A CDT Hamiltonian from Hořava-Lifshitz gravity

Yuki Sato (KEK)

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J. Ambjørn, L. Glaser, Y. S. and Y. Watabiki, Physics Letters B 722, 2013

Quantum Gravity

Question: How did the Universe at the very beginning look like?

Its energy was about 10^{19} GeV (short scale).

Its dynamics was governed by strong gravity.

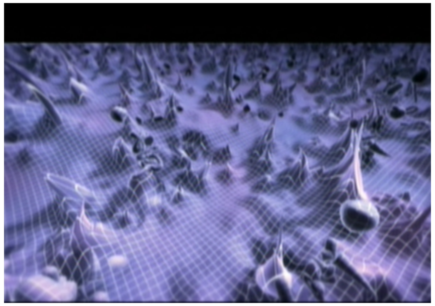
Physics at short scale → quantum mechanics

Gravitational physics at large scale → general relativity

To answer the question,
one needs to combine two theories in a consistent manner…
→ Quantum gravity

However, naïve construction doesn't work
because quantum fluctuations cannot be handled.

Non-renormalizabe



Wilsonian renormalization

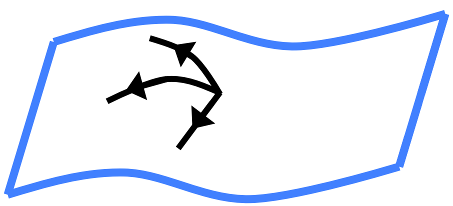
If and only if a cutoff theory is fundamental, it should possess

(1) Infinite UV cutoff limit characterized via a fixed point in RG flow:

(A) Gaussian fixed point (GF) → canonical scaling dim.

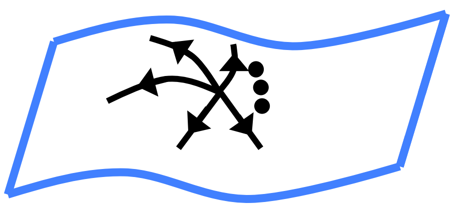
(B) Non-Gaussian fixed point (NGF) → non-canonical scaling dim.

(2) Predictability



Finite number of couplings

→ Predictable!!



Infinite number of couplings

→ Non predictable

0. INTRODUCTION

What is the IR description of CDT?

CDT looks like Horava-Lifshitz gravity (HL)…

CDT: Lattice quantum gravity w/ causality (**foliation)**

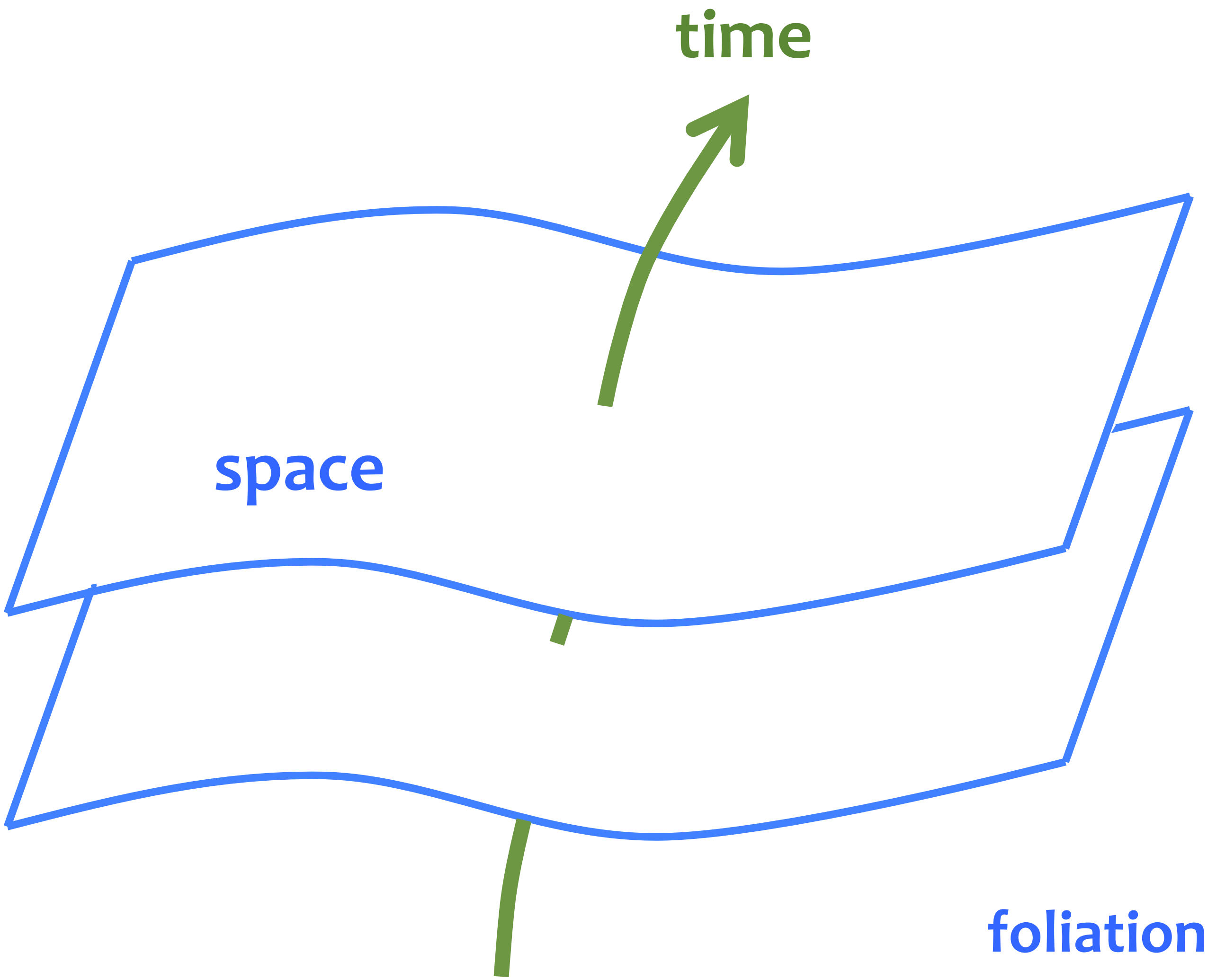
HL: Quantum gravity w/ anisotropic scaling (**foliation**)

