Equivalence of the field theory approach and the microscopic approach to the topological insulators in odd dimensions

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arXiv:1903.11852

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July 17, 2019, Seminar @ Dept. of Phys. Kyoto U.

1. Introduction

Topological insulator is interesting

(particularly to lattice theorists)

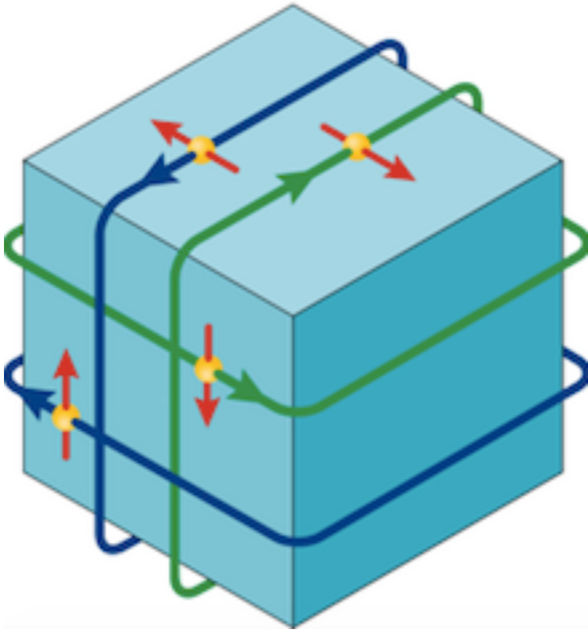


Figure from Tokura et al.
Nature Reviews Physics **vol 1**, 126 (2019)

- Interesting physics from non-trivial topology

  Bulk: insulator
  Surface: metal

  Topology guarantees edge modes
  (Bulk-Edge correspondence)

- Close relationship to domain-wall fermion

  New knowledge of topological matter

  → new hints to lattice fermions by Domain-wall fermion

  example: Gapped symmetric phase by 4-fermi interaction

  → Chiral gauge theory on the lattice

Characterizaton of topological insulator

Microscopic approach

TKNN formula

Thouless, Kohmoto, . Nightingale, . den Nijs

- Study the wavefunction of free fermion
- Applied to various different free systems (higher dim, higher symmetry)
- Looks rather technical (at least to me)
- Applicable only to free fermion systems

Field theory approach

K. Ishikawa 1984, H. So 1985, Golterman, Jansen, Kaplan 1993

- Study the effective action with gauge field
- Conceptually simple :
  bulk-edge correspondence =anomaly cancellation
- Applicable also to interacting fermion systems

Question

Two topological characterizations are identical?

In some specific cases, yes.

How generally identical and why ?

We prove the equivalence for general Hamiltonians bilinear in fermion in D=2+1 and D=4+1 dimensions.

Outline

✔☐ 1. Introduction
2. Review of the microscopic approach
3. Review of the field theory approach
4. Equivalence for general Hamiltonian
    1. The Setup and outline of the proof
    2. Fermion-loop expression → Winding number expression
    3. Winding number expression → TKNN formula
5. Summary

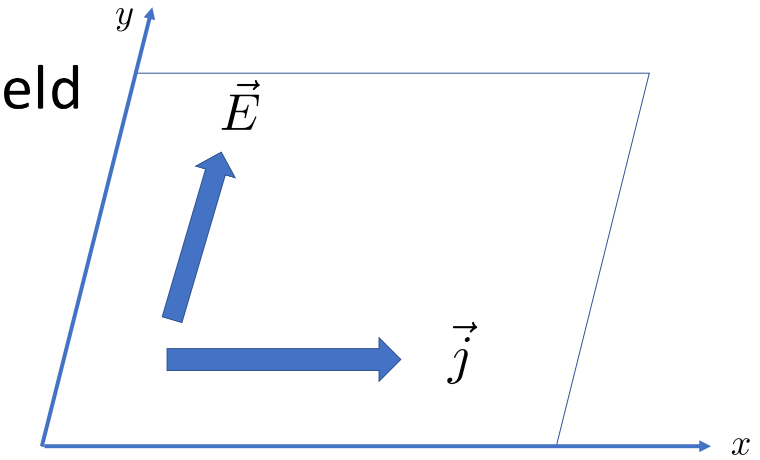
2. Review of the microscopic approach

Anomalous Hall effect

2+1 dim system with Parity Violation

Hall current perpendicular to Electric field

$$\langle j_x \rangle_E = \sigma_{xy} E_y$$



Hall conductivity can be expressed by topological quantity using

1) Kubo formula from perturbation theory
2) Formulae in quantum mechanics

Electron states under electric field (perturbation theory)

$$|n\rangle_E = |n\rangle + \sum_{m\neq n} |m\rangle \frac{\langle m|eE_y y|n\rangle}{E_n - E_m}$$

