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Entanglement Entropy after Multiple excitations in Rational CFTs

Tokiro Numasawa

Osaka University Particle Physics Theory

Based on

(1)arXiv:1403.0702 (Phys. Rev D 90 041701(2014))with Song He, Tadashi Takayanagi and Kento Watanabe

(2)arXiv:1610.06181(JHEP (2016) 061)

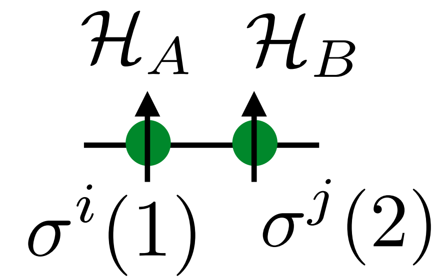
Contents of this talk

(1)Introduction
(2)How to calculate (Renyi) in QFTs
(3)Results for single excitation in 2d RCFTs
(4)Results for multiple excitation
(5)Conclusion and Future problems

(1)Introduction

・Entangled states

Let's consider two spins.



(1) $|\psi\rangle = |\uparrow\rangle\,|\uparrow\rangle$ (no entanglement)

$$\langle\psi|\,\sigma^i(1)\sigma^j(2)\,|\psi\rangle = \langle\psi|\,\sigma^i(1)\,|\psi\rangle \cdot \langle\psi|\,\sigma^j(2)\,|\psi\rangle$$

(2) $|\psi'\rangle = \dfrac{1}{\sqrt{2}}(|\uparrow\rangle\,|\uparrow\rangle + |\downarrow\rangle\,|\downarrow\rangle)$ (entangled)

$$\langle\psi'|\,\sigma^i(1)\sigma^j(2)\,|\psi'\rangle - \langle\psi'|\,\sigma^i(1)\,|\psi'\rangle \cdot \langle\psi'|\,\sigma^j(2)\,|\psi'\rangle \neq 0$$

non local correlation (consider i=j= z)

Definition of Entanglement Entropy

Entanglement Entropy (EE) is a quantification of quantum entanglement and is defined as follows.

Divide a quantum system into two subsystems A and B (represent a total Hilbert space as a tensor product of two Hilbert spaces):

$$\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Define the reduced density matrix ρ_A by

$$\rho_A = \mathrm{Tr}_B \rho_{tot} = \mathrm{Tr}_B \left|\Psi\right\rangle \left\langle\Psi\right|$$

The entanglement entropy is defined as a von-Neumann entropy of ρ_A:

$$S_A = -\mathrm{Tr}_A \rho_A \log \rho_A$$

There is a related quantity with Entanglement Entropy called (n-th) Renyi Entanglement entropy (REE) :

$$S_A^{(n)} = \frac{1}{1-n} \log \mathrm{Tr}_A \rho_A^n$$

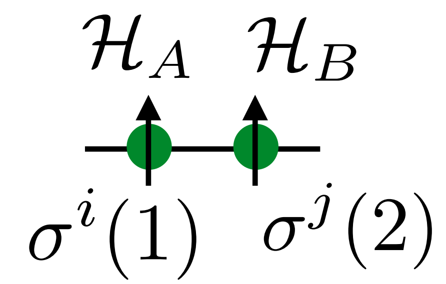
If we take the limit of n → 1、REE reduces to EE:

$$\lim_{n \to 1} S_A^{(n)} = S_A$$

It is easier to compute $\mathrm{Tr}_A \rho_A^n$ than to compute $\mathrm{Tr}_A \rho_A \log \rho_A$ (We see this in detail later).

Entanglement Entropy represents the loss of quantum information when we assume the system H_B

(1) $|\Psi\rangle = |\uparrow\rangle \otimes |\uparrow\rangle$



$\Rightarrow$ $\rho_A = |\uparrow\rangle\langle\uparrow|$ and $S_A^{(n)} = 0$.

Especially $S_A = 0$

(2) $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle)$

$\Rightarrow$ $\rho_A = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|)$ and $S_A^{(n)} = \log 2$.

Especially $S_A = \log 2$

Entanglement Entropy in QFTs

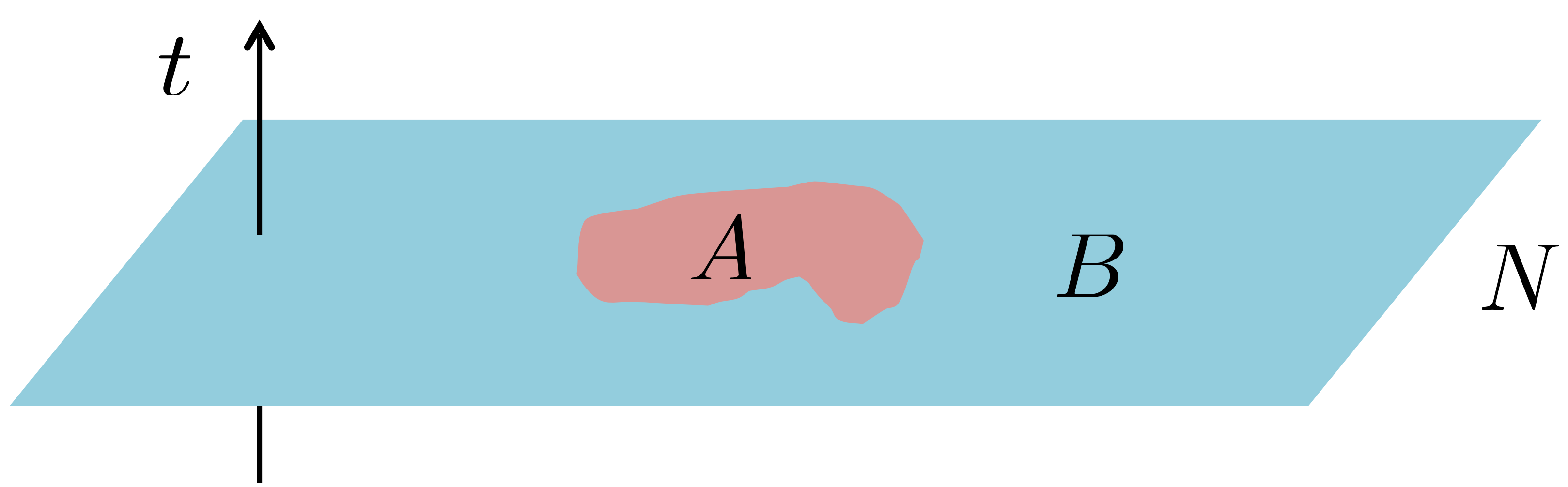
We consider a QFT on a d+1 dim. manifold $\mathbf{R} \times N$.

$$\langle 0| \phi(x)\phi(y) |0\rangle \neq 0$$

⇒ Ground states of QFTs should have entanglement !

To decompose the total Hilbert space,
We choose the subspace $A \subset N$.

In QFTs , We divide the total Hilbert space into $H_A \otimes H_B$
accompanied with the division of manifold N into $A \cup B$.



Use of Entanglement Entropy

E.E. is useful to characterize the quantum correlation of the states.

(1)A quantum Order parameter

➡ Classification of quantum phases.

(for example, Entanglement Entropy detect a topological order)

[Kitaev-Preskill 05,Levin-Wen 05]

(2)Relation with gravity

Ryu-Takayanagi formula: $$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N}$$ [Ryu-Takayanagi 06]

(3)Characterization of Excited states [Calabrese-Cardy 05,07]

Global quenches $S_A \propto c \cdot t$

Local quenches $S_A \propto c \cdot \log t$

➡ ・they are characterized by the time dependence.

In this talk, we consider the local operator excited states:

$$|\mathcal{O}(x)\rangle \equiv \mathcal{O}(x)\,|0\rangle$$ (single excitation)

or more generically

$$\mathcal{O}_{a_1}(x_1)\mathcal{O}_{a_2}(x_2)\cdots\mathcal{O}_{a_k}(x_k)\,|0\rangle$$ (multiple excitations)

・Decomposition of primary operator to Chiral Vertex Operator(CVO)

(in 2 dim)

$$\mathcal{O}_a(z,\bar{z}) = \sum_{bc} \underbrace{\psi\left[{a \atop bc}\right](z)}_{\text{CVO}} \otimes \underbrace{\bar{\psi}\left[{a \atop bc}\right](\bar{z})}_{\text{CVO}}$$

CVO

→ generally a primary operator is "Entangled".

・Time evolution reflects these entanglement !

Class of 2d CFTs

- Rational (Minimal Models, WZW models)
- Non-Rational (Holographic CFTs, generic point of moduli of CY3s)

・ Today I will talk about Rational CFTs

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Calculation of Entanglement Entropy

・Replica method

We consider to calculate $\mathrm{Tr}_A(\rho_A^n)$ instead of $\mathrm{Tr}_A\ \rho_A \log \rho_A$.

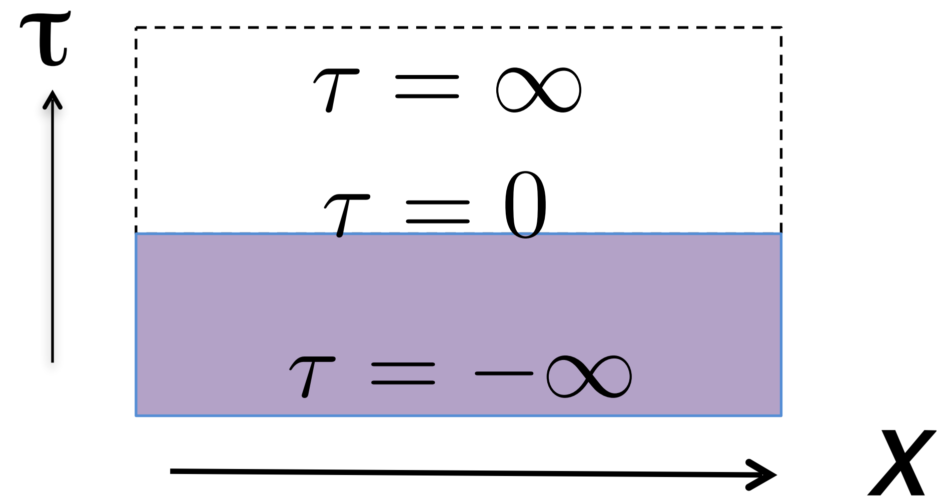
Then, we can get EE in the following way:

$$S_A = -\mathrm{Tr}\rho_A \log \rho_A = \frac{1}{1-n} \log \mathrm{Tr}_A \rho_A^n \bigg|_{n=1}$$:Replica method

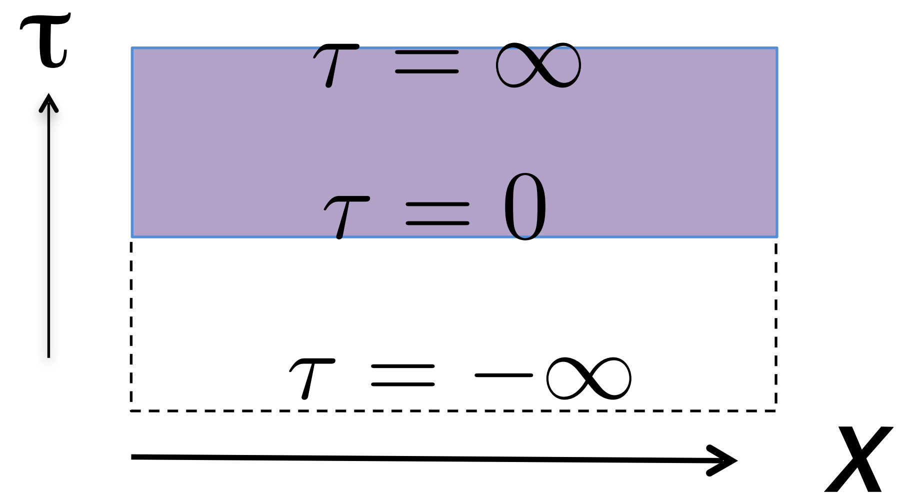
First, we consider the ground states cases.