→ **common**

In our work, we have determined that

2D CDT is 2D projectable HL quantum gravity!!

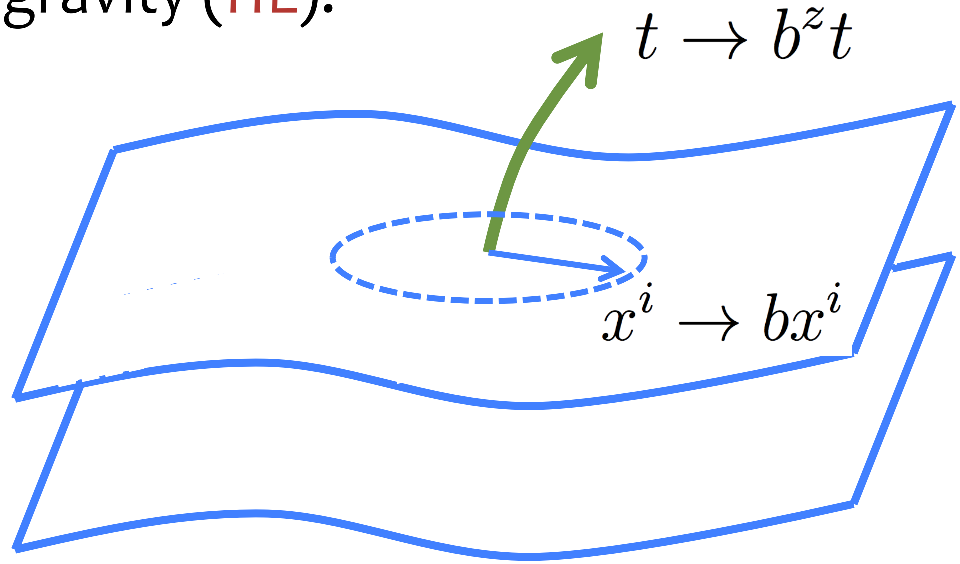
space-time → space [time]



foliation (葉層)