$|n\rangle$: eigenstate in free theory

$|n\rangle_E$: perturbed state

Hall current under the electric field $\langle j_x\rangle_E \equiv \sum_{n, E_n<0} \langle n|_E \frac{-ev_x}{L^2} |n\rangle_E$

Kubo formula

$$\sigma_{xy} = -\frac{ie^2}{L^2} \sum_{\vec{p}} \sum_{a} \sum_{b\neq a} \epsilon^{ij} \frac{\langle a,\vec{p}|v_i|b,\vec{p}\rangle\langle b,\vec{p}|v_j|a,\vec{p}\rangle}{(E_a(\vec{p}) - E_b(\vec{p}))^2}$$

where we have used

- Translational invariance: $n \Rightarrow (a, \vec{p})$
- Heisenberg equation: $[y, H] = iv_y$

a : band label,

$\vec{p}$: bloch momentum

Derivation of Useful formula from

$$v_i = \frac{\partial}{\partial p^i} H(\vec{p})$$

$$H(\vec{p})|a,\vec{p}\rangle = E_a(\vec{p})|a,\vec{p}\rangle$$

$$\langle a,\vec{p}|b,\vec{p}\rangle = 0 \qquad (a \neq b)$$

$$\langle a,\vec{p}|v_i|b,\vec{p}\rangle = (E_a(\vec{p}) - E_b(\vec{p}))\langle a,\vec{p}|\frac{\partial}{\partial p^i}|b,\vec{p}\rangle \qquad (a \neq b)$$

Combining with Kubo formula and defining $\mathcal{A}_i^{(a)}(\vec{p}) \equiv -i\langle a,\vec{p}|\frac{\partial}{\partial p^i}|a,\vec{p}\rangle$

Berry connection

$$\sigma_{xy} = \frac{e^2}{L^2}\sum_{\vec{p}}\sum_{a}\epsilon^{ij}\frac{\partial}{\partial p^i}\mathcal{A}_j^{(a)}(\vec{p})$$

$$= \frac{e^2}{2\pi}\int\frac{d^2p}{2\pi}\sum_{a}\epsilon^{ij}\frac{\partial}{\partial p^i}\mathcal{A}_j^{(a)}(\vec{p})$$

Chern number c_1 !

TKNN formula

3. Review of field theory approach

Effective gauge action

Integrating out massive fermions in 3-dimensions

$$S_{\text{eff}}(A) \equiv \ln\left[\int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-\int\bar{\psi}(D+m)\psi}\right]$$

Well-known example

Parity anomaly S. Deser, R. Jackiw, S. Templeton 1982, N. Redlich 1984

$$S_{\text{eff}}(A) = ic_{cs}S_{cs}(A) + \cdots$$

$$S_{\text{cs}}(A) \equiv \int d^3x\ \epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \qquad c_{cs} = -\frac{1}{8\pi}\frac{m}{|m|}$$

Parity violation of fermion induces Chern-Simons action

Anomalous Hall conductivity from Chern-Simons action

$$S_{\text{eff}}(A) = ic_{cs} \int d^3x \; \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$$

Hall conductivity is given by the Chern-Simons coupling c_cs

$$\langle j_i \rangle \equiv \frac{\partial}{\partial A_i} S_{\text{eff}}(A)$$
$$= 2c_{cs}\epsilon^{i\nu\lambda}\partial_\nu A_\lambda = 2c_{cs}\epsilon^{ij}E_j$$

$\Rightarrow$ $\sigma_{xy} \propto c_{cs}$

Expression of Chern-Simons coupling

c_{cs} can be obtained by differentiating S_eff with gauge fields and its momentum

$$c_{cs} = -\frac{\epsilon_{\alpha_0\beta_1\alpha_1}}{2!3!}\left(\frac{\partial}{\partial q_1}\right)_{\beta_1} \times \int d^3x_1 e^{iq_1x_1}\left.\frac{\delta^2 S_{\text{eff}}(A)}{\delta A_{\alpha_0}(x_0)\delta A_{\alpha_1}(x_1)}\right|_{A=0,q=0,x_0=0}$$

Current-Current correlator

↓

fermion 1-loop diagram

Fermion 1-loop diagram

Assuming multi-photon vertex does not contribute

True for continuum theory and Wilson fermion on the lattice

$$c_{cs} = \frac{\epsilon^{\alpha_0\beta_1\alpha_1}}{2\cdot 3!}\left(\frac{\partial}{\partial q_1}\right)_{\beta_1}\int \frac{d^3\ p}{(2\pi)^3} Tr\left[S(p)\Gamma^{(1)}_{\alpha_0}[q_1;p-q_1]S(p-q_1)\Gamma^{(1)}_{\alpha_1}(-q_1;p)\right]$$

