What is the gravitational theory that string theory predicts?

DFT = O(D, D) completion of GR with non-Riemannian bonus

박 정 혁 (朴 廷 爀) **Jeong-Hyuck Park** **Sogang University**

Sabbatival Visitor to YITP, Kyoto, This Year

19th February 2020

Introduction

– Surely, General Relativity is based on Riemannian geometry, where the only geometric and gravitational field is the Riemannian metric, $g_{\mu\nu}$. Other fields are meant to be extra matter.

– However, string theory suggests to put a two-form gauge potential, $B_{\mu\nu}$, and a scalar dilaton, ϕ, on an equal footing along with the metric:

They form the closed string massless (NS-NS) sector, being ubiquitous in all string theories,

$$\int \mathrm{d}^D x \, \sqrt{-g} e^{-2\phi} \left(R_g + 4\partial_\mu \phi \partial^\mu \phi - \tfrac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} \right) \qquad \text{where} \qquad H = \mathrm{d}B \,.$$

This action hides $\mathbf{O}(D, D)$ symmetry of T-duality which transforms g, B, ϕ into one another. Buscher 1987

– T-duality hints at a natural extension of GR where the entire closed string massless sector, $\{g, B, \phi\}$, constitutes the gravitational multiplet.

Double Field Theory (DFT), initiated by Siegel 1993 & Hull, Zwiebach 2009-2010 and further developed over the last decade, turns out to give the $\mathbf{O}(D, D)$ completion of GR or a novel form of 'pure gravity'.

Introduction

- Surely, General Relativity is based on Riemannian geometry, where the only geometric and gravitational field is the Riemannian metric, $g_{\mu\nu}$. Other fields are meant to be extra matter.

- However, string theory suggests to put a two-form gauge potential, $B_{\mu\nu}$, and a scalar dilaton, ϕ, on an equal footing along with the metric:

  They form the closed string massless (NS-NS) sector, being ubiquitous in all string theories,

$$\int \mathrm{d}^D x\, \sqrt{-g} e^{-2\phi} \left(R_g + 4\partial_\mu \phi \partial^\mu \phi - \tfrac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} \right) \qquad \text{where} \qquad H = \mathrm{d}B\,.$$

  This action hides $\mathbf{O}(D,D)$ symmetry of T-duality which transforms g, B, ϕ into one another. Buscher 1987

- T-duality hints at a natural extension of GR where the entire closed string massless sector, $\{g, B, \phi\}$, constitutes the gravitational multiplet.

  Double Field Theory (DFT), initiated by Siegel 1993 & Hull, Zwiebach 2009-2010 and further developed over the last decade, turns out to give the $\mathbf{O}(D,D)$ completion of GR or a novel form of 'pure gravity'.

Take-home message of this talk would be

- **DFT = O(D, D) completion of GR** : the pure gravitational theory that string theory predicts.

- DFT assumes the whole closed-string massless (NS-NS) sector as the gravitational multiplet. The **O**(D, D) Symmetry Principle then fixes its coupling to extra matter unambiguously.

- The previous Lagrangian itself is identified as a scalar curvature in novel differential geometry,

$$R_g + 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} \quad \Rightarrow \quad S_{(0)} \quad : \quad \text{Pure Gravity}$$

- The EOM of $\{g, B, \phi\}$ are unified into a single master formula,

$$G_{AB} = 8\pi G T_{AB} \quad : \quad \text{Einstein Double Field Equations}$$

  which is the **O**(D, D) completion of Einstein Field Equations, as A, B are **O**(D, D) indices.

  $\Rightarrow$ Stringy Newton Gravity : $\nabla^2\Phi = 4\pi G\rho + \mathbf{H}\cdot\mathbf{H}\,, \qquad \nabla\cdot\mathbf{H} = 0\,, \qquad \nabla\times\mathbf{H} = 4\pi G\mathbf{K}\,.$

- Further, taking **O**(D, D) covariant field variables as its truely fundamental constituents, DFT can accommodate not only conventional supergravity but also various non-Riemannian gravities where string becomes chiral, *e.g.* Newton–Cartan, Carroll, or Gomis–Ooguri.

- The theory appears to be defined on 'doubled-yet-gauged spacetime': the doubled coordinates are gauged such that a gauge orbit corresponds to a single physical point.

Take-home message of this talk would be

- **DFT = O(D,D) completion of GR** : the pure gravitational theory that string theory predicts.
- DFT assumes the whole closed-string massless (NS-NS) sector as the gravitational multiplet. The **O**(D,D) Symmetry Principle then fixes its coupling to extra matter unambiguously.
- The previous Lagrangian itself is identified as a scalar curvature in novel differential geometry,

$$R_g + 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} \quad \Rightarrow \quad S_{(0)} \quad : \quad \text{Pure Gravity}$$

- The EOM of $\{g, B, \phi\}$ are unified into a single master formula,

$$G_{AB} = 8\pi G T_{AB} \quad : \quad \text{Einstein Double Field Equations}$$

  which is the **O**(D,D) completion of Einstein Field Equations, as A, B are **O**(D,D) indices.

$$\Rightarrow \text{ Stringy Newton Gravity } : \quad \nabla^2\Phi = 4\pi G\rho + \mathbf{H}\cdot\mathbf{H}\,, \qquad \nabla\cdot\mathbf{H} = 0\,, \qquad \nabla\times\mathbf{H} = 4\pi G\mathbf{K}\,.$$

- Further, taking **O**(D,D) covariant field variables as its truely fundamental constituents, DFT can accommodate not only conventional supergravity but also various non-Riemannian gravities where string becomes chiral, *e.g.* Newton–Cartan, Carroll, or Gomis–Ooguri.
- The theory appears to be defined on 'doubled-yet-gauged spacetime': the doubled coordinates are gauged such that a gauge orbit corresponds to a single physical point.

Plan

I. Classification of DFT-geometries in terms of two non-negative integers, $(n, \bar{n})$.

II. Doubled-yet-gauged spacetime and sigma models.

III. Review of covariant derivatives, ∇_A, and curvatures, $S_{(0)}$, S_{AB}, G_{AB} in DFT.

IV. Derivation of the Einstein Double Field Equations, $G_{AB} = 8\pi G T_{AB}$,

$$G_{AB} := 4 V_{[A}{}^{p} \bar{V}_{B]}{}^{\bar{q}} S_{p\bar{q}} - \tfrac{1}{2} \mathcal{J}_{AB} S_{(0)} \,, \qquad \nabla_A G^{AB} = 0 \,,$$

$$T_{AB} := 4 V_{[A}{}^{p} \bar{V}_{B]}{}^{\bar{q}} K_{p\bar{q}} - \tfrac{1}{2} \mathcal{J}_{AB} T_{(0)} \,, \qquad \nabla_A T^{AB} = 0 \,.$$

V. Physical Implications
- $D = 4$ spherical solution : ‘stringy star’ *c.f.* Schwarzschild geometry
- $\mathbf{O}(D, D)$ completion of the Friedmann equations
- Stringy Newton Gravity (large c limit)

This talk is an overview of speaker’s collaborative works over the last decade, thanks to
Stephen Angus (2), Kevin Morand (3), Kyungho Cho (5), Thomas Basile, Shinji Mukohyama, Yuho Sakatani, Euihun Joung, Guilherme Franzmann, ... as well as earlier Imtak Jeon (8), Kanghoon Lee (8).

Notation

Index	Representation	Metric (raising/lowering indices)
$A, B, \cdots, M, N, \cdots$	$\mathbf{O}(D, D)$ vector	$\mathcal{J}_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$p, q, \cdots$	$\mathbf{Spin}(1, D{-}1)_L$ vector	$\eta_{pq} = \mathrm{diag}(- + + \cdots +)$
$\alpha, \beta, \cdots$	$\mathbf{Spin}(1, D{-}1)_L$ spinor	$C_{\alpha\beta}, \qquad (\gamma^p)^T = C\gamma^p C^{-1}$
$\bar{p}, \bar{q}, \cdots$	$\mathbf{Spin}(D{-}1, 1)_R$ vector	$\bar{\eta}_{\bar{p}\bar{q}} = \mathrm{diag}(+ - - \cdots -)$
$\bar{\alpha}, \bar{\beta}, \cdots$	$\mathbf{Spin}(D{-}1, 1)_R$ spinor	$\bar{C}_{\bar{\alpha}\bar{\beta}}, \qquad (\bar{\gamma}^{\bar{p}})^T = \bar{C}\bar{\gamma}^{\bar{p}}\bar{C}^{-1}$

- Further, the $\mathbf{O}(D, D)$ metric, $\mathcal{J}_{AB}$, decomposes the doubled coordinates into two parts,

$$x^A = (\tilde{x}_\mu, x^\nu), \qquad \partial_A = (\tilde{\partial}^\mu, \partial_\nu).$$

  where μ, ν are D-dimensional curved indices.

- The twofold local Lorentz symmetries, $\mathbf{Spin}(1, D{-}1)_L \times \mathbf{Spin}(D{-}1, 1)_R$, indicate two distinct locally inertial frames for the left and right moving sectors $\Rightarrow$ Unification of IIA and IIB.

Closed-string massless sector as ‘Gravitational Fields’

The gravitational fields consist of the DFT-dilaton, d, and DFT-metric, $\mathcal{H}_{MN}$:

$$\mathcal{H}_{MN} = \mathcal{H}_{NM}\,, \qquad\qquad \mathcal{H}_{K}{}^{L}\mathcal{H}_{M}{}^{N}\mathcal{J}_{LN} = \mathcal{J}_{KM}\,.$$

Combining $\mathcal{J}_{MN}$ and $\mathcal{H}_{MN}$, we get a pair of symmetric projection matrices,

$$P_{MN} = P_{NM} = \tfrac{1}{2}(\mathcal{J}_{MN} + \mathcal{H}_{MN})\,, \qquad\qquad P_{L}{}^{M}P_{M}{}^{N} = P_{L}{}^{N}\,,$$

$$\bar{P}_{MN} = \bar{P}_{NM} = \tfrac{1}{2}(\mathcal{J}_{MN} - \mathcal{H}_{MN})\,, \qquad\qquad \bar{P}_{L}{}^{M}\bar{P}_{M}{}^{N} = \bar{P}_{L}{}^{N}\,,$$

which are orthogonal and complete,

$$P_{L}{}^{M}\bar{P}_{M}{}^{N} = 0\,, \qquad\qquad P_{M}{}^{N} + \bar{P}_{M}{}^{N} = \delta_{M}{}^{N}\,.$$

Further, taking the “square roots" of the projectors,

$$P_{MN} = V_{M}{}^{p}V_{N}{}^{q}\eta_{pq}\,, \qquad\qquad \bar{P}_{MN} = \bar{V}_{M}{}^{\bar{p}}\bar{V}_{N}{}^{\bar{q}}\bar{\eta}_{\bar{p}\bar{q}}\,,$$

we get a pair of DFT-vielbeins satisfying their own defining properties,

$$V_{Mp}V^{M}{}_{q} = \eta_{pq}\,, \qquad\qquad \bar{V}_{M\bar{p}}\bar{V}^{M}{}_{\bar{q}} = \bar{\eta}_{\bar{p}\bar{q}}\,, \qquad\qquad V_{Mp}\bar{V}^{M}{}_{\bar{q}} = 0\,,$$

or equivalently

$$V_{M}{}^{p}V_{Np} + \bar{V}_{M}{}^{\bar{p}}\bar{V}_{N\bar{p}} = \mathcal{J}_{MN}\,.$$

Solution to the defining relation, $\mathcal{H}_{MN} = \mathcal{H}_{NM},\ \mathcal{H}_{K}{}^{L}\mathcal{H}_{M}{}^{N}\mathcal{J}_{LN} = \mathcal{J}_{KM}$?

$$\mathcal{H}_{MN} = \left(\begin{array}{cc} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{array}\right) \qquad \text{or} \qquad \mathcal{H}_{MN} = \mathcal{J}_{MN} = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right)$$

The left one is well-known: it contains a Riemannian metric and reduces DFT to SUGRA.

The right one is a *flat* background which admits no Riemannian nor SUGRA interpretation.

Thus, DFT describes not only Riemannian SUGRA but also non-Riemannian novel geometries.

Classification

Kevin Morand & JHP 1707.03713

The most general form of the DFT-metric, $\mathcal{H}_{MN}=\mathcal{H}_{NM}$, $\mathcal{H}_{K}{}^{L}\mathcal{H}_{M}{}^{N}\mathcal{J}_{LN}=\mathcal{J}_{KM}$, is characterized by two non-negative integers, $(n,\bar{n})$, $0\leq n+\bar{n}\leq D$:

$$\mathcal{H}_{AB}=\left(\begin{array}{cc} H^{\mu\nu} & -H^{\mu\sigma}B_{\sigma\lambda}+Y_{i}^{\mu}X_{\lambda}^{i}-\bar{Y}_{\bar{\imath}}^{\mu}\bar{X}_{\lambda}^{\bar{\imath}} \\ B_{\kappa\rho}H^{\rho\nu}+X_{\kappa}^{i}Y_{i}^{\nu}-\bar{X}_{\kappa}^{\bar{\imath}}\bar{Y}_{\bar{\imath}}^{\nu} & K_{\kappa\lambda}-B_{\kappa\rho}H^{\rho\sigma}B_{\sigma\lambda}+2X_{(\kappa}^{i}B_{\lambda)\rho}Y_{i}^{\rho}-2\bar{X}_{(\kappa}^{\bar{\imath}}B_{\lambda)\rho}\bar{Y}_{\bar{\imath}}^{\rho}\end{array}\right)$$

$$=\left(\begin{array}{cc}1&0\\B&1\end{array}\right)\left(\begin{array}{cc}H & Y_{i}(X^{i})^{T}-\bar{Y}_{\bar{\imath}}(\bar{X}^{\bar{\imath}})^{T}\\ X^{i}(Y_{i})^{T}-\bar{X}^{\bar{\imath}}(\bar{Y}_{\bar{\imath}})^{T} & K\end{array}\right)\left(\begin{array}{cc}1&-B\\0&1\end{array}\right)$$

i) Symmetric and skew-symmetric fields : $\quad H^{\mu\nu}=H^{\nu\mu}\,,\quad K_{\mu\nu}=K_{\nu\mu}\,,\quad B_{\mu\nu}=-B_{\nu\mu}\,;$

ii) Two kinds of zero eigenvectors: with $i,j=1,2,\cdots,n$ & $\bar{\imath},\bar{\jmath}=1,2,\cdots,\bar{n}$,

$$H^{\mu\nu}X_{\nu}^{i}=0\,,\qquad H^{\mu\nu}\bar{X}_{\nu}^{\bar{\imath}}=0\,,\qquad K_{\mu\nu}Y_{j}^{\nu}=0\,,\qquad K_{\mu\nu}\bar{Y}_{\bar{\jmath}}^{\nu}=0\,;$$

iii) Completeness relation: $\qquad H^{\mu\rho}K_{\rho\nu}+Y_{i}^{\mu}X_{\nu}^{i}+\bar{Y}_{\bar{\imath}}^{\mu}\bar{X}_{\nu}^{\bar{\imath}}=\delta^{\mu}{}_{\nu}\,.$

- **The trace is $\mathcal{H}_{A}{}^{A}=2(n-\bar{n})$ which is $\mathbf{O}(D,D)$ invariant.**
- **The coset is $\frac{\mathbf{O}(D,D)}{\mathbf{O}(t+n,s+n)\times\mathbf{O}(s+\bar{n},t+\bar{n})}$ with dimensions $D^{2}-(n-\bar{n})^{2}$ as Nambu–Goldstone moduli.**

Berman-Blair-Otsuki, Cho-JHP 2019

Classification

Kevin Morand & JHP 1707.03713

The most general form of the DFT-metric, $\mathcal{H}_{MN} = \mathcal{H}_{NM}$, $\mathcal{H}_K{}^L\mathcal{H}_M{}^N\mathcal{J}_{LN} = \mathcal{J}_{KM}$, is characterized by two non-negative integers, $(n, \bar{n})$, $0 \leq n + \bar{n} \leq D$:

$$\mathcal{H}_{AB} = \begin{pmatrix} H^{\mu\nu} & -H^{\mu\sigma}B_{\sigma\lambda} + Y_i^\mu X_\lambda^i - \bar{Y}_{\bar{\imath}}^\mu \bar{X}_\lambda^{\bar{\imath}} \\ B_{\kappa\rho}H^{\rho\nu} + X_\kappa^i Y_i^\nu - \bar{X}_\kappa^{\bar{\imath}} \bar{Y}_{\bar{\imath}}^\nu & K_{\kappa\lambda} - B_{\kappa\rho}H^{\rho\sigma}B_{\sigma\lambda} + 2X_{(\kappa}^i B_{\lambda)\rho} Y_i^\rho - 2\bar{X}_{(\kappa}^{\bar{\imath}} B_{\lambda)\rho} \bar{Y}_{\bar{\imath}}^\rho \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} H & Y_i(X^i)^T - \bar{Y}_{\bar{\imath}}(\bar{X}^{\bar{\imath}})^T \\ X^i(Y_i)^T - \bar{X}^{\bar{\imath}}(\bar{Y}_{\bar{\imath}})^T & K \end{pmatrix} \begin{pmatrix} 1 & -B \\ 0 & 1 \end{pmatrix}$$

i) Symmetric and skew-symmetric fields: $H^{\mu\nu} = H^{\nu\mu}$, $K_{\mu\nu} = K_{\nu\mu}$, $B_{\mu\nu} = -B_{\nu\mu}$;

ii) Two kinds of zero eigenvectors: with $i, j = 1, 2, \cdots, n$ & $\bar{\imath}, \bar{\jmath} = 1, 2, \cdots, \bar{n}$,

$$H^{\mu\nu}X_\nu^i = 0\,, \qquad H^{\mu\nu}\bar{X}_\nu^{\bar{\imath}} = 0\,, \qquad K_{\mu\nu}Y_j^\nu = 0\,, \qquad K_{\mu\nu}\bar{Y}_{\bar{\jmath}}^\nu = 0\,;$$

iii) Completeness relation: $H^{\mu\rho}K_{\rho\nu} + Y_i^\mu X_\nu^i + \bar{Y}_{\bar{\imath}}^\mu \bar{X}_\nu^{\bar{\imath}} = \delta^\mu{}_\nu$.

