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Integrability of BPS equations in ABJM theory

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The worldvolume theory of M2-branes was not known until recently.

The theory should be a 3D CFT with no adjustable coupling, but people could not construct such a CFT with sufficient amount of supersymmetries.

ABJM theory (Aharony-Bergman-Jafferis-Maldacena '08)

$\mathcal{N} = 6$ $\mathrm{U}(N) \times \mathrm{U}(N)$ super Chern-Simons theory with two bifundamental hypermultiplets

Chern-Simons levels are chosen to be $k_1 = -k_2 = k$

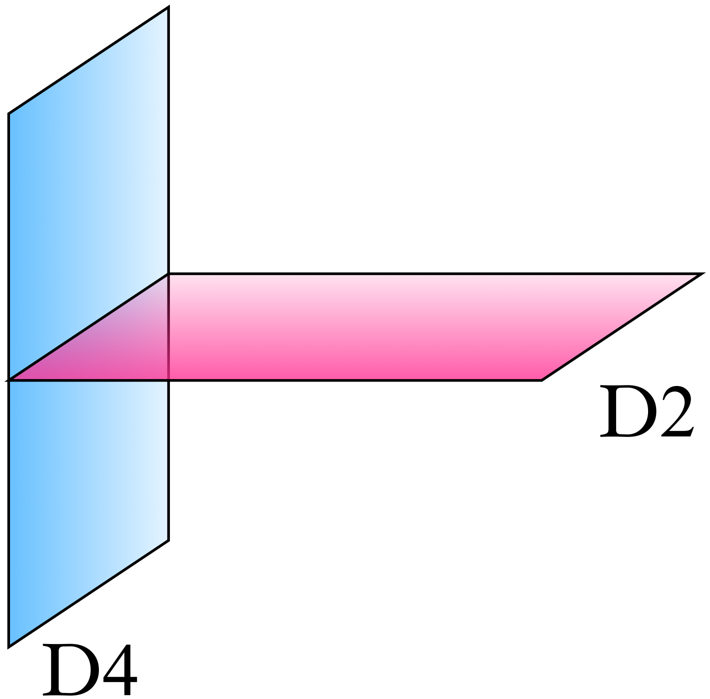
The theory describes the low energy effective theory of the worldvolume theory of N M2-branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity.

$k = 1$: M2-branes in flat space?

The worldvolume theory of M5-branes is still mysterious.
(There may not be a covariant Lagrangian description.)

Two descriptions of BPS D2-D4 bound states

(Diaconescu '96)



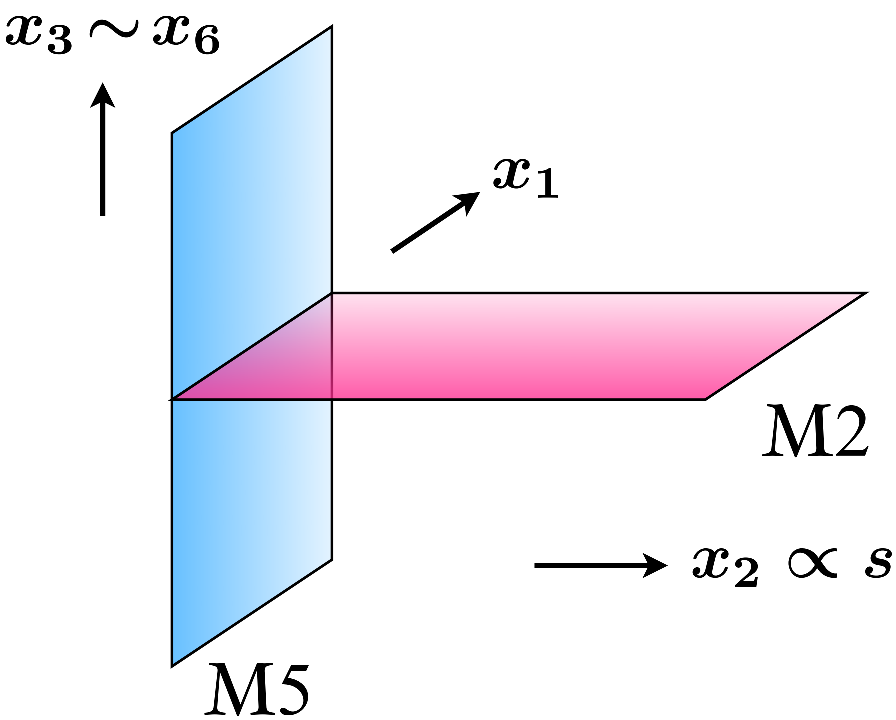
In the worldvolume theory of D2-branes, the bound states are described as solutions of the Nahm equations (Nahm data).