$\Gamma^{(1)}_{\mu}[q;p]$: fermion-fermion-photon vertex

p: incoming fermion momentum,

q: incoming photon momentum

Assuming derivative of vertex function does not contribute

True for continuum theory and Wilson fermion on the lattice

$$c_{cs} = -\frac{\epsilon^{\alpha_0\beta_1\alpha_1}}{2\cdot 3!}\int \frac{d^3\ p}{(2\pi)^3} Tr\left[S(p)\Gamma^{(1)}_{\alpha_0}[0;p]\frac{\partial S(p)}{\partial p^{\beta_1}}\Gamma^{(1)}_{\alpha_1}(0;p)\right]$$

Ward-Takahashi identity

$$\Gamma_{\mu}^{(1)}[0;p] = -i\frac{\partial S^{-1}(p)}{\partial p^{\mu}}$$

$$c_{cs} = -\frac{\epsilon^{\alpha_0\beta_1\alpha_1}}{2\cdot 3!}\int \frac{d^3\ p}{(2\pi)^3}\mathrm{Tr}\left[S(p)\partial_{\alpha_0}S^{-1}(p)S(p)\partial_{\beta_1}S^{-1}(p)S(p)\partial_{\alpha_1}S^{-1}(p)\right]$$

"Winding number" expression of Chern-Simons coupling

K. Ishikawa 1984, H. So 1985
Golterman, Jansen, Kaplan 1993

- Topological in S(p) has no singularity (true for gapped system)
- Winding number of a map $T^3 \to S^3$ for Wilson fermion

4. Equivalence for general Hamiltonian

Fukaya, T.O., Yamaguchi, Xi
arXiv:1903.11852

4-1. The Setup

2-1. The Setup

Gapped fermion system in D=2n+1 dimensions.

Fermions on 2n dim lattice with continuous time in Euclidean space

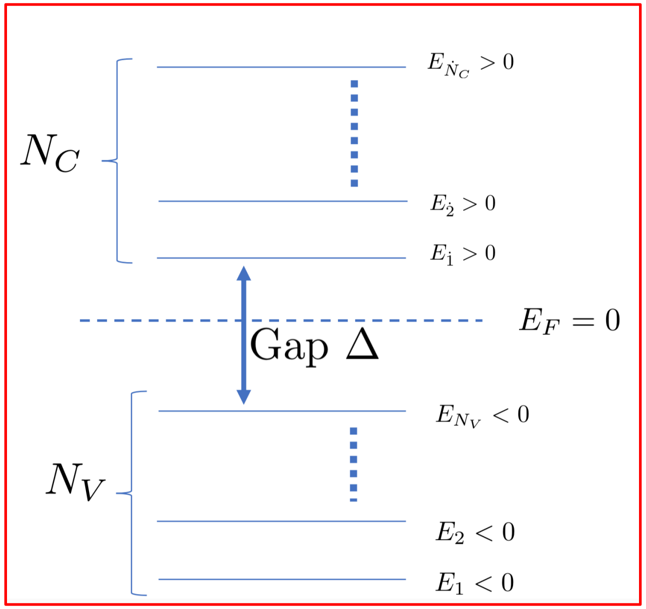
$$S_E = \int dt \sum_{\vec{r}} \psi^\dagger(t,\vec{r}) \left[\frac{\partial}{\partial t} + iA_0 + H(\vec{A}) \right] \psi(t,\vec{r})$$

$H(\vec{A})\Big|_{\vec{A}=0}$: translational inv. ➔ band structure

$\psi, \psi^\dagger$ can have many internal DOF
➔ Many bands

No particular structure is assumed
such as relativistic fermion, or Wilson fermion, …..

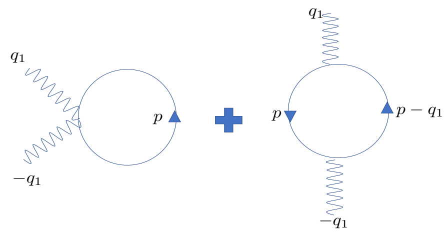
Energy eigenstates for fixed $\vec{p}$



Nv Valence bands
Nc Conduction bands
Δ: Gap

Outline of the proof

fermion loop expression of c_{cs}



Generalized Ward-Takahashi identities

winding number expression of c_{cs}

$$c_{\rm cs} = \frac{(-i)^2 \epsilon_{\alpha_0\beta_1\alpha_1}}{2!3!} \int \frac{dp_0}{2\pi} \int_{\rm BZ} \frac{d^2p}{(2\pi)^2} \times {\rm Tr}\left[S_F(p) \frac{\partial S_F^{-1}(p)}{\partial p_{\alpha_0}} S_F(p) \frac{\partial S_F^{-1}(p)}{\partial p_{\beta_1}} S_F(p) \frac{\partial S_F^{-1}(p)}{\partial p_{\alpha_1}} \right]$$