In the path integral formalism, the ground state wave functional $|\Psi\rangle$ can be represented as follows:

$$|\Psi\rangle = \int_{\tau=-\infty}^{\tau=0} \mathcal{D}\phi \; e^{-S[\phi]} =$$

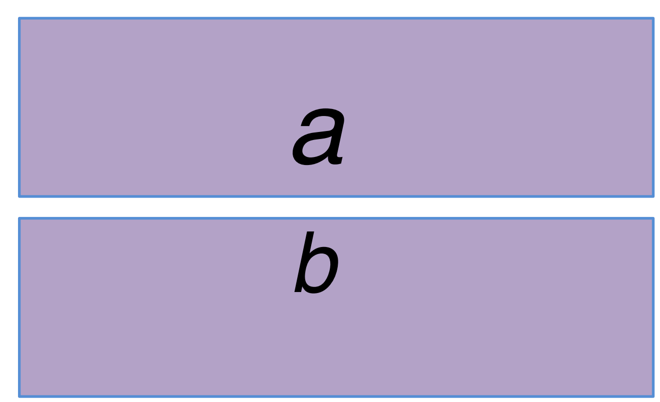


$$\langle\Psi| = \int_{\tau=0}^{\tau=\infty} \mathcal{D}\phi \; e^{-S[\phi]} =$$



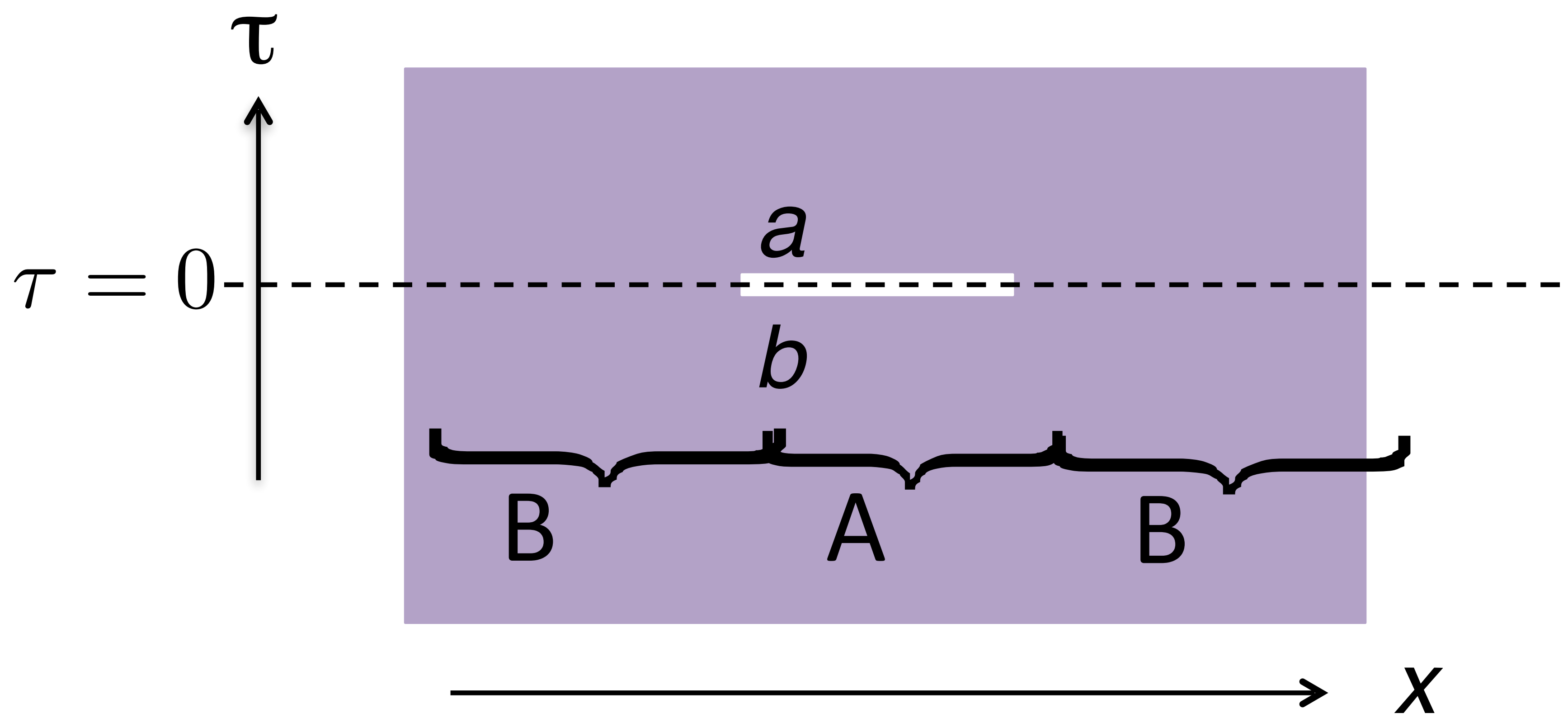
So we can express the total density matrix $\rho_{tot} = |\Psi\rangle \langle\Psi|$ as follows:

$$[\rho_{tot}]_{ab} =$$



Now, we can express the reduced density matrix $\rho_A = \mathrm{Tr}_B \left|\Psi\right\rangle \left\langle\Psi\right|$ as follows:

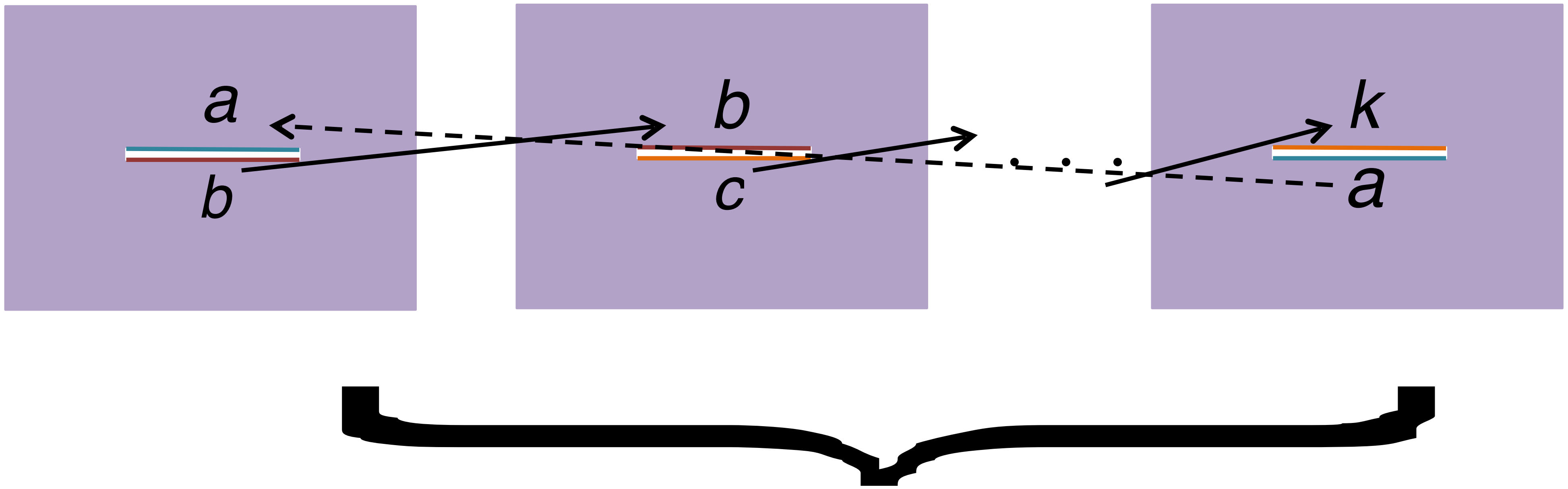
$$[\rho_A]_{ab} =$$



Finally, we can get a representation of $Tr_A \rho_A^n = \sum_{a,b,\cdots k} [\rho_A]_{ab} [\rho_A]_{bc} \cdots [\rho_A]_{ka}$

in the formalism of path integral as follows:

$$\mathrm{Tr}_A \rho_A^n =$$



Σ_n : branched covering of spacetime manifold (branched at ∂A)

$$= Z_n / (Z_1^n)$$

Z_n : Partition function of the geometry of Σ_n .

Final Result

$$\mathrm{Tr}_A \rho_A^n = Z_n / (Z_1)^n$$

Replica method for Local operator Excited states

In this case, the total density matrix is given by

$$\rho_{tot}(t) = e^{-iHt} e^{-\varepsilon H} \mathcal{O}(x) |0\rangle \langle 0| \mathcal{O}^{\dagger}(x) e^{-\varepsilon H} e^{iHt}$$

$$= O(\tau_e, x) |0\rangle \langle 0| \mathcal{O}^{\dagger}(\tau_l, x) \quad (\tau_e \equiv -\varepsilon - it, \tau_l \equiv -\varepsilon + it)$$

$(x : \text{coordinate of space})$

(ε :cutoff)

where ε is the UV regulator for the operator.(This is not equal to the lattice space.)

τ denotes the Euclidean time. To compute the time evolution, first we compute physical quantities considering τ as real parameter and then analytically continue to complex value .

