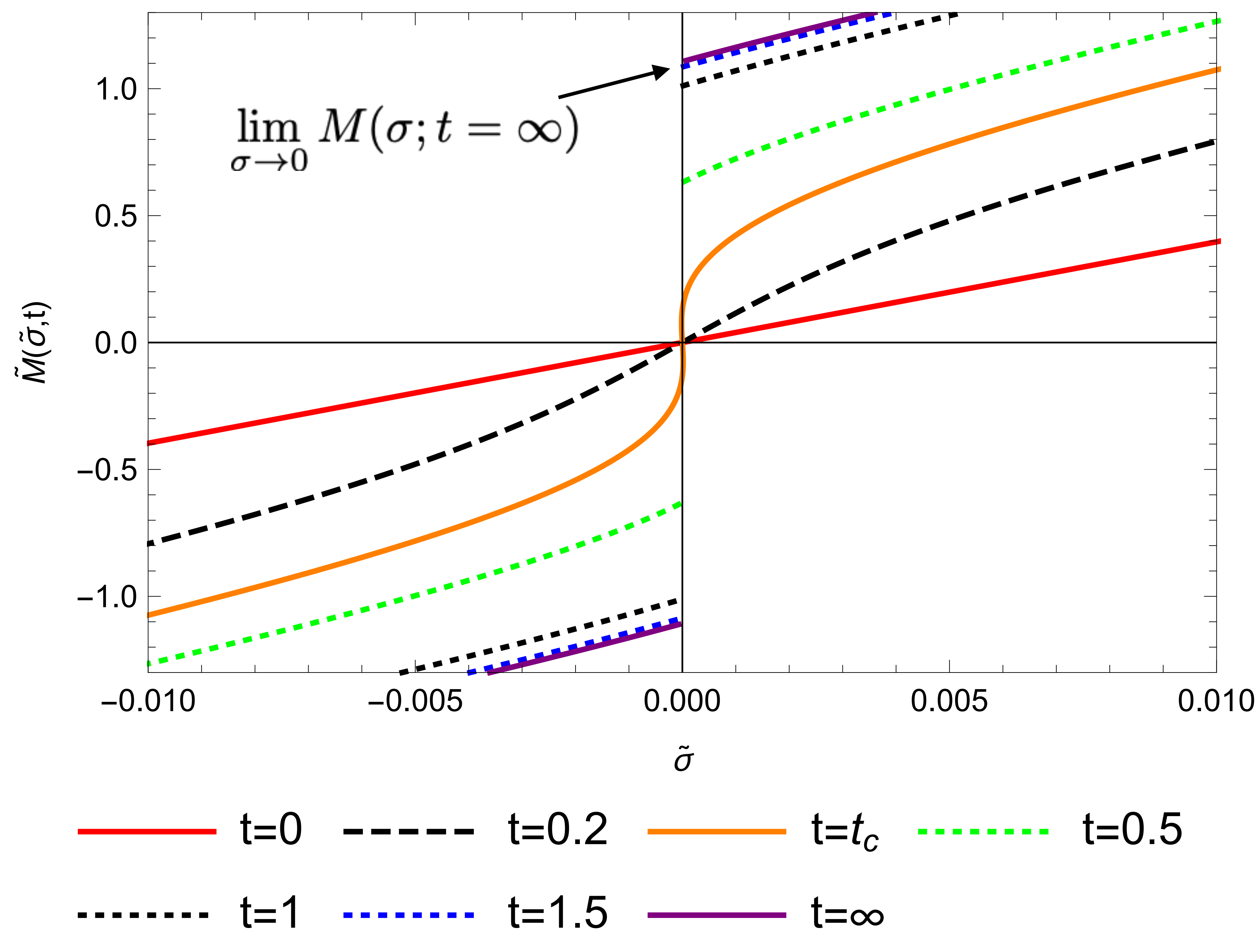
Mass function



Weak solution

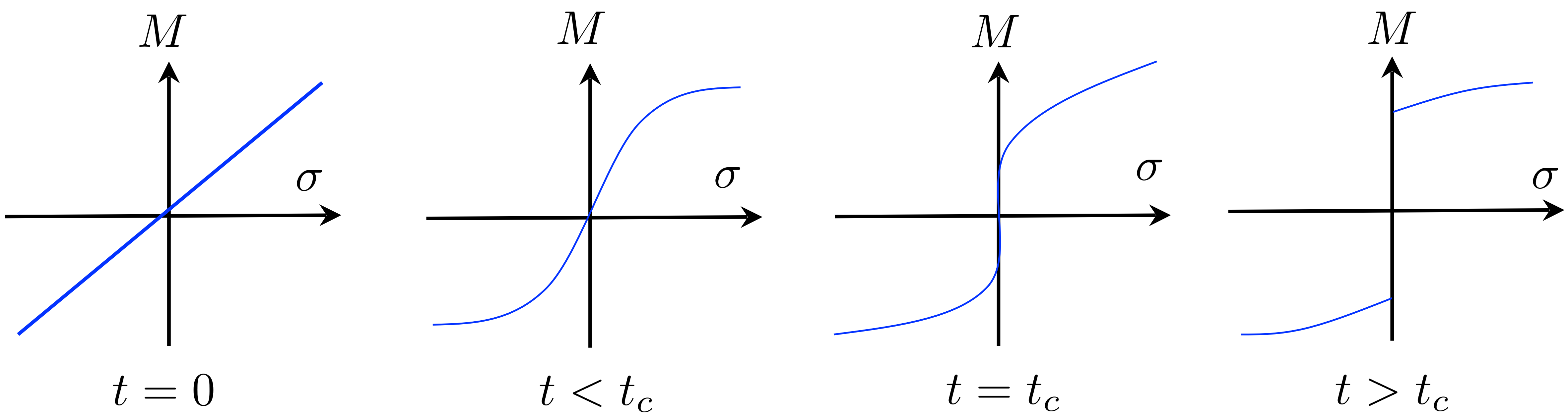


Weak solution



Evolution of RG flow

- Evolution of mass function

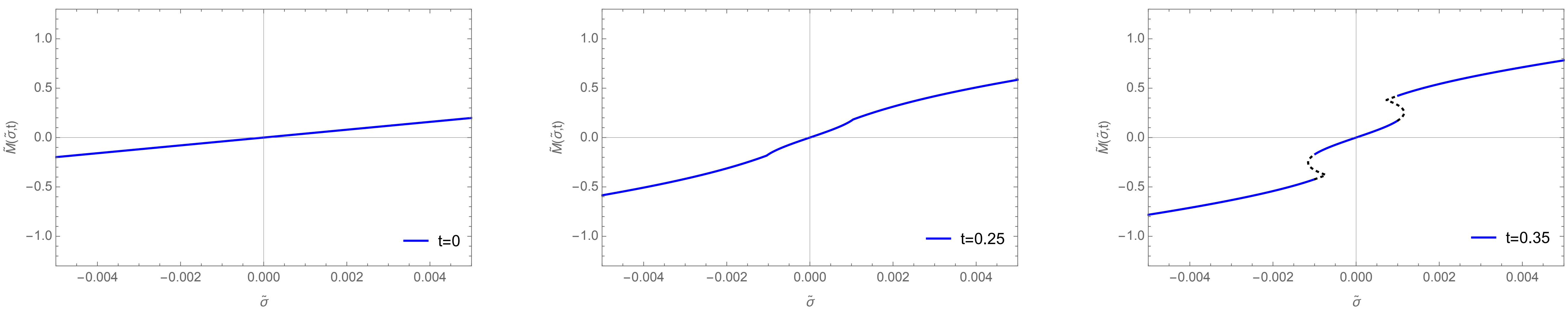


- At a scale the slope of the mass function becomes infinity.
- This corresponds to the second-order phase transition.

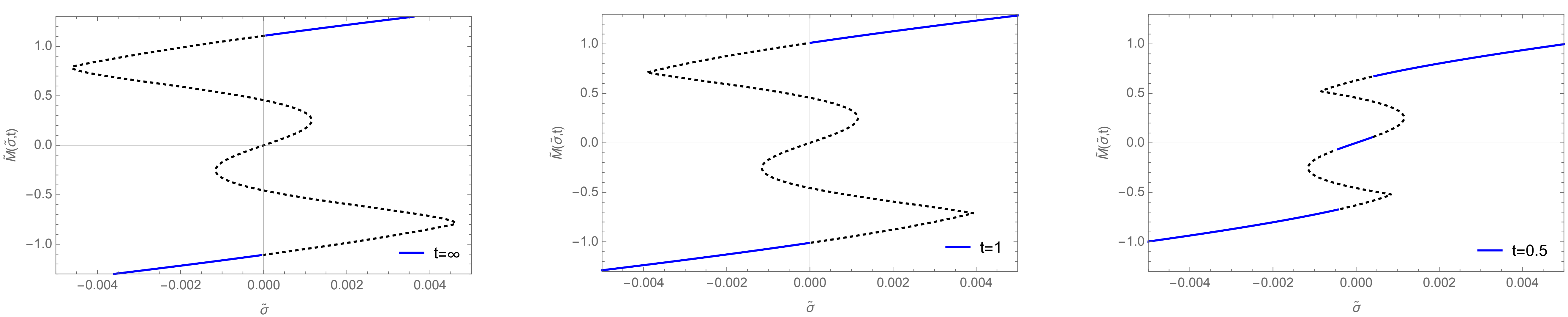
$$M(\sigma; t) = 2\pi^2 \tilde{G}(t)\, \sigma + \cdots$$

In case of first-order phase transition

$k \to \Lambda$



$k \to 0$



Summary so far

- The Solutions for RG equations with singularities is defined as a weak solutions
- Characteristic method: PDE to coupled ODE.
- In 2nd-order PT, the “Shock” arises at the origin $\sigma=0$.
- In 1st-order PT, the shocks arise at $\sigma\neq0$.

