Superstring theory and integration over the moduli space

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Oct, 9th, 2013@Kyoto

arXiv:1303.7299 [KO,Yuji Tachikawa]

Section 1

introduction

Simplified history of the superstring theory

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- '12: "Superstring Perturbation Theory Revisited" [E.Witten]:
  Superstring perturbation theory and supergeometry

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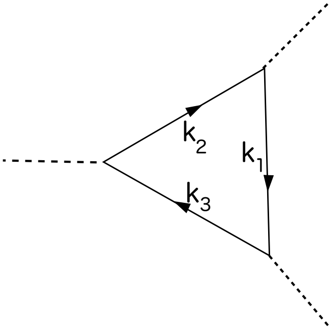
- $\Rightarrow$ Feynman diagram



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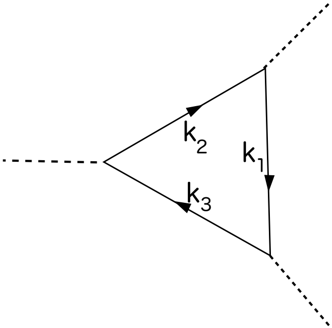

- $\Rightarrow \int \mathrm{d}^{4\mathbf{g}}\mathbf{k}\frac{\mathbf{1}}{\mathbf{k}^2} \times \cdots$

Schwinger parameter

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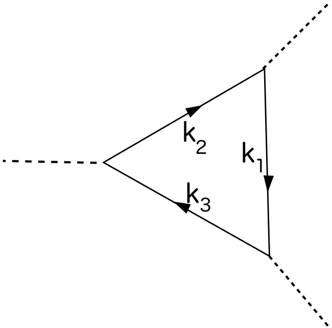

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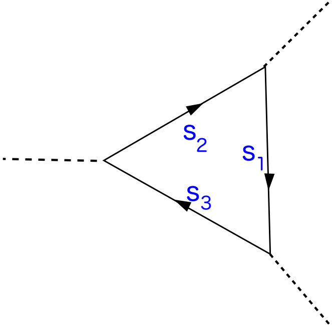

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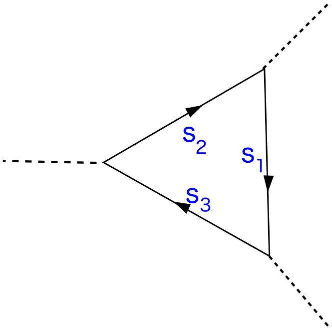

- Feynman parameter: $\mathbf{x_i} = \mathbf{s_i} / \sum_\mathbf{j} \mathbf{s_j}$

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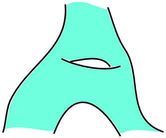

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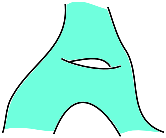

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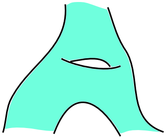

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- $\Rightarrow \mathcal{A} = \int_{\mathcal{M}_{bos,g}} \mathrm{d}m\,\mathrm{d}\bar{m}\, F(m, \bar{m})$

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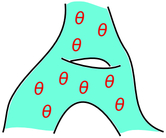

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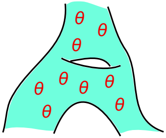

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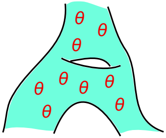

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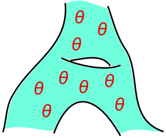

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- $\overset{?}{\Rightarrow} \mathcal{A} = \int_{\mathcal{M}_{\mathrm{bos,g}}} \mathrm{d}\mathbf{m}\mathrm{d}\bar{\mathbf{m}}\mathbf{F}'(\mathbf{m}, \bar{\mathbf{m}})$