- **The trace is $\mathcal{H}_A{}^A = 2(n - \bar{n})$ which is $\mathbf{O}(D, D)$ invariant.**
- **The coset is $\frac{\mathbf{O}(D,D)}{\mathbf{O}(t+n,s+n)\times\mathbf{O}(s+\bar{n},t+\bar{n})}$ with dimensions $D^2 - (n - \bar{n})^2$ as Nambu–Goldstone moduli.**

Berman-Blair-Otsuki, Cho-JHP 2019

Classification

Kevin Morand & JHP 1707.03713

I. $(n, \bar{n}) = (0, 0)$ corresponds to the Riemannian case or Generalized Geometry à la Hitchin :

$$\mathcal{H}_{MN} \equiv \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}, \quad e^{-2d} \equiv \sqrt{|g|}e^{-2\phi}$$

Giveon, Rabinovici, Veneziano '89, Duff '90

II. Generically, on worldsheet, string becomes chiral and anti-chiral over the n and $\bar{n}$ dimensions:

$$X^i_\mu \, \partial_+ x^\mu(\tau, \sigma) \equiv 0 \,, \qquad \bar{X}^{\bar{\imath}}_\mu \, \partial_- x^\mu(\tau, \sigma) \equiv 0 \,,$$

as we shall see shortly.

Non-Riemannian examples include

- $(1, 0)$ Newton-Cartan gravity $\quad (\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2, \ \lim_{c\to\infty} g^{-1}$ is finite & degenerate)
- $(1, 1)$ Gomis-Ooguri non-relativistic string — Melby-Thompson, Meyer, Ko, JHP 2015, Blair 2019
- $(D{-}1, 0)$ ultra-relativistic Carroll gravity
- $(D, 0)$ is uniquely given by $\mathcal{H} = \mathcal{J}$: maximally non-Riemannian with trivial coset, $\frac{\mathbf{O}(D,D)}{\mathbf{O}(D,D)}$.

  This is the completely $\mathbf{O}(D, D)$-symmetric vacuum of DFT with no moduli, *c.f.* Siegel's chiral string.

  "*Spacetime emerges after SSB of* $\mathbf{O}(D, D)$*, identifying* $\{g, B\}$ *as Nambu–Goldstone boson moduli.*"

  Berman, Blair, and Otsuki 2019

  Further, taken as an internal space, it gives a 'moduli-free' (Scherk-Schwarz twistable) Kaluza-Klein reduction of pure DFT to heterotic DFT : Heterotic string has higher dimensional non-Riemannian origin.

  Cho, Morand, JHP 2018

Classification

Kevin Morand & JHP 1707.03713

I. $(n,\bar{n})=(0,0)$ corresponds to the Riemannian case or Generalized Geometry à la Hitchin :

$$\mathcal{H}_{MN}\equiv\begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g-Bg^{-1}B \end{pmatrix},\qquad e^{-2d}\equiv\sqrt{|g|}e^{-2\phi}$$

Giveon, Rabinovici, Veneziano '89, Duff '90

II. Generically, on worldsheet, string becomes chiral and anti-chiral over the n and $\bar{n}$ dimensions:

$$X_{\mu}^{i}\,\partial_{+}x^{\mu}(\tau,\sigma)\equiv 0\,,\qquad\qquad \bar{X}_{\mu}^{\bar{\imath}}\,\partial_{-}x^{\mu}(\tau,\sigma)\equiv 0\,,$$

as we shall see shortly.

Non-Riemannian examples include

- $(1,0)$ Newton-Cartan gravity $\qquad(\mathrm{d}s^2=-c^2\mathrm{d}t^2+\mathrm{d}\mathbf{x}^2,\ \lim_{c\to\infty}g^{-1}$ is finite & degenerate$)$
- $(1,1)$ Gomis-Ooguri non-relativistic string $\qquad$ Melby-Thompson, Meyer, Ko, JHP 2015, Blair 2019
- $(D-1,0)$ ultra-relativistic Carroll gravity
- $(D,0)$ is uniquely given by $\mathcal{H}=\mathcal{J}$: maximally non-Riemannian with trivial coset, $\frac{\mathbf{O}(D,D)}{\mathbf{O}(D,D)}$.

  This is the completely $\mathbf{O}(D,D)$-symmetric vacuum of DFT with no moduli, *c.f.* Siegel's chiral string.

  "*Spacetime emerges after SSB of* $\mathbf{O}(D,D)$*, identifying* $\{g,B\}$ *as Nambu–Goldstone boson moduli.*"

  Berman, Blair, and Otsuki 2019

  Further, taken as an internal space, it gives a 'moduli-free' (Scherk-Schwarz twistable) Kaluza-Klein reduction of pure DFT to heterotic DFT : Heterotic string has higher dimensional non-Riemannian origin.

  Cho, Morand, JHP 2018

Classification

Kevin Morand & JHP 1707.03713

I. $(n,\bar{n})=(0,0)$ corresponds to the Riemannian case or Generalized Geometry à la Hitchin :

$$\mathcal{H}_{MN}\equiv\begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g-Bg^{-1}B\end{pmatrix},\quad e^{-2d}\equiv\sqrt{|g|}e^{-2\phi}$$

Giveon, Rabinovici, Veneziano '89, Duff '90

II. Generically, on worldsheet, string becomes chiral and anti-chiral over the n and $\bar{n}$ dimensions:

$$X^{i}_{\mu}\,\partial_{+}x^{\mu}(\tau,\sigma)\equiv 0\,,\qquad\qquad \bar{X}^{\bar{\imath}}_{\mu}\,\partial_{-}x^{\mu}(\tau,\sigma)\equiv 0\,,$$

as we shall see shortly.

Non-Riemannian examples include

- $(1,0)$ Newton-Cartan gravity $\quad(\mathrm{d}s^2=-c^2\mathrm{d}t^2+\mathrm{d}\mathbf{x}^2,\ \lim_{c\to\infty}g^{-1}$ is finite & degenerate$)$
- $(1,1)$ Gomis-Ooguri non-relativistic string $\quad$ Melby-Thompson, Meyer, Ko, JHP 2015, Blair 2019
- $(D-1,0)$ ultra-relativistic Carroll gravity
- $(D,0)$ is uniquely given by $\mathcal{H}=\mathcal{J}$: maximally non-Riemannian with trivial coset, $\frac{\mathbf{O}(D,D)}{\mathbf{O}(D,D)}$.
  This is the completely $\mathbf{O}(D,D)$-symmetric vacuum of DFT with no moduli, *c.f.* Siegel's chiral string.
  "*Spacetime emerges after SSB of* $\mathbf{O}(D,D)$*, identifying* $\{g,B\}$ *as Nambu–Goldstone boson moduli.*"
  Berman, Blair, and Otsuki 2019

  Further, taken as an internal space, it gives a 'moduli-free' (Scherk-Schwarz twistable) Kaluza-Klein reduction of pure DFT to heterotic DFT : Heterotic string has higher dimensional non-Riemannian origin.
  Cho, Morand, JHP 2018

Section condition

- **Diffeomorphisms** are generated by “generalized Lie derivative": Siegel 1993

$$\hat{\mathcal{L}}_\xi T_{A_1\cdots A_n} := \xi^B\partial_B T_{A_1\cdots A_n} + \omega_T\,\partial_B\xi^B T_{A_1\cdots A_n} + \sum_{i=1}^{n}(\partial_{A_i}\xi_B - \partial_B\xi_{A_i})T_{A_1\cdots A_{i-1}}{}^{B}{}_{A_{i+1}\cdots A_n}\,,$$

  where ω_T is the weight, *e.g.* $\delta e^{-2d} = \partial_B(\xi^B e^{-2d})$, $\delta V_{Ap} = \xi^B\partial_B V_{Ap} + (\partial_A\xi_B - \partial_B\xi_A)V^B{}_p$.

- For consistency of closure, the so-called ‘section condition’ should be imposed: $\partial_M\partial^M = 0$.

  From $\partial_M\partial^M = 2\partial_\mu\tilde{\partial}^\mu$, the section condition can be easily solved by letting $\tilde{\partial}^\mu = 0$.

  The general solutions are then generated by the $\mathbf{O}(D,D)$ rotation of it.

- The section condition is mathematically equivalent to a certain translational invariance:

$$\Phi_s(x) = \Phi_s(x+\Delta)\,, \qquad \Delta^M = \Phi_t\partial^M\Phi_u\,,$$

  where $\Phi_s,\Phi_t,\Phi_u \in \{\,d,\mathcal{H}_{MN}\,,\xi^M\,,\partial_N d\,,\partial_L\mathcal{H}_{MN},\cdots\}$, arbitrary functions appearing in DFT,

  and Δ^M is said to be derivative-index-valued. JHP 2013

▸ ‘Physics’ should be invariant under such shifts of the doubled coordinates.

Section condition

- **Diffeomorphisms** are generated by "generalized Lie derivative": Siegel 1993

$$\hat{\mathcal{L}}_\xi T_{A_1\cdots A_n} := \xi^B\partial_B T_{A_1\cdots A_n} + \omega_T\,\partial_B\xi^B T_{A_1\cdots A_n} + \sum_{i=1}^{n}(\partial_{A_i}\xi_B - \partial_B\xi_{A_i})T_{A_1\cdots A_{i-1}}{}^{B}{}_{A_{i+1}\cdots A_n}\,,$$

  where ω_T is the weight, *e.g.* $\delta e^{-2d} = \partial_B(\xi^B e^{-2d})$, $\delta V_{Ap} = \xi^B\partial_B V_{Ap} + (\partial_A\xi_B - \partial_B\xi_A)V^B{}_p$.

- For consistency of closure, the so-called 'section condition' should be imposed: $\partial_M\partial^M = 0$.

  From $\partial_M\partial^M = 2\partial_\mu\tilde{\partial}^\mu$, the section condition can be easily solved by letting $\tilde{\partial}^\mu = 0$.

  The general solutions are then generated by the $\mathbf{O}(D,D)$ rotation of it.

- The section condition is mathematically equivalent to a certain translational invariance:

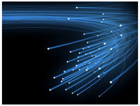
$$\Phi_s(x) = \Phi_s(x+\Delta)\,, \qquad\qquad \Delta^M = \Phi_t\partial^M\Phi_u\,,$$

  where $\Phi_s, \Phi_t, \Phi_u \in \{\, d, \mathcal{H}_{MN}, \xi^M, \partial_N d, \partial_L\mathcal{H}_{MN}, \cdots\}$, arbitrary functions appearing in DFT,

  and Δ^M is said to be derivative-index-valued. JHP 2013

▶ 'Physics' should be invariant under such shifts of the doubled coordinates.

Doubled-yet-gauged coordinates, $(D+D)\times(1-1/2)$



Doubled coordinates, $x^{M}=(\tilde{x}_{\mu},x^{\nu})$, are gauged through an equivalence relation,

$$x^{M}\ \sim\ x^{M}+\Delta^{M}(x)\,,$$

where Δ^{M} is derivative-index-valued.

Each equivalence class, or gauge orbit in $\mathbb{R}^{D+D}$, corresponds to a single physical point in $\mathbb{R}^{D}$.

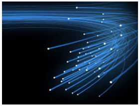
- If we solve the section condition by letting $\tilde{\partial}^{\mu}\equiv 0$, and further choose $\Delta^{M}=c_{\mu}\partial^{M}x^{\mu}$, we note

$$(\tilde{x}_{\mu}\,,\,x^{\nu})\ \sim\ (\tilde{x}_{\mu}+c_{\mu}\,,\,x^{\nu})$$

- Then, $\mathbf{O}(D,D)$ rotates the gauged directions and hence the section.

Doubled-yet-gauged coordinates, $(D+D)\times(1-1/2)$

JHP 1304.5946



Doubled coordinates, $x^M=(\tilde{x}_\mu\,,x^\nu)$, are gauged through an equivalence relation,

$$x^M \sim x^M+\Delta^M(x)\,,$$

where Δ^M is derivative-index-valued.

Each equivalence class, or gauge orbit in $\mathbb{R}^{D+D}$, corresponds to a single physical point in $\mathbb{R}^D$.

- If we solve the section condition by letting $\tilde{\partial}^\mu\equiv 0$, and further choose $\Delta^M=c_\mu\partial^M x^\mu$, we note

$$(\tilde{x}_\mu\,,\,x^\nu) \sim (\tilde{x}_\mu+c_\mu\,,\,x^\nu)$$

- Then, **O**(D,D) rotates the gauged directions and hence the section.

Doubled-yet-gauged coordinates, $(D+D)\times(1-1/2)$

JHP 1304.5946

Doubled coordinates, $x^M = (\tilde{x}_\mu\,, x^\nu)$, are gauged through an equivalence relation,

$$x^M \;\sim\; x^M + \Delta^M(x)\,,$$

where Δ^M is derivative-index-valued.

Each equivalence class, or gauge orbit in $\mathbb{R}^{D+D}$, corresponds to a single physical point in $\mathbb{R}^D$.

- In DFT, the usual coordinate basis of one-forms, $\mathrm{d}x^A$, is not covariant:

  Neither diffeomorphic covariant,

$$\delta x^M = \xi^M\,, \qquad\qquad \delta(\mathrm{d}x^M) = \mathrm{d}x^N\partial_N\xi^M \;\neq\; \mathrm{d}x^N(\partial_N\xi^M - \partial^M\xi_N)$$

  Nor invariant under the coordinate gauge symmetry,

$$\mathrm{d}x^M \quad\longrightarrow\quad \mathrm{d}\big(x^M + \Delta^M\big) \;\neq\; \mathrm{d}x^M\,.$$

$\Rightarrow$ The naive contraction with the DFT metric, $\mathrm{d}x^M\mathrm{d}x^N\mathcal{H}_{MN}$, is not an invariant scalar, and thus cannot lead to any sensible definition of the 'proper length' in DFT.

Doubled coordinates, $x^{M}=(\tilde{x}_{\mu}\,,\,x^{\nu})$, are gauged through an equivalence relation,

$$x^{M}~\sim~x^{M}+\Delta^{M}(x)\,,$$

where Δ^{M} is derivative-index-valued.

Each equivalence class, or gauge orbit in $\mathbb{R}^{D+D}$, corresponds to a single physical point in $\mathbb{R}^{D}$.

- In DFT, the usual coordinate basis of one-forms, $\mathrm{d}x^{A}$, is not covariant:

  Neither diffeomorphic covariant,

$$\delta x^{M}=\xi^{M}\,,\qquad\qquad \delta(\mathrm{d}x^{M})=\mathrm{d}x^{N}\partial_{N}\xi^{M}~\neq~\mathrm{d}x^{N}(\partial_{N}\xi^{M}-\partial^{M}\xi_{N})$$

  Nor invariant under the coordinate gauge symmetry,

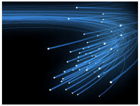
$$\mathrm{d}x^{M}\quad\longrightarrow\quad\mathrm{d}\big(x^{M}+\Delta^{M}\big)~\neq~\mathrm{d}x^{M}\,.$$

$\Rightarrow$ The naive contraction with the DFT metric, $\mathrm{d}x^{M}\mathrm{d}x^{N}\mathcal{H}_{MN}$, is not an invariant scalar,

and thus cannot lead to any sensible definition of the 'proper length' in DFT.

Doubled-yet-gauged coordinates, $(D+D)\times(1-1/2)$

JHP 1304.5946



Doubled coordinates, $x^{M}=(\tilde{x}_{\mu}\,,x^{\nu})$, are gauged through an equivalence relation,

$$x^{M}\ \sim\ x^{M}+\Delta^{M}(x)\,,$$

where Δ^{M} is derivative-index-valued.

Each equivalence class, or gauge orbit in $\mathbb{R}^{D+D}$, corresponds to a single physical point in $\mathbb{R}^{D}$.

