What is the gravitational theory that string theory predicts?

Double Field Theory as the $\mathbf{O}(D,D)$ completion of GR

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Einstein Double Field Equations [1]

Core idea: string theory predicts its own gravity rather than GR

In General Relativity the metric $g_{\mu\nu}$ is the only geometric and gravitational field, whereas in string theory the closed-string massless sector comprises a two-form potential $B_{\mu\nu}$ and the string dilaton ϕ in addition to the metric $g_{\mu\nu}$. Furthermore, these three fields transform into each other under T-duality. This hints at a natural augmentation of GR: upon treating the whole closed string massless sector as stringy graviton fields, Double Field Theory [3, 4] may evolve into 'Stringy Gravity'. Equipped with an $\mathbf{O}(D,D)$ covariant differential geometry beyond Riemann [5], we spell out the definitions of the stringy Einstein curvature tensor and the stringy Energy-Momentum tensor. Equating them, all the equations of motion of the closed string massless sector are unified into a single expression [1],

$$G_{AB}=\frac{8\pi G}{c^4}T_{AB}$$

which we dub the Einstein Double Field Equations.

• Built-in symmetries & Notation:

– $\mathbf{O}(D,D)$ T-duality

– DFT diffeomorphisms (ordinary diffeomorphisms plus B-field gauge symmetry)

– Twofold local Lorentz symmetries, $\mathbf{Spin}(1,D{-}1)\times\mathbf{Spin}(D{-}1,1)$

$\Rightarrow$ Two locally inertial frames exist separately for the left and the right modes.

Index	Representation	Metric (raising/lowering indices)
$A,B,\cdots,M,N,\cdots$	$\mathbf{O}(D,D)$ vector	$\mathcal{J}_{AB}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$
$p,q,\cdots$	$\mathbf{Spin}(1,D{-}1)$ vector	$\eta_{pq}=\mathrm{diag}(-++\cdots+)$
$\alpha,\beta,\cdots$	$\mathbf{Spin}(1,D{-}1)$ spinor	$C_{\alpha\beta},\quad(\gamma^p)^T=C\gamma^pC^{-1}$
$\bar{p},\bar{q},\cdots$	$\mathbf{Spin}(D{-}1,1)$ vector	$\bar{\eta}_{\bar{p}\bar{q}}=\mathrm{diag}(+--\cdots-)$
$\bar{\alpha},\bar{\beta},\cdots$	$\mathbf{Spin}(D{-}1,1)$ spinor	$\bar{C}_{\bar{\alpha}\bar{\beta}},\quad(\bar{\gamma}^{\bar{p}})^T=\bar{C}\bar{\gamma}^{\bar{p}}\bar{C}^{-1}$

The $\mathbf{O}(D,D)$ metric $\mathcal{J}_{AB}$ divides doubled coordinates into two: $x^A=(\tilde{x}_\mu,x^\nu),\partial_A=(\tilde{\partial}^\mu,\partial_\nu)$.

• Doubled-yet-gauged spacetime:

The doubled coordinates are 'gauged' through a certain equivalence relation, $x^A\sim x^A+\Delta^A$, such that each equivalence class, or gauge orbit in $\mathbb{R}^{D+D}$, corresponds to a single physical point in $\mathbb{R}^D$ [6]. This implies a section condition, $\partial_A\partial^A=0$, which can be conveniently solved by setting $\tilde{\partial}^\mu\equiv 0$.