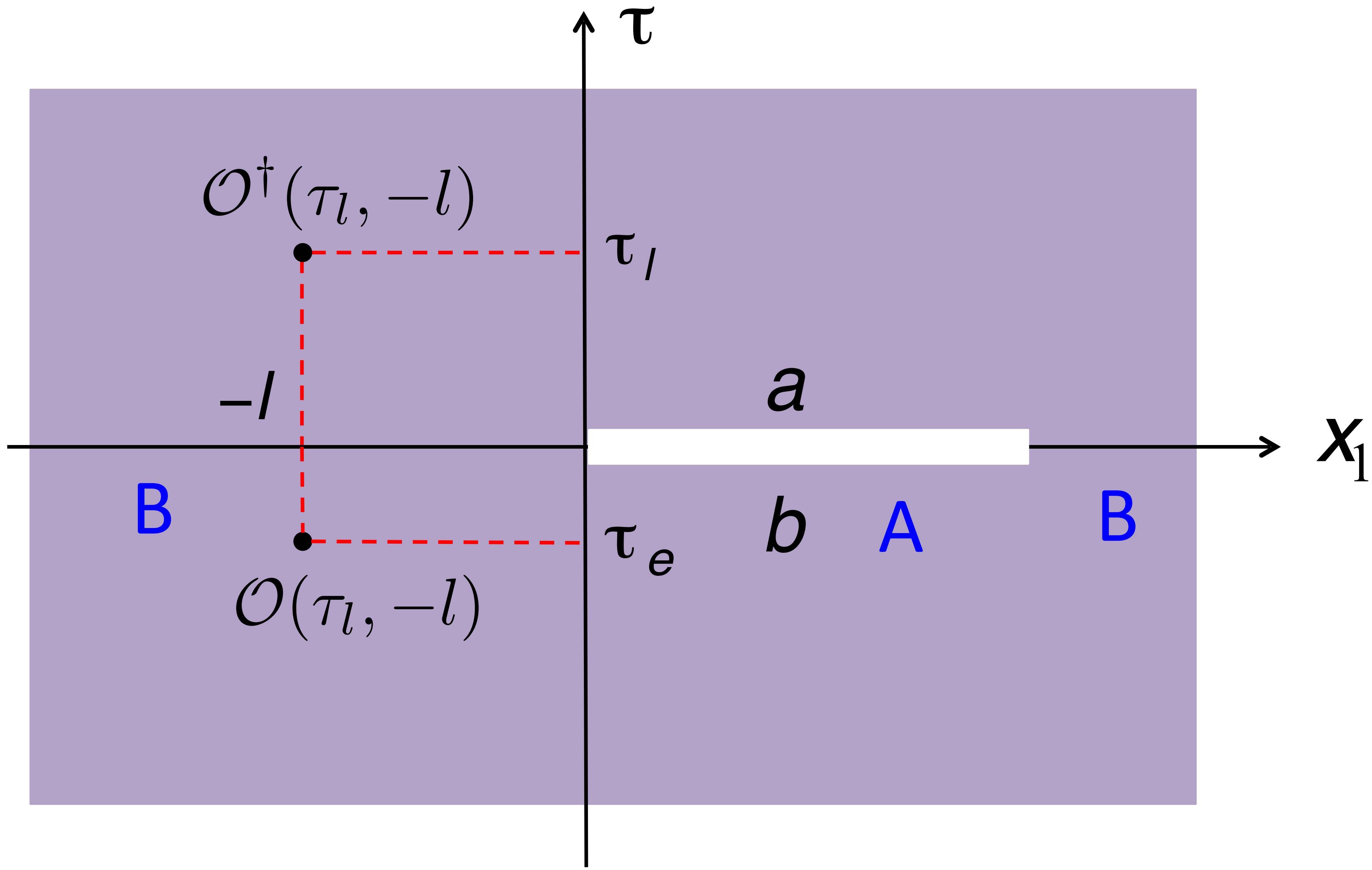
Reduced matrix for excited states

Consider 1+1 dim CFT on $\mathbf{R}^2$.

We take the coordinate as $(\tau, x) \in \mathbf{R}^2$.

Then, the reduced matrix is represented as follows:

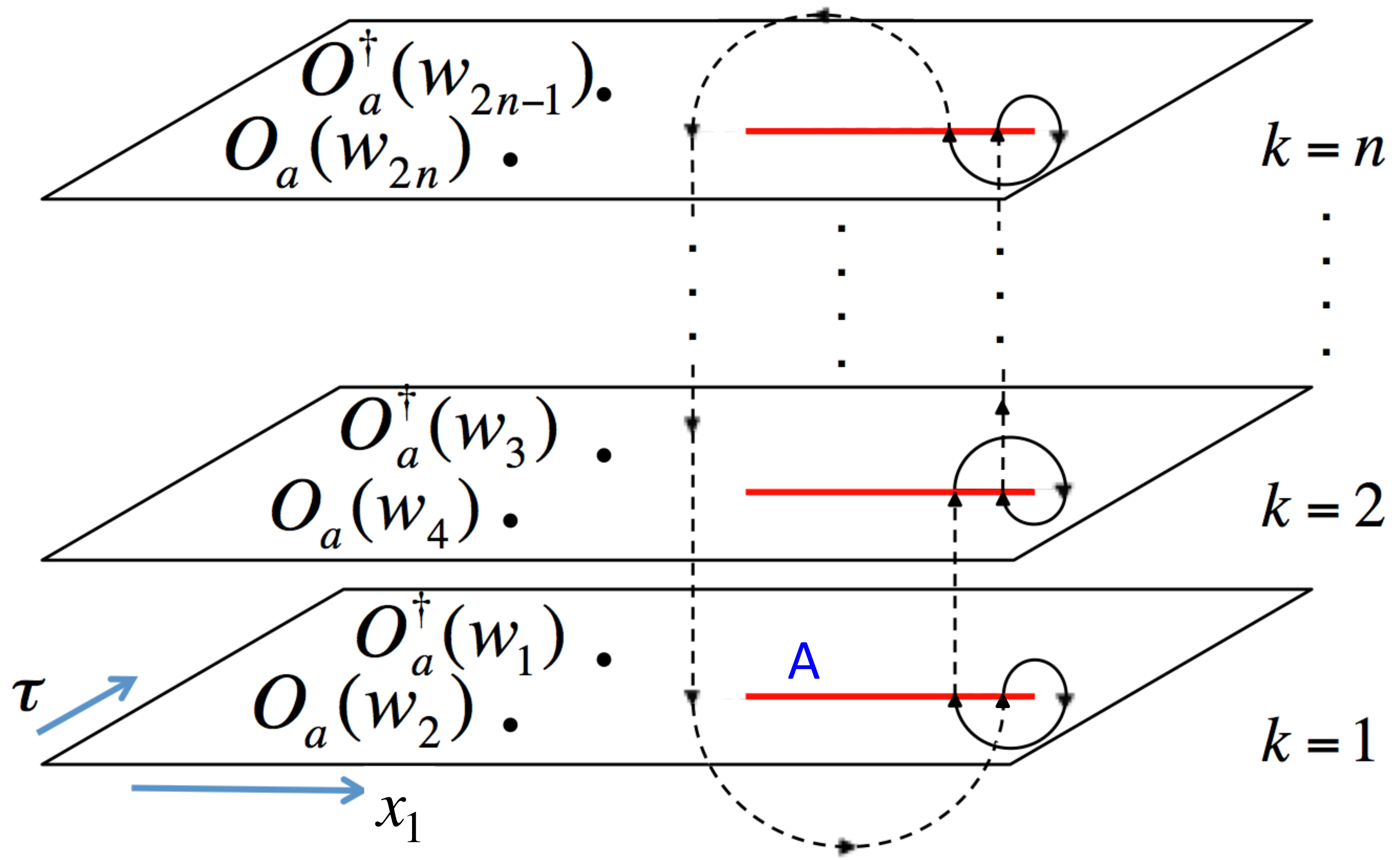
$[\rho_A]_{ab} =$



Finally, we can express the $\mathrm{Tr}_A \rho_A^n$ in terms of 2n-pt correlation function on Σ_n :

$$\Delta S_A^{(n)} = \frac{1}{1-n}\left[\log \langle O^\dagger(w_1)O(w_2)\cdots O^\dagger(w_{2n-1})O(w_{2n})\rangle_{\Sigma_n} - n\log\langle O^\dagger(w_1)O(w_2)\rangle_{\Sigma_1}\right]$$

$$\Delta S_A^{(n)} = S_A^{(n)} - S_A^{(n)}(|GS\rangle)$$



w_{2k-1}, w_{2k} :the coordinate of the inserted local operator on the k-th sheet .

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massless free scalar in 2d

[Nozaki-TN-Takayanagi 14]

In the case $\mathcal{O}_1 =: e^{i\alpha\phi}:$ (vertex op) , we find the result is trivial:

$$\Delta_S^{(n)} = \log 2$$

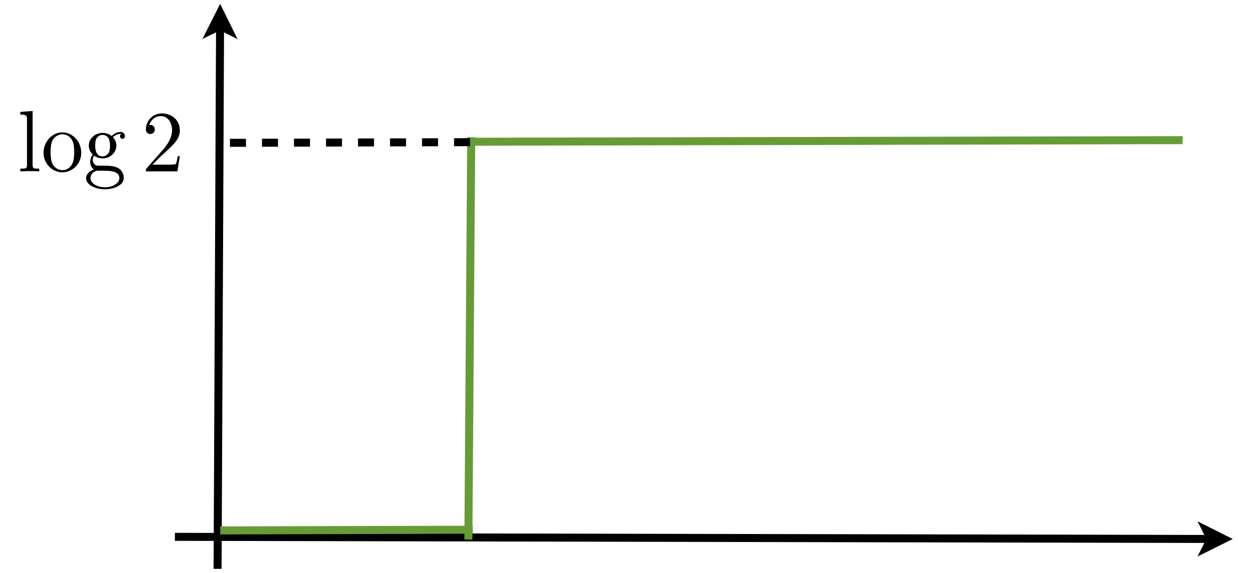
We consider the following (EPR like)operators:

$$\mathcal{O}_2 =: e^{i\alpha\phi} : + : e^{-i\alpha\phi} : \quad , \alpha \in \mathbf{R}.$$

if we take the operator $\mathcal{O}_2 =: e^{i\alpha\phi} : + : e^{-i\alpha\phi} :$
we can get a non-trivial value under time evolution.

If $t > l$

$$\Delta S_A^{(n)} = \log 2$$

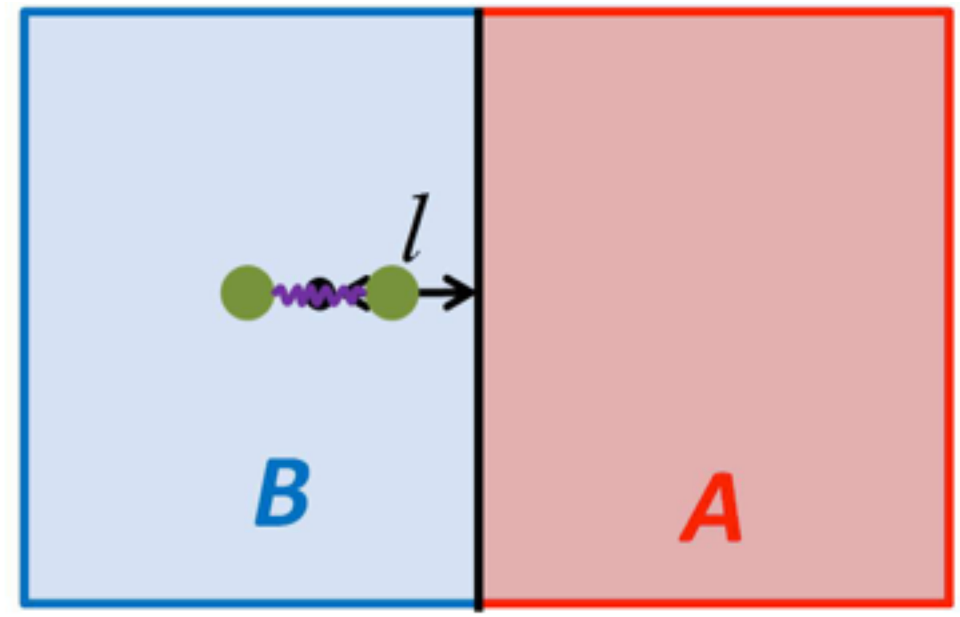


We can interpret this as follows.