- These problems can be all cured by gauging the coordinate basis of one-forms, $\mathrm{d}x^{A}$, explicitly,

$$Dx^{M}:=\mathrm{d}x^{M}-\mathcal{A}^{M}\,,\qquad \mathcal{A}^{M}\partial_{M}=0\quad\text{(derivative-index-valued)}\,.$$

Dx^{M} is covariant:

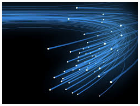
$$\delta x^{M}=\Delta^{M}\,,\quad \delta\mathcal{A}^{M}=\mathrm{d}\Delta^{M}\qquad\Longrightarrow\qquad \delta(Dx^{M})=0\,;$$

$$\delta x^{M}=\xi^{M}\,,\quad \delta\mathcal{A}^{M}=\partial^{M}\xi_{N}(\mathrm{d}x^{N}-\mathcal{A}^{N})\quad\Longrightarrow\quad \delta(Dx^{M})=Dx^{N}(\partial_{N}\xi^{M}-\partial^{M}\xi_{N})\,.$$

If we set $\tilde{\partial}^{\mu}\equiv0$, we have $\mathcal{A}^{M}=A_{\lambda}\partial^{M}x^{\lambda}=(A_{\mu}\,,0)$, $\quad Dx^{M}=(\mathrm{d}\tilde{x}_{\mu}-A_{\mu}\,,\mathrm{d}x^{\nu})\,.$

Doubled-yet-gauged coordinates, $(D+D)\times(1-1/2)$

JHP 1304.5946



Doubled coordinates, $x^M=(\tilde{x}_\mu\,,x^\nu)$, are gauged through an equivalence relation,

$$x^M \;\sim\; x^M+\Delta^M(x)\,,$$

where Δ^M is derivative-index-valued.

Each equivalence class, or gauge orbit in $\mathbb{R}^{D+D}$, corresponds to a single physical point in $\mathbb{R}^D$.

- These problems can be all cured by gauging the coordinate basis of one-forms, $\mathrm{d}x^A$, explicitly,

$$Dx^M:=\mathrm{d}x^M-\mathcal{A}^M\,,\qquad\qquad \mathcal{A}^M\partial_M=0\quad\text{(derivative-index-valued)}\,.$$

Dx^M is covariant:

$$\delta x^M=\Delta^M\,,\quad \delta\mathcal{A}^M=\mathrm{d}\Delta^M\qquad\Longrightarrow\qquad \delta(Dx^M)=0\,;$$

$$\delta x^M=\xi^M\,,\quad \delta\mathcal{A}^M=\partial^M\xi_N(\mathrm{d}x^N-\mathcal{A}^N)\quad\Longrightarrow\quad \delta(Dx^M)=Dx^N(\partial_N\xi^M-\partial^M\xi_N)\,.$$

If we set $\tilde{\partial}^\mu\equiv 0$, we have $\mathcal{A}^M=A_\lambda\partial^M x^\lambda=(A_\mu\,,0)$, $\;Dx^M=(\mathrm{d}\tilde{x}_\mu-A_\mu\,,\mathrm{d}x^\nu)\,.$

Doubled-yet-gauged coordinates, $(D+D)\times(1-1/2)$

JHP 1304.5946

Doubled coordinates, $x^{M}=(\tilde{x}_{\mu}\,,x^{\nu})$, are gauged through an equivalence relation,

$$x^{M}~\sim~x^{M}+\Delta^{M}(x)\,,$$

where Δ^{M} is derivative-index-valued.

Each equivalence class, or gauge orbit in $\mathbb{R}^{D+D}$, corresponds to a single physical point in $\mathbb{R}^{D}$.

- With $Dx^{M}=\mathrm{d}x^{M}-\mathcal{A}^{M}$, it is possible to define the 'proper length' through a path integral,

$$\textbf{Proper Length}:=-\ln\left[\int\mathcal{D}\mathcal{A}~\exp\left(-\int\sqrt{Dx^{M}Dx^{N}\mathcal{H}_{MN}}\right)\right].$$

  - For the (0, 0) Riemannian DFT-metric, with $\tilde{\partial}^{\mu}\equiv0$, $\mathcal{A}^{M}=(A_{\mu}\,,0)$, and from

$$Dx^{M}Dx^{N}\mathcal{H}_{MN}\equiv\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}g_{\mu\nu}+\left(\mathrm{d}\tilde{x}_{\mu}-A_{\mu}+\mathrm{d}x^{\rho}B_{\rho\mu}\right)\left(\mathrm{d}\tilde{x}_{\nu}-A_{\nu}+\mathrm{d}x^{\sigma}B_{\sigma\nu}\right)g^{\mu\nu}$$

    after integrating out A_{μ}, the proper length reduces to the conventional one,

$$\textbf{Proper Length}~\Longrightarrow~\int\sqrt{\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}g_{\mu\nu}(x)}\,.$$

  - Since it is independent of $\tilde{x}_{\mu}$, indeed it measures the distance between two gauge orbits, as desired.

Doubled-yet-gauged coordinates, $(D+D)\times(1-1/2)$

JHP 1304.5946

Doubled coordinates, $x^{M}=(\tilde{x}_{\mu},x^{\nu})$, are gauged through an equivalence relation,

$$x^{M}\ \sim\ x^{M}+\Delta^{M}(x)\,,$$

where Δ^{M} is derivative-index-valued.

Each equivalence class, or gauge orbit in $\mathbb{R}^{D+D}$, corresponds to a single physical point in $\mathbb{R}^{D}$.

- With $Dx^{M}=\mathrm{d}x^{M}-\mathcal{A}^{M}$, it is possible to define the 'proper length' through a path integral,

$$\textbf{Proper Length}:=-\ln\left[\int\mathcal{D}\mathcal{A}\ \exp\left(-\int\sqrt{Dx^{M}Dx^{N}\mathcal{H}_{MN}}\right)\right].$$

  - For the $(0,0)$ Riemannian DFT-metric, with $\tilde{\partial}^{\mu}\equiv 0$, $\mathcal{A}^{M}=(A_{\mu},0)$, and from

$$Dx^{M}Dx^{N}\mathcal{H}_{MN}\equiv\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}g_{\mu\nu}+\left(\mathrm{d}\tilde{x}_{\mu}-A_{\mu}+\mathrm{d}x^{\rho}B_{\rho\mu}\right)\left(\mathrm{d}\tilde{x}_{\nu}-A_{\nu}+\mathrm{d}x^{\sigma}B_{\sigma\nu}\right)g^{\mu\nu}$$

    after integrating out A_{μ}, the proper length reduces to the conventional one,

$$\textbf{Proper Length}\ \Longrightarrow\ \int\sqrt{\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}g_{\mu\nu}(x)}\,.$$

  - Since it is independent of $\tilde{x}_{\mu}$, indeed it measures the distance between two gauge orbits, as desired.

Doubled-yet-gauged coordinates, $(D+D)\times(1-1/2)$

JHP 1304.5946

Doubled coordinates, $x^{M}=(\tilde{x}_{\mu}\,,x^{\nu})$, are gauged through an equivalence relation,

$$x^{M}~\sim~x^{M}+\Delta^{M}(x)\,,$$

where Δ^{M} is derivative-index-valued.

Each equivalence class, or gauge orbit in $\mathbb{R}^{D+D}$, corresponds to a single physical point in $\mathbb{R}^{D}$.

- With $Dx^{M}=\mathrm{d}x^{M}-\mathcal{A}^{M}$, it is possible to define the 'proper length' through a path integral,

$$\textbf{Proper Length}:=-\ln\left[\int\mathcal{D}\mathcal{A}~\exp\left(-\int\sqrt{Dx^{M}Dx^{N}\mathcal{H}_{MN}}\right)\right].$$

  – For the $(0,0)$ Riemannian DFT-metric, with $\tilde{\partial}^{\mu}\equiv 0$, $\mathcal{A}^{M}=(A_{\mu}\,,0)$, and from

$$Dx^{M}Dx^{N}\mathcal{H}_{MN}\equiv\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}g_{\mu\nu}+\left(\mathrm{d}\tilde{x}_{\mu}-A_{\mu}+\mathrm{d}x^{\rho}B_{\rho\mu}\right)\left(\mathrm{d}\tilde{x}_{\nu}-A_{\nu}+\mathrm{d}x^{\sigma}B_{\sigma\nu}\right)g^{\mu\nu}$$

  after integrating out A_{μ}, the proper length reduces to the conventional one,

$$\textbf{Proper Length}~~\Longrightarrow~\int\sqrt{\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}g_{\mu\nu}(x)}\,.$$

  – Since it is independent of $\tilde{x}_{\mu}$, indeed it measures the distance between two gauge orbits, as desired.

Doubled-yet-gauged sigma models

The definition of the proper length readily leads to 'completely covariant' actions:

I. Particle action — Ko-JHP-Suh 2016

$$\mathcal{S}_{\mathrm{particle}} = \int \mathrm{d}\tau \; \tfrac{1}{2} e^{-1} D_\tau x^M D_\tau x^N \mathcal{H}_{MN}(x) - \tfrac{1}{2} m^2 e$$

II. String action — Hull 2006, Lee-JHP 2013, Arvanitakis-Blair 2017

$$S_{\mathrm{string}} = \tfrac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \; -\tfrac{1}{2}\sqrt{-h}h^{ij} D_i x^M D_j x^N \mathcal{H}_{MN}(x) - \epsilon^{ij} D_i x^M \mathcal{A}_{jM}$$

With the $(0,0)$ Riemannian DFT-metric plugged, after integrating out the auxiliary fields, the above actions reduce to the conventional ones:

$$S_{\mathrm{particle}} \Rightarrow \int \mathrm{d}\tau \; \tfrac{1}{2} e^{-1} \dot{x}^\mu \dot{x}^\nu g_{\mu\nu} - \tfrac{1}{2} m^2 e\,,$$

$$S_{\mathrm{string}} \Rightarrow \tfrac{1}{2\pi\alpha'} \int \mathrm{d}^2\sigma \; -\tfrac{1}{2}\sqrt{-h}h^{ij}\partial_i x^\mu \partial_j x^\nu g_{\mu\nu} + \tfrac{1}{2}\epsilon^{ij}\partial_i x^\mu \partial_j x^\nu B_{\mu\nu} + \tfrac{1}{2}\epsilon^{ij}\partial_i \tilde{x}_\mu \partial_j x^\mu\,.$$

III. κ-symmetric Green-Schwarz doubled-yet-gauged superstring, unifying IIA & IIB — JHP 2016

$$S_{\mathrm{GS}} = \tfrac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \; -\tfrac{1}{2}\sqrt{-h}h^{ij}\Pi_i^M \Pi_j^N \mathcal{H}_{MN} - \epsilon^{ij} D_i x^M \left(\mathcal{A}_{jM} - i\Sigma_{jM}\right),$$

where $\Pi_i^M := D_i x^M - i\Sigma_i^M$ and $\Sigma_i^M := \bar{\theta}\gamma^M \partial_i \theta + \bar{\theta}'\bar{\gamma}^M \partial_i \theta'$.

Doubled-yet-gauged sigma models

The definition of the proper length readily leads to 'completely covariant' actions:

I. Particle action Ko-JHP-Suh 2016

$$\mathcal{S}_{\rm particle}=\int{\rm d}\tau~\tfrac{1}{2}e^{-1}D_{\tau}x^{M}D_{\tau}x^{N}\mathcal{H}_{MN}(x)-\tfrac{1}{2}m^{2}e$$

II. String action Hull 2006, Lee-JHP 2013, Arvanitakis-Blair 2017

$$S_{\rm string}=\tfrac{1}{4\pi\alpha'}\int{\rm d}^{2}\sigma~-\tfrac{1}{2}\sqrt{-h}h^{ij}D_{i}x^{M}D_{j}x^{N}\mathcal{H}_{MN}(x)-\epsilon^{ij}D_{i}x^{M}\mathcal{A}_{jM}$$

With the $(0,0)$ Riemannian DFT-metric plugged, after integrating out the auxiliary fields, the above actions reduce to the conventional ones:

$$S_{\rm particle}\Rightarrow\int{\rm d}\tau~\tfrac{1}{2}e^{-1}\dot{x}^{\mu}\dot{x}^{\nu}g_{\mu\nu}-\tfrac{1}{2}m^{2}e\,,$$

$$S_{\rm string}\Rightarrow\tfrac{1}{2\pi\alpha'}\int{\rm d}^{2}\sigma~-\tfrac{1}{2}\sqrt{-h}h^{ij}\partial_{i}x^{\mu}\partial_{j}x^{\nu}g_{\mu\nu}+\tfrac{1}{2}\epsilon^{ij}\partial_{i}x^{\mu}\partial_{j}x^{\nu}B_{\mu\nu}+\tfrac{1}{2}\epsilon^{ij}\partial_{i}\tilde{x}_{\mu}\partial_{j}x^{\mu}\,.$$

III. κ-symmetric Green-Schwarz doubled-yet-gauged superstring, unifying IIA & IIB JHP 2016

$$\mathcal{S}_{\rm GS}=\tfrac{1}{4\pi\alpha'}\int{\rm d}^{2}\sigma~-\tfrac{1}{2}\sqrt{-h}h^{ij}\Pi_{i}^{M}\Pi_{j}^{N}\mathcal{H}_{MN}-\epsilon^{ij}D_{i}x^{M}\left(\mathcal{A}_{jM}-i\Sigma_{jM}\right)\,,$$

where $\Pi_{i}^{M}:=D_{i}x^{M}-i\Sigma_{i}^{M}$ and $\Sigma_{i}^{M}:=\bar{\theta}\gamma^{M}\partial_{i}\theta+\bar{\theta}'\bar{\gamma}^{M}\partial_{i}\theta'$.

Doubled-yet-gauged sigma models

The definition of the proper length readily leads to 'completely covariant' actions:

I. Particle action Ko-JHP-Suh 2016

$$\mathcal{S}_{\rm particle} = \int {\rm d}\tau \ \tfrac{1}{2} e^{-1} D_\tau x^M D_\tau x^N \mathcal{H}_{MN}(x) - \tfrac{1}{2} m^2 e$$

II. String action Hull 2006, Lee-JHP 2013, Arvanitakis-Blair 2017

$$S_{\rm string} = \tfrac{1}{4\pi\alpha'} \int {\rm d}^2\sigma \ - \tfrac{1}{2}\sqrt{-h} h^{ij} D_i x^M D_j x^N \mathcal{H}_{MN}(x) - \epsilon^{ij} D_i x^M \mathcal{A}_{jM}$$

With the $(0,0)$ Riemannian DFT-metric plugged, after integrating out the auxiliary fields, the above actions reduce to the conventional ones:

$$S_{\rm particle} \Rightarrow \int {\rm d}\tau \ \frac{1}{2} e^{-1} \dot{x}^\mu \dot{x}^\nu g_{\mu\nu} - \frac{1}{2} m^2 e \,,$$

$$S_{\rm string} \quad \Rightarrow \frac{1}{2\pi\alpha'} \int {\rm d}^2\sigma \ - \frac{1}{2}\sqrt{-h} h^{ij} \partial_i x^\mu \partial_j x^\nu g_{\mu\nu} + \frac{1}{2}\epsilon^{ij}\partial_i x^\mu \partial_j x^\nu B_{\mu\nu} + \frac{1}{2}\epsilon^{ij}\partial_i \tilde{x}_\mu \partial_j x^\mu \,.$$

III. κ-symmetric Green-Schwarz doubled-yet-gauged superstring, unifying IIA & IIB JHP 2016

$$\mathcal{S}_{\rm GS} = \tfrac{1}{4\pi\alpha'} \int {\rm d}^2\sigma \ - \tfrac{1}{2}\sqrt{-h} h^{ij} \Pi_i^M \Pi_j^N \mathcal{H}_{MN} - \epsilon^{ij} D_i x^M \left(\mathcal{A}_{jM} - i\Sigma_{jM}\right) \,,$$

where $\Pi_i^M := D_i x^M - i\Sigma_i^M$ and $\Sigma_i^M := \bar{\theta}\gamma^M \partial_i \theta + \bar{\theta}' \bar{\gamma}^M \partial_i \theta'$.

On the other hand, upon a generic $(n, \bar{n})$ non-Riemannian backgrounds, the auxiliary gauge potential decomposes into three parts:

$$A_\mu = K_{\mu\rho} H^{\rho\nu} A_\nu + X^i_\mu Y^\nu_i A_\nu + \bar{X}^{\bar{\imath}}_\mu \bar{Y}^\nu_{\bar{\imath}} A_\nu \,.$$

– The first part appears quadratically, which leads to Gaussian integral.