Nahm transformation

In the worldvolume theory of D4-branes, the bound states are described as monopole solutions.

It would be very interesting if one could promote this picture to M-theory.

Two descriptions of BPS M2-M5 bound states



In the worldvolume theory of M2-branes (ABJM theory), the bound states are described as solutions of the BPS equations.

analog of Nahm transformation?

In the worldvolume theory of M5-branes, the bound states should be given as some solutions.

BPS equations in the ABJM theory

(Terashima '08)

(Gomis, Rodriguez-Gomez, Van Ramsdonk and Verlinde '08)

$$\dot{Y}^a = Y^b Y^{b\dagger} Y^a - Y^a Y^{b\dagger} Y^b$$

$Y^a(s)\ (a = 1, 2):\ N \times N$ complex matrices

s : a real coordinate $\qquad \dot{Y}^a := \frac{d}{ds} Y^a$

Automorphism

$$Y^a \to Y'^a = e^{i\varphi} \Lambda^a{}_b U Y^b V^\dagger$$

$$U, V \in \mathrm{SU}(N), \quad (\Lambda^a{}_b) \in \mathrm{SU}(2), \quad e^{i\varphi} \in \mathrm{U}(1)$$

Y'^a again satisfy the above equations

We argue that the BPS equations are classically integrable.

The BPS equations admit a Lax representation

$$\dot{\boldsymbol{A}} = [\boldsymbol{A}, \boldsymbol{B}]$$

$$\boldsymbol{A}(s;\lambda) = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{Y}^1 + \lambda \boldsymbol{Y}^2 \\ \boldsymbol{Y}^{1\dagger} - \lambda^{-1}\boldsymbol{Y}^{2\dagger} & \boldsymbol{O} \end{pmatrix}$$

$$\boldsymbol{B}(s;\lambda) = \begin{pmatrix} \lambda^{-1}\boldsymbol{Y}^1\boldsymbol{Y}^{2\dagger} + \lambda \boldsymbol{Y}^2\boldsymbol{Y}^{1\dagger} & \boldsymbol{O} \\ \boldsymbol{O} & \lambda \boldsymbol{Y}^{1\dagger}\boldsymbol{Y}^2 + \lambda^{-1}\boldsymbol{Y}^{2\dagger}\boldsymbol{Y}^1 \end{pmatrix}$$

$\lambda \in \mathbb{C}$: the spectral parameter

Involution structures

$$\{A, \Gamma\} = 0, \qquad [B, \Gamma] = 0$$

$$\Gamma := \begin{pmatrix} 1_N & 0 \\ 0 & -1_N \end{pmatrix}$$

$$A^\star = A, \qquad B^\star = -B$$

$$\mathcal{M}^\star(\lambda) := \mathcal{M}(-\bar{\lambda}^{-1})^\dagger$$ "star-involution"

Another useful property

$$B = \lambda \frac{\partial}{\partial \lambda} A^2$$

Auxiliary linear problem

- The Lax equation is regarded as the compatibility condition of the following auxiliary linear problem:

$$\boldsymbol{A}(\boldsymbol{s};\boldsymbol{\lambda})\psi(\boldsymbol{s};\boldsymbol{\lambda}) = \eta(\boldsymbol{\lambda})\psi(\boldsymbol{s};\boldsymbol{\lambda})$$

$$\boldsymbol{B}(\boldsymbol{s};\boldsymbol{\lambda})\psi(\boldsymbol{s};\boldsymbol{\lambda}) = -\dot{\psi}(\boldsymbol{s};\boldsymbol{\lambda})$$

For a system with Lax representation, there are several powerful techniques to restrict the form of $\psi(\boldsymbol{s};\boldsymbol{\lambda})$ and construct a class of general solutions.

There is an even better way to construct solutions: this is done by making use of the relation between the BPS equations and the Nahm equations.

Nahm equations

$$\dot{T}^I = i\epsilon_{IJK} T^J T^K$$

$$T^I \ (I = 1, 2, 3) \ : \ N \times N \text{ hermitian matrices}$$

- Relation between the BPS equations and the Nahm equations

$$T_1^I := (\sigma^I)_{ab} Y^a Y^{b\dagger}, \qquad T_2^I := (\sigma^I)_{ab} Y^{b\dagger} Y^a$$

$$\sigma^I \ (I = 1, 2, 3) : \text{Pauli matrices}$$

If Y^a are solutions to the BPS equations,

both T_1^I and T_2^I satisfy the Nahm equations.