At $t=0$, entangled (quasi) particles are created at $x=-l$,
and they are propagate with the velocity of light.

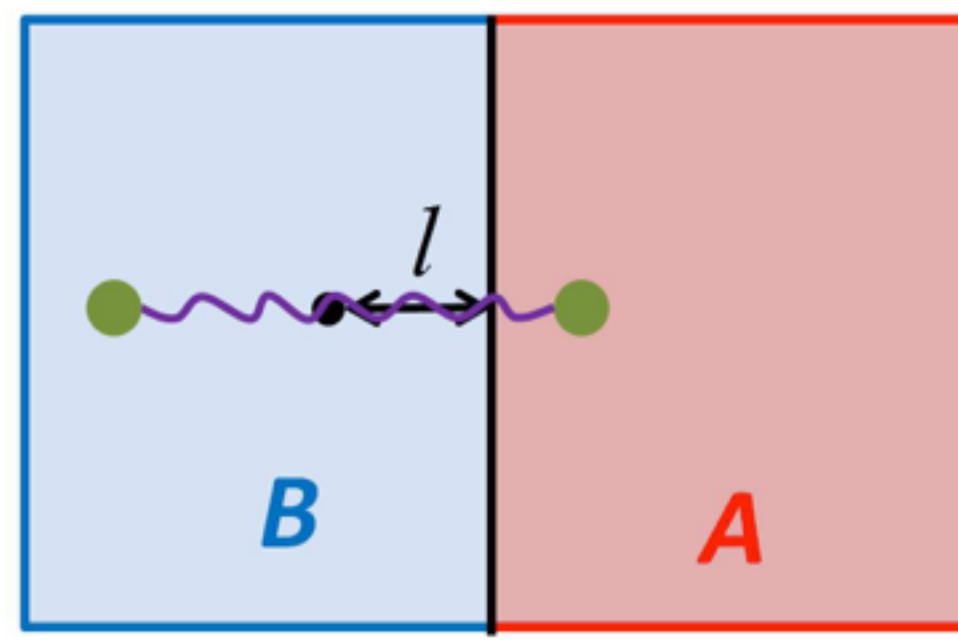
If $t<l$, quasi particles don't leach at entangling surface , so REE don't change:

$$\Delta S_A^{(n)}=0$$



$t < l$

If $t>l$, quasi particles pass the entangling surface, so the value of REE increase.



$t > l$

In 2d CFT, free boson is decomposed into chiral and anti-chiral component:

$$\phi(t,x) = \phi_L(x-t) + \phi_R(x+t)$$

In the case $\mathcal{O}_1 =: e^{i\alpha\phi}:$, the local operator excited state is tensor product state:

$$\mathcal{O}_1 |0\rangle = e^{i\alpha\phi_L} |0_L\rangle \otimes e^{i\alpha\phi_R} |0_R\rangle$$

Reduced density matrix becomes pure state $e^{i\alpha\phi_L} |0_L\rangle \langle 0_L| e^{-i\alpha\phi_L}$

On the other hand ,

$$\mathcal{O}_2 = e^{i\alpha\phi_L} |0_L\rangle \otimes e^{i\alpha\phi_L} |0_R\rangle + e^{-i\alpha\phi_L} |0_L\rangle \otimes e^{-i\alpha\phi_L} |0_R\rangle$$

$$\approx |\uparrow\rangle_L \otimes |\uparrow\rangle_R + |\downarrow\rangle_L \otimes |\downarrow\rangle_R$$

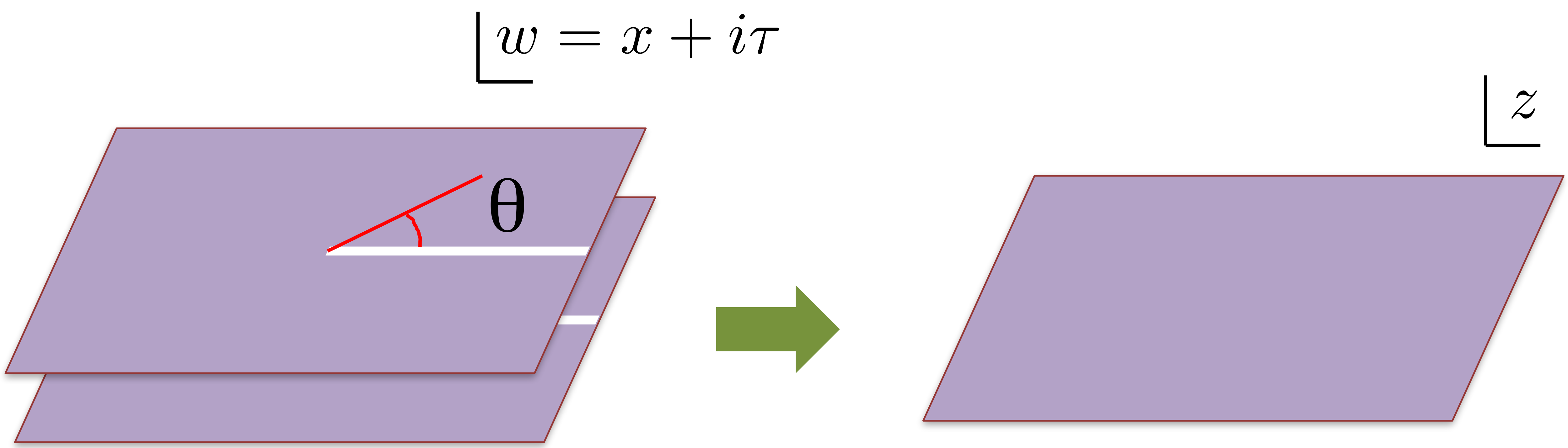
This is interpreted as EPR state and EE becomes $\log 2$!

(4)Results for 2d RCFTs

・ n=2 REE

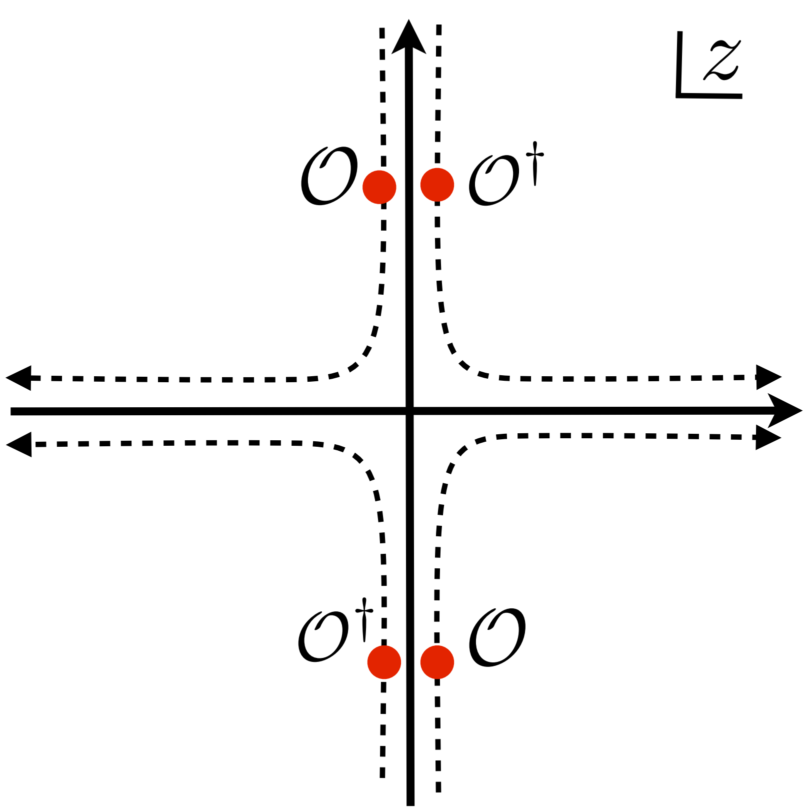
By conformal mapping, we can map 2-sheet Riemann surface Σ_2 to 1 sheet Riemann surface $\Sigma_1 = \mathrm{C}$ (complex plane) :

$$z = \sqrt{w} = \sqrt{r} e^{i\frac{\theta}{2}} \qquad (0 \leq \theta \leq 4\pi)$$