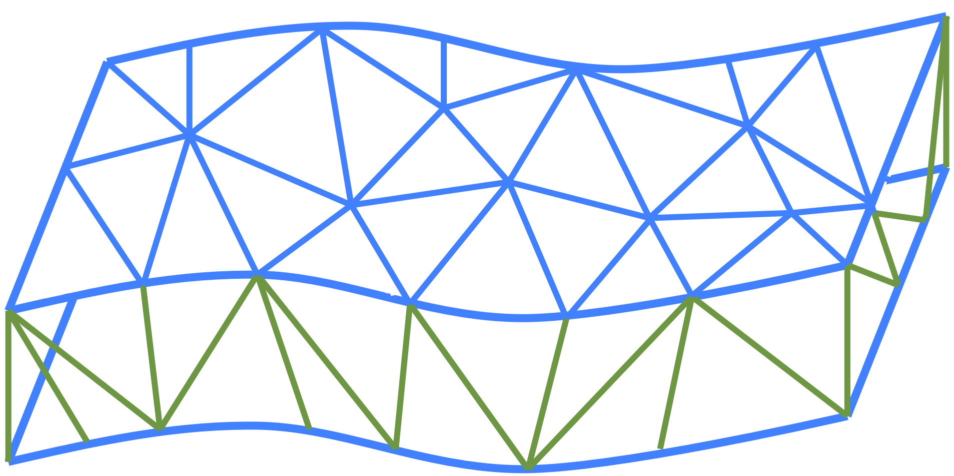
Horava-Lifshitz gravity (HL):

anisotropy



Causal Dynamical Triangulations (CDT):

causality



0. INTRODUCTION

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OUTLINE

1. 2D CDT

→ (1) CDT quantum Hamiltonian (known)

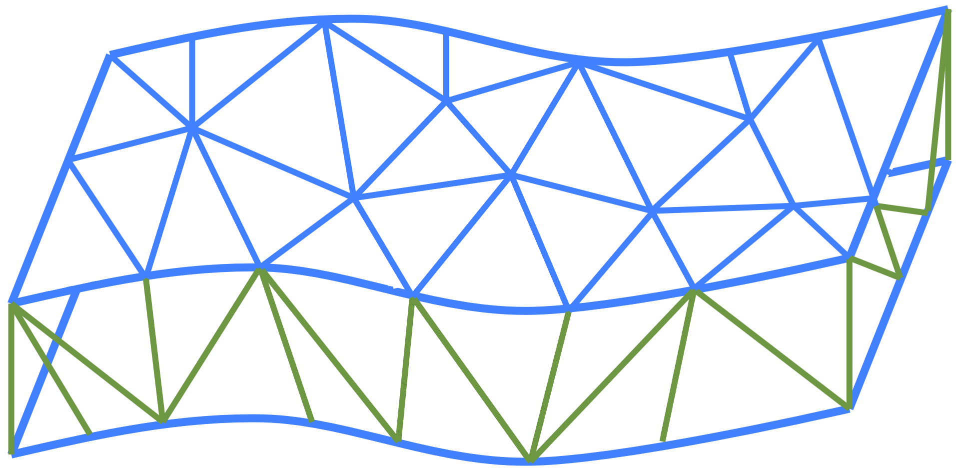
2. 2D HL

→ (2) HL quantum Hamiltonian (our work)