(Nosaka-Terashima '12)

Lax representation for the Nahm equations

(Hitchin '83)

$$\dot{A}_\alpha = [A_\alpha, B_\alpha]$$

$$A_\alpha := T_\alpha^3 + \frac{\lambda}{2}\left(T_\alpha^1 - iT_\alpha^2\right) - \frac{1}{2\lambda}\left(T_\alpha^1 + iT_\alpha^2\right)$$

$$B_\alpha := \frac{\lambda}{2}\left(T_\alpha^1 - iT_\alpha^2\right) + \frac{1}{2\lambda}\left(T_\alpha^1 + iT_\alpha^2\right)$$

The above Lax forms are related to those of the BPS equations in a remarkably simple way:

$$A^2 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

We assume that $\boldsymbol{A}$ has $\boldsymbol{2N}$ linearly independent eigenvectors and express the linear problem for the BPS equations as

$$\boldsymbol{A\Psi} = \boldsymbol{\Psi D}$$

$$\boldsymbol{B\Psi} = -\dot{\boldsymbol{\Psi}}$$

$$\boldsymbol{\Psi} = \frac{1}{\sqrt{2}}\begin{pmatrix} \boldsymbol{\Psi_1} & \boldsymbol{\Psi_1} \\ \boldsymbol{\Psi_2} & -\boldsymbol{\Psi_2} \end{pmatrix} \qquad \boldsymbol{D} = \begin{pmatrix} \boldsymbol{H} & \boldsymbol{O} \\ \boldsymbol{O} & -\boldsymbol{H} \end{pmatrix}$$

$$\boldsymbol{H} = \mathbf{diag}(\boldsymbol{\eta_1, \ldots, \eta_N})$$

Due to the relation between the Lax representations, $\boldsymbol{\Psi_\alpha}$ automatically give solutions to the linear problems for the Nahm equations

$$\boldsymbol{A_\alpha \Psi_\alpha} = \boldsymbol{\Psi_\alpha H^2}$$

$$\boldsymbol{B_\alpha \Psi_\alpha} = -\dot{\boldsymbol{\Psi}}_{\boldsymbol{\alpha}}$$

with a common eigenvalue matrix $\boldsymbol{H^2}$

We can use the relation the other way around:

Prepare a pair of Nahm data T^I_α sharing the same eigenvalues and compute eigenvectors Ψ_α for the linear problem.

Then $\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \Psi_1 & \Psi_1 \\ \Psi_2 & -\Psi_2 \end{pmatrix}$ gives a solution

to the linear problem for the BPS equations

and the operator A for the BPS equations is expressed as

$$A = \begin{pmatrix} O & \Psi_1 H \mathcal{N}_2^{-1} \Psi_2^\star \\ \Psi_2 H \mathcal{N}_1^{-1} \Psi_1^\star & O \end{pmatrix}, \qquad (\mathcal{N}_\alpha := \Psi_\alpha^\star \Psi_\alpha)$$

if the following conditions are satisfied:

$$\frac{\partial^2}{\partial \lambda^2} \left[\Psi_1 H \mathcal{N}_2^{-1} \Psi_2^\star \right] = 0$$

$$H \mathcal{N}_1 = \mathcal{N}_2 H$$

Recall that the operator $\boldsymbol{A}$ for the BPS equations is expressed as

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{Y}^1 + \boldsymbol{\lambda} \boldsymbol{Y}^2 \\ \boldsymbol{Y}^{1\dagger} - \boldsymbol{\lambda}^{-1} \boldsymbol{Y}^{2\dagger} & \boldsymbol{O} \end{pmatrix}$$

The solutions $\boldsymbol{Y}^a(\boldsymbol{s})$ to the BPS equations are thus obtained as

$$\boldsymbol{Y}^1 + \boldsymbol{\lambda} \boldsymbol{Y}^2 = \boldsymbol{\Psi}_1 \boldsymbol{H} \boldsymbol{\mathcal{N}}_2^{-1} \boldsymbol{\Psi}_2^\star$$