The bosonic and super moduli space

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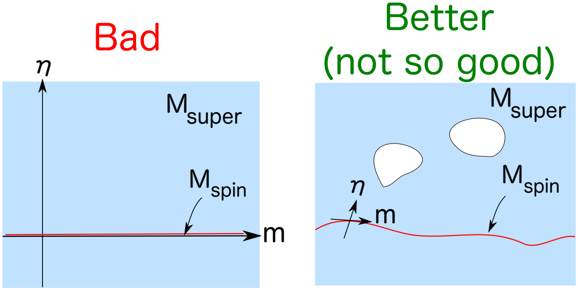
- $\mathcal{M}_{\text{super}} \supset \mathcal{M}_{\text{bos}}$



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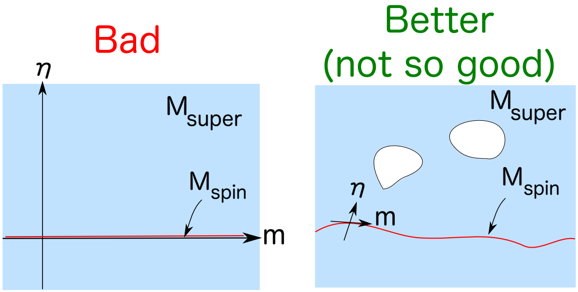

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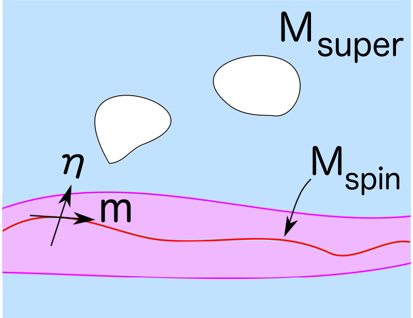

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- Topological amplitudes
  [I. Antoniadis, E. Gava, K.S. Narain, T.R. Taylor '94]
  [M. Bershadsky, S. Cecotti, H. Ooguri, C. Vafa '94]

Section 2

Supergeometry and Superstring

Supermanifold

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- $\mathbf{M}$: supermanifold of dimension $\mathbf{p|q}$
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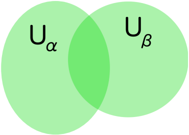

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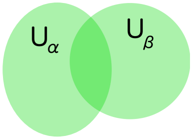
- $x_i' = f_i(x|\theta)$
- $\theta_\mu' = \psi_\mu(x|\theta)$
- $\mathbf{M}_{red} = \{(x_i, \cdots, x_p)\} \hookrightarrow \mathbf{M}$



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  $\theta' = \zeta'(\mathsf{z}|\eta) + \theta\sqrt{\partial_{\mathsf{z}}\mathsf{u}(\mathsf{z}|\eta) + \zeta(\mathsf{z}|\eta)\partial_{\mathsf{z}}\zeta(\mathsf{z}|\eta)}$
- $\mathsf{D}_\theta = \partial_\theta + \theta\partial_{\mathsf{z}} = \mathsf{F}(\mathsf{z}|\theta)(\partial_{\theta'} + \theta'\partial_{\mathsf{z}'})$

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Split Super Riemann Surface

- $\chi = 0$ (globally)
- $z' = u(z)$
  $\theta' = \theta\sqrt{\partial_z u(z)}$
- $\theta \in H^0(\Sigma, T\Sigma_{\text{red}}^{*1/2})$
  $\Rightarrow$ split SRS $\leftrightarrow$ Riemann surface with spin str.