Plan

1. Four-Fermi coupling diverges.

2. Functional renormalization group with weak solution (Weak renormalisation group)

3. Chiral phase diagram

Finite temperature and density

- Effective action

$$\Gamma_k = \int d^4x \left[\bar{\psi}\partial\!\!\!/\psi - V(\sigma;t)\right]$$

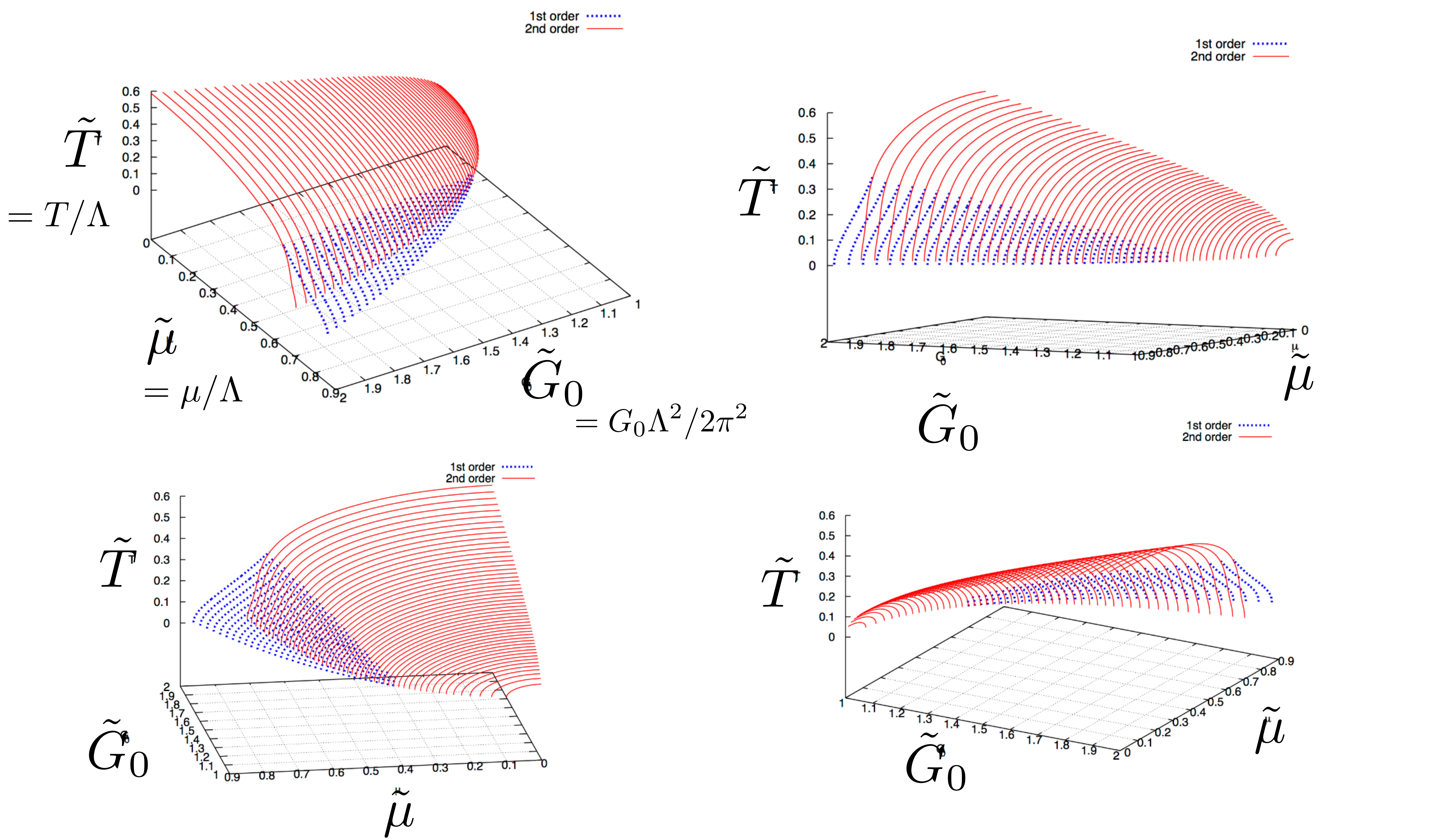
$$\Gamma_k = \int_0^{1/T} d\tau \int d^3x \left[\bar{\psi}\partial\!\!\!/\psi - V(\sigma;t) + \mu\bar{\psi}\gamma^0\psi\right]$$