3. SUMMARY

→ (1) and (2) are the same

2D Causal Dynamical Triangulation

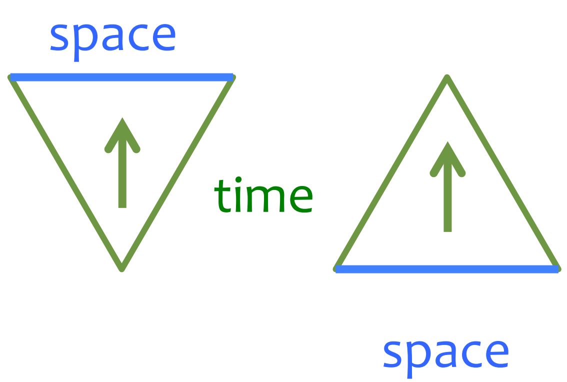


1. 2D CDT

CDT

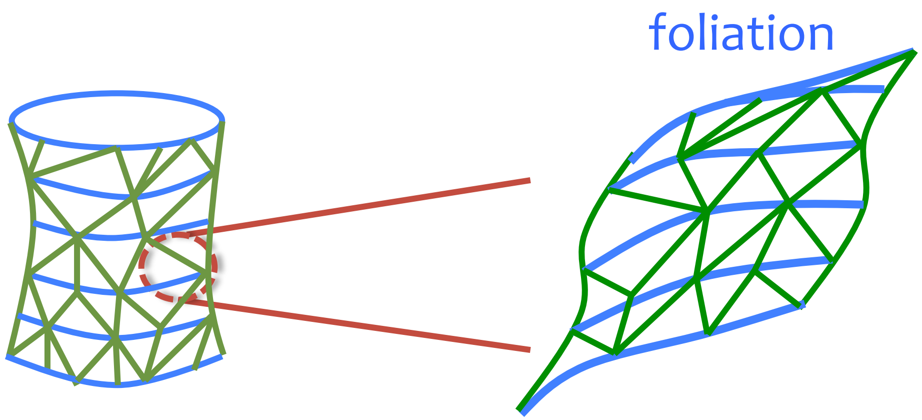
→ A tool to conduct the gravitational path-integral non-perturbatively.

Triangle lattice (UV cutoff):

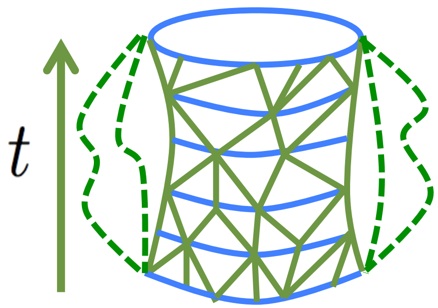


fixed lattice spacing ε

Discretised geometry:



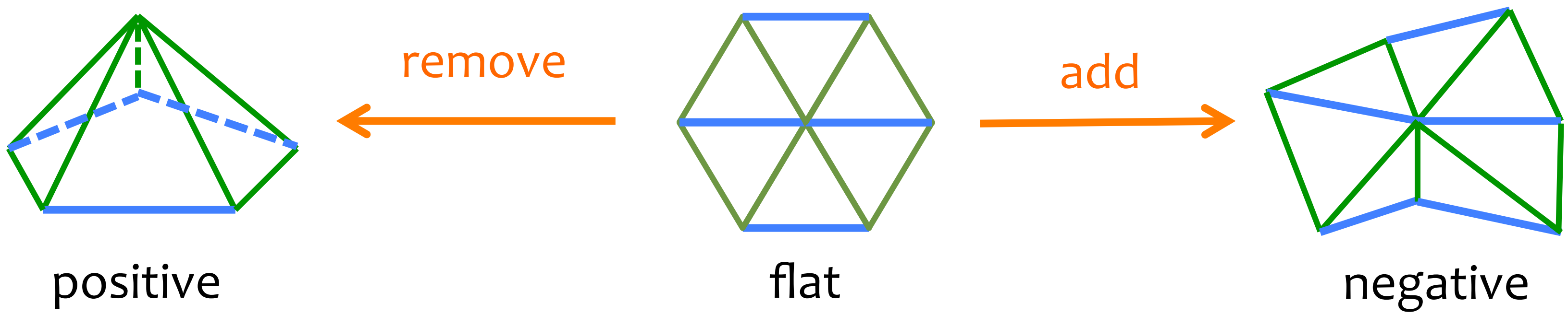
Triangulations := how to divide geometry by triangles



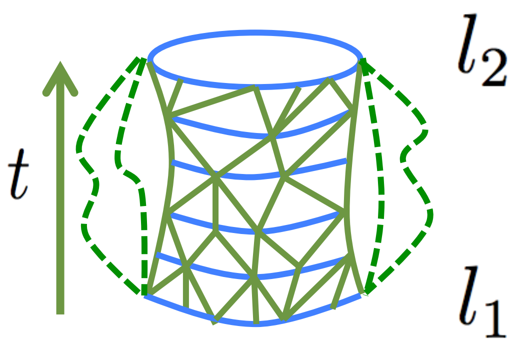
CDT path-integral := sum over triangulations

1. 2D CDT

Change triangulations by adding or removing triangles



CDT path-integral:



propagator

$$G(l_2, l_1; t) = \sum_{T(l_1,l_2)} e^{-\lambda n(T)} \equiv \sum_{T(l_1,l_2)} g^{n(T)}$$