Energy eigenstate expression & p_0 integration

TKNN formula

$$c_{cs} \propto \sum_a \int \frac{d^2\vec{p}}{2\pi} \epsilon^{ij} \partial_i \mathcal{A}_j^{(a)}$$

4-2 Chern-Simons level → Winding number

2-2. Fermion-loop expression → Winding number expression

$$S_{\text{eff}}(A) = \cdots + i c_{cs} S_{cs}(A) + \cdots$$

$$S_{cs}(A) = \int d^{2n+1}x \epsilon_{\alpha_0 \beta_1 \alpha_1 \cdots \beta_n \alpha_n} A_{\alpha_0} \partial_{\beta_1} A_{\alpha_1} \cdots \partial_{\beta_n} A_{\alpha_n}$$

c_{cs} can be obtained by differentiating the effective action as

$$c_{cs} = \frac{(-i)^{n+1} \epsilon_{\alpha_0 \beta_1 \alpha_1 \cdots \beta_n \alpha_n}}{(n+1)!(2n+1)!} \left(\frac{\partial}{\partial q_1} \right)_{\beta_1} \cdots \left(\frac{\partial}{\partial q_n} \right)_{\beta_n} \times \prod_{i=1}^{n} \int d^{2n+1}x_i e^{i q_i x_i} \left. \frac{\delta^{n+1} S_{\text{eff}}(A)}{\delta A_{\alpha_0}(x_0) \delta A_{\alpha_1}(x_1) \cdots \delta A_{\alpha_n}(x_n)} \right|_{A=0, q_i=0}$$

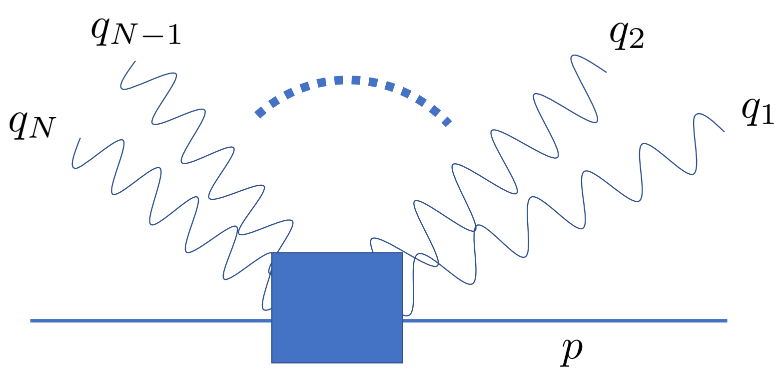
⇕

Fermion 1-loop diagram with n+1 external photons

For general Hamiltonian,

Feynman rule can have fermion-fermion-multiphoton vertices

$$\Gamma^{(n)}[q_N, \alpha_N; \cdots ; q_1, \alpha_1; p]$$



➔ 1-loop n-point function from several diagrams in general.

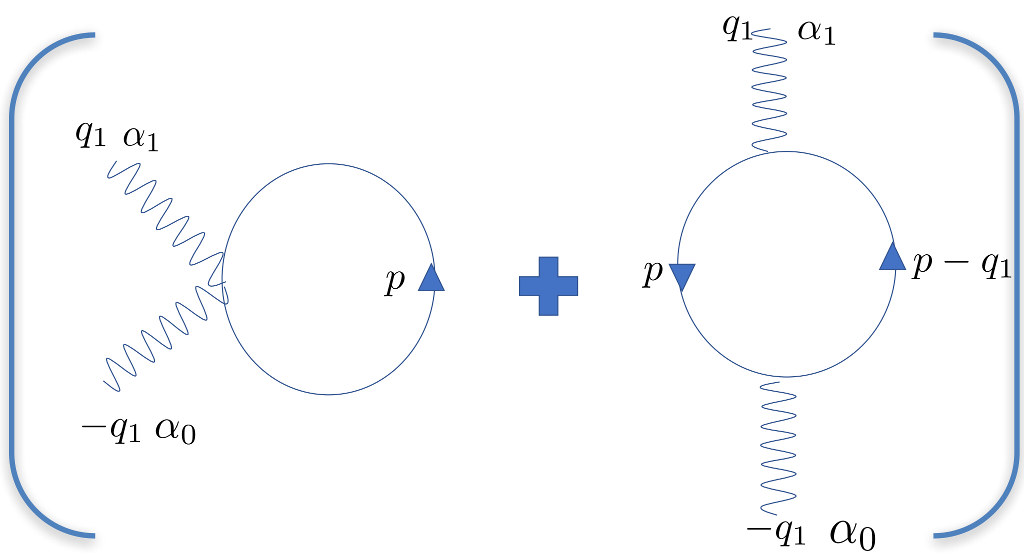
D=2+1 case

$$c_{\rm cs} = -\frac{(-i)^2 \epsilon_{\alpha_0 \beta_1 \alpha_1}}{2!3!} \int \frac{d^3 p}{(2\pi)^3} \left(\frac{\partial}{\partial q_1} \right)_{\beta_1}$$