• Geometric notation for DFT or Stringy gravity

Integral measure	e^{-2d} (*weight one scalar density*)
Generalized metric	$\mathcal{H}_{MN}=\mathcal{H}_{NM}\,,\qquad \mathcal{H}_K{}^L\mathcal{H}_M{}^N\mathcal{J}_{LN}=\mathcal{J}_{KM}\,,$
Projectors	$P_{AB}=P_{BA}=\frac{1}{2}(\mathcal{J}_{AB}+\mathcal{H}_{AB})\,,\qquad \bar{P}_{AB}=\bar{P}_{BA}=\frac{1}{2}(\mathcal{J}_{AB}-\mathcal{H}_{AB})$ $P_A{}^BP_B{}^C=P_A{}^C\,,\qquad \bar{P}_A{}^B\bar{P}_B{}^C=\bar{P}_A{}^C\,,\qquad P_A{}^B\bar{P}_B{}^C=0$
Christoffel symbols	$\Gamma_{CAB}=2\left(P\partial_CP\bar{P}\right)_{[AB]}+2\left(\bar{P}_{[A}{}^D\bar{P}_{B]}{}^E-P_{[A}{}^DP_{B]}{}^E\right)\partial_DP_{EC}$ $-4\left(\frac{1}{P_M{}^M-1}P_{C[A}P_{B]}{}^D+\frac{1}{\bar{P}_M{}^M-1}\bar{P}_{C[A}\bar{P}_{B]}{}^D\right)\left(\partial_Dd+(P\partial^EP\bar{P})_{[ED]}\right)$
Covariant derivatives	$P_A{}^C\bar{P}_B{}^D\nabla_CV_D,\qquad \bar{P}_A{}^CP_B{}^D\nabla_CV_D,\qquad P^{AB}\nabla_AV_B,\qquad \bar{P}^{AB}\nabla_AV_B$
Semi-covariant derivative	$\nabla_CV_D=\partial_CV_D-\omega_V\Gamma^E{}_{EC}V_D+\Gamma_{CD}{}^EV_E$
Compatibility	$\nabla_CP_{AB}=\nabla_C\bar{P}_{AB}=\nabla_C\mathcal{J}_{AB}=0,\qquad \nabla_Cd=-\frac{1}{2}e^{2d}\nabla_C\left(e^{-2d}\right)=0$
Scalar curvature	$S_{(0)}=\mathcal{H}^{AB}S_{AB}$
Ricci curvature	$(PS\bar{P})_{AB}=P_A{}^C\bar{P}_B{}^DS_{CD}$
Einstein curvature	$G_{AB}=4P_{[A}{}^C\bar{P}_{B]}{}^DS_{CD}-\frac{1}{2}\mathcal{J}_{AB}S_{(0)}$
Semi-covariant curvature	$S_{AB}=2\partial_A\partial_Bd-e^{2d}\partial_C\left(e^{-2d}\Gamma_{(AB)}{}^C\right)+\frac{1}{2}\Gamma_{ACD}\Gamma_B{}^{CD}-\frac{1}{2}\Gamma_{CDA}\Gamma^{CD}{}_B$
Variational property	$\delta S_{AB}=\nabla_{[A}\delta\Gamma_{C]B}{}^C+\nabla_{[B}\delta\Gamma_{C]A}{}^C$
Energy-Momentum tensor	$T^{AB}=e^{2d}\left(8\bar{P}^{[A}{}_CP^{B]}{}_D\frac{\delta\mathcal{L}_{\mathrm{matter}}}{\delta\mathcal{H}_{CD}}-\frac{1}{2}\mathcal{J}^{AB}\frac{\delta\mathcal{L}_{\mathrm{matter}}}{\delta d}\right)$
Conservation	$\nabla_AG^{AB}=0$ (*off-shell*), $\qquad\nabla_AT^{AB}=0$ (*on-shell*)

• The most general form of the DFT-metric is classified by two non-negative integers $(n,\bar{n})$ [7]

$$\mathcal{H}_{MN}=\begin{pmatrix}H^{\mu\nu} & -H^{\mu\sigma}B_{\sigma\lambda}+Y_i^\mu X_\lambda^i-\bar{Y}_{\bar{\imath}}^\mu\bar{X}_\lambda^{\bar{\imath}}\\ B_{\kappa\rho}H^{\rho\nu}+X_\kappa^iY_i^\nu-\bar{X}_\kappa^{\bar{\imath}}\bar{Y}_{\bar{\imath}}^\nu & K_{\kappa\lambda}-B_{\kappa\rho}H^{\rho\sigma}B_{\sigma\lambda}+2X^i_{(\kappa}B_{\lambda)\rho}Y_i^\rho-2\bar{X}^{\bar{\imath}}_{(\kappa}B_{\lambda)\rho}\bar{Y}_{\bar{\imath}}^\rho\end{pmatrix},$$

where $1\leq i\leq n$, $1\leq\bar{\imath},i\leq\bar{n}$ and

$$H^{\mu\nu}X_\nu^i=0\,,\quad H^{\mu\nu}\bar{X}_\nu^{\bar{\imath}}=0\,,\quad K_{\mu\nu}Y_i^\nu=0\,,\quad K_{\mu\nu}\bar{Y}_{\bar{\imath}}^\nu=0\,,\quad H^{\mu\rho}K_{\rho\nu}+Y_i^\mu X_\nu^i+\bar{Y}_{\bar{\imath}}^\mu\bar{X}_\nu^{\bar{\imath}}=\delta^\mu{}_\nu\,.$$

Strings become chiral and anti-chiral over n and $\bar{n}$ directions: $X^i_\mu\partial_+x^\mu=0$, $\bar{X}^{\bar{\imath}}_\mu\partial_-x^\mu=0$.