- Beta function

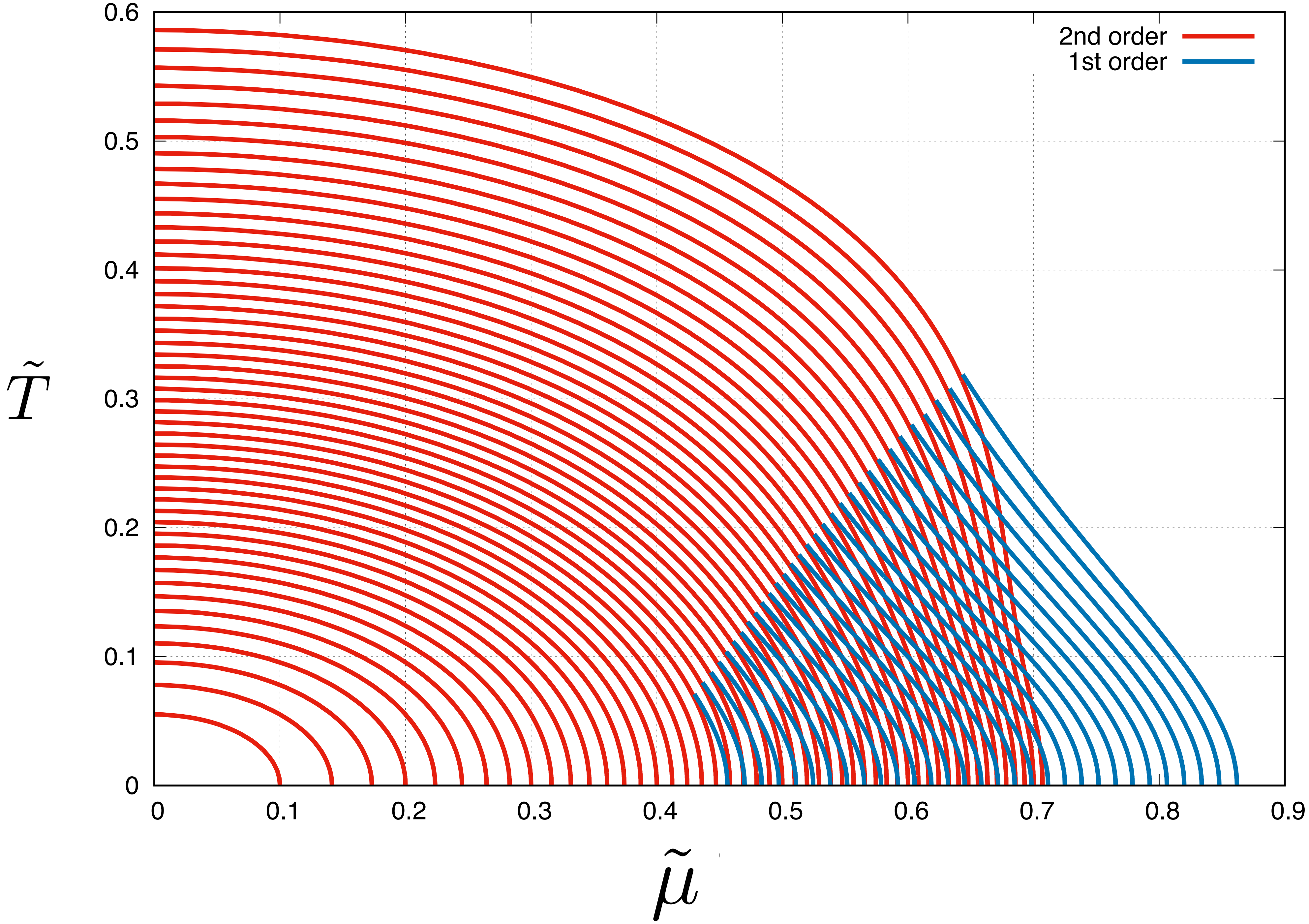
$$F(M;t) \qquad\qquad F(M;t;T,\mu)$$

- Parameters: $G_0,\ T,\ \mu,\ \Lambda$

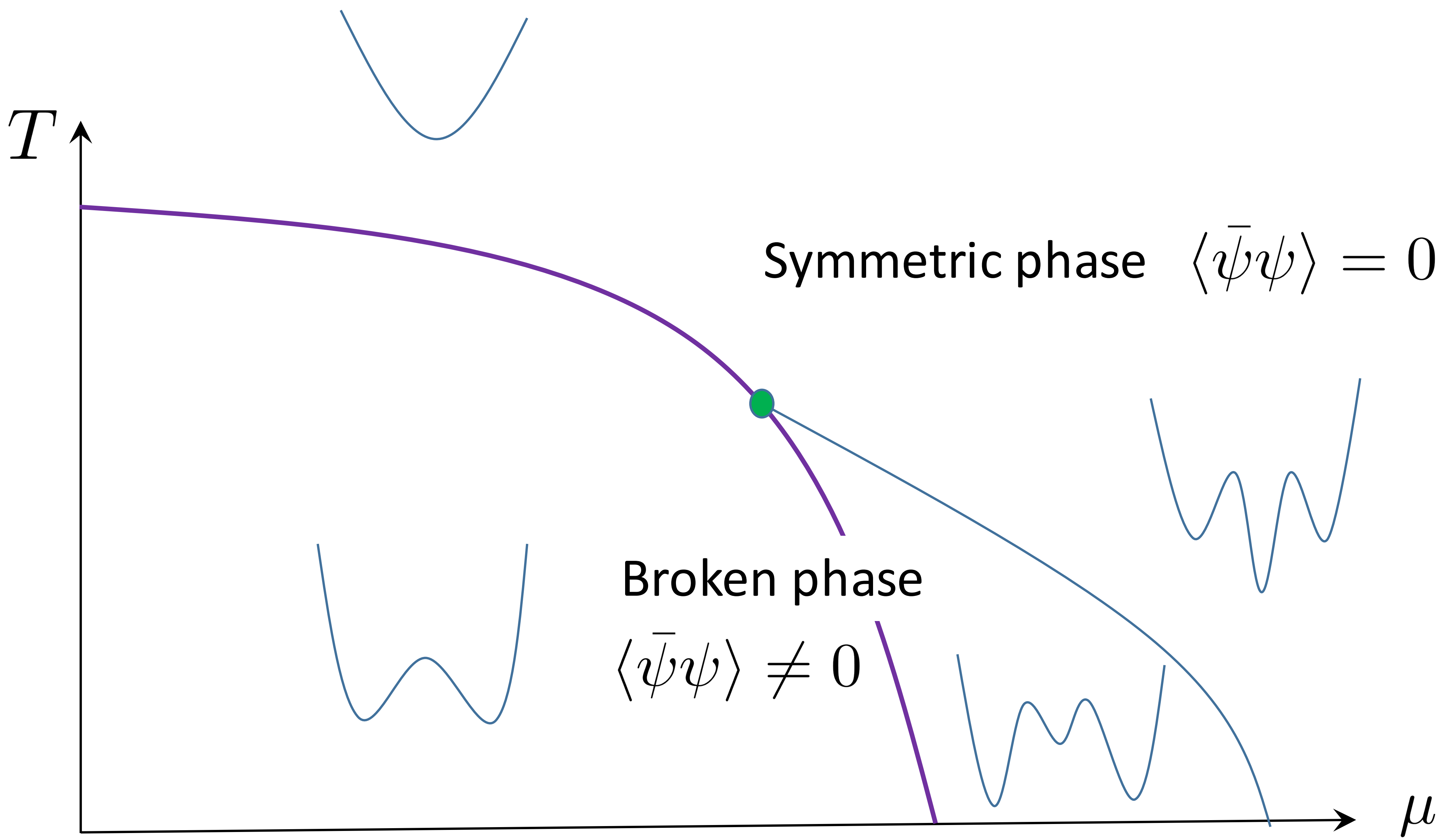
Phase diagram



Phase diagram on T-μ plane



Schematic figure of phase diagram

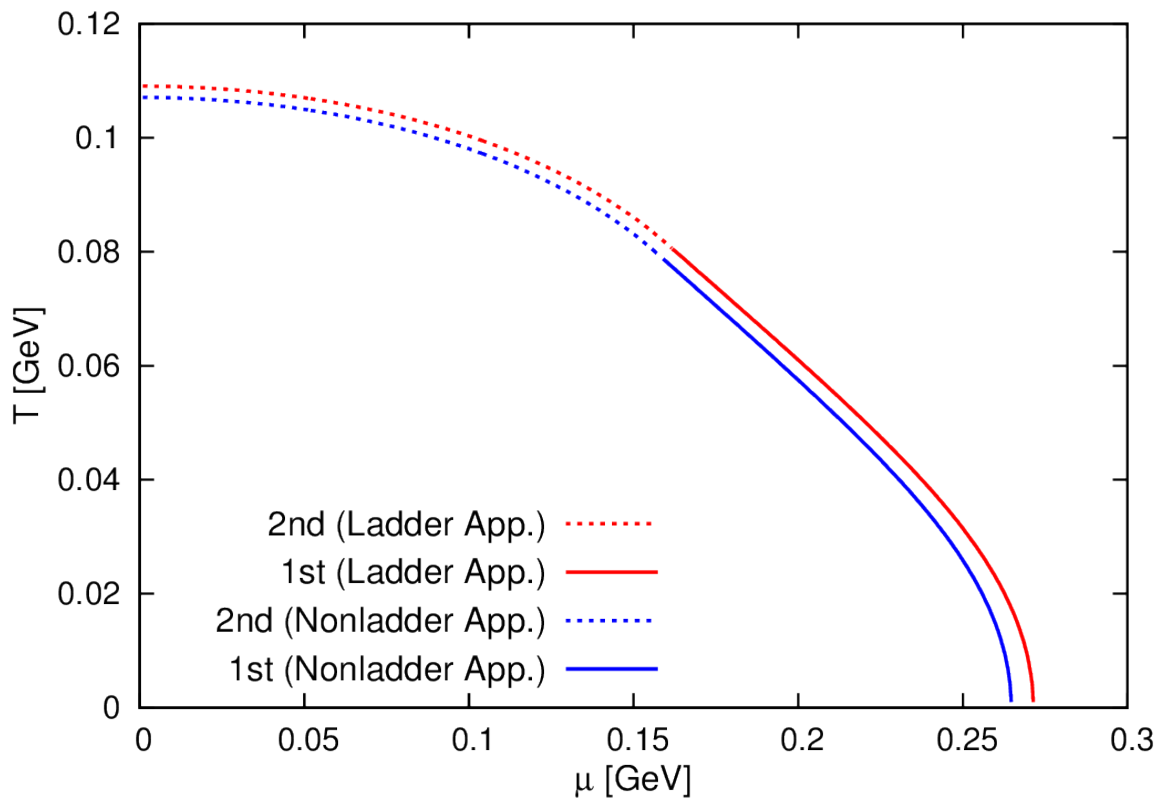


Summary

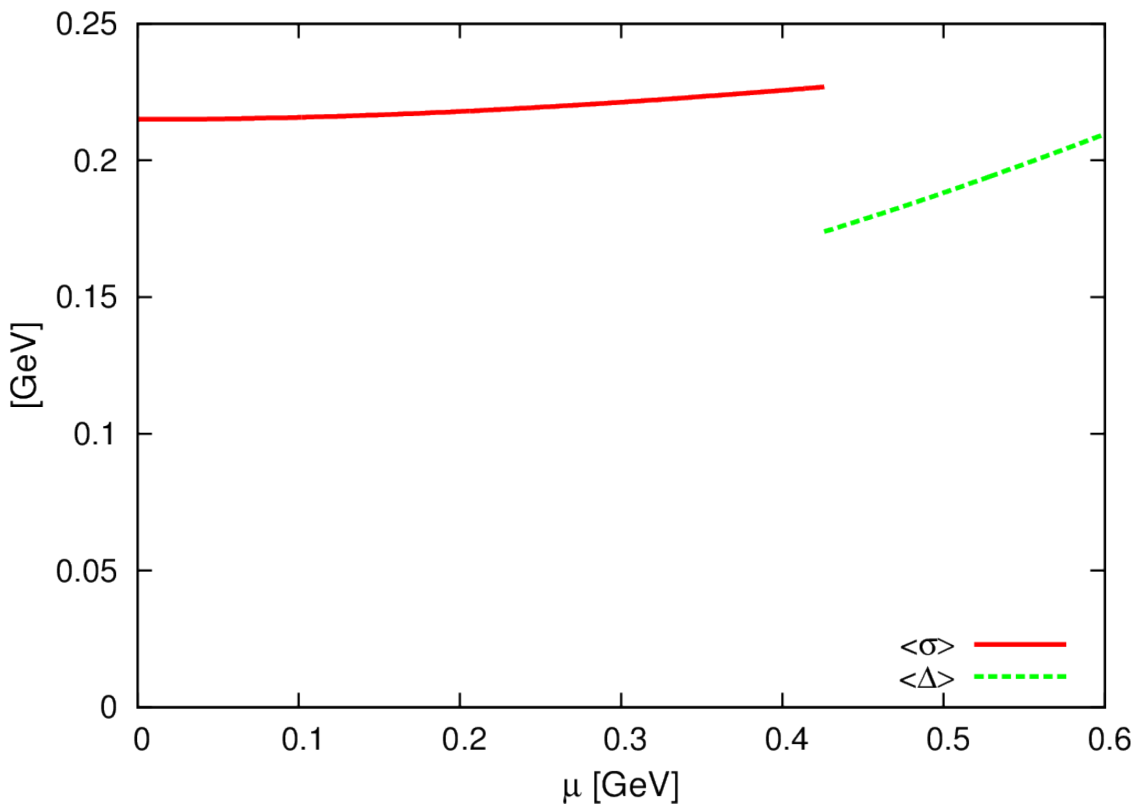
- Dynamical Chiral Symmetry Breaking with weak renormalization group
- Mathematically define the solution of RG equation with singularity.
- Phase diagram of NJL model

Prospects

- Phase diagram of QCD



supercolor conductivity(T=0)



- Improving the approximation

- Relationship with dynamical bosonization?

Improvement of approximation⋯