$$\left\{ \mathrm{Tr} \left[S_F(p) \Gamma^{(2)} [-q_1, \alpha_0; q_1, \alpha_1; p] \right] \right.$$

$$\left. + \ \mathrm{Tr} \left[S_F(p - q_1) \Gamma^{(1)} [-q_1, \alpha_0; p] S_F(p) \Gamma^{(1)} [q_1, \alpha_1; p - q_1] \right] \right\} \Bigg|_{q_1 = 0}$$

$$X \equiv \epsilon^{\alpha_0 \beta_1 \alpha_1} \left(\frac{\partial}{\partial q_1} \right)_{\beta_1}$$



Naively, simplest Ward-Takahashi identity reduce the fermion loop expression into winding number expression

$$\partial_\mu S_F^{-1}(p) = -i\Gamma_\mu(k,p)\mid_{k=0}$$

K. Ishikawa 1984, H. So 1985, Golterman, Jansen, Kaplan 1993

However, two new contributions in general case

1. Multi-photon vertex contribution → non-zero
2. Momentum derivative of the vertex function → non-zero

corrections to the winding number expression!

New Ward-Takahashi identities

Gauge invariant lattice action can be formally expanded by infinite series of covariant derivatives.

Example: $\psi^\dagger(t,\vec{x}) e^{i\int_{\vec{x}}^{\vec{x}+a\vec{\mu}} d\vec{r}'\cdot\vec{A}(\vec{r}')}\psi(t,\vec{x}+a\vec{\mu}) = \psi^\dagger(t,\vec{x})\sum_{n=0}^{\infty}\frac{a^n}{n!}\left(D_\mu^n\psi\right)(t,\vec{x})$

Therefore, formally action can be expressed as

$$S = \int dt \sum_{\vec{x}} \sum_{n=0}^{\infty} \psi^\dagger(t,\vec{x}) M_{\mu_1\cdots\mu_n} (D_{\mu_1}\cdots D_{\mu_n}\psi)(t,\vec{x})$$

Same coefficient M appear in propagator and vertices

Formal expansions of propagator and vertices

Using the coefficients M,

$$S_F^{-1}(p) = \sum_{n=0}^{\infty} M_{\mu_1\cdots\mu_n} \prod_{i=1}^{n} (ip_{\mu_i})$$

$$\Gamma^{(1)}[k,\mu;p] = -i \sum_{n=1}^{\infty} \sum_{a=1}^{n} M_{\mu_1\cdots\mu_{a-1}\mu\mu_{a+1}\cdots\mu_n} \prod_{i=1}^{a-1} (i(p+k)_{\mu_i}) \prod_{i=a+1}^{n} (ip_{\mu_i})$$

$$\begin{aligned}
&\Gamma^{(2)}[k,\mu;l,\nu;p] \\
&= -i^2 \sum_{n=1}^{\infty} \sum_{\substack{a,b=1 \\ a<b}}^{n} M_{\mu_1\cdots\mu_{a-1}\mu\mu_{a+1}\cdots\mu_{b-1}\nu\mu_{b+1}\cdots\mu_n} \prod_{i=1}^{a-1} (i(p+k+l)_{\mu_i}) \prod_{i=a+1}^{b-1} (i(p+l)_{\mu_i}) \prod_{i=b+1}^{n} (ip_{\mu_i}) \\
&- i^2 \sum_{n=1}^{\infty} \sum_{\substack{a,b=1 \\ a<b}}^{n} M_{\mu_1\cdots\mu_{a-1}\nu\mu_{a+1}\cdots\mu_{b-1}\mu\mu_{b+1}\cdots\mu_n} \prod_{i=1}^{a-1} (i(p+k+l)_{\mu_i}) \prod_{i=a+1}^{b-1} (i(p+k)_{\mu_i}) \prod_{i=b+1}^{n} (ip_{\mu_i})
\end{aligned}$$

New Ward-Takahashi identity

The formal expression reproduces usual Ward-Takahashi identities.

In addition, one also obtains the following 2nd order W-T identity

$$\left.\frac{\partial^2 \Gamma^{(1)}[k,\mu;p]}{\partial k_\nu \partial p_\lambda}\right|_{k=0} = \left.\frac{\partial \Gamma^{(2)}[k,\mu;0,\lambda;p]}{\partial k_\nu}\right|_{k=0} = \left.\frac{\partial \Gamma^{(2)}[0,\lambda;l,\mu;p]}{\partial l_\nu}\right|_{l=0}$$

1st derivative of the two-photon vertex with respect to momentum is related to 2nd derivative of the single photon vertex.

Correction terms to 1-loop expression is shown to be total derivatives and vanish.