– The second and third parts appear linearly, as Lagrange multipliers, to prescribe

i) Particle freezes over the $(n + \bar{n})$ dimensions

$$X^i_\mu \dot{x}^\mu \equiv 0 \,, \qquad \bar{X}^{\bar{\imath}}_\mu \dot{x}^\mu \equiv 0 \,.$$

Remaining orthogonal directions are described by a reduced action:

$$S_{\rm particle} \;\Rightarrow\; \int {\rm d}\tau \; \tfrac{1}{2} e^{-1} \dot{x}^\mu \dot{x}^\nu K_{\mu\nu} - \tfrac{1}{2} m^2 e \,.$$

ii) String becomes chiral over the n dimensions and anti-chiral over the $\bar{n}$ dimensions

$$X^i_\mu \left(\partial_\alpha x^\mu + \tfrac{1}{\sqrt{-h}} \epsilon_\alpha{}^\beta \partial_\beta x^\mu \right) \equiv 0 \,, \qquad \bar{X}^{\bar{\imath}}_\mu \left(\partial_\alpha x^\mu - \tfrac{1}{\sqrt{-h}} \epsilon_\alpha{}^\beta \partial_\beta x^\mu \right) \equiv 0 \,.$$

$$S_{\rm string} \;\Rightarrow\; \tfrac{1}{2\pi\alpha'} \int {\rm d}^2\sigma \; - \tfrac{1}{2} \sqrt{-h} h^{ij} \partial_i x^\mu \partial_j x^\nu K_{\mu\nu} + \tfrac{1}{2} \epsilon^{ij} \partial_i x^\mu \partial_j x^\nu B_{\mu\nu} + \tfrac{1}{2} \epsilon^{ij} \partial_i \tilde{x}_\mu \partial_j x^\mu \,.$$

On the other hand, upon a generic $(n,\bar{n})$ non-Riemannian backgrounds, the auxiliary gauge potential decomposes into three parts:

$$A_{\mu} = K_{\mu\rho}H^{\rho\nu}A_{\nu} + X^{i}_{\mu}Y^{\nu}_{i}A_{\nu} + \bar{X}^{\bar{\imath}}_{\mu}\bar{Y}^{\nu}_{\bar{\imath}}A_{\nu}\,.$$

- The first part appears quadratically, which leads to Gaussian integral.
- The second and third parts appear linearly, as Lagrange multipliers, to prescribe

i) **Particle freezes over the $(n+\bar{n})$ dimensions**

$$X^{i}_{\mu}\dot{x}^{\mu} \equiv 0\,, \qquad\qquad \bar{X}^{\bar{\imath}}_{\mu}\dot{x}^{\mu} \equiv 0\,.$$

Remaining orthogonal directions are described by a reduced action:

$$S_{\rm particle} \;\Rightarrow\; \int {\rm d}\tau\;\tfrac{1}{2}e^{-1}\dot{x}^{\mu}\dot{x}^{\nu}K_{\mu\nu} - \tfrac{1}{2}m^{2}e\,.$$

ii) **String becomes chiral over the n dimensions and anti-chiral over the $\bar{n}$ dimensions**

$$X^{i}_{\mu}\left(\partial_{\alpha}x^{\mu} + \tfrac{1}{\sqrt{-h}}\epsilon_{\alpha}{}^{\beta}\partial_{\beta}x^{\mu}\right) \equiv 0\,, \qquad \bar{X}^{\bar{\imath}}_{\mu}\left(\partial_{\alpha}x^{\mu} - \tfrac{1}{\sqrt{-h}}\epsilon_{\alpha}{}^{\beta}\partial_{\beta}x^{\mu}\right) \equiv 0\,.$$

$$S_{\rm string} \;\Rightarrow\; \tfrac{1}{2\pi\alpha'}\int {\rm d}^{2}\sigma\; -\tfrac{1}{2}\sqrt{-h}h^{ij}\partial_{i}x^{\mu}\partial_{j}x^{\nu}K_{\mu\nu} + \tfrac{1}{2}\epsilon^{ij}\partial_{i}x^{\mu}\partial_{j}x^{\nu}B_{\mu\nu} + \tfrac{1}{2}\epsilon^{ij}\partial_{i}\tilde{x}_{\mu}\partial_{j}x^{\mu}\,.$$

On the other hand, upon a generic $(n, \bar{n})$ non-Riemannian backgrounds, the auxiliary gauge potential decomposes into three parts:

$$A_\mu = K_{\mu\rho} H^{\rho\nu} A_\nu + X_\mu^i Y_i^\nu A_\nu + \bar{X}_\mu^{\bar{\imath}} \bar{Y}_{\bar{\imath}}^\nu A_\nu \,.$$

- The first part appears quadratically, which leads to Gaussian integral.
- The second and third parts appear linearly, as Lagrange multipliers, to prescribe

i) Particle freezes over the $(n + \bar{n})$ dimensions

$$X_\mu^i \dot{x}^\mu \equiv 0 \,, \qquad \bar{X}_\mu^{\bar{\imath}} \dot{x}^\mu \equiv 0 \,.$$

Remaining orthogonal directions are described by a reduced action:

$$S_{\rm particle} \;\Rightarrow\; \int \mathrm{d}\tau \; \tfrac{1}{2} e^{-1} \dot{x}^\mu \dot{x}^\nu K_{\mu\nu} - \tfrac{1}{2} m^2 e \,.$$

ii) String becomes chiral over the n dimensions and anti-chiral over the $\bar{n}$ dimensions

$$X_\mu^i \left(\partial_\alpha x^\mu + \frac{1}{\sqrt{-h}} \epsilon_\alpha{}^\beta \partial_\beta x^\mu\right) \equiv 0 \,, \qquad \bar{X}_\mu^{\bar{\imath}} \left(\partial_\alpha x^\mu - \frac{1}{\sqrt{-h}} \epsilon_\alpha{}^\beta \partial_\beta x^\mu\right) \equiv 0 \,.$$

$$S_{\rm string} \;\Rightarrow\; \frac{1}{2\pi\alpha'} \int \mathrm{d}^2\sigma \; - \tfrac{1}{2}\sqrt{-h} h^{ij} \partial_i x^\mu \partial_j x^\nu K_{\mu\nu} + \tfrac{1}{2} \epsilon^{ij} \partial_i x^\mu \partial_j x^\nu B_{\mu\nu} + \tfrac{1}{2} \epsilon^{ij} \partial_i \tilde{x}_\mu \partial_j x^\mu \,.$$

Comment: Relation to Graded Poisson Geometry Basile-Joung-JHP 1910.13120

- In 2016, **Deser and Sämann** formulates the generalized Lie derivative using a graded Poisson bracket:

$$\Big[T(x,\theta)\,,\;[p_A\theta^A,\xi_B\theta^B\}\Big\} = \hat{\mathcal{L}}_\xi T(x,\theta)\,,\qquad [F,G\} := \frac{\partial F}{\partial x^A}\frac{\partial G}{\partial p_A} - \frac{\partial F}{\partial p_A}\frac{\partial G}{\partial x^A} - (-1)^{\deg(F)}\frac{\partial F}{\partial\theta^A}\frac{\partial G}{\partial\theta_A}$$

where $T(x,\theta) = \frac{1}{p!}T_{C_1C_2\cdots A_p}(x)\theta^{C_1}\theta^{C_2}\cdots\theta^{C_p}$.

- Recently, we have identified this graded Poisson bracket as the Dirac bracket in the Hamiltonian formulation of the Faddeev–Popov doubled-yet-gagued particle action,

$$S_{\rm F.P.} = \int {\rm d}\tau\ \tfrac{1}{2}e^{-1}D_\tau x^A D_\tau x^B\mathcal{H}_{AB}(x) - \tfrac{1}{2}m^2 e + k_A\mathcal{A}^A + k(e-1) + \tfrac{1}{2}\theta_A\dot{\theta}^A + \sum_{\alpha=1}^{2}\tfrac{1}{2}\vartheta_\alpha\dot{\vartheta}^\alpha\,,$$

where $\theta^A = B^A + C^A$ and C^A is the derivative-index-valued ghost for the coordinate gauge symmetry,

$$\tfrac{1}{2}\theta_A\dot{\theta}^A = B_A\dot{C}^A + \tfrac{1}{2}\tfrac{{\rm d}}{{\rm d}\tau}\Big(C^AB_A\Big)\,.$$

Further, intringuingly, the *bc* ghost system for the worldline diffeomorphisms has also $\mathbf{O}(1,1)$ symmetry,

$$\vartheta_1 = \vartheta^2 = b\,,\qquad \vartheta_2 = \vartheta^1 = c\,,\qquad \sum_{\alpha=1}^{2}\tfrac{1}{2}\vartheta_\alpha\dot{\vartheta}^\alpha = b\dot{c} + \tfrac{1}{2}\tfrac{{\rm d}}{{\rm d}\tau}(cb)\,.$$

- Requiring target-spacetime DFT-diffeomorphisms on the wave function subject to Hamiltonian constraint,

$$\Big(\hat{p}_A\mathcal{H}^{AB}(\hat{x})\hat{p}_B + m^2\Big)\Psi(\hat{x})|0\rangle = 0\,,\qquad \hat{p}_A = -i\hbar\partial_A \;\Longrightarrow\; -i\hbar\nabla_A\,,$$

one can obtain quantum corrections to the classical action, analogously to the Fradkin–Tseytlin term,

$$S_\hbar = \int {\rm d}\tau\ \tfrac{1}{2}e^{-1}\Big(D_\tau x^A - i\hbar e\mathcal{H}^{AC}\partial_C d\Big)\Big(D_\tau x^B - i\hbar e\mathcal{H}^{BD}\partial_D d\Big)\mathcal{H}_{AB} - \tfrac{1}{2}m^2 e\,.$$

Comment: Relation to Graded Poisson Geometry Basile-Joung-JHP 1910.13120

- In 2016, **Deser and Sämann** formulates the generalized Lie derivative using a graded Poisson bracket:

$$\Big[T(x,\theta)\,,\ [p_A\theta^A,\xi_B\theta^B\}\Big\} = \hat{\mathcal{L}}_\xi T(x,\theta)\,,\qquad [F,G\} := \frac{\partial F}{\partial x^A}\frac{\partial G}{\partial p_A} - \frac{\partial F}{\partial p_A}\frac{\partial G}{\partial x^A} - (-1)^{\deg(F)}\frac{\partial F}{\partial\theta^A}\frac{\partial G}{\partial\theta_A}$$

where $T(x,\theta) = \frac{1}{p!}T_{C_1C_2\cdots A_p}(x)\theta^{C_1}\theta^{C_2}\cdots\theta^{C_p}$.

- Recently, we have identified this graded Poisson bracket as the Dirac bracket in the Hamiltonian formulation of the Faddeev–Popov doubled-yet-gagued particle action,

$$S_{\rm F.P.} = \int \mathrm{d}\tau\ \tfrac{1}{2}e^{-1}D_\tau x^A D_\tau x^B\mathcal{H}_{AB}(x) - \tfrac{1}{2}m^2 e + k_A\mathcal{A}^A + k(e-1) + \tfrac{1}{2}\theta_A\dot{\theta}^A + \sum_{\alpha=1}^{2}\tfrac{1}{2}\vartheta_\alpha\dot{\vartheta}^\alpha\,,$$

where $\theta^A = B^A + C^A$ and C^A is the derivative-index-valued ghost for the coordinate gauge symmetry,

$$\tfrac{1}{2}\theta_A\dot{\theta}^A = B_A\dot{C}^A + \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(C^A B_A\Big)\ .$$

Further, intringuingly, the *bc* ghost system for the worldline diffeomorphisms has also $\mathbf{O}(1,1)$ symmetry,

$$\vartheta_1 = \vartheta^2 = b\,,\qquad \vartheta_2 = \vartheta^1 = c\,,\qquad \sum_{\alpha=1}^{2}\tfrac{1}{2}\vartheta_\alpha\dot{\vartheta}^\alpha = b\dot{c} + \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}(cb)\,.$$

- Requiring target-spacetime DFT-diffeomorphisms on the wave function subject to Hamiltonian constraint,

$$\Big(\hat{p}_A\mathcal{H}^{AB}(\hat{x})\hat{p}_B + m^2\Big)\Psi(\hat{x})|0\rangle = 0\,,\qquad \hat{p}_A = -i\hbar\partial_A \implies -i\hbar\nabla_A\,,$$

one can obtain quantum corrections to the classical action, analogously to the Fradkin–Tseytlin term,

$$S_\hbar = \int \mathrm{d}\tau\ \tfrac{1}{2}e^{-1}\Big(D_\tau x^A - i\hbar e\mathcal{H}^{AC}\partial_C d\Big)\Big(D_\tau x^B - i\hbar e\mathcal{H}^{BD}\partial_D d\Big)\mathcal{H}_{AB} - \tfrac{1}{2}m^2 e\,.$$

Comment: Relation to Graded Poisson Geometry Basile-Joung-JHP 1910.13120

- In 2016, **Deser and Sämann** formulates the generalized Lie derivative using a graded Poisson bracket:

$$\Big[T(x,\theta)\,,\ [p_A\theta^A,\xi_B\theta^B\}\Big\} = \hat{\mathcal{L}}_\xi T(x,\theta)\,, \qquad [F,G\} := \frac{\partial F}{\partial x^A}\frac{\partial G}{\partial p_A} - \frac{\partial F}{\partial p_A}\frac{\partial G}{\partial x^A} - (-1)^{\mathrm{deg}(F)}\frac{\partial F}{\partial\theta^A}\frac{\partial G}{\partial\theta_A}$$

where $T(x,\theta) = \frac{1}{p!}T_{C_1C_2\cdots A_p}(x)\theta^{C_1}\theta^{C_2}\cdots\theta^{C_p}$.

- Recently, we have identified this graded Poisson bracket as the Dirac bracket in the Hamiltonian formulation of the Faddeev–Popov doubled-yet-gagued particle action,

$$S_{\mathrm{F.P.}} = \int \mathrm{d}\tau\ \tfrac{1}{2}e^{-1}D_\tau x^A D_\tau x^B\mathcal{H}_{AB}(x) - \tfrac{1}{2}m^2e + k_A\mathcal{A}^A + k(e-1) + \tfrac{1}{2}\theta_A\dot{\theta}^A + \sum_{\alpha=1}^{2}\tfrac{1}{2}\vartheta_\alpha\dot{\vartheta}^\alpha\,,$$

where $\theta^A = B^A + C^A$ and C^A is the derivative-index-valued ghost for the coordinate gauge symmetry,

$$\tfrac{1}{2}\theta_A\dot{\theta}^A = B_A\dot{C}^A + \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}\tau}\Big(C^AB_A\Big)\,.$$

Further, intringuingly, the *bc* ghost system for the worldline diffeomorphisms has also $\mathbf{O}(1,1)$ symmetry,

$$\vartheta_1 = \vartheta^2 = b\,, \qquad \vartheta_2 = \vartheta^1 = c\,, \qquad \sum_{\alpha=1}^{2}\tfrac{1}{2}\vartheta_\alpha\dot{\vartheta}^\alpha = b\dot{c} + \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}\tau}(cb)\,.$$

- Requiring target-spacetime DFT-diffeomorphisms on the wave function subject to Hamiltonian constraint,

$$\Big(\hat{p}_A\mathcal{H}^{AB}(\hat{x})\hat{p}_B + m^2\Big)\Psi(\hat{x})|0\rangle = 0\,, \qquad \hat{p}_A = -i\hbar\partial_A \implies -i\hbar\nabla_A\,,$$

one can obtain quantum corrections to the classical action, analogously to the Fradkin–Tseytlin term,

$$S_\hbar = \int \mathrm{d}\tau\ \tfrac{1}{2}e^{-1}\Big(D_\tau x^A - i\hbar e\mathcal{H}^{AC}\partial_C d\Big)\Big(D_\tau x^B - i\hbar e\mathcal{H}^{BD}\partial_D d\Big)\mathcal{H}_{AB} - \tfrac{1}{2}m^2e\,.$$

Einstein Double Field Equations

$$G_{AB} = 8\pi G T_{AB}$$

where A, B **are** $\mathbf{O}(D, D)$ **indices**

Semi-covariant formalism

Imtak Jeon, Kanghoon Lee, & JHP 2010, 2011

- Semi-covariant derivative :

$$\nabla_C T_{A_1 A_2 \cdots A_n} := \partial_C T_{A_1 A_2 \cdots A_n} - \omega_T \Gamma^B{}_{BC} T_{A_1 A_2 \cdots A_n} + \sum_{i=1}^{n} \Gamma_{CA_i}{}^B T_{A_1 \cdots A_{i-1} B A_{i+1} \cdots A_n} \,,$$

for which the 'DFT-Christoffel' connection can be uniquely fixed,

$$\Gamma_{CAB} = 2\left(P\partial_C P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^D \bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E\right)\partial_D P_{EC} - \frac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^D + P_{C[A}P_{B]}{}^D\right)\left(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\right)$$

by demanding compatibility with $\{\mathcal{J}_{AB}, \mathcal{H}_{AB}, d\}$, torsionless condition, and projection property,

$$\nabla_A P_{BC} = \nabla_A \bar{P}_{BC} = \nabla_A d = 0 \,, \qquad \hat{\mathcal{L}}_\xi^\partial = \hat{\mathcal{L}}_\xi^\nabla \;\Leftrightarrow\; \Gamma_{[ABC]} = 0 \,, \qquad (\mathcal{P} + \bar{\mathcal{P}})_{ABC}{}^{DEF}\Gamma_{DEF} = 0 \,,$$

where multi-indexed projectors are

$$\mathcal{P}_{ABC}{}^{DEF} := P_A{}^D P_{[B}{}^{[E} P_{C]}{}^{F]} + \frac{2}{P_M{}^M - 1} P_{A[B} P_{C]}{}^{[E} P^{F]D} \,, \qquad \text{same for } \bar{\mathcal{P}}_{ABC}{}^{DEF} \text{ with } P_{AB} \leftrightarrow \bar{P}_{AB} \,.$$

– In particular, DFT-Killing equations can be defined from

$$\hat{\mathcal{L}}_\xi^\nabla \mathcal{H}_{AB} = 8\bar{P}_{(A}{}^{[C} P_{B)}{}^{D]} \nabla_C \xi_D \,, \qquad \hat{\mathcal{L}}_\xi^\nabla d = -\tfrac{1}{2}\nabla_A \xi^A \,.$$

– There are no normal coordinates where Γ_{CAB} would vanish point-wise: Equivalence Principle is broken for string (*i.e.* extended object), but recoverable when coupled to point particle (or scalar field).