- Introduce the test function $\varphi(\sigma;t)$

  Smooth and satisfying $\varphi(\pm\infty;t)=\varphi(\sigma;\infty)=0$

$$\int_0^\infty dt\int_{-\infty}^\infty d\sigma\left[\frac{\partial M}{\partial t}+\frac{\partial F(M;t)}{\partial\sigma}\right]\varphi(\sigma;t)=0$$

integration by parts

$$\int_{-\infty}^\infty d\sigma\left[(M\varphi)|_{t=0}^{t=\infty}-\int_0^\infty dt\,M\frac{\partial\varphi}{\partial t}\right]+\int_0^\infty dt\left[(\varphi F(M;t))|_{\sigma=-\infty}^{\sigma=\infty}-\int_{-\infty}^\infty d\sigma\,M\frac{\partial\varphi}{\partial\sigma}\right]=0$$

Weak RG equation

$$\int_0^\infty dt\int_{-\infty}^\infty d\sigma\left[M\frac{\partial\varphi}{\partial t}+F(M;t)\frac{\partial\varphi}{\partial\sigma}\right]+\int_{-\infty}^\infty d\sigma\,M(\sigma;0)\,\varphi(\sigma;0)=0$$

When there is a higher derivative terms? $F(M,\partial_\sigma M;t)$

Appendix

Characteristics

RG equation

$$\partial_t M(\sigma;t) + \partial_\sigma F(M;t) = 0$$

$$\begin{pmatrix} \partial_M F & 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_\sigma M \\ \partial_t M \\ -1 \end{pmatrix} = 0$$

$$\partial_\sigma F = (\partial_M F)(\partial_\sigma M)$$

total derivative: $dM = \partial_\sigma M d\sigma + \partial_t M dt$

$$\begin{pmatrix} d\sigma & dt & dM \end{pmatrix} \begin{pmatrix} \partial_\sigma M \\ \partial_t M \\ -1 \end{pmatrix} = 0$$