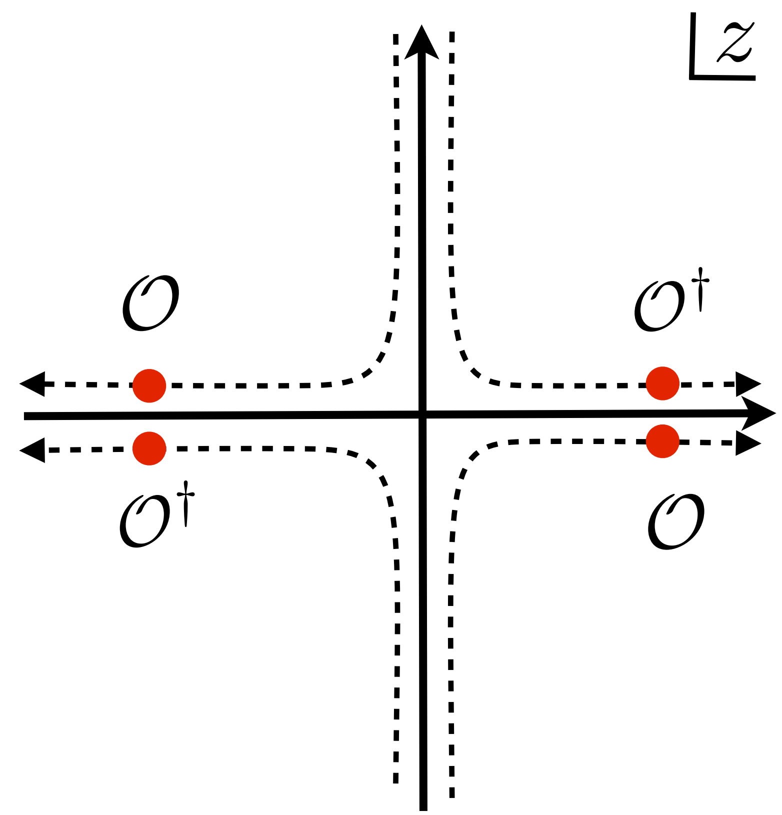


The orbit of each coordinate

- **Holomorphic part**



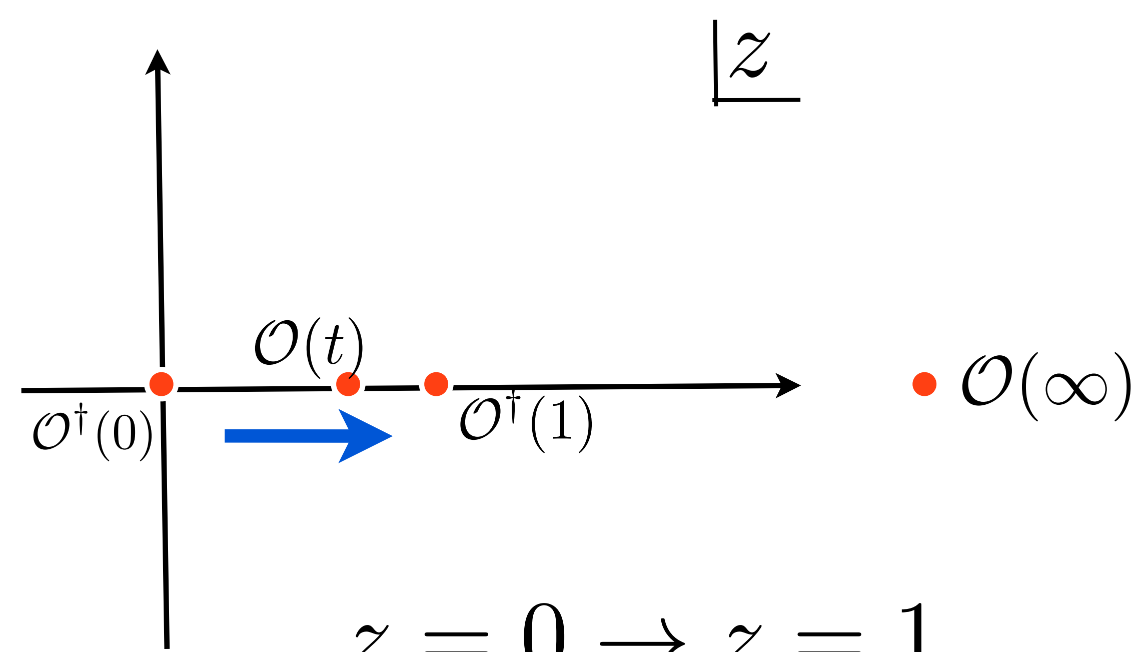
time evolution



Using $SL(2,C)$ symmetry

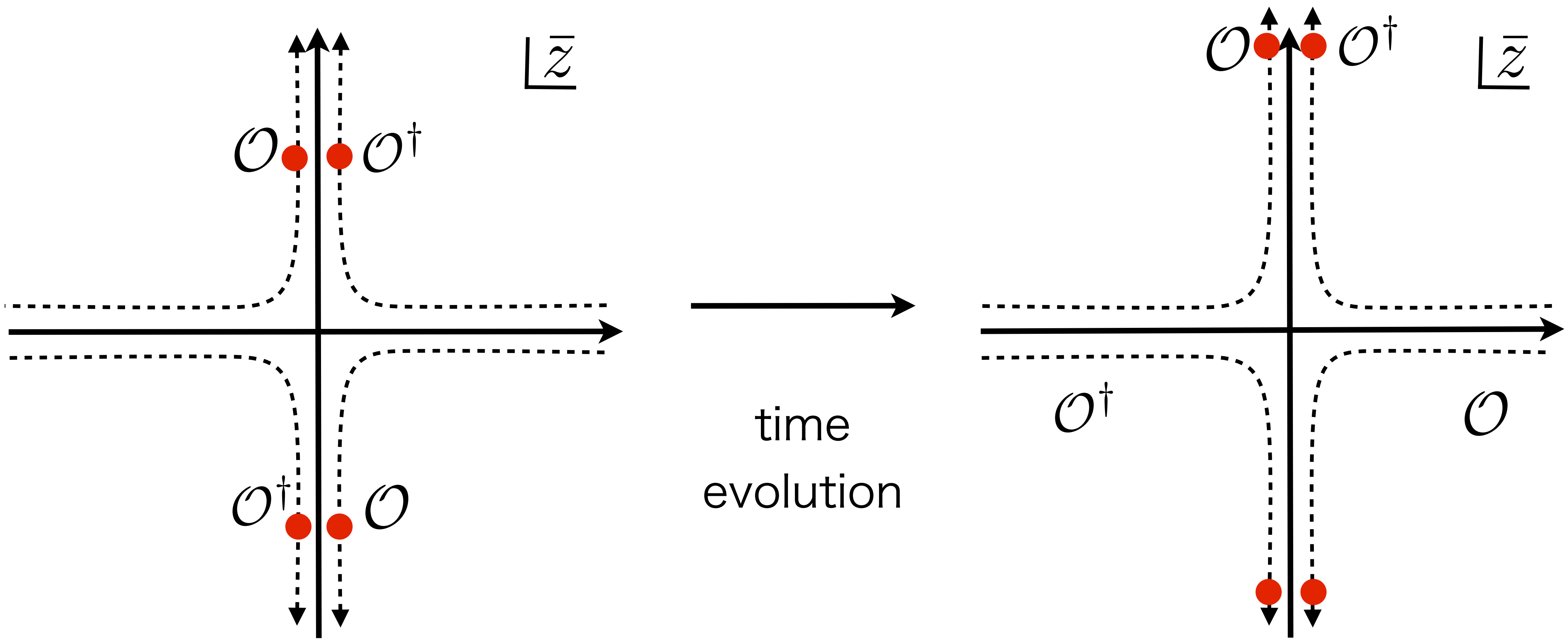
(Global Conformal Sym)

$$z(t) = \frac{-(l-t) + \sqrt{(l-t)^2 + \epsilon^2}}{2\sqrt{(l-t)^2 + \epsilon^2}}$$

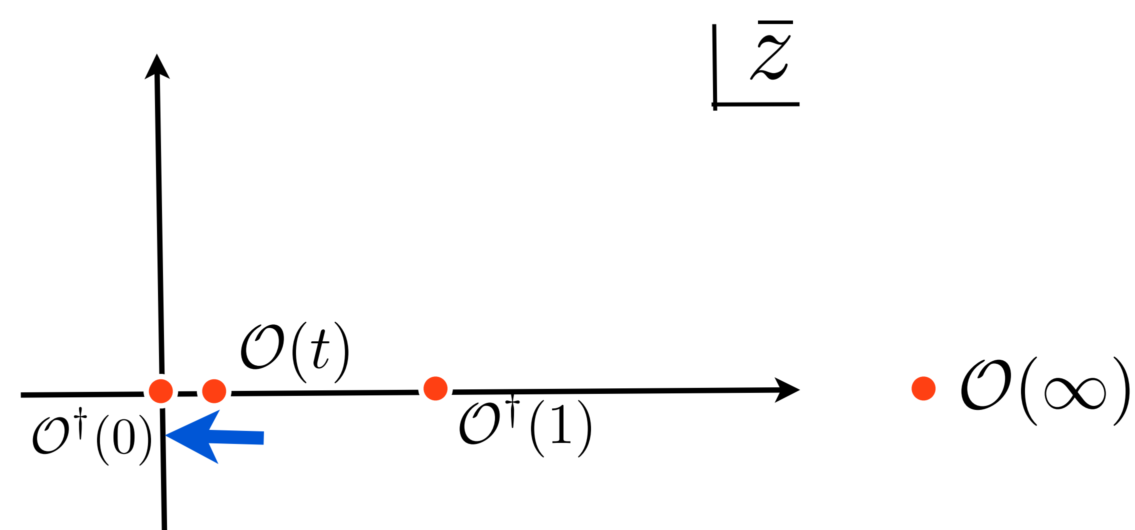


$$z = 0 \to z = 1$$

- **Anti-Holomorphic part**



$$\bar{z}(t) = \frac{-(l+t)+\sqrt{(l+t)^2+\epsilon^2}}{2\sqrt{(l+t)^2+\epsilon^2}}$$



$$z = 0 \to z = 0$$

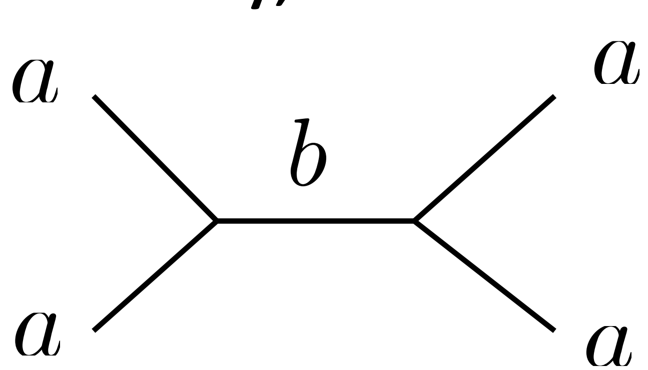
Then , in terms of conformal block, we find at late time as follows:

$$
\begin{aligned}
G_a(z,\bar{z}) &= \sum_b (C^b_{aa})^2 F_a(b|z)\bar{F}_a(b|\bar{z}) \\
&\underset{(z,\bar{z})\to(1,0)}{\simeq} F_{00}[a]\cdot F_a(0|1-z)\bar{F}_a(0|z) \\
&\simeq F_{00}[a]\cdot(1-z)^{-2\Delta_a}\,\bar{z}^{-2\Delta_a}
\end{aligned}
$$

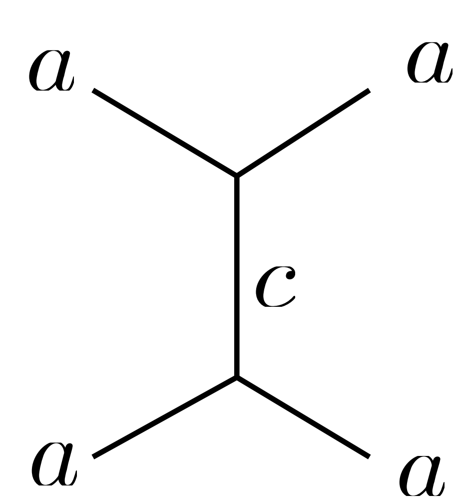
where $F_{bc}[a]$ is the fusion matrix defined by

$$
F_a(b|1-z) = \sum_c F_{bc}[a]\cdot F_a(c|z)\ .
$$

Pictorially,



$$
= \sum_c F_{bc}[a]
$$



In this way, we can show that the late time n = 2 REE becomes

$$\Delta S_A^{(2)} = -\log F_{00}[a].$$

In the 2d RCFTs , $F_{00}[a]$ is the inverse of quantum dimension d_a:

$$F_{00}[a] = \frac{S_{00}}{S_{0a}} = \frac{1}{d_a}.$$

[Moore-Seiberg 89]

Finally, we can write the n=2 REE as follows:

$$\Delta S_A^{(2)} = \log d_a.$$

※) The behavior

$$G_a(z,\bar{z}) \simeq F_{00}[a] \cdot (1-z)^{-2\Delta_a} \bar{z}^{-2\Delta_a}$$

is peculiar to RCFTs (theory dependent). [Asplund-Bernamonti-Galli-Hartman15]

In general, the behavior is less singular in $(z,\bar{z}) \to (1,0)$ limit.