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- $\mathcal{M}_{\text{super,red}} \simeq \mathcal{M}_{\text{spin}}$

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  $= \mathbf{3g - 3 + n_{\mathrm{NS}} + n_{\mathrm{R}} | 2g - 2 + n_{\mathrm{NS}} + n_{\mathrm{R}}/2 \quad (g \geq 2)}$

Integration cycle for string theory

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- $(t^1 \cdots t^{2\Delta_e} | \zeta^1 \cdots \zeta^{\Delta_o + \widetilde{\Delta_o}}) \in \Gamma \subset \mathcal{M}_R \times \mathcal{M}_L$

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- $\mathbf{m}^1 = \mathbf{t}^1 + \mathbf{i}\mathbf{t}^2 + \text{nilp.}, \ \bar{\mathbf{m}}^1 = \mathbf{t}^1 - \mathbf{i}\mathbf{t}^2 + \text{nilp.},$
  $\zeta^{\mathbf{i}} = \eta^{\mathbf{i}}, \ \zeta^{\mathbf{i}+\Delta_o} = \bar{\eta}^{\bar{\mathbf{i}}}$

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- $\mathcal{M}_R \ni (m, \eta), \quad \mathcal{M}_L \ni (\bar{m}, \bar{\eta})$
- $m^1 = t^1 + it^2 +$ nilp., $\bar{m}^1 = t^1 - it^2 +$ nilp.,

  $\zeta^i = \eta^i,\ \zeta^{i+\Delta_o} = \bar{\eta}^{\bar{i}}$
- Integration over Γ is unique

  [E.Witten , arXiv:1209.2199]

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- $(t^1 \cdots t^{2\Delta_e} | \zeta^1 \cdots \zeta^{\Delta_o + \widetilde{\Delta}_o}) \in \Gamma \subset \mathcal{M}_R \times \mathcal{M}_L$
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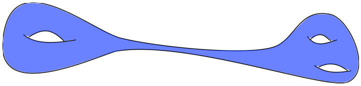
- $\mathcal{M}_R \ni (m, \eta), \quad \mathcal{M}_L \ni (\bar{m}, \bar{\eta})$
- $m^1 = t^1 + it^2 +$ nilp., $\bar{m}^1 = t^1 - it^2 +$ nilp.,
  $\zeta^i = \eta^i,\ \zeta^{i+\Delta_o} = \bar{\eta}^{\bar{i}}$
- Integration over Γ is unique
  [E.Witten , arXiv:1209.2199]
- if we fix the behavior of Γ near boundary.



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- Valid where $[\delta(\mathbf{z} - \mathbf{p}_\sigma)]$ spans odd deformations
  $\Rightarrow$ Coordinates given by picture changing formalism is not global!

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- non-holomorphic projection destroys holomorphic factorization.

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  (with holomorphic factorization)

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Section 3

Reduction to integration over moduli

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- ω does not depend on the locations of picture changing operators

Topological amplitudes in string theory

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- F-term of 4d effective field theory is related to topological string.

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- $\mathbf{F_g}$: $\mathbf{g}$ loop vacuum amplitude

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Vertex operators

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- Graviphoton: $\mathbf{V_T = \Sigma}$ $\times$ spacetime $\times$ **(**ghost**)**
  $\mathbf{\Sigma = \exp(i\frac{\sqrt{3}}{2}(H(z) \mp \widetilde{H}(z)))} \quad \mathbf{H : U(1)_R}$ boson
  $\mathbf{\Sigma}$: $\mathbf{U(1)_R}$ charge $(3/2, \mp 3/2)$

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- Graviton: $\mathbf{V_R} = \int_{\mathbf{\Sigma}_{\rm red}}$ spacetime,ghost

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- $\mathcal{A}_g =$
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  $\prod_{\sigma=1}^{\Delta_o} \delta(\int_{\Sigma_{red}} d^2z \beta\chi^\sigma) \prod_{\tau=1}^{\tilde{\Delta}_o} \delta(\int_{\Sigma_{red}} d^2z \tilde{\beta}\tilde{\chi}^\tau)$
  $\exp(-\int_{\Sigma_{red}} (T_F\chi + \tilde{T}_F\tilde{\chi} + A\tilde{\chi}\chi)) \Big\rangle_m$
- $T_F = T_{F,\text{spacetime,ghost}} + G^+ + G^-$
- $G^\pm$: $U(1)_R$ charge ± 1
- $A = A_{++} + A_{+-} + A_{-+} + A_{--} + A_{\text{spacetime}}$
  $A_{\pm\pm}$: $U(1)_R$ charge $(\pm 1, \pm 1)$
  $\Leftarrow$ Coupling A to $\mathcal{N} = (2,2)$ supergravity:
  $\int_{\Sigma_{red}} A_{\pm\pm}\chi^\mp\tilde{\chi}^\mp \to \chi = \chi^+ = \chi^-$

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- Another application of supergeometry

End.