Semi-covariant formalism

Imtak Jeon, Kanghoon Lee, & JHP 2010, 2011

- Semi-covariant derivative :

$$\nabla_C T_{A_1 A_2 \cdots A_n} := \partial_C T_{A_1 A_2 \cdots A_n} - \omega_T \Gamma^B{}_{BC} T_{A_1 A_2 \cdots A_n} + \sum_{i=1}^{n} \Gamma_{CA_i}{}^B T_{A_1 \cdots A_{i-1} B A_{i+1} \cdots A_n} ,$$

for which the 'DFT-Christoffel' connection can be uniquely fixed,

$$\Gamma_{CAB} = 2\big(P\partial_C P\bar{P}\big)_{[AB]} + 2\big(\bar{P}_{[A}{}^D \bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E\big)\partial_D P_{EC} - \tfrac{4}{D-1}\big(\bar{P}_{C[A}\bar{P}_{B]}{}^D + P_{C[A}P_{B]}{}^D\big)\big(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\big)$$

by demanding compatibility with $\{\mathcal{J}_{AB}, \mathcal{H}_{AB}, d\}$, torsionless condition, and projection property,

$$\nabla_A P_{BC} = \nabla_A \bar{P}_{BC} = \nabla_A d = 0 , \qquad \hat{\mathcal{L}}_\xi^\partial = \hat{\mathcal{L}}_\xi^\nabla \;\; \Leftrightarrow \;\; \Gamma_{[ABC]} = 0 , \qquad (\mathcal{P} + \bar{\mathcal{P}})_{ABC}{}^{DEF}\Gamma_{DEF} = 0 ,$$

where multi-indexed projectors are

$$\mathcal{P}_{ABC}{}^{DEF} := P_A{}^D P_{[B}{}^{[E} P_{C]}{}^{F]} + \tfrac{2}{P_M{}^M - 1} P_{A[B} P_{C]}{}^{[E} P^{F]D} , \qquad \text{same for } \bar{\mathcal{P}}_{ABC}{}^{DEF} \text{ with } P_{AB} \leftrightarrow \bar{P}_{AB} .$$

– In particular, DFT-Killing equations can be defined from

$$\hat{\mathcal{L}}_\xi^\nabla \mathcal{H}_{AB} = 8\bar{P}_{(A}{}^{[C} P_{B)}{}^{D]} \nabla_C \xi_D , \qquad \hat{\mathcal{L}}_\xi^\nabla d = -\tfrac{1}{2}\nabla_A \xi^A .$$

– There are no normal coordinates where Γ_{CAB} would vanish point-wise: Equivalence Principle is broken for string (*i.e.* extended object), but recoverable when coupled to point particle (or scalar field).

- Semi-covariant Riemann curvature :

$$S_{ABCD}=S_{[AB][CD]}=S_{CDAB}:=\tfrac{1}{2}\left(R_{ABCD}+R_{CDAB}-\Gamma^{E}{}_{AB}\Gamma_{ECD}\right)\,,\qquad S_{[ABC]D}=0\,,$$

where R_{ABCD} denotes the ordinary "field strength", $R_{CDAB}=\partial_A\Gamma_{BCD}-\partial_B\Gamma_{ACD}+\Gamma_{AC}{}^{E}\Gamma_{BED}-\Gamma_{BC}{}^{E}\Gamma_{AED}$.

By construction, it varies as 'total derivative',

$$\delta S_{ABCD}=\nabla_{[A}\delta\Gamma_{B]CD}+\nabla_{[C}\delta\Gamma_{D]AB}\,,$$

which is useful for Lagrangian variation, *i.e.* action principle.

- Semi-covariant 'Master' derivative :

$$\mathcal{D}_A:=\partial_A+\Gamma_A+\Phi_A+\bar{\Phi}_A=\nabla_A+\Phi_A+\bar{\Phi}_A\,.$$

The two spin connections are determined in terms of the DFT-Christoffel connection,

$$\Phi_{Apq}=V^{B}{}_{p}\nabla_A V_{Bq}\,,\qquad \bar{\Phi}_{A\bar{p}\bar{q}}=\bar{V}^{B}{}_{\bar{p}}\nabla_A\bar{V}_{B\bar{q}}\,,$$

by requiring the compatibility with the vielbeins,

$$\mathcal{D}_A V_{Bp}=\nabla_A V_{Bp}+\Phi_{Ap}{}^{q}V_{Bq}=0\,,\qquad \mathcal{D}_A\bar{V}_{B\bar{p}}=\nabla_A\bar{V}_{B\bar{p}}+\bar{\Phi}_{A\bar{p}}{}^{\bar{q}}\bar{V}_{B\bar{q}}=0\,.$$

- Semi-covariant Riemann curvature :

$$S_{ABCD}=S_{[AB][CD]}=S_{CDAB}:=\tfrac{1}{2}\left(R_{ABCD}+R_{CDAB}-\Gamma^{E}{}_{AB}\Gamma_{ECD}\right)\,,\qquad S_{[ABC]D}=0\,,$$

where R_{ABCD} denotes the ordinary "field strength", $R_{CDAB}{=}\partial_A\Gamma_{BCD}{-}\partial_B\Gamma_{ACD}{+}\Gamma_{AC}{}^{E}\Gamma_{BED}{-}\Gamma_{BC}{}^{E}\Gamma_{AED}$.

By construction, it varies as 'total derivative',

$$\delta S_{ABCD}=\nabla_{[A}\delta\Gamma_{B]CD}+\nabla_{[C}\delta\Gamma_{D]AB}\,,$$

which is useful for Lagrangian variation, *i.e.* action principle.

- Semi-covariant 'Master' derivative :

$$\mathcal{D}_A:=\partial_A+\Gamma_A+\Phi_A+\bar{\Phi}_A=\nabla_A+\Phi_A+\bar{\Phi}_A\,.$$

The two spin connections are determined in terms of the DFT-Christoffel connection,

$$\Phi_{Apq}=V^{B}{}_{p}\nabla_A V_{Bq}\,,\qquad\qquad \bar{\Phi}_{A\bar{p}\bar{q}}=\bar{V}^{B}{}_{\bar{p}}\nabla_A\bar{V}_{B\bar{q}}\,,$$

by requiring the compatibility with the vielbeins,

$$\mathcal{D}_A V_{Bp}=\nabla_A V_{Bp}+\Phi_{Ap}{}^{q}V_{Bq}=0\,,\qquad \mathcal{D}_A\bar{V}_{B\bar{p}}=\nabla_A\bar{V}_{B\bar{p}}+\bar{\Phi}_{A\bar{p}}{}^{\bar{q}}\bar{V}_{B\bar{q}}=0\,.$$

Anomaly is under control through the six-indexed projectors

- Semi-covariance:

$$\delta_\xi\left(\nabla_C T_{A_1\cdots A_n}\right)=\hat{\mathcal{L}}_\xi\left(\nabla_C T_{A_1\cdots A_n}\right)+\sum_{i=1}^{n}2(\mathcal{P}+\bar{\mathcal{P}})_{CA_i}{}^{BDEF}\partial_D\partial_E\xi_F\,T_{A_1\cdots A_{i-1}BA_{i+1}\cdots A_n}\,,$$

$$\delta_\xi S_{ABCD}=\hat{\mathcal{L}}_\xi S_{ABCD}+2\nabla_{[A}\Big((\mathcal{P}+\bar{\mathcal{P}})_{B][CD]}{}^{EFG}\partial_E\partial_F\xi_G\Big)+2\nabla_{[C}\Big((\mathcal{P}+\bar{\mathcal{P}})_{D][AB]}{}^{EFG}\partial_E\partial_F\xi_G\Big)\,.$$

- This is due to

$$\delta_\xi\Gamma_{CAB}=\hat{\mathcal{L}}_\xi\Gamma_{CAB}+2\Big[(\mathcal{P}+\bar{\mathcal{P}})_{CAB}{}^{FDE}-\delta_C^F\delta_A^D\delta_B^E\Big]\partial_F\partial_{[D}\xi_{E]}\,.$$

Ideally one might desire to cancel these red-colored anomalies by adding extra terms to Γ_{CAB}.

But, since

$$\delta\mathcal{H}_{AB}=(P\delta\mathcal{H}\bar{P})_{AB}+(\bar{P}\delta\mathcal{H}P)_{AB}\,,\qquad \delta_\xi(\partial_C\mathcal{H}_{AB})=\hat{\mathcal{L}}_\xi(\partial_C\mathcal{H}_{AB})+8\bar{P}_{(A}{}^{D}P_{B)}{}^{E}\partial_C\partial_{[D}\xi_{E]}\,,$$

it is impossible to construct such compensating terms out of the derivatives of $\mathcal{H}_{AB}$.

- However, we can easily project out the anomalies.

Complete covariantization: fixing the O(D,D) coupling to matter

– Tensors:

$$P_C{}^D\bar{P}_{A_1}{}^{B_1}\cdots\bar{P}_{A_n}{}^{B_n}\nabla_D T_{B_1\cdots B_n}\quad\Longrightarrow\quad\mathcal{D}_p T_{\bar{q}_1\bar{q}_2\cdots\bar{q}_n}\,,$$

$$\bar{P}_C{}^D P_{A_1}{}^{B_1}\cdots P_{A_n}{}^{B_n}\nabla_D T_{B_1\cdots B_n}\quad\Longrightarrow\quad\mathcal{D}_{\bar{p}} T_{q_1 q_2\cdots q_n}\,,$$

$$\mathcal{D}^p T_{p\bar{q}_1\bar{q}_2\cdots\bar{q}_n}\,,\qquad\mathcal{D}^{\bar{p}}T_{\bar{p}q_1q_2\cdots q_n}\,;\qquad\mathcal{D}_p\mathcal{D}^p T_{\bar{q}_1\bar{q}_2\cdots\bar{q}_n}\,,\qquad\mathcal{D}_{\bar{p}}\mathcal{D}^{\bar{p}}T_{q_1q_2\cdots q_n}\,.$$

– Yang-Mills:

$$\mathcal{F}_{p\bar{q}}:=\mathcal{F}_{AB}V^A{}_p\bar{V}^B{}_{\bar{q}}\qquad\text{where}\qquad\mathcal{F}_{AB}:=\nabla_A W_B-\nabla_B W_A-i\,[W_A,W_B]\,.$$

– Spinors, ρ^α, $\psi^\alpha_{\bar{p}}$:

$$\gamma^p\mathcal{D}_p\rho\,,\qquad\mathcal{D}_{\bar{p}}\rho\,,\qquad\gamma^p\mathcal{D}_p\psi_{\bar{q}}\,,\qquad\mathcal{D}_{\bar{p}}\psi^{\bar{p}}\,,$$

– RR sector, $\mathcal{C}^\alpha{}_{\bar{\alpha}}$:

$$\mathcal{D}_\pm\mathcal{C}:=\gamma^p\mathcal{D}_p\mathcal{C}\pm\gamma^{(D+1)}\mathcal{D}_{\bar{p}}\mathcal{C}\bar{\gamma}^{\bar{p}}\,,\quad(\mathcal{D}_\pm)^2=0\qquad\Longrightarrow\qquad\mathcal{F}:=\mathcal{D}_+\mathcal{C}\qquad(\text{RR flux})\,.$$

– Curvatures:

$$S_{p\bar{q}}:=S_{AB}V^A{}_p\bar{V}^B{}_{\bar{q}}\quad(\text{Ricci})\,,\qquad S_{(0)}:=(P^{AC}P^{BD}-\bar{P}^{AC}\bar{P}^{BD})S_{ABCD}\quad(\text{scalar}\ \Rightarrow\ \text{'pure' DFT})\,.$$

Complete covariantization: fixing the O(D, D) coupling to matter

– Tensors:

$$P_C{}^D \bar{P}_{A_1}{}^{B_1} \cdots \bar{P}_{A_n}{}^{B_n} \nabla_D T_{B_1 \cdots B_n} \quad \Longrightarrow \quad \mathcal{D}_p T_{\bar{q}_1 \bar{q}_2 \cdots \bar{q}_n} \,,$$

$$\bar{P}_C{}^D P_{A_1}{}^{B_1} \cdots P_{A_n}{}^{B_n} \nabla_D T_{B_1 \cdots B_n} \quad \Longrightarrow \quad \mathcal{D}_{\bar{p}} T_{q_1 q_2 \cdots q_n} \,,$$

$$\mathcal{D}^p T_{p \bar{q}_1 \bar{q}_2 \cdots \bar{q}_n} \,, \qquad \mathcal{D}^{\bar{p}} T_{\bar{p} q_1 q_2 \cdots q_n} \,; \qquad \mathcal{D}_p \mathcal{D}^p T_{\bar{q}_1 \bar{q}_2 \cdots \bar{q}_n} \,, \qquad \mathcal{D}_{\bar{p}} \mathcal{D}^{\bar{p}} T_{q_1 q_2 \cdots q_n} \,.$$

– Yang-Mills:

$$\mathcal{F}_{p\bar{q}} := \mathcal{F}_{AB} V^A{}_p \bar{V}^B{}_{\bar{q}} \qquad \text{where} \qquad \mathcal{F}_{AB} := \nabla_A W_B - \nabla_B W_A - i\,[W_A, W_B] \,.$$

– Spinors, ρ^α, $\psi^\alpha_{\bar{p}}$:

$$\gamma^p \mathcal{D}_p \rho \,, \qquad \mathcal{D}_{\bar{p}} \rho \,, \qquad \gamma^p \mathcal{D}_p \psi_{\bar{q}} \,, \qquad \mathcal{D}_{\bar{p}} \psi^{\bar{p}} \,,$$

– RR sector, $\mathcal{C}^\alpha{}_{\bar{\alpha}}$:

$$\mathcal{D}_\pm \mathcal{C} := \gamma^p \mathcal{D}_p \mathcal{C} \pm \gamma^{(D+1)} \mathcal{D}_{\bar{p}} \mathcal{C} \bar{\gamma}^{\bar{p}} \,, \quad (\mathcal{D}_\pm)^2 = 0 \qquad \Longrightarrow \qquad \mathcal{F} := \mathcal{D}_+ \mathcal{C} \qquad (\,\text{RR flux}\,) \,.$$

– Curvatures:

$$S_{p\bar{q}} := S_{AB} V^A{}_p \bar{V}^B{}_{\bar{q}} \quad (\,\text{Ricci}\,) \,, \qquad S_{(0)} := (P^{AC} P^{BD} - \bar{P}^{AC} \bar{P}^{BD}) S_{ABCD} \quad (\,\text{scalar} \;\Rightarrow\; \text{‘pure’ DFT}\,) \,.$$

Complete covariantization: fixing the $\mathbf{O}(D,D)$ coupling to matter

– Tensors:

$$P_C{}^D \bar{P}_{A_1}{}^{B_1} \cdots \bar{P}_{A_n}{}^{B_n} \nabla_D T_{B_1 \cdots B_n} \quad \Longrightarrow \quad \mathcal{D}_p T_{\bar{q}_1 \bar{q}_2 \cdots \bar{q}_n} \,,$$

$$\bar{P}_C{}^D P_{A_1}{}^{B_1} \cdots P_{A_n}{}^{B_n} \nabla_D T_{B_1 \cdots B_n} \quad \Longrightarrow \quad \mathcal{D}_{\bar{p}} T_{q_1 q_2 \cdots q_n} \,,$$

$$\mathcal{D}^p T_{p \bar{q}_1 \bar{q}_2 \cdots \bar{q}_n} \,, \qquad \mathcal{D}^{\bar{p}} T_{\bar{p} q_1 q_2 \cdots q_n} \,; \qquad \mathcal{D}_p \mathcal{D}^p T_{\bar{q}_1 \bar{q}_2 \cdots \bar{q}_n} \,, \qquad \mathcal{D}_{\bar{p}} \mathcal{D}^{\bar{p}} T_{q_1 q_2 \cdots q_n} \,.$$

– Yang-Mills:

$$\mathcal{F}_{p\bar{q}} := \mathcal{F}_{AB} V^A{}_p \bar{V}^B{}_{\bar{q}} \qquad \text{where} \qquad \mathcal{F}_{AB} := \nabla_A W_B - \nabla_B W_A - i\,[W_A, W_B] \,.$$

– Spinors, ρ^α, $\psi^\alpha_{\bar{p}}$:

$$\gamma^p \mathcal{D}_p \rho \,, \qquad \mathcal{D}_{\bar{p}} \rho \,, \qquad \gamma^p \mathcal{D}_p \psi_{\bar{q}} \,, \qquad \mathcal{D}_{\bar{p}} \psi^{\bar{p}} \,,$$

– RR sector, $\mathcal{C}^\alpha{}_{\bar{\alpha}}$:

$$\mathcal{D}_\pm \mathcal{C} := \gamma^p \mathcal{D}_p \mathcal{C} \pm \gamma^{(D+1)} \mathcal{D}_{\bar{p}} \mathcal{C} \bar{\gamma}^{\bar{p}} \,, \quad (\mathcal{D}_\pm)^2 = 0 \qquad \Longrightarrow \qquad \mathcal{F} := \mathcal{D}_+ \mathcal{C} \qquad (\text{RR flux}) \,.$$

– Curvatures:

$$S_{p\bar{q}} := S_{AB} V^A{}_p \bar{V}^B{}_{\bar{q}} \quad (\text{Ricci}) \,, \qquad S_{(0)} := (P^{AC} P^{BD} - \bar{P}^{AC} \bar{P}^{BD}) S_{ABCD} \quad (\text{scalar} \;\Rightarrow\; \text{‘pure’ DFT}) \,.$$

$\mathbf{O}(D,D)$ coupling to other superstring sectors or the Standard Model

- $D=10$ Maximally Supersymmetric DFT — Jeon-Lee-JHP-Suh 2012 [**Full order construction**]

$$\mathcal{L}_{\rm type\,II}=e^{-2d}\Big[\,\tfrac{1}{8}S_{(0)}+\tfrac{1}{2}{\rm Tr}(\mathcal{F}\bar{\mathcal{F}})+i\bar{\rho}\mathcal{F}\rho'+i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q}+i\tfrac{1}{2}\bar{\rho}\gamma^{p}\mathcal{D}_{p}\rho-i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\rho'$$
$$-i\bar{\psi}^{\bar{p}}\mathcal{D}_{\bar{p}}\rho-i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^{q}\mathcal{D}_{q}\psi_{\bar{p}}+i\bar{\psi}'^{p}\mathcal{D}_{p}\rho'+i\tfrac{1}{2}\bar{\psi}'^{p}\bar{\gamma}^{\bar{q}}\mathcal{D}_{\bar{q}}\psi'_{p}\,\Big]$$

which unifies IIA & IIB SUGRAs, and Gomis-Ooguri gravity as different solution/parametrization sectors.