For example, in holographic CFTs, we obtain

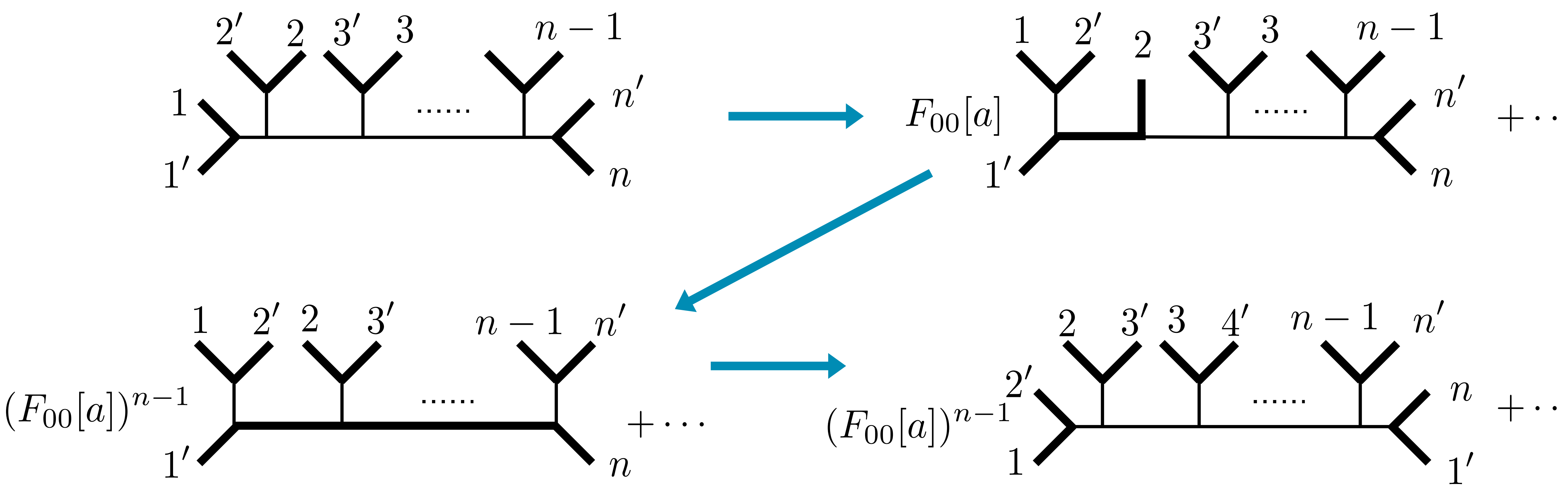
$$G_a(z,\bar{z}) \simeq \bar{z}^{-2\Delta_a}$$

and the time evolution of E.E becomes [Caputa-Nozaki-Takayanagi 14]

$$\Delta S_A^{(2)} \simeq 4\Delta_a \log \frac{2t}{\epsilon} - \log 2$$

(do not reach any finite values)

For generic n-th Renyi entropy, we can apply the sequences of fusion rules



(thin line = propagation of Identity operator $\mathbb{I}$)
(thick line = propagation of primary operator $\mathcal{O}_a$)

and derive

$$\Delta S_A^{(n)} = \log d_a$$

[He-TN-Watanabe-Takayanagi 14]

・Ising Model case

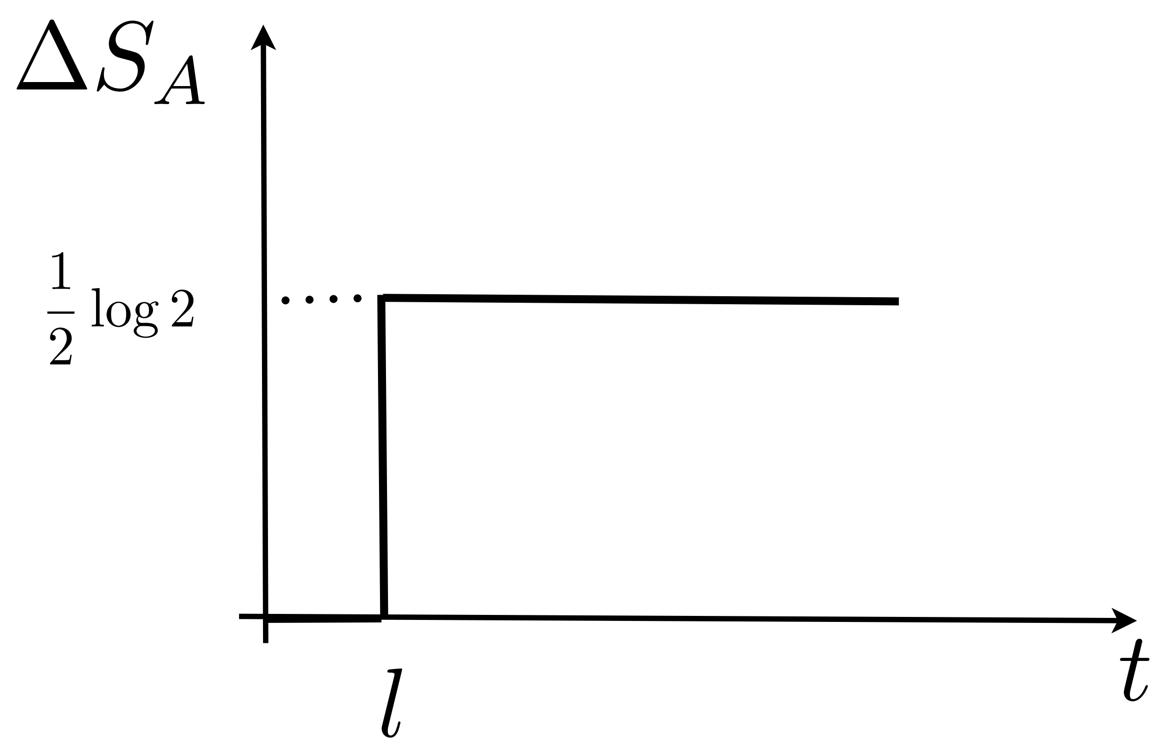
There are 3 primary fields: $\mathbb{I}\ ,\ \sigma\ ,\ \epsilon$

(Identity, Spin op, Energy op)

$$\log d_\sigma = \log\sqrt{2}$$
$$\log d_\epsilon = 0 \quad \text{(no entanglement)}$$

⇒Time evolution of (Renyi) EE for spin op. is given by

$$\Delta S_A^{(n)}(t) = \begin{cases} 0 & (t < l), \\ \frac{1}{2}\log 2 & (t > l) \end{cases}$$



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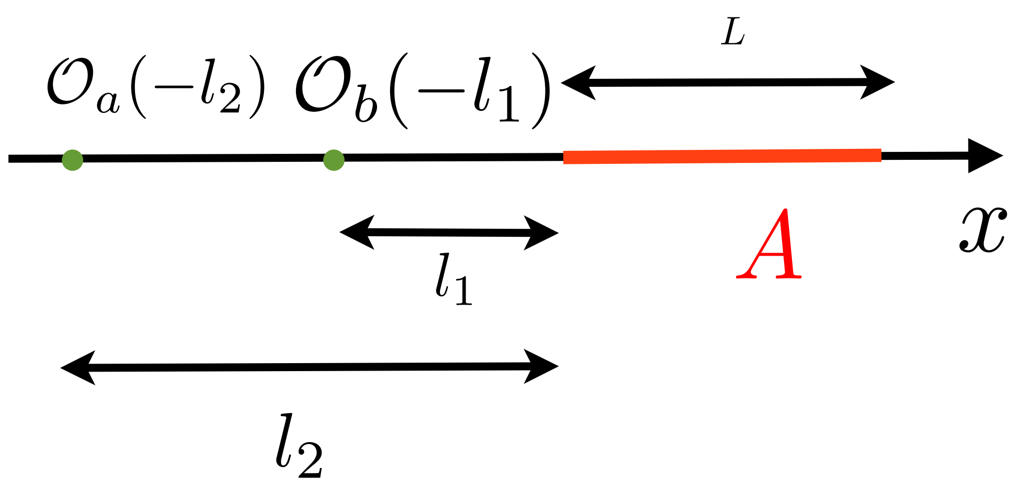
(4)Multiple excitations [TN, 16]

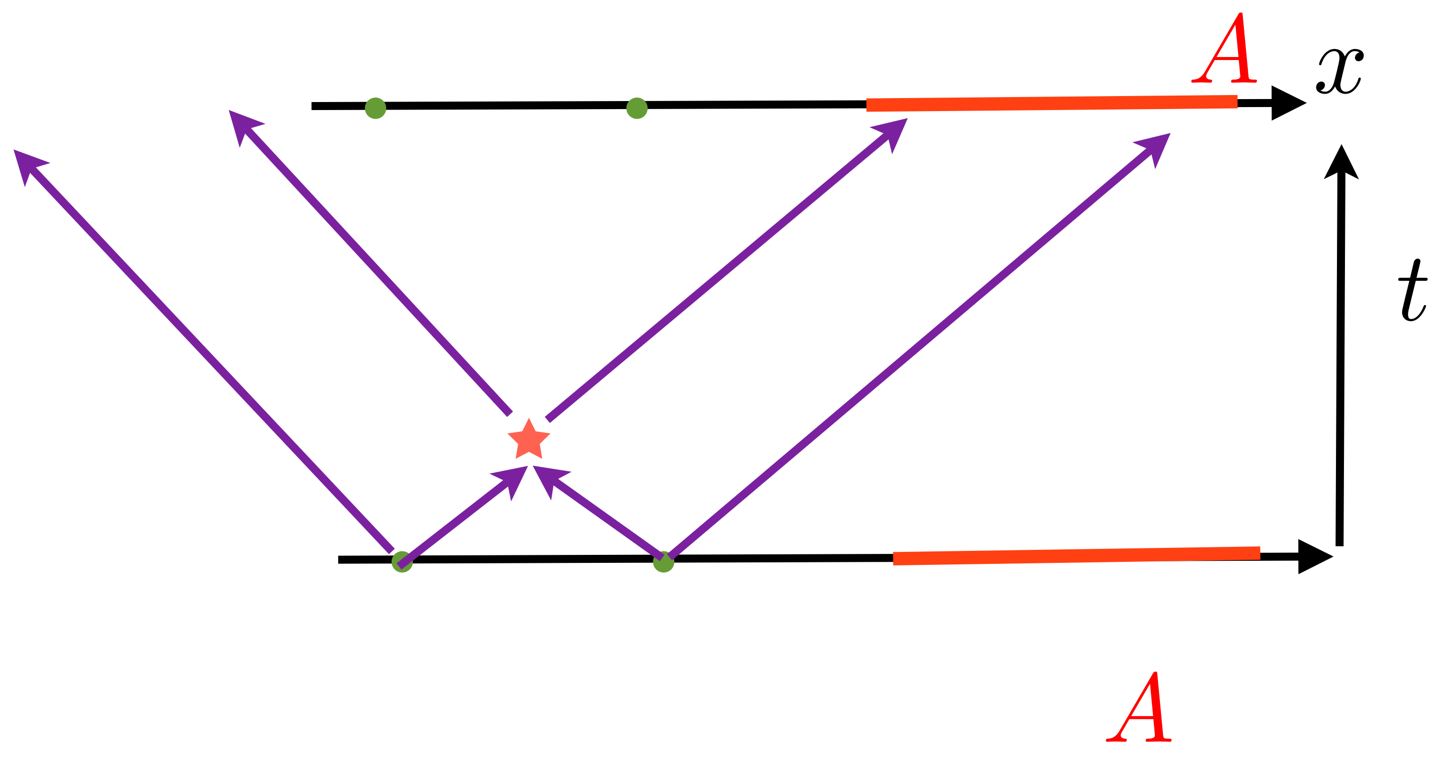
two excitations

insert two local operators:

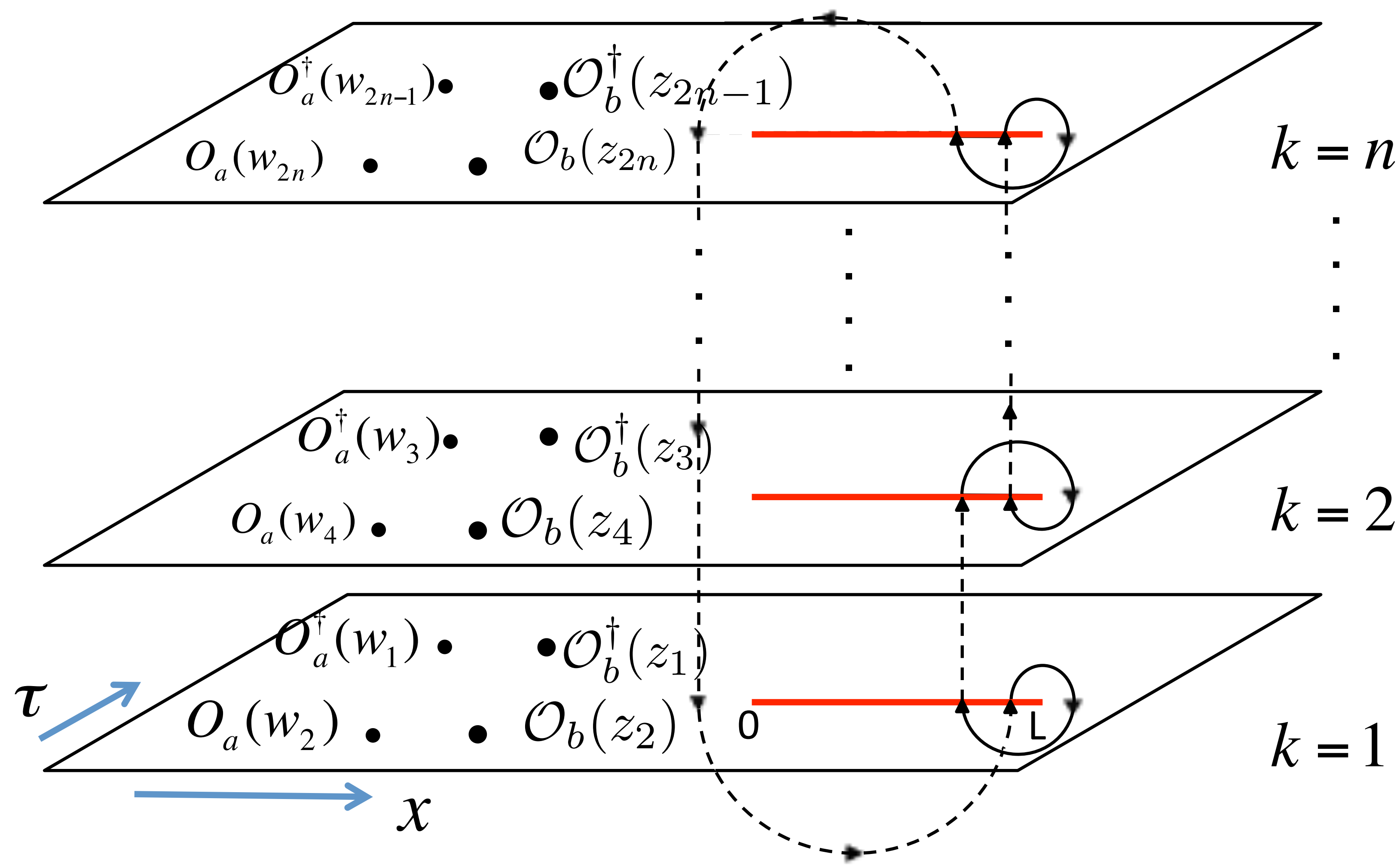
$$|\Psi\rangle \propto \mathcal{O}_a(-l_2)\mathcal{O}_b(-l_1)\,|0\rangle$$



scattering at $t = \frac{l_2 - l_1}{2}$



・Calculation of (Renyi) EE



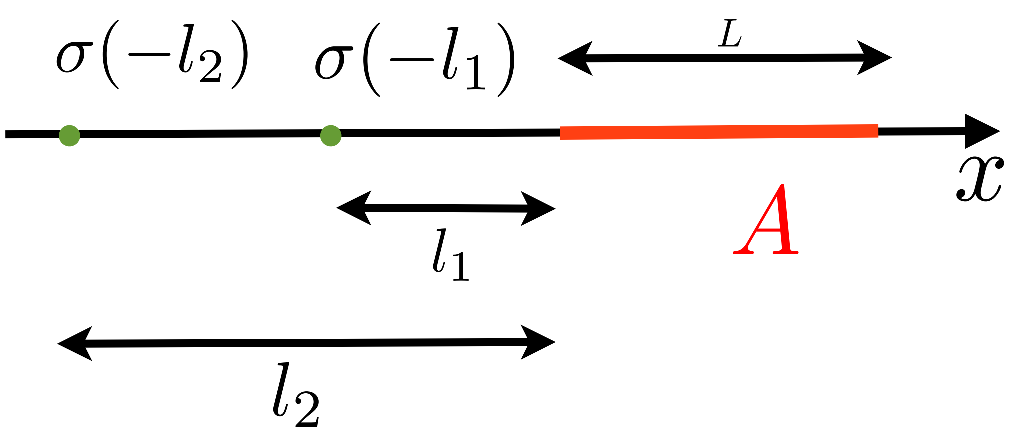
For example, we need 8pt function to calculate 2nd Renyi

⇒ First consider Ising CFT (all correlation function are known)

ex)Ising model (arbitrary n-point functions are known)