$$= \sum_{n} w_{l_1,l_2,n} g^n$$

λ: cosmological constant   # () = n #(triangulation) = $w_{l_1,l_2,n}$

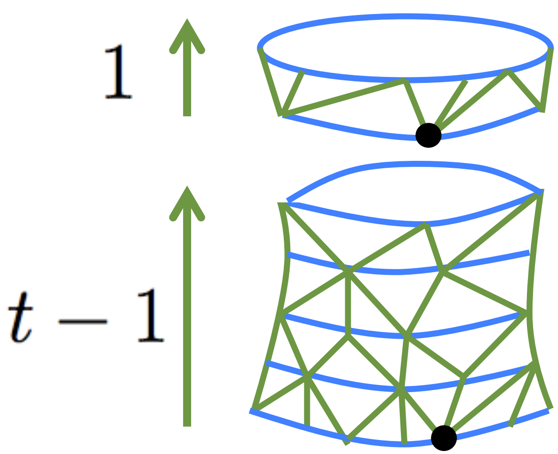
1. 2D CDT

Tool 1: Generating fun for boundary lengths

$$G(y,x;t) = \sum_{l_1,l_2} y^{l_2} x^{l_1} G(l_2,l_1;t)$$
$$= \sum_{l_1,l_2,n} w_{l_1,l_2,n}\ x^{l_1} y^{l_2} g^n$$

g → c.c.
x → c.c. for l_1
y → c.c. for l_2

Tool 2: Composition rule



for propagator:

$$G(l_2,l_1;t) = \sum_{l} G(l_2,l,1)G(l,l_1;t-1)$$

for generating fun:

$$G(y,x;t) = \oint \frac{dz}{2\pi i z} G(y,z^{-1};1)G(z,x;t-1)$$

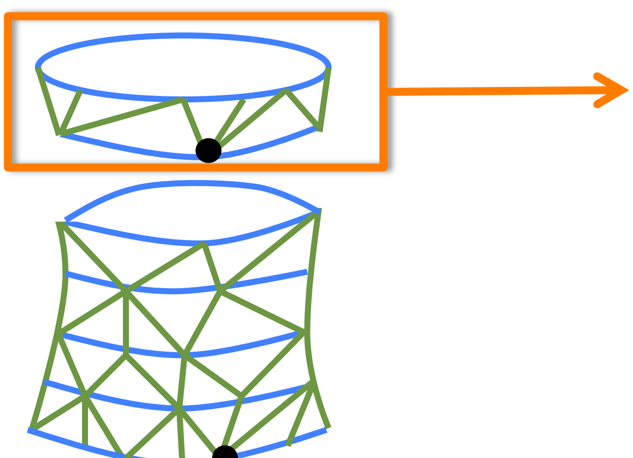
1. 2D CDT

Composition rule:

$$G(y,x;t) = \oint \frac{dz}{2\pi i z} \boxed{G(y,z^{-1};1)} G(z,x;t-1)$$

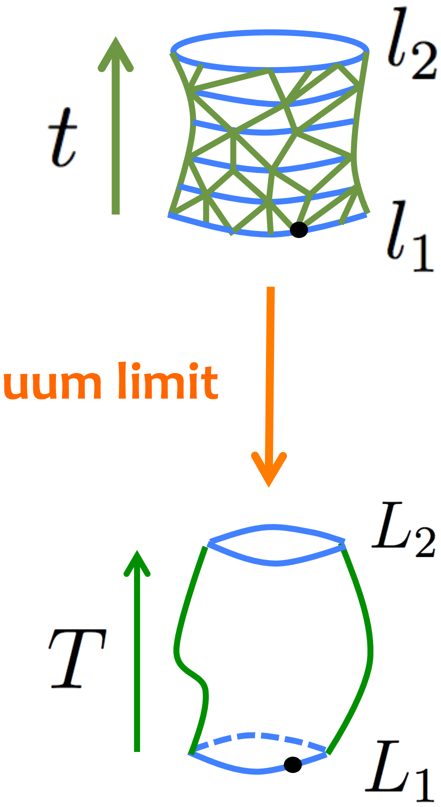
$$= \frac{gx}{1-gx} G\left(\frac{g}{1-gx}, y; t-1\right)$$

→ one-time-step recursion



$$G(y,x;1) = \frac{g^2 xy}{(1-gx)(1-gx-gy)}$$

1. 2D CDT



Continuum limit:

(1) Fine-tuning couplings:

$$(g, x, y) \rightarrow (g_c, x_c, y_c) = (1/2, 1, 1)$$

(△) → infinity

(2) lattice spacing → zero:

$$\varepsilon \rightarrow 0$$

under the fixed volume ($V = \varepsilon^2 n$)

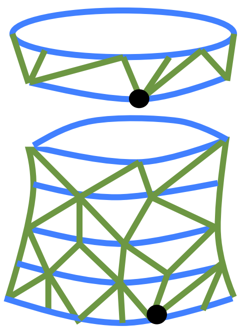
$$L_1 := \varepsilon l_1 \quad L_2 := \varepsilon l_2, \quad T := \varepsilon t$$

→ Physical variables

c.c. for bulk	c.c. for L_1	c.c. for L_2
$g = g_c e^{-\varepsilon^2 \boxed{\Lambda}}$,	$x = x_c e^{-\varepsilon \boxed{X}}$,	$y = y_c e^{-\varepsilon \boxed{Y}}$

→ Physical c.c.