- $\mathbf{O}(4,4)$ coupling to the $D=4$ Standard Model, — Kangsin Choi & JHP 2015

$$\mathcal{L}_{\rm SM}\;=\;e^{-2d}\left[\begin{array}{l}\frac{1}{16\pi G_{N}}S_{(0)}\\ +\sum_{\mathcal{V}}{\rm Tr}(\mathcal{F}_{p\bar{q}}\mathcal{F}^{p\bar{q}})+\sum_{\psi}\bar{\psi}\gamma^{a}\mathcal{D}_{a}\psi+\sum_{\psi'}\bar{\psi}'\bar{\gamma}^{\bar{a}}\mathcal{D}_{\bar{a}}\psi'\\ -\mathcal{H}^{AB}(\mathcal{D}_{A}\phi)^{\dagger}\mathcal{D}_{B}\phi\,-\,V(\phi)\,+y_{d}\,\bar{q}\cdot\phi\,d+y_{u}\,\bar{q}\cdot\tilde{\phi}\,u+y_{e}\,\bar{l}'\cdot\phi\,e'\end{array}\right]$$

⋆ Every single term above is completely covariant, w.r.t. $\mathbf{O}(D,D)$, DFT-diffeomorphisms, and twofold local Lorentz symmetries. Leptons are for **Spin**$(1,3)$ and quarks are for **Spin**$(3,1)$?!!

Derivation of EDFE from General Covariance 1/2

- Henceforth, we consider a general DFT action coupled to matter fields, Υ_a,

$$\text{Action} = \int_\Sigma e^{-2d}\Big[\tfrac{1}{16\pi G}S_{(0)} + L_{\mathrm{matter}}(\Upsilon_a, \mathcal{D}_A\Upsilon_b)\Big]\,,$$

  and seek the variation of the action induced by all the fields, d, V_{Ap}, $\bar{V}_{A\bar{p}}$, Υ_a.

Note $\delta V_{Ap} = (\bar{P}+P)_A{}^B\delta V_{Bp} = \bar{V}_{A\bar{q}}\bar{V}^{B\bar{q}}\delta V_{Bp} + (\delta V_{B[p}V^B{}_{q]})V_A{}^q$. The 2nd term is a local Lorentz rotation and can be absorbed into $\delta\Upsilon_a$. Thus, only the projected variation, $\bar{V}^B{}_{\bar{q}}\delta V_{Bp} = -V^B{}_p\delta\bar{V}_{B\bar{q}}$, appears.

- Firstly, the 'pure' DFT part transforms, up to total derivatives ($\simeq$), as

$$\delta\Big(e^{-2d}S_{(0)}\Big) \simeq 4e^{-2d}\Big(\bar{V}^{B\bar{q}}\delta V_B{}^p S_{p\bar{q}} - \tfrac{1}{2}\delta d\, S_{(0)}\Big)\,.$$

- Secondly, the variation of the matter part,

$$\delta\Big(e^{-2d}L_{\mathrm{matter}}\Big) \simeq -2e^{-2d}\Big(\bar{V}^{A\bar{q}}\delta V_A{}^p K_{p\bar{q}} - \tfrac{1}{2}\delta d\,T_{(0)} - \tfrac{1}{2}\delta\Upsilon_a\frac{\delta L_{\mathrm{matter}}}{\delta\Upsilon_a}\Big)$$

  naturally defines

$$K_{p\bar{q}} := \frac{1}{2}\left(V_{Ap}\frac{\delta L_{\mathrm{matter}}}{\delta\bar{V}_A{}^{\bar{q}}} - \bar{V}_{A\bar{q}}\frac{\delta L_{\mathrm{matter}}}{\delta V_A{}^p}\right)\,,\qquad T_{(0)} := e^{2d}\times\frac{\delta\Big(e^{-2d}L_{\mathrm{matter}}\Big)}{\delta d}\,.$$

- Henceforth, we consider a general DFT action coupled to matter fields, Υ_a,

$$\text{Action} = \int_\Sigma e^{-2d}\Big[\,\tfrac{1}{16\pi G}S_{(0)} + L_{\rm matter}\big(\Upsilon_a, \mathcal{D}_A\Upsilon_b\big)\Big]\,,$$

and seek the variation of the action induced by all the fields, d, V_{Ap}, $\bar{V}_{Ap}$, Υ_a.

Note $\delta V_{Ap} = (\bar{P}+P)_A{}^B\delta V_{Bp} = \bar{V}_{A\bar{q}}\bar{V}^{B\bar{q}}\delta V_{Bp} + (\delta V_{B[p}V^B{}_{q]})V_A{}^q$. The 2nd term is a local Lorentz rotation and can be absorbed into $\delta\Upsilon_a$. Thus, only the projected variation, $\bar{V}^B{}_{\bar{q}}\delta V_{Bp} = -V^B{}_p\delta\bar{V}_{B\bar{q}}$, appears.

– Firstly, the 'pure' DFT part transforms, up to total derivatives ($\simeq$), as

$$\delta\Big(e^{-2d}S_{(0)}\Big) \simeq 4e^{-2d}\Big(\bar{V}^{B\bar{q}}\delta V_B{}^p S_{p\bar{q}} - \tfrac{1}{2}\delta d\, S_{(0)}\Big)\,.$$

– Secondly, the variation of the matter part,

$$\delta\Big(e^{-2d}L_{\rm matter}\Big) \simeq -2e^{-2d}\Big(\bar{V}^{A\bar{q}}\delta V_A{}^p K_{p\bar{q}} - \tfrac{1}{2}\delta d\, T_{(0)} - \tfrac{1}{2}\delta\Upsilon_a\frac{\delta L_{\rm matter}}{\delta\Upsilon_a}\Big)$$

naturally defines

$$K_{p\bar{q}} := \frac{1}{2}\left(V_{Ap}\frac{\delta L_{\rm matter}}{\delta\bar{V}_A{}^{\bar{q}}} - \bar{V}_{A\bar{q}}\frac{\delta L_{\rm matter}}{\delta V_A{}^p}\right)\,, \qquad T_{(0)} := e^{2d}\times\frac{\delta\Big(e^{-2d}L_{\rm matter}\Big)}{\delta d}\,.$$

Derivation of EDFE from General Covariance 1/2

Angus-Cho-JHP 1804.00964

- Henceforth, we consider a general DFT action coupled to matter fields, Υ_a,

$$\text{Action} = \int_\Sigma e^{-2d}\Big[\tfrac{1}{16\pi G}S_{(0)} + L_{\mathrm{matter}}(\Upsilon_a, \mathcal{D}_A\Upsilon_b)\Big]\,,$$

and seek the variation of the action induced by all the fields, d, V_{Ap}, $\bar{V}_{Ap}$, Υ_a.

Note $\delta V_{Ap} = (\bar{P}+P)_A{}^B\delta V_{Bp} = \bar{V}_{A\bar{q}}\bar{V}^{B\bar{q}}\delta V_{Bp} + (\delta V_{B[p}V^B{}_{q]})V_A{}^q$. The 2nd term is a local Lorentz rotation and can be absorbed into $\delta\Upsilon_a$. Thus, only the projected variation, $\bar{V}^B{}_{\bar{q}}\delta V_{Bp} = -V^B{}_p\delta\bar{V}_{B\bar{q}}$, appears.

  – Firstly, the 'pure' DFT part transforms, up to total derivatives ($\simeq$), as

$$\delta\Big(e^{-2d}S_{(0)}\Big) \simeq 4e^{-2d}\Big(\bar{V}^{B\bar{q}}\delta V_B{}^p S_{p\bar{q}} - \tfrac{1}{2}\delta d\, S_{(0)}\Big)\,.$$

  – Secondly, the variation of the matter part,

$$\delta\Big(e^{-2d}L_{\mathrm{matter}}\Big) \simeq -2e^{-2d}\Big(\bar{V}^{A\bar{q}}\delta V_A{}^p K_{p\bar{q}} - \tfrac{1}{2}\delta d\, T_{(0)} - \tfrac{1}{2}\delta\Upsilon_a\frac{\delta L_{\mathrm{matter}}}{\delta\Upsilon_a}\Big)$$

naturally defines

$$K_{p\bar{q}} := \frac{1}{2}\Big(V_{Ap}\frac{\delta L_{\mathrm{matter}}}{\delta\bar{V}_A{}^{\bar{q}}} - \bar{V}_{A\bar{q}}\frac{\delta L_{\mathrm{matter}}}{\delta V_A{}^p}\Big)\,, \qquad T_{(0)} := e^{2d}\times\frac{\delta\Big(e^{-2d}L_{\mathrm{matter}}\Big)}{\delta d}\,.$$

Derivation of EDFE from General Covariance 1/2

- Henceforth, we consider a general DFT action coupled to matter fields, Υ_a,

$$\text{Action} = \int_\Sigma e^{-2d}\Big[\tfrac{1}{16\pi G}S_{(0)} + L_{\rm matter}\big(\Upsilon_a,\mathcal{D}_A\Upsilon_b\big)\Big]\,,$$

and seek the variation of the action induced by all the fields, d, V_{Ap}, $\bar{V}_{Ap}$, Υ_a.

Note $\delta V_{Ap} = (\bar{P}+P)_A{}^B\delta V_{Bp} = \bar{V}_{A\bar{q}}\bar{V}^{B\bar{q}}\delta V_{Bp} + (\delta V_{B[p}V^B{}_{q]})V_A{}^q$. The 2nd term is a local Lorentz rotation and can be absorbed into $\delta\Upsilon_a$. Thus, only the projected variation, $\bar{V}^B{}_{\bar{q}}\delta V_{Bp} = -V^B{}_p\delta\bar{V}_{B\bar{q}}$, appears.

  – Firstly, the 'pure' DFT part transforms, up to total derivatives ($\simeq$), as

$$\delta\Big(e^{-2d}S_{(0)}\Big) \simeq 4e^{-2d}\Big(\bar{V}^{B\bar{q}}\delta V_B{}^pS_{p\bar{q}} - \tfrac{1}{2}\delta d\,S_{(0)}\Big)\,.$$

  – Secondly, the variation of the matter part,

$$\delta\Big(e^{-2d}L_{\rm matter}\Big) \simeq -2e^{-2d}\left(\bar{V}^{A\bar{q}}\delta V_A{}^pK_{p\bar{q}} - \tfrac{1}{2}\delta d\,T_{(0)} - \tfrac{1}{2}\delta\Upsilon_a\frac{\delta L_{\rm matter}}{\delta\Upsilon_a}\right)$$

  naturally defines

$$K_{p\bar{q}} := \frac{1}{2}\left(V_{Ap}\frac{\delta L_{\rm matter}}{\delta\bar{V}_A{}^{\bar{q}}} - \bar{V}_{A\bar{q}}\frac{\delta L_{\rm matter}}{\delta V_A{}^p}\right)\,, \qquad T_{(0)} := e^{2d}\times\frac{\delta\Big(e^{-2d}L_{\rm matter}\Big)}{\delta d}\,.$$

Derivation of EDFE from General Covariance 2/2

- Combining the two results, the variation of the action reads

$$\delta\text{Action} = \int_\Sigma e^{-2d}\left[\tfrac{1}{4\pi G}\bar{V}^{A\bar{q}}\delta V_A{}^p(S_{p\bar{q}} - 8\pi G K_{p\bar{q}}) - \tfrac{1}{8\pi G}\delta d(S_{(0)} - 8\pi G T_{(0)}) + \delta\Upsilon_a \frac{\delta L_{\rm matter}}{\delta\Upsilon_a}\right].$$

- Specifically when the variation is generated by diffeomorphisms, we have $\delta_\xi\Upsilon_a = \hat{\mathcal{L}}_\xi\Upsilon_a$ and

$$\bar{V}^{A\bar{q}}\delta_\xi V_A{}^p = \bar{V}^{A\bar{q}}\hat{\mathcal{L}}_\xi V_A{}^p = 2\mathcal{D}_{[A}\xi_{B]}\bar{V}^{A\bar{q}}V^{Bp}\,, \qquad \delta_\xi d = -\tfrac{1}{2}e^{2d}\hat{\mathcal{L}}_\xi\left(e^{-2d}\right) = -\tfrac{1}{2}\mathcal{D}_A\xi^A\,.$$

- The Diffeomorphic General Covariance of the Action then implies

$$0 = \int_\Sigma e^{-2d}\left[\tfrac{1}{8\pi G}\xi^B\mathcal{D}^A\left[4V_{[A}{}^p\bar{V}_{B]}{}^{\bar{q}}(S_{p\bar{q}} - 8\pi G K_{p\bar{q}}) - \tfrac{1}{2}\mathcal{J}_{AB}(S_{(0)} - 8\pi G T_{(0)})\right] + \delta_\xi\Upsilon_a\frac{\delta L_{\rm matter}}{\delta\Upsilon_a}\right].$$

This gives the $\mathbf{O}(D,D)$ completion of Einstein curvature, JHP-Rey-Rim-Sakatani 2015

$$G_{AB} := 4V_{[A}{}^p\bar{V}_{B]}{}^{\bar{q}}S_{p\bar{q}} - \tfrac{1}{2}\mathcal{J}_{AB}S_{(0)}\,, \qquad \mathcal{D}_A G^{AB} = 0 \qquad \text{(off-shell)}\,,$$

the $\mathbf{O}(D,D)$ completion of Energy-Momentum tensor, Angus-Cho-JHP 2018

$$T_{AB} := 4V_{[A}{}^p\bar{V}_{B]}{}^{\bar{q}}K_{p\bar{q}} - \tfrac{1}{2}\mathcal{J}_{AB}T_{(0)}\,, \qquad \mathcal{D}_A T^{AB} = 0 \qquad \text{(on-shell)}\,,$$

and hence, the $\mathbf{O}(D,D)$ completion of Einstein Field Equations, $G_{AB} = 8\pi G T_{AB}$.

Derivation of EDFE from General Covariance 2/2

Angus-Cho-JHP 1804.00964

- Combining the two results, the variation of the action reads

$$\delta\text{Action} = \int_\Sigma e^{-2d}\left[\tfrac{1}{4\pi G}\bar{V}^{A\bar{q}}\delta V_A{}^p(S_{p\bar{q}} - 8\pi G K_{p\bar{q}}) - \tfrac{1}{8\pi G}\delta d(S_{(0)} - 8\pi G T_{(0)}) + \delta\Upsilon_a \frac{\delta L_{\text{matter}}}{\delta\Upsilon_a}\right].$$

- Specifically when the variation is generated by diffeomorphisms, we have $\delta_\xi\Upsilon_a = \hat{\mathcal{L}}_\xi\Upsilon_a$ and

$$\bar{V}^{A\bar{q}}\delta_\xi V_A{}^p = \bar{V}^{A\bar{q}}\hat{\mathcal{L}}_\xi V_A{}^p = 2\mathcal{D}_{[A}\xi_{B]}\bar{V}^{A\bar{q}}V^{Bp}\,, \qquad \delta_\xi d = -\tfrac{1}{2}e^{2d}\hat{\mathcal{L}}_\xi\left(e^{-2d}\right) = -\tfrac{1}{2}\mathcal{D}_A\xi^A\,.$$

- The Diffeomorphic General Covariance of the Action then implies

$$0 = \int_\Sigma e^{-2d}\left[\tfrac{1}{8\pi G}\xi^B\mathcal{D}^A\left[4V_{[A}{}^p\bar{V}_{B]}{}^{\bar{q}}(S_{p\bar{q}} - 8\pi G K_{p\bar{q}}) - \tfrac{1}{2}\mathcal{J}_{AB}(S_{(0)} - 8\pi G T_{(0)})\right] + \delta_\xi\Upsilon_a\frac{\delta L_{\text{matter}}}{\delta\Upsilon_a}\right].$$

This gives the $\mathbf{O}(D,D)$ completion of Einstein curvature, JHP-Rey-Rim-Sakatani 2015

$$G_{AB} := 4V_{[A}{}^p\bar{V}_{B]}{}^{\bar{q}}S_{p\bar{q}} - \tfrac{1}{2}\mathcal{J}_{AB}S_{(0)}\,, \qquad \mathcal{D}_A G^{AB} = 0 \qquad \text{(off-shell)}\,,$$

the $\mathbf{O}(D,D)$ completion of Energy-Momentum tensor, Angus-Cho-JHP 2018

$$T_{AB} := 4V_{[A}{}^p\bar{V}_{B]}{}^{\bar{q}}K_{p\bar{q}} - \tfrac{1}{2}\mathcal{J}_{AB}T_{(0)}\,, \qquad \mathcal{D}_A T^{AB} = 0 \qquad \text{(on-shell)}\,,$$

and hence, the $\mathbf{O}(D,D)$ completion of Einstein Field Equations, $G_{AB} = 8\pi G T_{AB}$.