Using these Ward-Takahashi identities, one can rewrite the integrand X as

$$X = \epsilon^{\alpha_0\beta_1\alpha_1}\frac{\partial}{\partial p_{\alpha_0}}\mathrm{Tr}\left(2S_F(p)\frac{\partial\Gamma^{(1)}[q_1,\alpha_1;p)}{\partial q_{\beta_1}}\right)\Bigg|_{q=0}$$

$$+\,\epsilon^{\alpha_0\beta_1\alpha_1}\left(S_F(p)\frac{\partial S_F^{-1}(p)}{\partial p_{\alpha_0}}S_F(p)\frac{\partial S_F^{-1}(p)}{\partial p_{\beta_1}}S_F(p)\frac{\partial S_F^{-1}(p)}{\partial p_{\alpha_1}}\right)$$

Therefore, additional contributions add up to a total derivative.

Thus, for general Hamiltonian, we obtain

$$c_{\mathrm{cs}} = \frac{(-i)^2\epsilon_{\alpha_0\beta_1\alpha_1}}{2!3!}\int\frac{dp_0}{2\pi}\int_{\mathrm{BZ}}\frac{d^2p}{(2\pi)^2}$$

$$\times\,\mathrm{Tr}\left[S_F(p)\frac{\partial S_F^{-1}(p)}{\partial p_{\alpha_0}}S_F(p)\frac{\partial S_F^{-1}(p)}{\partial p_{\beta_1}}S_F(p)\frac{\partial S_F^{-1}(p)}{\partial p_{\alpha_1}}\right]$$

D=4+1 case

Chern-Simons coupling is obtained from 3-point 1-loop diagrams.

For general Hamiltonian,

[diagram with a 3-photon vertex] ,

[diagrams with a two-photon vertex + a single photon vertex],

[diagram with 3 single-photon vertices] can contribute.

$$
\begin{aligned}
c_{\rm cs} = & -\frac{(-i)^3 \epsilon_{\alpha_0\beta_1\alpha_1\beta_2\alpha_2}}{3!5!} \int \frac{d^5p}{(2\pi)^5} \left(\frac{\partial}{\partial q_1}\right)_{\beta_1} \left(\frac{\partial}{\partial q_2}\right)_{\beta_2} \\
& \{ \ {\rm Tr} \left[S_F(p) \cdot \Gamma^{(3)}[-(q_1+q_2),\alpha_0; q_1,\alpha_1; q_2,\alpha_2; p] \right] \\
& + 2{\rm Tr} \left[S_F(p-q_2)\Gamma^{(2)}[-(q_1+q_2),\alpha_0; q_1,\alpha_1; p] S_F(p)\Gamma^{(1)}[q_2,\alpha_2; p-q_2] \right] \\
& + \ {\rm Tr} \left[S_F(p+q_1+q_2)\Gamma^{(2)}[q_1,\alpha_1; q_2,\alpha_2; p] S_F(p)\Gamma^{(1)}[-(q_1+q_2),\alpha_0; p+q_1+q_2] \right] \Big\} , \\
& + \ 2{\rm Tr} \left[S_F(p+q_1)\Gamma^{(1)}[q_1,\alpha_1; p] S_F(p)\Gamma^{(1)}[q_2,\alpha_2; p-q_2] \ S_F(p-q_2)\Gamma^{(1)}[-(q_1+q_2),\alpha_0; p+q_1] \right] \Big\} \Big|_{q_1=q_2=0} ,
\end{aligned}
$$

Also, momentum derivative of vertex functions do not vanish.

However, one can derive new 3rd order WT-identity from formal expansion of the action in terms of covariant derivatives as

$$\left.\frac{\partial^2 \Gamma^{(3)}[q,\mu;r,\nu;s,\lambda;p]}{\partial q_\alpha \partial r_\beta}\right|_{q,r,s=0} = \left.\frac{\partial^3 \Gamma^{(2)}[q,\mu;r,\nu;p]}{\partial q_\alpha \partial r_\beta \partial p_\lambda}\right|_{q,r=0}$$

Using previous 2^{nd} order WT-identity and new 3^{rd} order WT-identity

One can show correction terms cancel and c_cs is given by winding number expressions as

$$
c_{\rm cs} = -\frac{(-i)^3\cdot 2}{3!5!}\int \frac{d^5p}{(2\pi)^5}\epsilon_{\alpha_0\beta_1\alpha_1\beta_2\alpha_2}
$$
$$
\times\,{\rm Tr}\left[S_F(p)\frac{\partial S_F^{-1}(p)}{\partial p_{\alpha_0}}S_F(p)\frac{\partial S_F^{-1}(p)}{\partial p_{\beta_1}}S_F(p)\frac{\partial S_F^{-1}(p)}{\partial p_{\alpha_1}}S_F(p)\frac{\partial S_F^{-1}(p)}{\partial p_{\beta_2}}S_F(p)\frac{\partial S_F^{-1}(p)}{\partial p_{\alpha_2}}\right].
$$