- Solutions of the Nahm equations are well studied (Ercolani-Sinha '89)
  (see also the textbook by Manton-Sutcliffe '04)
- It is straightforward to compute the eigenvectors
- The only nontrivial point we have to consider is the condition

$$\frac{\partial^2}{\partial \lambda^2} \left[\boldsymbol{\Psi}_1 \boldsymbol{H} \boldsymbol{\mathcal{N}}_2^{-1} \boldsymbol{\Psi}_2^\star \right] = 0$$

Semi-infinite solutions with $N = 2$

- The Nahm data

$$T_\alpha^1 = \frac{c}{\sinh(x - x_\alpha)} \frac{\sigma^1}{2} + t^1 1_2, \qquad T_\alpha^2 = \frac{c}{\sinh(x - x_\alpha)} \frac{\sigma^2}{2} + t^2 1_2,$$

$$T_\alpha^3 = \frac{c}{\tanh(x - x_\alpha)} \frac{\sigma^3}{2} + t^3 1_2$$

$$x = cs, \qquad c \geq 0$$

σ^I : Pauli matrices

$$x_1 = 0, \qquad x_2 = -l, \qquad l \geq 0$$

$t^I \in \mathbb{R}$

$$\frac{\partial^2}{\partial\lambda^2}\left[\Psi_1 H \mathcal{N}_2^{-1} \Psi_2^\star\right] = 0$$

$$\Rightarrow \quad (t^1)^2 + (t^2)^2 + \frac{(t^3)^2}{\cosh^2 l} = \frac{c^2}{4\sinh^2 l}$$

This is solved as

$$t^1 = \frac{c}{2\sinh l} n_1, \qquad t^2 = \frac{c}{2\sinh l} n_2, \qquad t^3 = \frac{c}{2\tanh l} n_3$$

with

$$(n_1, n_2, n_3) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$Y^1 = \sqrt{\frac{c}{2\sinh l \sinh x \sinh(x+l)}} \begin{pmatrix} \sinh(x+l)\cos\frac{\theta}{2}\, e^{i\phi} & \sinh l \sin\frac{\theta}{2} \\ 0 & \sinh x \cos\frac{\theta}{2}\, e^{i\phi} \end{pmatrix}$$

$$Y^2 = \sqrt{\frac{c}{2\sinh l \sinh x \sinh(x+l)}} \begin{pmatrix} \sinh x \sin\frac{\theta}{2} & 0 \\ \sinh l \cos\frac{\theta}{2}\, e^{i\phi} & \sinh(x+l)\sin\frac{\theta}{2} \end{pmatrix}$$

The most general solution with $N = 2$

- The semi-infinite solution in the last slide can be expressed as

$$Y^1 = \frac{1}{2}\left(f_1 \sin\frac{\theta}{2}\,\sigma^1 + f_2 \sin\frac{\theta}{2}\,i\sigma^2 + f_3 e^{i\phi}\cos\frac{\theta}{2}\,\sigma^3 - f_0 e^{i\phi}\cos\frac{\theta}{2}\,1_2\right)$$
$$Y^2 = \frac{1}{2}\left(f_1 e^{i\phi}\cos\frac{\theta}{2}\,\sigma^1 - f_2 e^{i\phi}\cos\frac{\theta}{2}\,i\sigma^2 - f_3 \sin\frac{\theta}{2}\,\sigma^3 - f_0 \sin\frac{\theta}{2}\,1_2\right)$$

with

$$f_1 = f_2 = \sqrt{\frac{c\sinh l}{2\sinh x \sinh(x+l)}},$$

$$f_3 = \frac{\cosh(x+l/2)}{\cosh(l/2)} f_1, \qquad f_0 = -\frac{\sinh(x+l/2)}{\sinh(l/2)} f_1$$

- Note that $f_i(s)$ are real functions and satisfy

$$\dot{f}_i = f_j f_k f_l$$

where the values of i, j, k, l are taken to be all distinct.

- One can check that matrices of the form

$$Y^1 = \frac{1}{2}\left(f_1 \sin\frac{\theta}{2}\,\sigma^1 + f_2 \sin\frac{\theta}{2}\, i\sigma^2 + f_3 e^{i\phi}\cos\frac{\theta}{2}\,\sigma^3 - f_0 e^{i\phi}\cos\frac{\theta}{2}\,1_2\right)$$
$$Y^2 = \frac{1}{2}\left(f_1 e^{i\phi}\cos\frac{\theta}{2}\,\sigma^1 - f_2 e^{i\phi}\cos\frac{\theta}{2}\, i\sigma^2 - f_3 \sin\frac{\theta}{2}\,\sigma^3 - f_0 \sin\frac{\theta}{2}\,1_2\right)$$

with any real functions $f_i(s)$ satisfying

$$\dot{f}_i = f_j f_k f_l$$

are solutions of the BPS equations.