- Combining the two results, the variation of the action reads

$$\delta\text{Action} = \int_\Sigma e^{-2d}\left[\tfrac{1}{4\pi G}\bar{V}^{A\bar{q}}\delta V_A{}^p(S_{p\bar{q}} - 8\pi GK_{p\bar{q}}) - \tfrac{1}{8\pi G}\delta d(S_{(0)} - 8\pi GT_{(0)}) + \delta\Upsilon_a\frac{\delta L_{\text{matter}}}{\delta\Upsilon_a}\right].$$

- Specifically when the variation is generated by diffeomorphisms, we have $\delta_\xi\Upsilon_a = \hat{\mathcal{L}}_\xi\Upsilon_a$ and

$$\bar{V}^{A\bar{q}}\delta_\xi V_A{}^p = \bar{V}^{A\bar{q}}\hat{\mathcal{L}}_\xi V_A{}^p = 2\mathcal{D}_{[A}\xi_{B]}\bar{V}^{A\bar{q}}V^{Bp}\,, \qquad \delta_\xi d = -\tfrac{1}{2}e^{2d}\hat{\mathcal{L}}_\xi\left(e^{-2d}\right) = -\tfrac{1}{2}\mathcal{D}_A\xi^A\,.$$

- The Diffeomorphic General Covariance of the Action then implies

$$0 = \int_\Sigma e^{-2d}\left[\tfrac{1}{8\pi G}\xi^B\mathcal{D}^A\left[4V_{[A}{}^p\bar{V}_{B]}{}^{\bar{q}}(S_{p\bar{q}} - 8\pi GK_{p\bar{q}}) - \tfrac{1}{2}\mathcal{J}_{AB}(S_{(0)} - 8\pi GT_{(0)})\right] + \delta_\xi\Upsilon_a\frac{\delta L_{\text{matter}}}{\delta\Upsilon_a}\right].$$

This gives the **O**(D,D) completion of Einstein curvature, JHP-Rey-Rim-Sakatani 2015

$$G_{AB} := 4V_{[A}{}^p\bar{V}_{B]}{}^{\bar{q}}S_{p\bar{q}} - \tfrac{1}{2}\mathcal{J}_{AB}S_{(0)}\,, \qquad \mathcal{D}_AG^{AB} = 0 \qquad \text{(off-shell)}\,,$$

the **O**(D,D) completion of Energy-Momentum tensor, Angus-Cho-JHP 2018

$$T_{AB} := 4V_{[A}{}^p\bar{V}_{B]}{}^{\bar{q}}K_{p\bar{q}} - \tfrac{1}{2}\mathcal{J}_{AB}T_{(0)}\,, \qquad \mathcal{D}_AT^{AB} = 0 \qquad \text{(on-shell)}\,,$$

and hence, the **O**(D,D) completion of Einstein Field Equations, $G_{AB} = 8\pi GT_{AB}$.

Examples of $T_{AB} := 4\,V_{[A}{}^{p}\,\bar{V}_{B]}{}^{\bar{q}}K_{p\bar{q}} - \frac{1}{2}\mathcal{J}_{AB}\,T_{(0)}$

- Scalar field,

$$L_{\varphi} = -\tfrac{1}{2}\mathcal{H}^{MN}\partial_{M}\varphi\partial_{N}\varphi - V(\varphi)\,, \qquad K_{p\bar{q}} = \partial_{p}\varphi\partial_{\bar{q}}\varphi\,, \qquad T_{(0)} = -2L_{\varphi}\,.$$

- Spinor field,

$$L_{\psi} = \bar{\psi}\gamma^{p}\mathcal{D}_{p}\psi + m_{\psi}\bar{\psi}\psi\,, \qquad K_{p\bar{q}} = -\tfrac{1}{4}(\bar{\psi}\gamma_{p}\mathcal{D}_{\bar{q}}\psi - \mathcal{D}_{\bar{q}}\bar{\psi}\gamma_{p}\psi)\,, \qquad T_{(0)} \equiv 0\,.$$

- RR sector,

$$L_{\rm RR} = \tfrac{1}{2}\mathrm{Tr}(\mathcal{F}\bar{\mathcal{F}})\,, \qquad K_{p\bar{q}} = -\tfrac{1}{4}\mathrm{Tr}(\gamma_{p}\mathcal{F}\bar{\gamma}_{\bar{q}}\bar{\mathcal{F}})\,, \qquad T_{(0)} = 0\,.$$

- Fundamental string: with $D_{i}y^{M} = \partial_{i}y^{M} - \mathcal{A}_{i}^{M}$ (doubled-yet-gauged),

$$e^{-2d}L_{\rm string} = \frac{1}{4\pi\alpha'}\int \mathrm{d}^{2}\sigma\left[-\tfrac{1}{2}\sqrt{-h}h^{ij}D_{i}y^{M}D_{j}y^{N}\mathcal{H}_{MN}(y) - \epsilon^{ij}D_{i}y^{M}\mathcal{A}_{jM}\right]\delta^{D}(x - y(\sigma))\,,$$

$$K_{p\bar{q}} = \frac{1}{4\pi\alpha'}\int \mathrm{d}^{2}\sigma\sqrt{-h}h^{ij}D_{i}y^{M}D_{j}y^{N}\,V_{Mp}\bar{V}_{N\bar{q}}\,e^{2d(x)}\delta^{D}(x - y(\sigma))\,, \qquad T_{(0)} = 0\,.$$

– More examples include Yang-Mills, point particle, Green-Schwarz superstring, *etc.* 1804.00964

DFT = O(D, D) completion of GR

- One single master formula unifies all the EOMs of the whole massless NS-NS sector,

$$G_{AB} = 8\pi G T_{AB} \quad : \quad \text{Einstein Double Field Equations (EDFEs)}$$

which is naturally consistent with our central idea that DFT treats the closed-string massless sector as the geometrical graviton multiplet.

- The $(0,0)$ Riemannian parametrization reduces EDFEs to

$$R_{\mu\nu} + 2\nabla_\mu(\partial_\nu\phi) - \tfrac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} = 8\pi G K_{(\mu\nu)}\,,$$

$$e^{2\phi}\nabla^\rho\left(e^{-2\phi}H_{\rho\mu\nu}\right) = 16\pi G K_{[\mu\nu]}\,,$$

$$R + 4\Box\phi - 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} = 8\pi G T_{(0)}\,.$$

- EDFEs should also govern the dynamics of generic $(n, \bar{n})$ non-Riemannian geometries, including Newton–Cartan, Carroll, Gomis–Ooguri, *etc.*

DFT = O(D, D) completion of GR

- One single master formula unifies all the EOMs of the whole massless NS-NS sector,

$$G_{AB} = 8\pi G T_{AB} \quad : \quad \text{Einstein Double Field Equations (EDFEs)}$$

which is naturally consistent with our central idea that DFT treats the closed-string massless sector as the geometrical graviton multiplet.

- The $(0,0)$ Riemannian parametrization reduces EDFEs to

$$\begin{aligned} R_{\mu\nu} + 2\nabla_\mu(\partial_\nu\phi) - \tfrac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} &= 8\pi G K_{(\mu\nu)}\,, \\ e^{2\phi}\nabla^\rho\left(e^{-2\phi}H_{\rho\mu\nu}\right) &= 16\pi G K_{[\mu\nu]}\,, \\ R + 4\Box\phi - 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} &= 8\pi G T_{(0)}\,. \end{aligned}$$

- EDFEs should also govern the dynamics of generic $(n, \bar{n})$ non-Riemannian geometries, including Newton–Cartan, Carroll, Gomis–Ooguri, *etc.*

Non-Riemannian $(n, \bar{n})$ sectors in DFT

Kyoungho Cho & JHP 1909.10711

- After $\tilde{\partial}^{\mu} \equiv 0$, the semi-covariant formalism naturally induces a 'upper-indexed' covariant derivative for the undoubled ordinary diffeomorphisms and **GL**$(n) \times$ **GL**$(\bar{n})$ local rotations,

$$\mathbb{D}^{\mu} = H^{\mu\rho}\partial_{\rho} + \Omega^{\mu} + \Upsilon^{\mu} + \bar{\Upsilon}^{\mu} \,,$$

$$\begin{aligned}\Omega^{\mu\nu}{}_{\lambda} \,=\, & -\tfrac{1}{2}\partial_{\lambda}H^{\mu\nu} - H^{\rho[\mu}\partial_{\rho}H^{\nu]\sigma}K_{\sigma\lambda} - H^{\rho[\mu}\partial_{\rho}Y_{i}^{\nu]}X_{\lambda}^{i} - H^{\rho[\mu}\partial_{\rho}\bar{Y}_{\bar{\imath}}^{\nu]}\bar{X}_{\lambda}^{\bar{\imath}} \\ & + \left(2H^{\rho[\mu}Y_{i}^{\nu]}\partial_{[\tau}X_{\rho]}^{i} - 2H^{\rho[\mu}\bar{Y}_{\bar{\imath}}^{\nu]}\partial_{[\tau}\bar{X}_{\rho]}^{\bar{\imath}}\right)\left(Y_{j}^{\tau}X_{\lambda}^{j} - \bar{Y}_{\bar{\jmath}}^{\tau}\bar{X}_{\lambda}^{\bar{\jmath}}\right),\end{aligned}$$

$$\Upsilon^{\mu\,i}{}_{j} \,=\, -2H^{\mu\rho}Y_{j}^{\sigma}\partial_{[\rho}X_{\sigma]}^{i}\,, \qquad \bar{\Upsilon}^{\mu\,\bar{\imath}}{}_{\bar{\jmath}} \,=\, -2H^{\mu\rho}\bar{Y}_{\bar{\jmath}}^{\sigma}\partial_{[\rho}\bar{X}_{\sigma]}^{\bar{\imath}}\,.$$

- For $(0,0)$, it reduces to $\mathbb{D}^{\mu} = g^{\mu\nu}\bigtriangledown_{\nu}$, while for generic $(n, \bar{n})$ it enables us to expand

$$\int e^{-2d}S_{(0)} = \int e^{-2d}\Big[R_{\Omega} + 4K_{\mu\nu}\mathbb{D}^{\mu}d\,\mathbb{D}^{\nu}d - \tfrac{1}{12}H^{\lambda\rho}H^{\mu\sigma}H^{\nu\tau}H_{\lambda\mu\nu}H_{\rho\sigma\tau} - H_{\lambda\mu\nu}H^{\lambda\rho}\Big(Y_{i}^{\mu}\widehat{\mathbb{D}}^{\nu}X_{\rho}^{i} - \bar{Y}_{\bar{\imath}}^{\mu}\widehat{\mathbb{D}}^{\nu}\bar{X}_{\rho}^{\bar{\imath}}\Big)\Big]$$

- This might have provided the action principle for each non-Riemannian gravity with fixed $(n, \bar{n})$, but it cannot produce the full EDFE, since $\delta\mathcal{H}_{AB}$ may involve $(n, \bar{n}) \rightarrow (n-1, \bar{n}-1)$.

⋆ Our conclusion is that the various non-Riemannian gravities should be better identified as different solution sectors of DFT rather than viewed as independent theories.

– Milne-shift as well as **GL**$(n) \times$ **GL**$(\bar{n})$ invariant H-flux has been also identified:

$$\widehat{\mathbb{H}}^{\lambda\mu\nu} := H^{\lambda\rho}H^{\mu\sigma}H^{\nu\tau}H_{\rho\sigma\tau} + 6H^{\rho[\lambda}Y_{i}^{\mu}\mathbb{D}^{\nu]}X_{\rho}^{i} - 6H^{\rho[\lambda}\bar{Y}_{\bar{\imath}}^{\mu}\mathbb{D}^{\nu]}\bar{X}_{\rho}^{\bar{\imath}}\,.$$

Non-Riemannian $(n,\bar{n})$ sectors in DFT

- After $\tilde{\partial}^{\mu}\equiv 0$, the semi-covariant formalism naturally induces a 'upper-indexed' covariant derivative for the undoubled ordinary diffeomorphisms and $\mathbf{GL}(n)\times\mathbf{GL}(\bar{n})$ local rotations,

$$\mathbb{D}^{\mu}=H^{\mu\rho}\partial_{\rho}+\Omega^{\mu}+\Upsilon^{\mu}+\bar{\Upsilon}^{\mu}\,,$$

$$\begin{aligned}\Omega^{\mu\nu}{}_{\lambda}\;=&\;-\tfrac{1}{2}\partial_{\lambda}H^{\mu\nu}-H^{\rho[\mu}\partial_{\rho}H^{\nu]\sigma}K_{\sigma\lambda}-H^{\rho[\mu}\partial_{\rho}Y_{i}^{\nu]}X_{\lambda}^{i}-H^{\rho[\mu}\partial_{\rho}\bar{Y}_{\bar{\imath}}^{\nu]}\bar{X}_{\lambda}^{\bar{\imath}}\\&+\Big(2H^{\rho[\mu}Y_{i}^{\nu]}\partial_{[\tau}X_{\rho]}^{i}-2H^{\rho[\mu}\bar{Y}_{\bar{\imath}}^{\nu]}\partial_{[\tau}\bar{X}_{\rho]}^{\bar{\imath}}\Big)\Big(Y_{j}^{\tau}X_{\lambda}^{j}-\bar{Y}_{\bar{\jmath}}^{\tau}\bar{X}_{\lambda}^{\bar{\jmath}}\Big),\end{aligned}$$

$$\Upsilon^{\mu\,i}{}_{j}=-2H^{\mu\rho}Y_{j}^{\sigma}\partial_{[\rho}X_{\sigma]}^{i}\,,\qquad\qquad\bar{\Upsilon}^{\mu\,\bar{\imath}}{}_{\bar{\jmath}}=-2H^{\mu\rho}\bar{Y}_{\bar{\jmath}}^{\sigma}\partial_{[\rho}\bar{X}_{\sigma]}^{\bar{\imath}}\,.$$

- For $(0,0)$, it reduces to $\mathbb{D}^{\mu}=g^{\mu\nu}\bigtriangledown_{\nu}$, while for generic $(n,\bar{n})$ it enables us to expand

$$\int e^{-2d}S_{(0)}=\int e^{-2d}\Big[R_{\Omega}+4K_{\mu\nu}\mathbb{D}^{\mu}d\,\mathbb{D}^{\nu}d-\tfrac{1}{12}H^{\lambda\rho}H^{\mu\sigma}H^{\nu\tau}H_{\lambda\mu\nu}H_{\rho\sigma\tau}-H_{\lambda\mu\nu}H^{\lambda\rho}\Big(Y_{i}^{\mu}\widehat{\mathbb{D}}^{\nu}X_{\rho}^{i}-\bar{Y}_{\bar{\imath}}^{\mu}\widehat{\mathbb{D}}^{\nu}\bar{X}_{\rho}^{\bar{\imath}}\Big)\Big]$$

- This might have provided the action principle for each non-Riemannian gravity with fixed $(n,\bar{n})$, but it cannot produce the full EDFE, since $\delta\mathcal{H}_{AB}$ may involve $(n,\bar{n})\rightarrow(n-1,\bar{n}-1)$.

⋆ Our conclusion is that the various non-Riemannian gravities should be better identified as different solution sectors of DFT rather than viewed as independent theories.

– Milne-shift as well as $\mathbf{GL}(n)\times\mathbf{GL}(\bar{n})$ invariant H-flux has been also identified:

$$\widehat{\mathbb{H}}^{\lambda\mu\nu}:=H^{\lambda\rho}H^{\mu\sigma}H^{\nu\tau}H_{\rho\sigma\tau}+6H^{\rho[\lambda}Y_{i}^{\mu}\mathbb{D}^{\nu]}X_{\rho}^{i}-6H^{\rho[\lambda}\bar{Y}_{\bar{\imath}}^{\mu}\mathbb{D}^{\nu]}\bar{X}_{\rho}^{\bar{\imath}}\,.$$

Physical implications

Stringy 'star' of radius, r_c :
$$G_{AB} = \begin{cases} 8\pi G T_{AB} & \text{for} \quad r \leq r_c \quad \text{(spherical)} \\ 0 & \text{for} \quad r > r_c \end{cases}$$

- Outside the star, $r \geq r_c$, the vacuum geometry is known — Burgess-Myers-Quevedo '94

$$e^{2\phi} = \gamma_+ \left(\frac{r-\alpha}{r+\beta}\right)^{\frac{b}{\sqrt{a^2+b^2}}} + \gamma_- \left(\frac{r+\beta}{r-\alpha}\right)^{\frac{b}{\sqrt{a^2+b^2}}}, \qquad H_{(3)} = h \sin\vartheta \, \mathrm{d}t \wedge \mathrm{d}\vartheta \wedge \mathrm{d}\varphi\,,$$
$$\mathrm{d}s^2 = e^{2\phi}\left[-\left(\frac{r-\alpha}{r+\beta}\right)^{\frac{a}{\sqrt{a^2+b^2}}} \mathrm{d}t^2 + \left(\frac{r+\beta}{r-\alpha}\right)^{\frac{a}{\sqrt{a^2+b^2}}} \left\{\mathrm{d}r^2 + (r-\alpha)(r+\beta)\mathrm{d}\Omega^2\right\}\right]$$

having four parameters, $\{\alpha, \beta, a, h\}$, while $b^2 = (\alpha+\beta)^2 - a^2$ and $\gamma_\pm = \frac{1}{2}(1 \pm \sqrt{1 - h^2/b^2})$.
If $b = h = 0$, it reduces to Schwarzschild geometry.