4-3 Winding number → TKNN formula

This part was essentially already given by

Qi, Hughes, Zhang , Phys. Rev. B78, 195424, 2008

Idea : Evaluate the winding number expression as follows

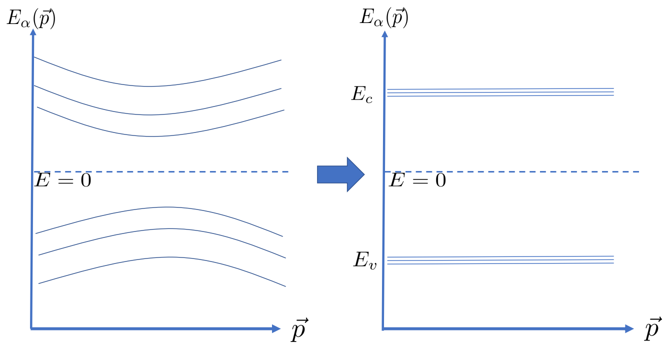
1. Rewrite the fermion propagator using eigenstates

$$S(p) = \sum_{\alpha} |\alpha, \vec{p}\rangle \frac{1}{ip^0 + E_\alpha(\vec{p})} \langle \alpha, \vec{p}|$$

2. Continuously deform only the eigenvalues to degenerate flat band

$$E_\alpha(\vec{p})(< 0) \longrightarrow E_v = \text{constant}$$
$$E_\alpha(\vec{p})(> 0) \longrightarrow E_c = \text{constant}$$

$$|a\rangle \quad (a = 1, \cdots, N_v) : \text{valence bands}$$
$$|\dot{a}\rangle \quad (\dot{a} = 1, \cdots, N_c) : \text{conduction bands}$$



3. Carry out momentum integral over p^0

Step 1

Inserting complete set of energies eigenstates, one obtains

$$c_{\mathrm{cs}} = \frac{n!(-i)^{n+2}}{(n+1)!(2n)!} \int \frac{d^{2n}p}{(2\pi)^{2n}} J$$

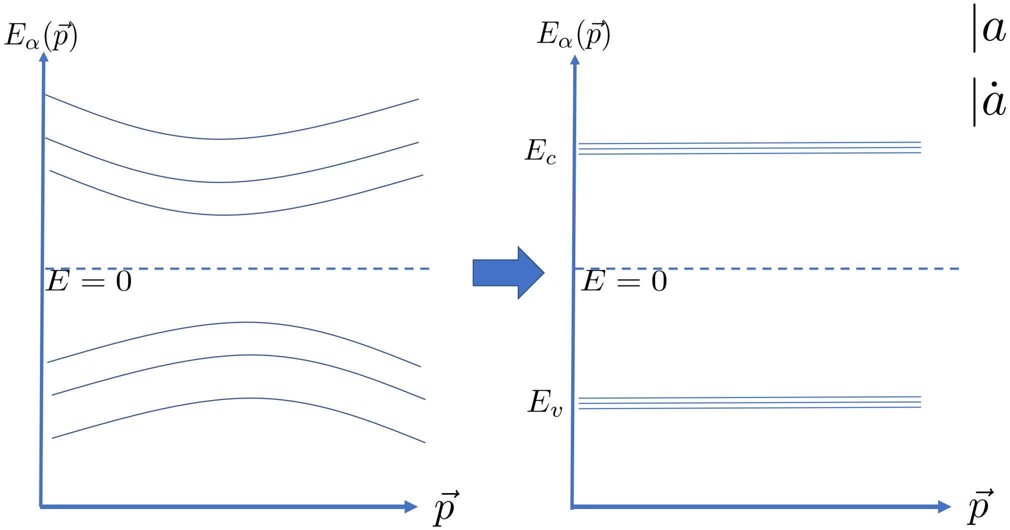
$$J = \sum_{\alpha_1, \cdots, \alpha_{2n}} \epsilon^{i_1 i_2 \cdots i_{2n}} \int \frac{dp^0}{2\pi} \frac{\langle \alpha_1 | \partial_{i_1} H | \alpha_2 \rangle \langle \alpha_2 | \partial_{i_2} H | \alpha_3 \rangle \cdots \langle \alpha_{2n} | \partial_l H | \alpha_1 \rangle}{(ip^0 + E_{\alpha_1})^2 (ip^0 + E_{\alpha_2}) \cdots (ip^0 + E_{\alpha_{2n}})}$$

Step 2

Continuously deform Hamiltonian by changing only the eigenvalues keeping the gap to degenerate flat band.