A sufficiently general solution is given by

$$f_i = \frac{\vartheta_{i+1}(u)}{\vartheta_{i+1}(u_*)}\sqrt{\frac{\pi}{2\omega_1}\,\frac{\vartheta_1(u_*)\vartheta_2(u_*)\vartheta_3(u_*)\vartheta_4(u_*)}{\vartheta_1(u_*+u)\vartheta_1(u_*-u)}}$$

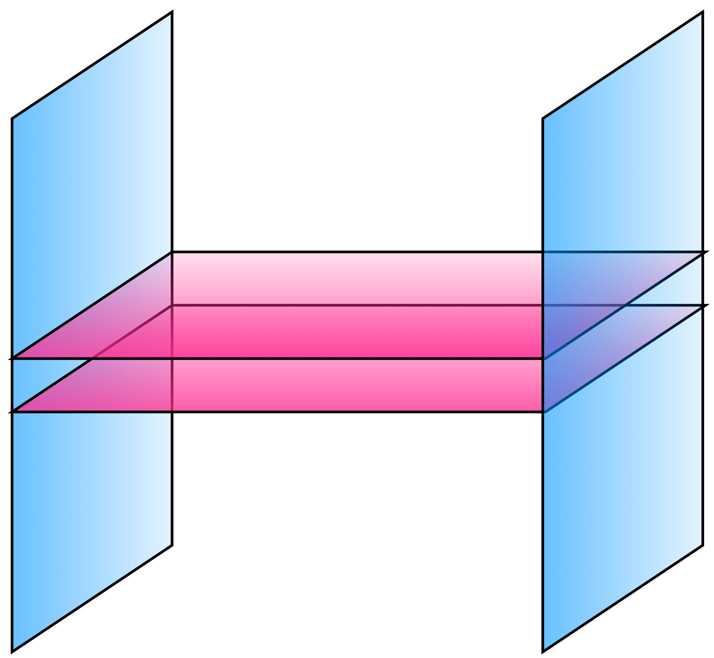
$$\vartheta_{i+1}(u) := \vartheta_{i+1}(u,\tau) \qquad (i = 0, 1, 2, 3)$$

$$u = \frac{s - s_0}{2\omega_1}, \qquad s_0 \in \mathbb{R}, \qquad 0 < u_* < \frac{1}{2}, \qquad \omega_1 \in \mathbb{R}_{>0}, \qquad \tau \in i\mathbb{R}_{>0}$$

The solution is defined over the region

$$-u_* < u < u_*$$

and f_i diverge at each boundary of the region.



Reduction in connection with the periodic Toda chain

- Let us make an ansatz of $Y^a(s)$ as follows:

$$(Y^1)_{mn} = g_m(s)\delta_{m,n}, \qquad (Y^2)_{mn} = h_n(s)\delta_{m,n+1}$$

$$(m, n = 1, \ldots, N)$$

The BPS equations become

$$\dot{g}_m = \left(h_{m-1}^2 - h_m^2\right) g_m, \qquad \dot{h}_m = \left(g_{m+1}^2 - g_m^2\right) h_m.$$

If we introduce

$$a_m := g_{m+1}h_m, \qquad \tilde{a}_m := g_m h_m,$$
$$b_m := g_m^2 - h_m^2, \qquad \tilde{b}_m := g_m^2 - h_{m-1}^2,$$

a_m, b_m satisfy (and the same is true for $\tilde{a}_m, \tilde{b}_m$)

$$\dot{a}_m = a_m(b_{m+1} - b_m), \qquad \dot{b}_m = -2(a_m^2 - a_{m-1}^2).$$

These are the equations for the periodic Toda chain!

Summary

- We have shown that the BPS equations in the ABJM theory is classically integrable.

- The integrable structure of the BPS equations is closely related to that of the Nahm equations.

- By making use of this fact, we have formulated an efficient way of constructing solutions of the BPS equations.

- By way of illustration we have constructed the most general solution describing two M2-branes.

Outlook

- What is the structure of the moduli space of the solutions?
- Are there any other integrable BPS equations?
- What is the analog of the Nahm construction?
- What is the role of integrability in the theory of M5-branes and in the whole M-theory?