- Inside the star, EDFEs fix all the constants, $\{\alpha, \beta, a, h\}$, in terms of T_{AB}, for example

$$a = \int_0^{r_c} \mathrm{d}r \int_0^{\pi} \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi \; e^{-2d} \left[\tfrac{1}{4\pi} H_{r\vartheta\varphi} H^{r\vartheta\varphi} + 2G\left(K_r{}^r + K_\vartheta{}^\vartheta + K_\varphi{}^\varphi - K_t{}^t - T_{(0)}\right)\right].$$

Various components of T_{AB} enrich the spherical geometry of DFT, beyond Schwarzschild.

Spherical solution to $D=4$ EDFEs

Angus-Cho-JHP 1804.00964

Stringy 'star' of radius, $r_{\rm c}$: $$G_{AB}=\begin{cases}8\pi GT_{AB} & \text{for}\quad r\leq r_{\rm c}\qquad\text{(spherical)}\\ 0 & \text{for}\quad r>r_{\rm c}\end{cases}$$

- Outside the star, $r\geq r_{\rm c}$, the vacuum geometry is known

Burgess-Myers-Quevedo '94

$$e^{2\phi}=\gamma_{+}\left(\frac{r-\alpha}{r+\beta}\right)^{\frac{b}{\sqrt{a^2+b^2}}}+\gamma_{-}\left(\frac{r+\beta}{r-\alpha}\right)^{\frac{b}{\sqrt{a^2+b^2}}}\,,\qquad H_{(3)}=h\sin\vartheta\,{\rm d}t\wedge{\rm d}\vartheta\wedge{\rm d}\varphi\,,$$
$${\rm d}s^2=e^{2\phi}\left[-\left(\frac{r-\alpha}{r+\beta}\right)^{\frac{a}{\sqrt{a^2+b^2}}}{\rm d}t^2+\left(\frac{r+\beta}{r-\alpha}\right)^{\frac{a}{\sqrt{a^2+b^2}}}\left\{{\rm d}r^2+(r-\alpha)(r+\beta){\rm d}\Omega^2\right\}\right]$$

having four parameters, $\{\alpha,\beta,a,h\}$, while $b^2=(\alpha+\beta)^2-a^2$ and $\gamma_{\pm}=\frac{1}{2}(1\pm\sqrt{1-h^2/b^2})$. If $b=h=0$, it reduces to Schwarzschild geometry.

- Inside the star, EDFEs fix all the constants, $\{\alpha,\beta,a,h\}$, in terms of T_{AB}, for example

$$a=\int_0^{r_{\rm c}}{\rm d}r\int_0^{\pi}{\rm d}\vartheta\int_0^{2\pi}{\rm d}\varphi\;e^{-2d}\left[\tfrac{1}{4\pi}H_{r\vartheta\varphi}H^{r\vartheta\varphi}+2G\left(K_r{}^r+K_\vartheta{}^\vartheta+K_\varphi{}^\varphi-K_t{}^t-T_{(0)}\right)\right].$$

Various components of T_{AB} enrich the spherical geometry of DFT, beyond Schwarzschild.

Spherical solution to $D=4$ EDFEs

Angus-Cho-JHP 1804.00964

Stringy 'star' of radius, $r_{\rm c}$: $$G_{AB}=\begin{cases}8\pi G T_{AB} & \text{for}\quad r\leq r_{\rm c}\qquad \text{(spherical)}\\ 0 & \text{for}\quad r>r_{\rm c}\end{cases}$$

- Outside the star, $r\geq r_{\rm c}$, the vacuum geometry is known Burgess-Myers-Quevedo '94

$$e^{2\phi}=\gamma_{+}\left(\frac{r-\alpha}{r+\beta}\right)^{\frac{b}{\sqrt{a^2+b^2}}}+\gamma_{-}\left(\frac{r+\beta}{r-\alpha}\right)^{\frac{b}{\sqrt{a^2+b^2}}}\,,\qquad H_{(3)}=h\sin\vartheta\,{\rm d}t\wedge{\rm d}\vartheta\wedge{\rm d}\varphi\,,$$
$${\rm d}s^2=e^{2\phi}\left[-\left(\frac{r-\alpha}{r+\beta}\right)^{\frac{a}{\sqrt{a^2+b^2}}}{\rm d}t^2+\left(\frac{r+\beta}{r-\alpha}\right)^{\frac{a}{\sqrt{a^2+b^2}}}\left\{{\rm d}r^2+(r-\alpha)(r+\beta){\rm d}\Omega^2\right\}\right]$$

having four parameters, $\{\alpha,\beta,a,h\}$, while $b^2=(\alpha+\beta)^2-a^2$ and $\gamma_{\pm}=\frac{1}{2}(1\pm\sqrt{1-h^2/b^2})$. If $b=h=0$, it reduces to Schwarzschild geometry.

- Inside the star, EDFEs fix all the constants, $\{\alpha,\beta,a,h\}$, in terms of T_{AB}, for example

$$a=\int_0^{r_{\rm c}}{\rm d}r\int_0^{\pi}{\rm d}\vartheta\int_0^{2\pi}{\rm d}\varphi\;e^{-2d}\left[\tfrac{1}{4\pi}H_{r\vartheta\varphi}H^{r\vartheta\varphi}+2G\left(K_r{}^r+K_\vartheta{}^\vartheta+K_\varphi{}^\varphi-K_t{}^t-T_{(0)}\right)\right].$$

Various components of T_{AB} enrich the spherical geometry of DFT, beyond Schwarzschild.

- In terms of Areal Radius, R: $\mathrm{d}s^2 = g_{tt}\mathrm{d}t^2 + g_{RR}\mathrm{d}R^2 + R^2\mathrm{d}\Omega^2$, the Newton potential reads

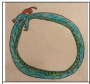
$$\Phi = -\tfrac{1}{2}(1+g_{tt}) = -\frac{MG}{R} + \left(\frac{2b^2 - h^2 + 2ab\sqrt{1-h^2/b^2}}{a^2+b^2-h^2+2ab\sqrt{1-h^2/b^2}}\right)\left(\frac{MG}{R}\right)^2 + \cdots$$

where the ellipses denote higher order terms in $\frac{MG}{R}$ which is 'dimensionless', and

$$MG = \tfrac{1}{2}\left(a + b\sqrt{1-h^2/b^2}\right) = \int_0^\infty \mathrm{d}r \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi \; e^{-2d}\left(-2GK_t{}^t + \tfrac{1}{8\pi}\left|H_{t\vartheta\varphi}H^{t\vartheta\varphi}\right|\right).$$

That is to say, DFT modifies GR for small $\frac{R}{MG}$. In particular, it can be *repulsive.*

- Intriguingly, the dark energy and matter problems arise from small $\frac{R}{MG}$ observations:

	Electron ($R\simeq 0$)	Proton	Hydrogen Atom	Billiard Ball	Earth	Solar System ($1\mathrm{AU}/M_\odot G$)	Milky Way (visible)	Galaxy Cluster	Universe ($M\propto R^3$)
$R/(MG)$	0^+	7.1×10^{38}	2.0×10^{43}	2.4×10^{26}	1.4×10^{9}	1.0×10^{8}	1.5×10^{6}	$\sim 10^{5}$	0^+

The observations of stars/galaxies far away may reveal the short-distance nature of gravity.

The repulsive force at short distance may explain the accelerating expansion of Universe?

- Since B-field does not couple to particle geodesics, from the mass formula above, we might speculate that electric H-flux is dark matter, while $K_t{}^t$ represents ordinary matter (baryonic).

Spherical solution to $D=4$ EDFEs

Angus-Cho-JHP 1804.00964

- In terms of Areal Radius, R: $\mathrm{d}s^2 = g_{tt}\mathrm{d}t^2 + g_{RR}\mathrm{d}R^2 + R^2\mathrm{d}\Omega^2$, the Newton potential reads

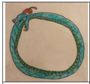
$$\Phi = -\tfrac{1}{2}(1+g_{tt}) = -\frac{MG}{R} + \left(\frac{2b^2-h^2+2ab\sqrt{1-h^2/b^2}}{a^2+b^2-h^2+2ab\sqrt{1-h^2/b^2}}\right)\left(\frac{MG}{R}\right)^2 + \cdots$$

where the ellipses denote higher order terms in $\frac{MG}{R}$ which is 'dimensionless', and

$$MG = \tfrac{1}{2}\left(a+b\sqrt{1-h^2/b^2}\right) = \int_0^\infty \mathrm{d}r \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi\; e^{-2d}\left(-2GK_t{}^t + \tfrac{1}{8\pi}\left|H_{t\vartheta\varphi}H^{t\vartheta\varphi}\right|\right).$$

That is to say, DFT modifies GR for small $\frac{R}{MG}$. In particular, it can be *repulsive.*

- Intriguingly, the dark energy and matter problems arise from small $\frac{R}{MG}$ observations:

	Electron $(R\simeq 0)$	Proton	Hydrogen Atom	Billiard Ball	Earth	Solar System $(1\mathrm{AU}/M_\odot G)$	Milky Way (visible)	Galaxy Cluster	Universe $(M\propto R^3)$
$R/(MG)$	0^+	7.1×10^{38}	2.0×10^{43}	2.4×10^{26}	1.4×10^{9}	1.0×10^{8}	1.5×10^{6}	$\sim 10^{5}$	0^+

The observations of stars/galaxies far away may reveal the short-distance nature of gravity.

The repulsive force at short distance may explain the accelerating expansion of Universe?

- Since B-field does not couple to particle geodesics, from the mass formula above, we might speculate that electric H-flux is dark matter, while $K_t{}^t$ represents ordinary matter (baryonic).

- **$\mathbf{O}(D,D)$ completion of the Friendamann equations:**

$$\frac{8\pi G}{3}\rho e^{2\phi}+\frac{h^2}{12a^6} = H^2-2\left(\frac{\phi'}{N}\right)H+\frac{2}{3}\left(\frac{\phi'}{N}\right)^2+\frac{k}{a^2}$$

$$\frac{4\pi G}{3}(\rho+3p)e^{2\phi}+\frac{h^2}{6a^6} = -H^2-\frac{H'}{N}+\left(\frac{\phi'}{N}\right)H-\frac{2}{3}\left(\frac{\phi'}{N}\right)^2+\frac{1}{N}\left(\frac{\phi'}{N}\right)'$$

$$\frac{8\pi G}{3}\left(\rho e^{2\phi}-\frac{1}{2}T_{(0)}\right) = -H^2-\frac{H'}{N}+\frac{2}{3N}\left(\frac{\phi'}{N}\right)'$$

which imply the conservation equation,

$$\rho'+3NH(\rho+p)+\phi' T_{(0)}e^{-2\phi}=0\,.$$

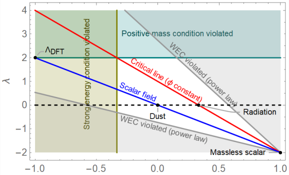
Here most general cosmological (homogeneous, isotropic, & Riemannian) ansatzes have been adopted:

$$\rho:=\left(-K^t{}_t+\tfrac{1}{2}T_{(0)}\right)e^{-2\phi}\,,\qquad p:=\left(K^r{}_r-\tfrac{1}{2}T_{(0)}\right)e^{-2\phi}\,,\qquad H_{(3)}=\frac{hr^2}{\sqrt{1-kr^2}}\sin\vartheta\,\mathrm{d}r\wedge\mathrm{d}\vartheta\wedge\mathrm{d}\varphi\,.$$

* This gives an enriched and novel framework beyond typical string cosmology, enjoying two equation-of-state parameters, $w=p/\rho$ (conventional) and $\lambda=T_{(0)}e^{-2\phi}/\rho$ (new).

In particular, de Sitter is unnatural as incompatible with DFT C.C. term, $e^{-2d}\Lambda_{\mathrm{DFT}}$. It might be an artifact of GR.

c.f. Swampland *a la* Vafa



It is straightforward to take the weak field approximation and non-relativistic limit of the $D=4$ EDFEs, $G_{AB}=8\pi GT_{AB}$, to obtain the string theory extention of Newton Gravity,

$$\nabla^2\Phi=4\pi G\rho+\mathbf{H}{\cdot}\mathbf{H}\,,\qquad \nabla\cdot\mathbf{H}=0\,,\qquad \nabla\times\mathbf{H}=4\pi G\mathbf{K}\,,$$

- Not only the mass density $\rho\propto K_{00}$ but also the current density $\mathbf{K}\propto(K_{[01]},K_{[02]},K_{[03]})$ is intrinsic to matter. Sourcing $\mathbf{H}\propto(H_{[023]},H_{[031]},H_{[012]})$, $\mathbf{K}$ is nontrivial if the matter is 'stringy'.
- Since $\mathbf{K}$ is divergenceless, we may introduce the notion of 'stringization' , analogous to magnetization, and note the 'stringy dipole',

$$\mathbf{K}=\nabla\times\mathbf{s}\,;\qquad \mathbf{H}\simeq G\,\frac{3\hat{\mathbf{x}}\,(\hat{\mathbf{x}}{\cdot}\mathbf{S}(t))-\mathbf{S}(t)}{|\mathbf{x}|^3}\,,\qquad \mathbf{S}(t)=\int\mathrm{d}^3x\,\mathbf{s}(t,\mathbf{x})\,.$$

- $\mathbf{H}$ contributes quadratically to the Newton potential, but otherwise is decoupled from the point particle dynamics nor electromagnetism (light),

$$\ddot{\mathbf{x}}=-\nabla\Phi\,,\qquad S_{\mathrm{photon}}=\int\mathrm{d}^4x\;-\tfrac{1}{4}\sqrt{-g}e^{-2\phi}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}\,.$$

$\Rightarrow$ H-flux behaves like a dark matter.

$\Rightarrow$ Light does not merely follow a null geodesic if ϕ is nontrivial.

It is straightforward to take the weak field approximation and non-relativistic limit of the $D = 4$ EDFEs, $G_{AB} = 8\pi G T_{AB}$, to obtain the string theory extention of Newton Gravity,

$$\nabla^2\Phi = 4\pi G\rho + \mathbf{H}\cdot\mathbf{H}\,, \qquad \nabla\cdot\mathbf{H} = 0\,, \qquad \nabla\times\mathbf{H} = 4\pi G\mathbf{K}\,,$$

- Not only the mass density $\rho \propto K_{00}$ but also the current density $\mathbf{K} \propto (K_{[01]}, K_{[02]}, K_{[03]})$ is intrinsic to matter. Sourcing $\mathbf{H} \propto (H_{[023]}, H_{[031]}, H_{[012]})$, $\mathbf{K}$ is nontrivial if the matter is 'stringy'.
- Since $\mathbf{K}$ is divergenceless, we may introduce the notion of 'stringization' , analogous to magnetization, and note the 'stringy dipole',

$$\mathbf{K} = \nabla\times\mathbf{s}\,; \qquad \mathbf{H} \simeq G\,\frac{3\hat{\mathbf{x}}\,(\hat{\mathbf{x}}\cdot\mathbf{S}(t)) - \mathbf{S}(t)}{|\mathbf{x}|^3}\,, \qquad \mathbf{S}(t) = \int \mathrm{d}^3x\ \mathbf{s}(t,\mathbf{x})\,.$$

- $\mathbf{H}$ contributes quadratically to the Newton potential, but otherwise is decoupled from the point particle dynamics nor electromagnetism (light),

$$\ddot{\mathbf{x}} = -\nabla\Phi\,, \qquad S_{\mathrm{photon}} = \int \mathrm{d}^4x\ -\tfrac{1}{4}\sqrt{-g}e^{-2\phi}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}\,.$$

$\Rightarrow$ *H-flux behaves like a dark matter.*

$\Rightarrow$ Light does not merely follow a null geodesic if ϕ is nontrivial.

Concluding Remark

DFT = O(D, D) completion of GR

$$G_{AB} = 8\pi G T_{AB}$$

EDFE as the master formula for massless NS-NS & non-Riemannian geometry.

- $c \to \infty$ **limit:** $\nabla^2\Phi = 4\pi G\rho + \mathbf{H}\cdot\mathbf{H}\,, \qquad \nabla\cdot\mathbf{H} = 0\,, \qquad \nabla\times\mathbf{H} = 4\pi G\mathbf{K}$

– ***H*-flux as dark matter?**

– **Repulsive gravitational force for small $\frac{R}{MG}$ as dark energy?**

– **Are leptons and quarks distinct kinds of spinors for Spin$(1,3) \times$ Spin$(3,1)$?**

– **O(D, D) Symmetry Principle: O(D, D) can be broken only spontaneously but never explicitly. Is this true in Nature?**

Concluding Remark

DFT = O(D,D) completion of GR

$$G_{AB} = 8\pi G T_{AB}$$

EDFE as the master formula for massless NS-NS & non-Riemannian geometry.

- $c \to \infty$ **limit:** $\nabla^2\Phi = 4\pi G\rho + \mathbf{H}\cdot\mathbf{H}\,, \qquad \nabla\cdot\mathbf{H} = 0\,, \qquad \nabla\times\mathbf{H} = 4\pi G\mathbf{K}$

– ***H*-flux as dark matter?**

– **Repulsive gravitational force for small $\frac{R}{MG}$ as dark energy?**

– **Are leptons and quarks distinct kinds of spinors for Spin$(1,3)$ $\times$ Spin$(3,1)$?**

– **O(D,D) Symmetry Principle: O(D,D) can be broken only spontaneously but never explicitly. Is this true in Nature?**

ありがとうございま