Restricting to the $(0,0)$ Riemannian background, the Einstein Double Field Equations reduce to

$$\begin{aligned}R_{\mu\nu}+2\nabla_\mu(\partial_\nu\phi)-\tfrac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma}&=\frac{8\pi G}{c^4}K_{(\mu\nu)}\,,\\ \nabla^\rho\left(e^{-2\phi}H_{\rho\mu\nu}\right)&=\frac{16\pi G}{c^4}e^{-2\phi}K_{[\mu\nu]}\,,\\ R+4\Box\phi-4\partial_\mu\phi\partial^\mu\phi-\tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu}&=\frac{8\pi G}{c^4}T_{(0)}\,,\end{aligned}$$

which imply the conservation law, $\nabla_AT^{AB}=0$, given explicitly by

$$\nabla^\mu K_{(\mu\nu)}-2\partial^\mu\phi\,K_{(\mu\nu)}+\tfrac{1}{2}H_\nu{}^{\lambda\mu}K_{[\lambda\mu]}-\tfrac{1}{2}\partial_\nu T_{(0)}=0\,,\qquad \nabla^\mu\left(e^{-2\phi}K_{[\mu\nu]}\right)=0\,.$$

The Einstein Double Field Equations also govern the dynamics of other non-Riemannian cases, $(n,\bar{n})\neq(0,0)$, where the Riemannian metric, $g_{\mu\nu}$, cannot be defined [8]. Restricted to the $D=4$ $(0,0)$ Riemannian case, one may analyze the most general spherical regular solution [1] and their cosmological applications [9].

Stringy Newton Graivty with $H-$flux [2]

Weak field approximation of EDFEs

Linearizing the metric $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$, around a flat Minkowskian background with trivial H-flux and dilaton ϕ using the gauge $\partial_\rho h^\rho{}_\mu-\frac{1}{2}\partial_\mu h^\rho{}_\rho+2\partial_\mu\phi=0$ and as well as the scale assumption

$$h_{\mu\nu}\sim K_{(\mu\nu)}\sim\phi\sim T_{(0)}\sim\left(H_{\lambda\mu\nu}\right)^2\sim\left(K_{[\mu\nu]}\right)^2,$$

we obtain the following linearized EDFEs

$$\begin{aligned}\partial^\rho H_{\rho\mu\nu}&=\partial_\rho\partial^\rho B_{\mu\nu}=\tfrac{16\pi G}{c^4}K_{[\mu\nu]}\,,\\ \partial_\rho\partial^\rho h_{\mu\nu}+\tfrac{1}{2}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma}&=-\tfrac{16\pi G}{c^4}K_{(\mu\nu)}\,,\\ \partial_\rho\partial_\sigma h^{\rho\sigma}+\tfrac{1}{12}H_{\rho\sigma\tau}H^{\rho\sigma\tau}&=-\tfrac{8\pi G}{c^4}T_{(0)}\,.\end{aligned}$$

These imply the following linearized conservation equations,

$$\partial^\rho K_{[\rho\mu]}=0\,,\qquad\partial^\rho K_{(\rho\mu)}+\tfrac{1}{2}H_\mu{}^{\rho\sigma}K_{[\rho\sigma]}-\tfrac{1}{2}\partial_\mu T_{(0)}=0\,,$$

and the linearized geodesic equation has the form,

$$\ddot{x}^\lambda+\tfrac{1}{2}\left(\partial_\mu h^\lambda{}_\nu+\partial_\nu h^\lambda{}_\mu-\partial^\lambda h_{\mu\nu}\right)\dot{x}^\mu\dot{x}^\nu=0\,.$$

Non-relativistic limit: String theory augmentation of Newton Gravity

In taking the non-relativistic limit, we focus on the Newton potential which is the only quantity directly relevant to the particle dynamics,

$$\Phi:=-\tfrac{1}{2}c^2h_{00}\,,\qquad\ddot{\mathbf{x}}=-\nabla\Phi\,.$$

We then identify all the quantities which can affect the Newton potential: the mass density ρ, the stringy current density $\mathbf{K}$, and B-field/H-flux vectors $\mathbf{B},\mathbf{H}$, as follows

$$\rho:=2c^2K_{00}\,,\qquad\mathbf{K}:=2\sqrt{2}c^3\left(K_{[01]},\,K_{[02]},\,K_{[03]}\right),$$
$$\mathbf{B}:=\tfrac{1}{\sqrt{2}c}\left(B_{10},\,B_{20},\,B_{30}\right),\qquad\mathbf{H}:=\nabla\times\mathbf{B}=\tfrac{1}{\sqrt{2}c}\left(H_{023},\,H_{031},\,H_{012}\right).$$

Crucially, $\{\rho,\mathbf{K},\Phi,\mathbf{H}\}$ forms an 'autonomy' of closed relations, *i.e.* Stringy Newton Gravity:

$$\nabla^2\Phi=4\pi G\rho+\mathbf{H}\cdot\mathbf{H}\,,\qquad\nabla\times\mathbf{H}=4\pi G\,\mathbf{K}\,,\qquad\boldsymbol{\nabla}\cdot\mathbf{K}=0\,,\qquad\boldsymbol{\nabla}\cdot\mathbf{H}=0\,.$$