$$H(\vec{p}) \equiv \sum_{a=1}^{N_v} E_a(\vec{p}) |a(\vec{p})\rangle\langle a(\vec{p})| + \sum_{\dot{b}=1}^{N_c} E_{\dot{b}}(\vec{p}) |\dot{b}(\vec{p})\rangle\langle \dot{b}(\vec{p})|$$

$$H_{\text{new}}(\vec{p}) = E_v \sum_{a=1}^{N_v} |a(\vec{p})\rangle\langle a(\vec{p})| + E_c \sum_{\dot{b}=1}^{N_c} |\dot{b}(\vec{p})\rangle\langle \dot{b}(\vec{p})|$$



$|a\rangle \quad (a = 1, \cdots, N_v)$: valence bands

$|\dot{a}\rangle \quad (\dot{a} = 1, \cdots, N_c)$: conduction bands

Useful formulae

$$\langle a(\vec{p})|\partial_\mu H(\vec{p})|b(\vec{p})\rangle = 0, \qquad \langle \dot{a}(\vec{p})|\partial_\mu H(\vec{p})|\dot{b}(\vec{p})\rangle = 0,$$
$$\langle a(\vec{p})|\partial_\mu H(\vec{p})|\dot{b}(\vec{p})\rangle = (E_c - E_v)\langle a|\partial_\mu \dot{b}\rangle,$$
$$\langle \dot{a}(\vec{p})|\partial_\mu H(\vec{p})|b(\vec{p})\rangle = -(E_c - E_v)\langle \dot{a}|\partial_\mu b\rangle,$$
$$(a, b = 1, \cdots, N_v, \quad \dot{a}, \dot{b} = 1, \cdots, N_c).$$

shows that inserted states should be valence and conduction band appearing alternately.

Step 3

p^0 integration can be easily carried out by Cauchy integral

$$J = \sum_{a_1,\cdots,a_n=1}^{N_v} \sum_{\dot{a}_1,\cdots,\dot{a}_n=1}^{N_c} \epsilon^{i_1 j_1 \cdots i_{2n} j_{2n}} (-1)^{n+1} \frac{(2n)!}{(n!)^2}$$
$$\times \langle a_1|\partial_{i_1}\dot{a}_1\rangle\langle\dot{a}_1|\partial_{j_1}a_2\rangle \times \cdots \times \langle a_n|\partial_{i_n}\dot{a}_n\rangle\langle\dot{a}_n|\partial_{j_n}a_1\rangle.$$

Berry connection

Define the Berry connection as

$$\mathcal{A}^{ab} \equiv \mathcal{A}^{ab}_{\mu} dp^{\mu} = -i\langle a|\partial_{\mu} b\rangle dp^{\mu} = -i\langle a|db\rangle$$

$$\mathcal{F}^{ab} \equiv (d\mathcal{A} + i\mathcal{A}\mathcal{A})^{ab}$$

$$= -\langle da|db\rangle + i\sum_{c=1}^{N_v}(-i)\langle a|dc\rangle(-i)\langle c|db\rangle$$

Inserting complete set $\mathbf{1} = \sum_{c=1}^{N_v} |c\rangle\langle c| + \sum_{\dot{c}=1}^{N_c} |\dot{c}\rangle\langle\dot{c}|$

$$\mathcal{F}^{ab} = i\sum_{\dot{c}=1}^{N_c}\langle a|d\dot{c}\rangle\langle\dot{c}|db\rangle$$

This product gives Berry curvature!

Using $\mathcal{F}^{ab} = i \sum_{\dot{c}=1}^{N_c} \langle a|d\dot{c}\rangle\langle\dot{c}|db\rangle$, one can rewrite the integrand J using only the product of Berry curvature

Inserting this expression into c_cs using J, and using the definition of the Berry curvature one obtains

$$c_{\mathrm{cs}} \equiv \frac{k}{(n+1)!(2\pi)^n} = \frac{(-1)^n}{(n+1)!(2\pi)^n} \int_{BZ} \mathrm{ch}_n(\mathcal{A}),$$

$$\mathrm{ch}_n(\mathcal{A}) = \frac{1}{n!}\frac{1}{(2\pi)^n}\mathrm{tr}(\mathcal{F}^n)$$

This result shows that

Chern-Simons level in field theory approach

and

Chern number in microscopic approach (TKNN)

are identical for general Hamiltonian bilinear in fermion for D=2+1, 4+1 dimensions.

5. Summary

- We have shown microscopic approach (TKNN) and field theory approach give identical topological number for general Hamiltonian bilinear in fermion.
- A series of Ward-Takahashi identities are crucial to show the equivalence.
- No other details beyond gauge symmetry (such as existence of relativistic field theory at low energy) is needed.

- In 4+1 dimensions, there are two independent Chern numbers. However, only a particular Chern number appeared.

- This means that topological classification in microscopic approach may be finer, or those detailed structure may not be robust.

- It would be interesting to see similar equivalence holds or not for other cases such as systems with higher symmetry or systems with interacting fermions.