1. 2D CDT



One-time-step recursion:

$$G(y,x;t) = \frac{gx}{1-gx} G\left(\frac{g}{1-gx}, y; t-1\right)$$

continuum limit

$$\frac{\partial}{\partial T} G(Y,X;T) = -\hat{H}(X) G(Y,X;T)$$

where

$$\hat{H}(X) = \frac{\partial}{\partial X}(X^2 - \Lambda)$$

$$G(Y,X;T) = \lim_{\varepsilon \to 0} \varepsilon \, G(y,x;t)$$

Inverse Laplace tr.

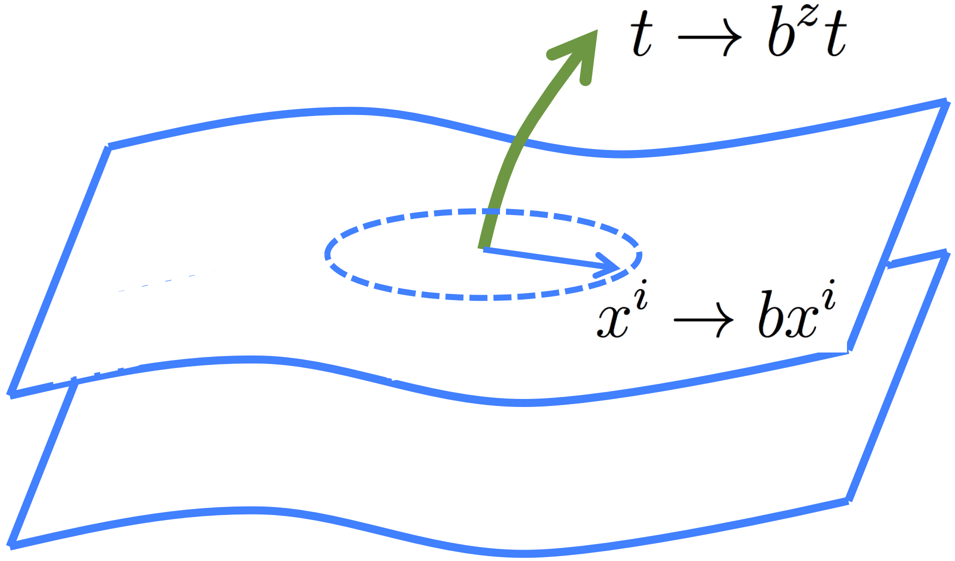
$$G(L_2, L_1; T) = \int_{-\infty}^{i\infty} dY \int_{-i\infty}^{i\infty} dX \; e^{YL_2} e^{XL_1} G(Y,X;T)$$

$$\frac{\partial}{\partial T} G(L_2, L_1; T) = -\boxed{\hat{H}(L_1)} G(L_2, L_1; T)$$

CDT Hamiltonian

$$\boxed{\hat{H}(L_1) = -L_1 \frac{\partial^2}{\partial L_1^2} + \Lambda L_1}$$

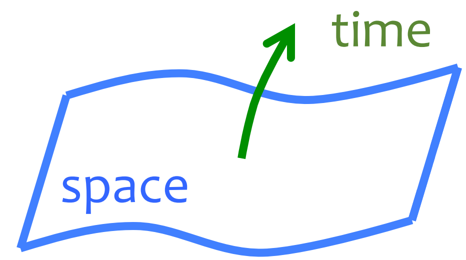
2D Horava-Lifshitz quantum gravity



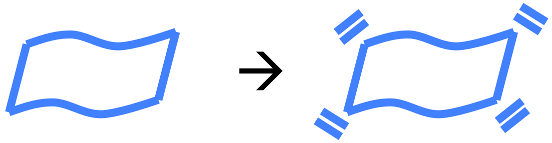
2. 2D HL

4d Einstein (covariant) gravity:

$$S_{\mathrm{ADM}} = \frac{1}{\kappa}\int dt d^3x \sqrt{h} N (K_{ij}K^{ij} - K^2 + R - 2\Lambda)$$