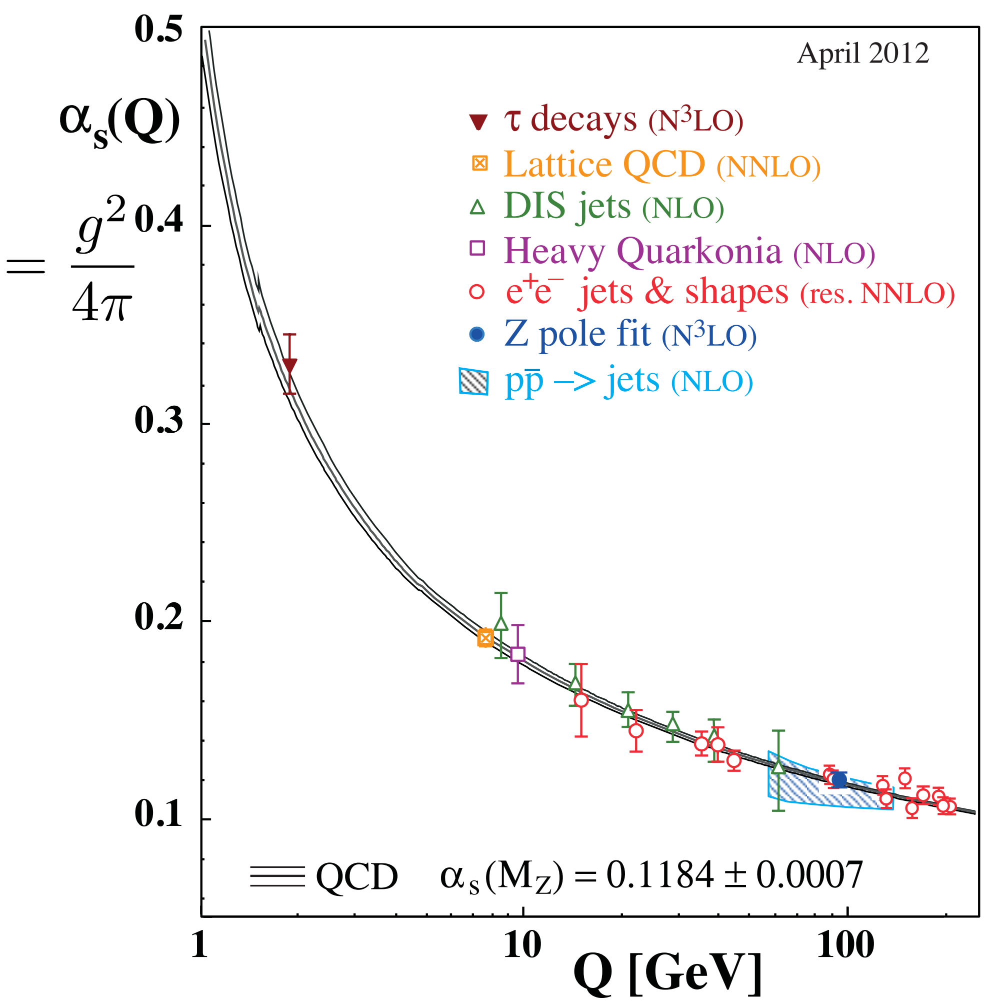
ds: constant of proportionality

$$\frac{d\sigma(s)}{ds} = \frac{\partial F}{\partial M}, \qquad \frac{dt(s)}{ds} = 1, \qquad \frac{dM(\sigma;s)}{ds} = 0$$

Quantum Chromodynamics (QCD)

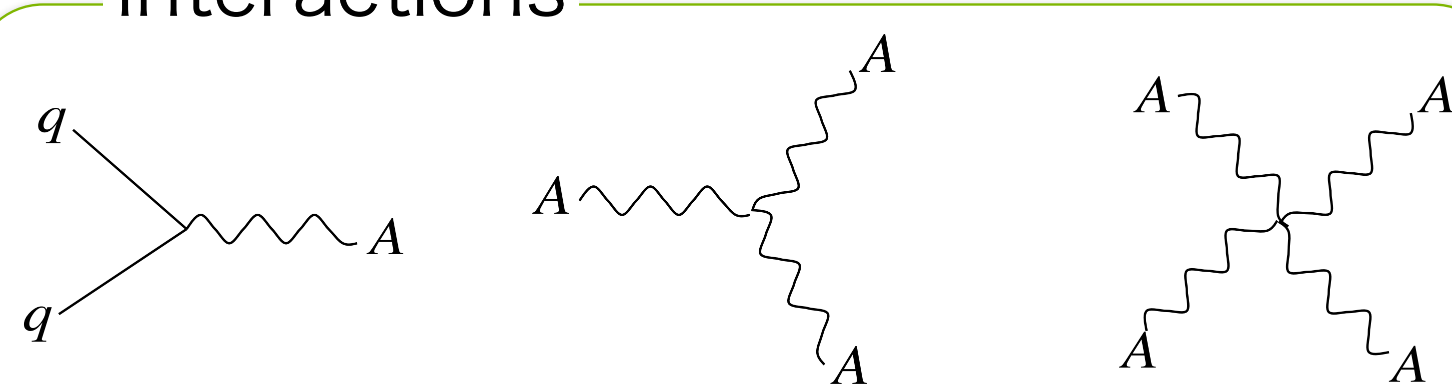
- SU(3) non-Abelian gauge theory

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \sum_{i=1}^{N_f} \bar{\psi}_{c,i} \left[i\gamma^\mu (\partial_\mu - igA^a_\mu \frac{\lambda^a}{2})_{cd} - m_i \delta_{cd} \right] \psi_{d,i}$$