Berkovits and Vafa embedding

[N.Berkovits, C.Vafa '94]

- Worldsheet supersymmetric model equivalent to bosonic string theory
  $\Rightarrow$ Bosonic string : An (unstable) vacuum of superstring theory
- $\langle \mathbf{V_1} \cdots \mathbf{V_n} \rangle_{\text{bos}} = \left\langle \mathbf{V'_1} \cdots \mathbf{V'_n} \right\rangle_{\text{super}}$
- $\Leftrightarrow \int_{\mathcal{M}_{\text{bos}}} \cdots = \int_{\mathcal{M}_{\text{super}}} \cdots$

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  $\mathbf{H}^1(\mathbf{\Sigma}_{\mathrm{red}}, \mathbf{T\Sigma}_{\mathrm{red}})+$ positions of punctures

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- $\dim \mathcal{M}_{\text{super}} = \boldsymbol{\Delta}_{\mathbf{e}} | \boldsymbol{\Delta}_{\mathbf{o}}$
  $= \mathbf{3g - 3 + n_{\text{NS}} + n_{\text{R}} | 2g - 2 + n_{\text{NS}} + n_{\text{R}}/2 \quad (g \geq 2)}$

Superconformal transformation

- $\nu_f = f(z)(\partial_\theta - \theta\partial_z)$
- $V_g = g(z)\partial_z + 1/2\partial_z g\theta\partial_\theta$
- $G_n = \nu_{z^{n+1/2}}$
- $L_m = -V_{z^{m+1}}$

Superconformal transformation near R vertex

- $\nu_f = f(z)(\partial_\theta - z\theta\partial_z)$
- $V_g = z(g(z)\partial_z + 1/2\partial_z g\theta\partial_\theta)$
- $G_r = \nu_{z^r}$
- $L_m = -V_{z^m}$

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- $\mathbf{D}^*_\theta = \partial_\theta + (\mathbf{z} - \mathbf{z_1})\theta\partial_\mathbf{z}$
- $\theta' = \sqrt{\mathbf{z} - \mathbf{z_1}}\,\theta \Rightarrow \mathbf{D}^*_\theta = \sqrt{\mathbf{z}}(\partial_{\theta'} + \theta'\partial_\mathbf{z})$
  θ':antiperiodic around $\mathbf{z_1}$

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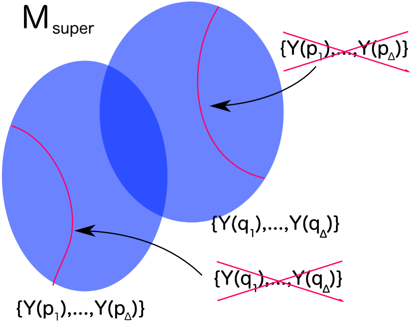

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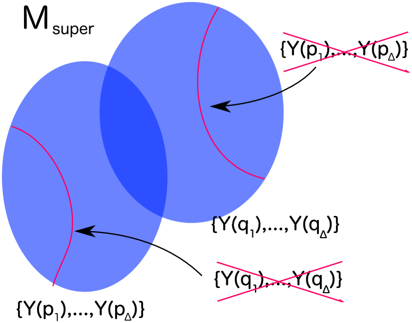

- movement of PCOs $\Rightarrow$ BRST exact term $\Leftrightarrow$ exact form on patches