ADM

[symmetry]  changeable

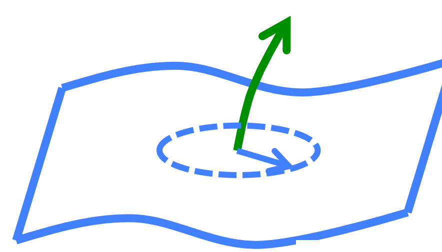
[Newton's constant] $[\kappa] = -2$

4d HL (anisotropic) gravity:

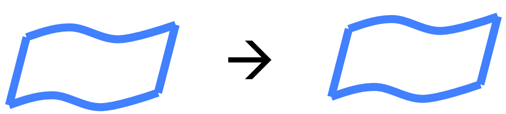
higher spatial curvature

$$S_{\mathrm{HL}} = \frac{1}{\kappa}\int dt d^3x \sqrt{h} N (K_{ij}K^{ij} - \lambda K^2 - \underline{\mathcal{V}[h_{ij}]})$$

$t \to b^z t$



$x^i \to b x^i$

[symmetry]  fixed

[Newton's constant] $[\kappa] = z - 3$ renormalizable

2. 2D HL

2D **projectable** HL:

$$S_{\rm HL} = \int dt\, dx\, N\gamma \left[(1-\lambda)K^2 - 2\Lambda\right]$$

projectable lapse

$$N = N(t)$$

where $\gamma := \sqrt{h}$ & $K = \frac{1}{N}\left(\frac{1}{\gamma}\partial_0\gamma - \frac{1}{\gamma^2}\partial_1 N_1 + \frac{N_1}{\gamma^3}\partial_1\gamma\right)$

Legendre tr.

$$\pi^\gamma = \frac{\partial\mathcal{L}}{\partial(\partial_0\gamma)} = 2(1-\lambda)K$$

$$\{\gamma(x,t), \pi^\gamma(y,t)\} = \delta(x-y)$$

$$H = \int dx\, [N\mathcal{H} + N_1\mathcal{H}^1]$$

"Hamiltonian constr."

$$\mathcal{H} = \gamma\frac{(\pi^\gamma)^2}{4(1-\lambda)} + 2\Lambda\gamma,$$

momentum constr.

$$\mathcal{H}^1 = -\frac{\partial_1\pi^\gamma}{\gamma}$$

2. 2D HL

Solve momentum constraint

$$H = \int dx \, [N\mathcal{H} + N_1 \mathcal{H}^1]$$

$$\mathcal{H}^1 = 0 \quad \text{i.e.} \quad \pi^\gamma(x,t) = \pi^\gamma(t)$$

Fix spatial Diff

$$H = N(t) \left(L(t) \frac{(\pi^\gamma(t))^2}{4(1-\lambda)} + 2\Lambda L(t) \right), \qquad L(t) := \int dx \, \gamma(x,t)$$

1d system

Solve Hamiltonian constraint

$$(\pi^\gamma)^2 = 8(\lambda - 1)\Lambda \quad \text{for} \quad (\lambda - 1)\Lambda > 0$$

no dynamics

or

$$L(t) = 0 \qquad \text{for} \quad \Lambda > 0 \quad \lambda < 1$$

If one wants to have a non-trivial classical dynamics,
one can add the constr. that the space-time volume is a constant.

2. 2D HL

Back to the Lagrangian formalism:

$$S = \int dt \left(\frac{\dot{L}^2}{4N(t)L(t)} - \tilde{\Lambda} N(t) L(t) \right), \qquad \tilde{\Lambda} = \frac{\Lambda}{2(1-\lambda)} \qquad \Lambda > 0 \quad \lambda < 1$$

Quantum amplitude (after a rotation to Euclidean signature):

$$G(L_2, L_1; T) = \int \frac{\mathcal{D}N(t)}{\mathrm{Diff}[0,1]} \int \mathcal{D}L(t)\, \mathrm{e}^{-S_E[N(t),L(t)]}$$

where $$S_E = \int dt \left(\frac{\dot{L}^2}{4N(t)L(t)} + \tilde{\Lambda} N(t) L(t) \right) \qquad \int_0^1 dt\, N(t) = T$$