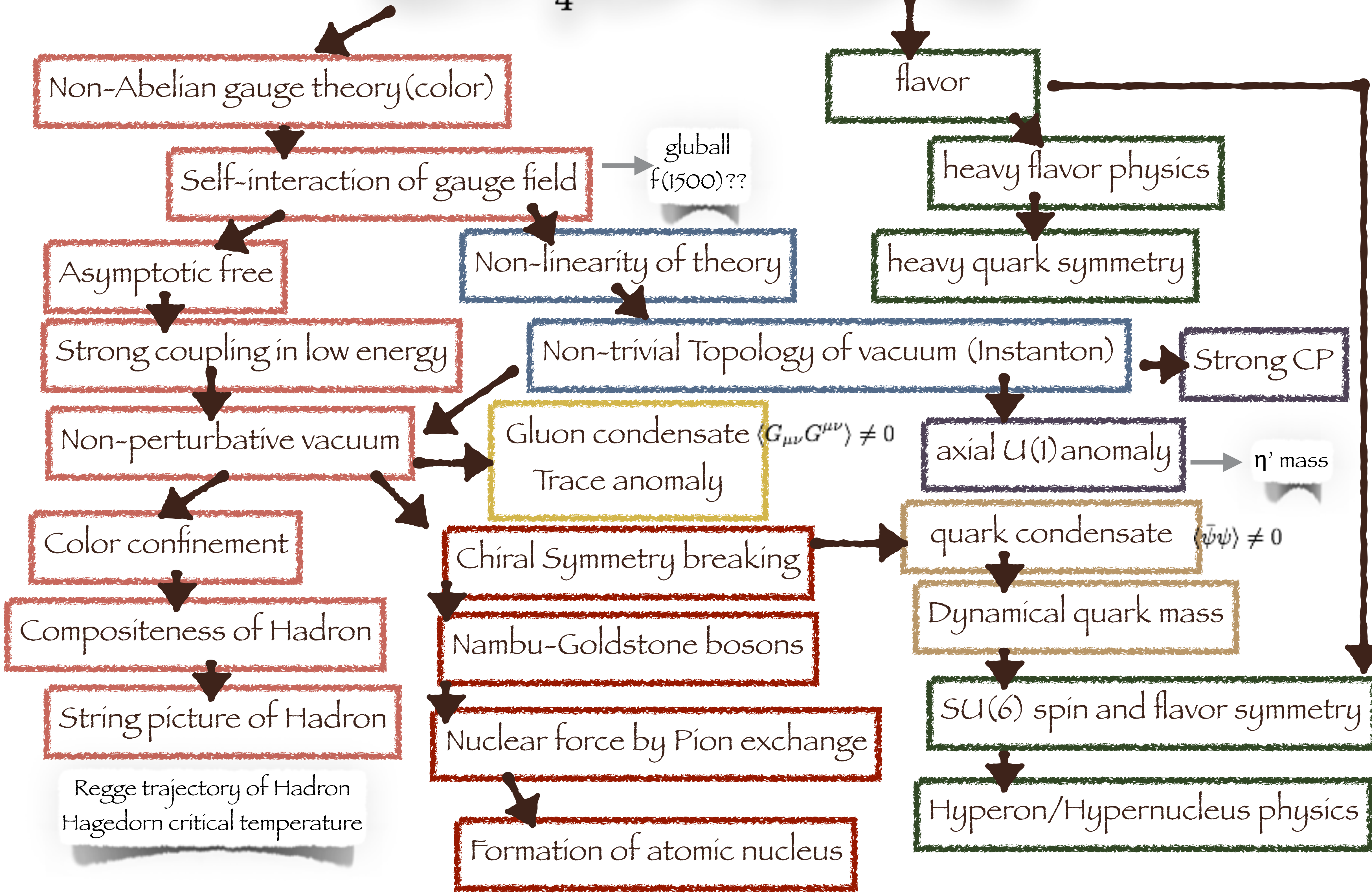
$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial^a_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

Interactions



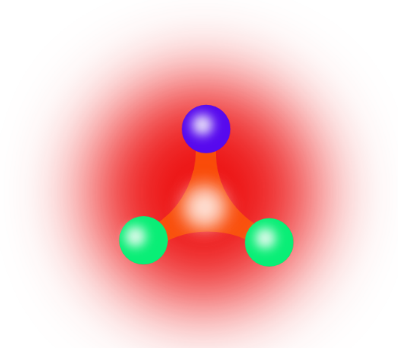
Physics from QCD

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \bar{\psi}(i\gamma_\mu D^\mu - m)\psi$$

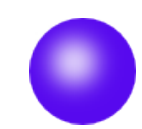


Dynamical Chiral Symmetry Breaking (D χ SB)

- Origin of Hadron mass



$m_q \sim 300\,\mathrm{MeV}$



$m_q \sim 5\,\mathrm{MeV}$

- 99% of mass of matter comes from the D χ SB.

Notation

- The dimensionless scale t

$$k = \Lambda e^{-t} \qquad \frac{d}{dt} = -k\frac{d}{dk}$$

- $k \to 0 \Longleftrightarrow t \to \infty$

At finite density