The Newton potential is fully determined by both the mass density and the stringy current density,

$$\begin{aligned}\Phi&=-G\int\mathrm{d}^3x'\,\frac{\rho_{eff}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,,\qquad\rho_{eff}:=\rho+\tfrac{1}{4\pi G}\mathbf{H}\cdot\mathbf{H}\,,\\ \mathbf{H}&=G\int\mathrm{d}^3x'\,\mathbf{K}(t,\mathbf{x}')\times\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}=\boldsymbol{\nabla}\times\mathbf{B}\,,\\ \mathbf{B}&=G\int\mathrm{d}^3x'\,\frac{\mathbf{K}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,,\qquad\boldsymbol{\nabla}\cdot\mathbf{B}=0\,.\end{aligned}$$

In analogy to the magnetization in electrodynamics, we introduce the notion of ***stringization*** for the stringy current density $\mathbf{K}$ which is divergence free, $\mathbf{K}(t,\mathbf{x})=\boldsymbol{\nabla}\times\mathbf{s}(t,\mathbf{x})$. The corresponding $\mathbf{B},\mathbf{H}$ are,

$$\begin{aligned}\mathbf{B}&=G\int\mathrm{d}^3x'\,\frac{\mathbf{s}(t,\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}+G\oint\mathrm{d}\mathbf{A}\times\frac{\mathbf{s}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,,\\ \mathbf{H}&=4\pi G\mathbf{s}(t,\mathbf{x})+G\int\mathrm{d}^3x'\,\frac{3\hat{\mathbf{n}}'\left(\hat{\mathbf{n}}'\cdot\mathbf{s}(t,\mathbf{x}')\right)-\mathbf{s}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}=4\pi G\mathbf{s}(t,\mathbf{x})-G\boldsymbol{\nabla}\Phi_s(t,\mathbf{x})\,,\end{aligned}$$

in which $\hat{\mathbf{n}}'=\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}$ and Φ_s is a *stringy scalar potential*,

$$\Phi_s(t,\mathbf{x})=\int\mathrm{d}^3x'\,\frac{\mathbf{s}(t,\mathbf{x}')\cdot(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}=-\boldsymbol{\nabla}\cdot\int\mathrm{d}^3x'\,\frac{\mathbf{s}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,.$$

Clearly, $\boldsymbol{\nabla}\times\mathbf{H}=4\pi G\boldsymbol{\nabla}\times\mathbf{s}$. Far away from a localized source, $|\mathbf{x}|>>|\mathbf{x}'|$, we observe a stringy dipole,

$$\mathbf{H}\simeq G\,\frac{3\hat{\mathbf{x}}\left(\hat{\mathbf{x}}\cdot\mathbf{S}(t)\right)-\mathbf{S}(t)}{|\mathbf{x}|^3}\,,\qquad\mathbf{S}(t)=\int\mathrm{d}^3x\,\mathbf{s}(t,\mathbf{x})\,.$$

Examples of Stringy Newton gravity with H-flux

- **Uniformly 'stringized' sphere of radius a, with constant ρ and s**

$$\begin{aligned}&\Phi_s=\tfrac{4\pi}{3}\,\mathbf{s}\cdot\mathbf{x}\,,\qquad\mathbf{H}=\tfrac{8\pi G}{3}\,\mathbf{s}\\ &\Phi_s=\tfrac{4\pi a^3}{3}\,\tfrac{\mathbf{s}\cdot\mathbf{x}}{|\mathbf{x}|^3}\,,\qquad\mathbf{H}=\tfrac{4\pi Ga^3}{3}\left(\tfrac{3\hat{\mathbf{x}}(\hat{\mathbf{x}}\cdot\mathbf{s})-\mathbf{s}}{|\mathbf{x}|^3}\right)\end{aligned}\quad\Rightarrow\quad\rho_{eff}(t,\mathbf{x})=\begin{cases}\rho+\frac{16\pi G}{9}\,|\mathbf{s}|^2 & \text{for }|\mathbf{x}|\leq a\\ \frac{4\pi G}{9}\,|\mathbf{s}|^2\,a^6\left(\frac{1+3\cos^2\theta}{|\mathbf{x}|^6}\right) & \text{for }|\mathbf{x}|>a\,.\end{cases}$$

- **Dirac monopole type**

$$\mathbf{B}=Gq\int_{\mathbf{x}'=\mathbf{0}}^{\infty}\mathrm{d}\mathbf{x}'\times\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\,,\qquad\mathbf{H}=Gq\frac{\mathbf{x}}{|\mathbf{x}|^3}\,,\qquad\Rightarrow\qquad\rho_{eff}=\frac{Gq^2}{4\pi\,|\mathbf{x}|^4}$$

where the path should not cross the point of $\mathbf{x}$. It resembles dark matter halos.

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