[Tactics] compute infinitesimal propagation

read off!! ← compute

$$\langle L_2 | e^{-\varepsilon \hat{H}} | \psi \rangle = \int [dL_1] \langle L_2 | e^{-\varepsilon \hat{H}} | L_1 \rangle \langle L_1 | \psi \rangle$$

Operator Path-integral

completeness cond.

$$\int [dL_1] | L_1 \rangle \langle L_1 | = 1$$

2. 2D HL

Infinitesimal propagation:

$$\int [dL_1] \langle L_2 | e^{-\varepsilon \hat{H}} | L_1 \rangle \langle L_1 | \psi \rangle = \langle L_2 | e^{-\varepsilon \hat{H}} | \psi \rangle$$

$$\int_0^\infty \left[\frac{(L_1)^a dL_1}{A} \right] G(L_2, L_1; \varepsilon) \psi(L_1) = \psi(L_2) - \varepsilon (\hat{H}\psi)(L_2) + \mathcal{O}(\varepsilon^{3/2}),$$

[dL1]

$$(\text{lhs}) = \int_0^\infty \left[\frac{(L_1)^a dL_1}{A} \right] \exp\left(-\frac{(L_2 - L_1)^2}{4\varepsilon L_2} - \varepsilon \tilde{\Lambda} L_2 \right) \psi(L_1)$$

$$= \psi(L_2) - \varepsilon \left(-L_2 \frac{\partial^2}{\partial L_2^2} - 2a \frac{\partial}{\partial L_2} - \frac{a(a-1)}{L_2} + \tilde{\Lambda} L_2 \right) \psi(L_2) + \cdots$$

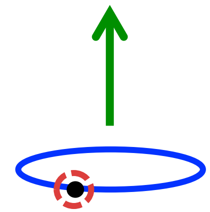
a = 1 or 0
A: normalisation

2. 2D HL

Quantum Hamiltonian for HL

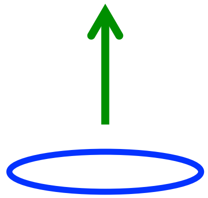
$$\hat{H} = -L\frac{\partial^2}{\partial L^2} - 2a\frac{\partial}{\partial L} - \frac{a(a-1)}{L} + \tilde{\Lambda}L \qquad \tilde{\Lambda} = \frac{\Lambda}{2(1-\lambda)}$$

$$a = 0 \leftrightarrow \underline{dL} \leftrightarrow \hat{H} = -L\frac{\partial^2}{\partial L^2} + \tilde{\Lambda}L$$



→ CDT Hamiltonian for **a marked** loop

$$a = 1 \leftrightarrow \underline{LdL} \leftrightarrow \hat{H} = -L\frac{\partial^2}{\partial L^2} - 2\frac{\partial}{\partial L} + \tilde{\Lambda}L$$



→ CDT Hamiltonian for an **unmarked** loop

$$\langle L_2|e^{-\varepsilon\hat{H}}|\psi\rangle = \int \underline{[dL_1]}\langle L_2|e^{-\varepsilon\hat{H}}|L_1\rangle\langle L_1|\psi\rangle$$

3. SUMMARY

2D CDT turns out to be the 2D projectable Horava-Lifshitz quantum gravity,

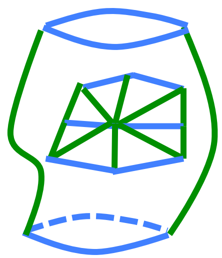
$$S_{\mathrm{HL}} = \int dt\, dx\, N\gamma \left[(1-\lambda)K^2 - 2\Lambda\right]$$

where

$$N = N(t)$$
$$\Lambda > 0 \quad \lambda < 1$$

DT

CDT



continuum

Liouville gravity

Infinite outgrowth of baby universes

Projectable Horava-Lifshitz gravity

No baby universe

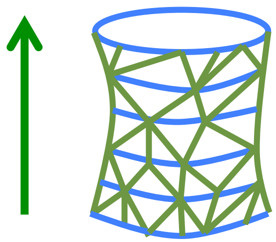
FAQ

Q1. Is the diffeomorphism broken in a lattice approach?

No. Because a lattice gravity is quantum gravity without coordinates.

Q2. Is CDT a background-independent formulation?

It seems No. Because there is a global time direction.



Q3. So, then what's the status of CDT?

Probably, an effective theory arising from integrating out baby universes.