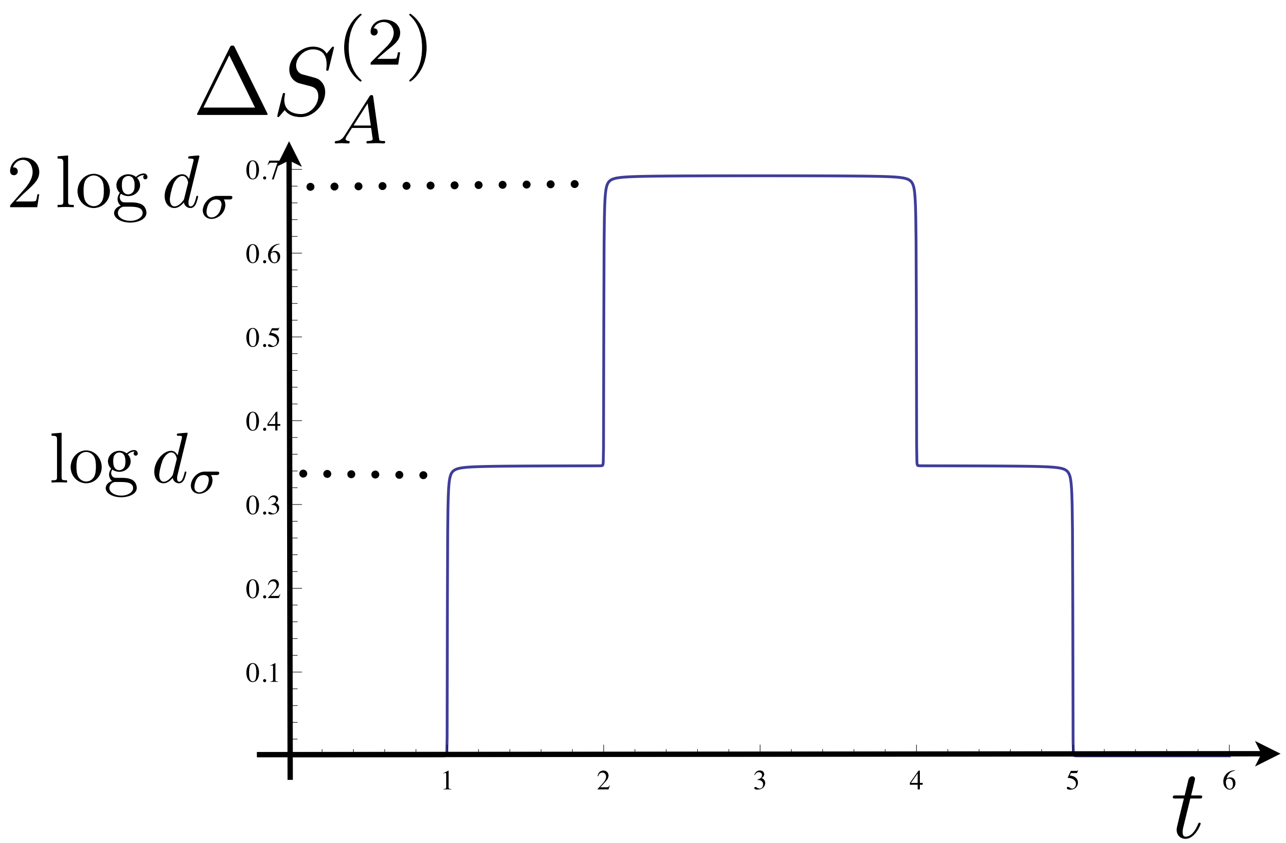
two spin operators

$$|\Psi\rangle \propto \sigma(-l_2)\sigma(-l_1)\,|0\rangle$$



2nd Renyi entropy

summation of constitutions from each operator

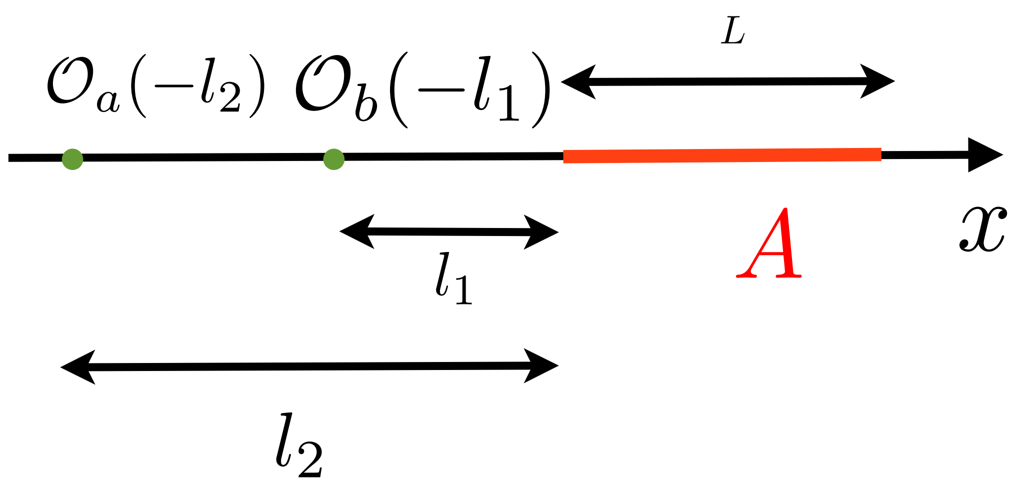


entanglement doesn't change after scattering

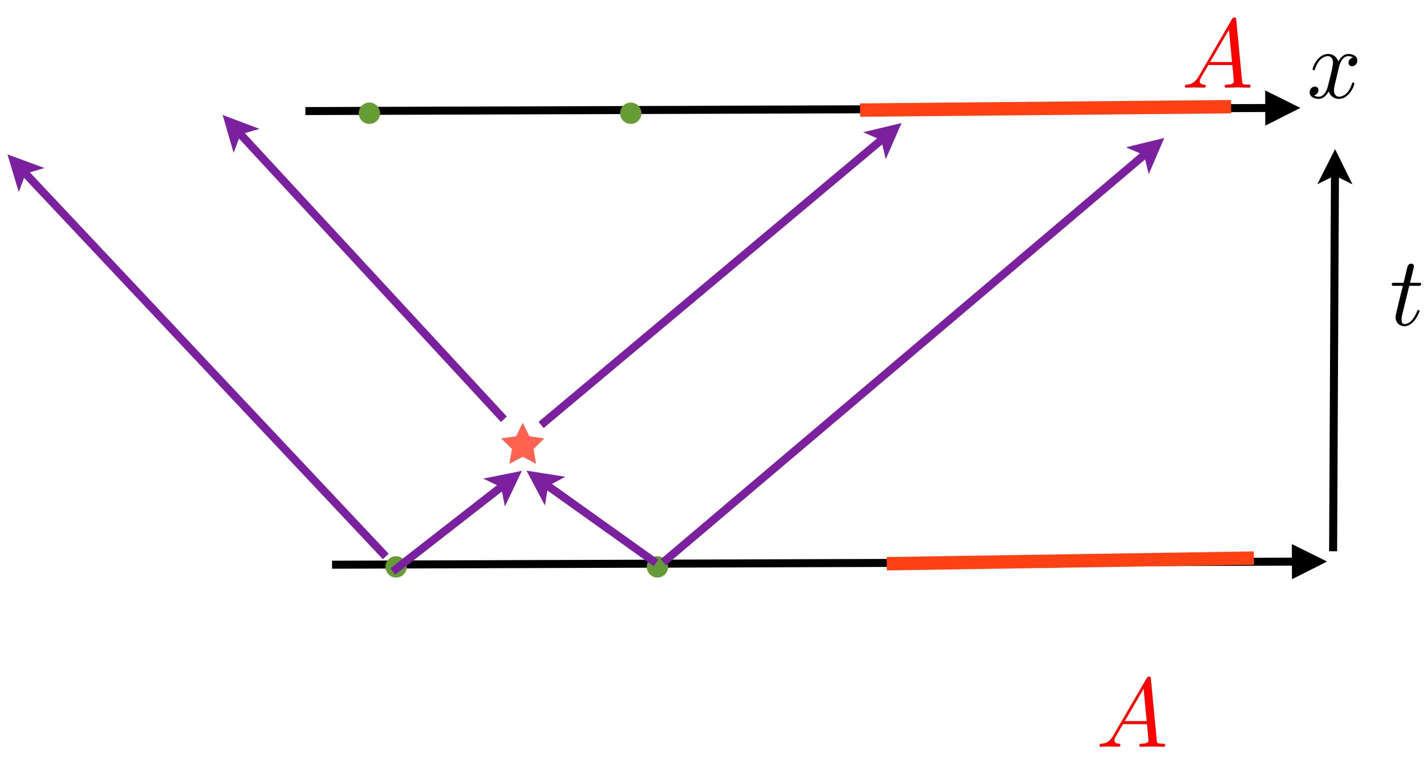
<u>general RCFT</u>

insert two local operators:

$$|\Psi\rangle \propto \mathcal{O}_a(-l_2)\mathcal{O}_b(-l_1)\,|0\rangle$$

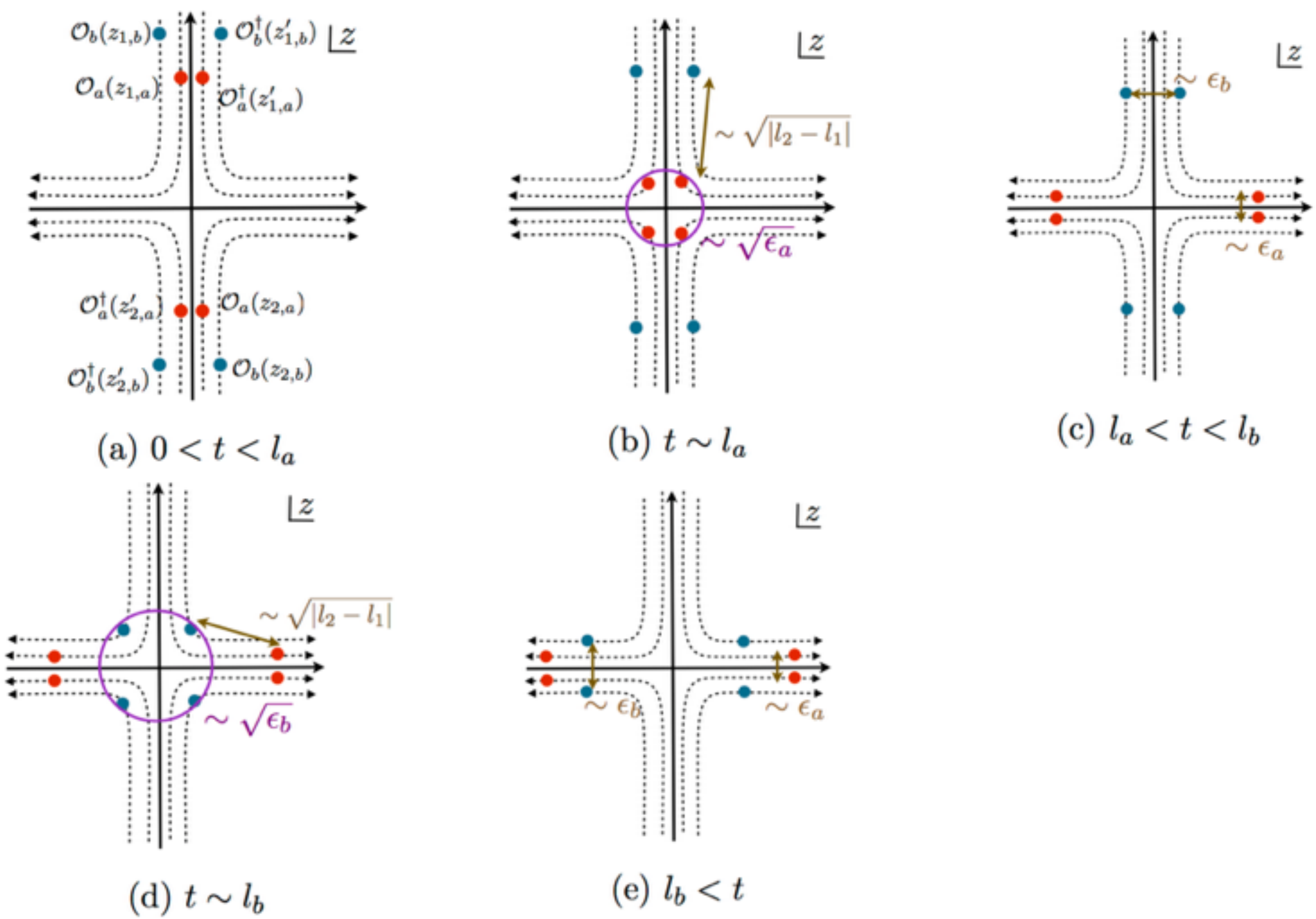


scattering at $t = \dfrac{l_2 - l_1}{2}$

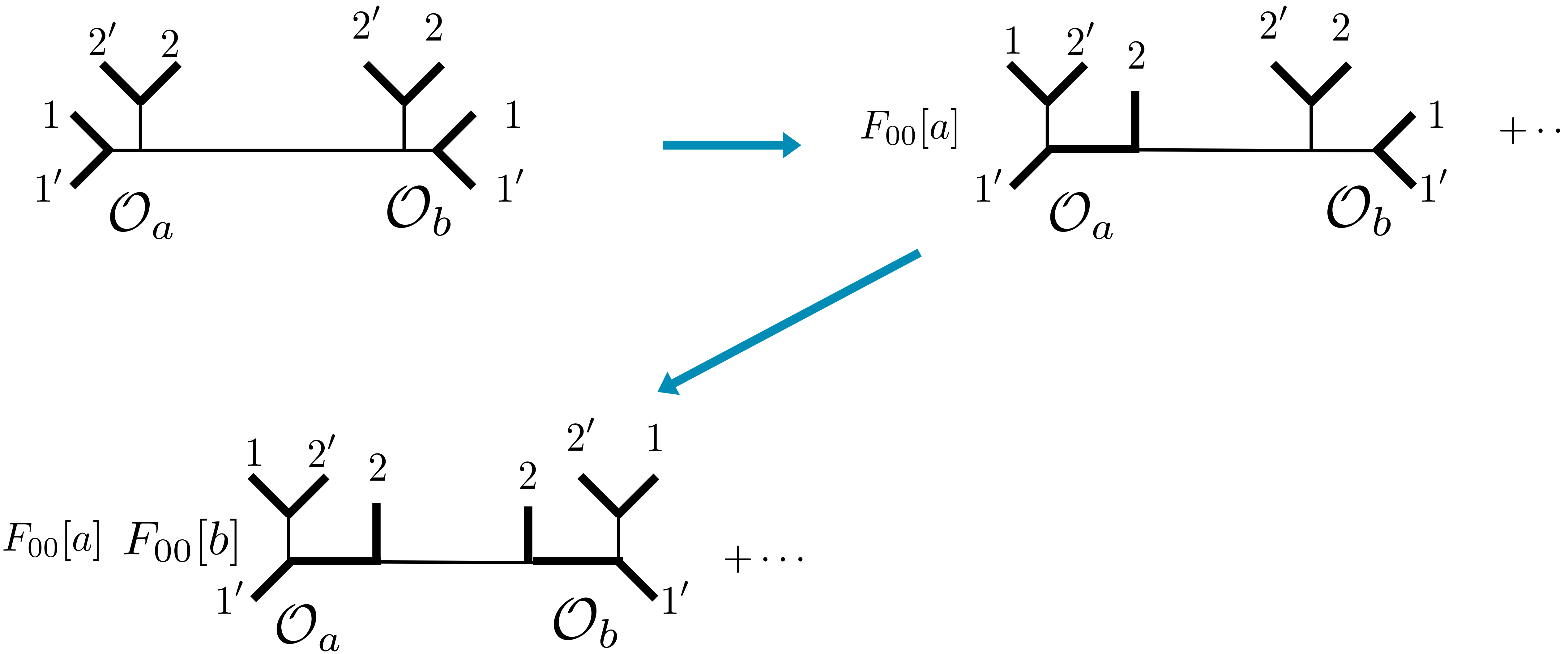


The orbit of each coordinate

After the conformal transformation $z = \sqrt{w} = \sqrt{r} e^{i\frac{\theta}{2}}$,



Fusion transformations for $\mathcal{O}_a$'s and $\mathcal{O}_b$'s are taken separately.



2′ 2
1
1′
$\mathcal{O}_a$
2′ 2
1
1′
$\mathcal{O}_b$
1 2′ 2
$F_{00}[a]$
1′
$\mathcal{O}_a$
2′ 2
1
1′
$\mathcal{O}_b$
$+\cdots$
1 2′ 2
$F_{00}[a]$ $F_{00}[b]$
1′
$\mathcal{O}_a$
2 2′ 1
1′
$\mathcal{O}_b$
$+\cdots$

Therefore, 2nd Renyi entropy is expressed as the summation of contributions from each operator:

$$\Delta S_A^{(2)} = \log d_a + \log d_b$$

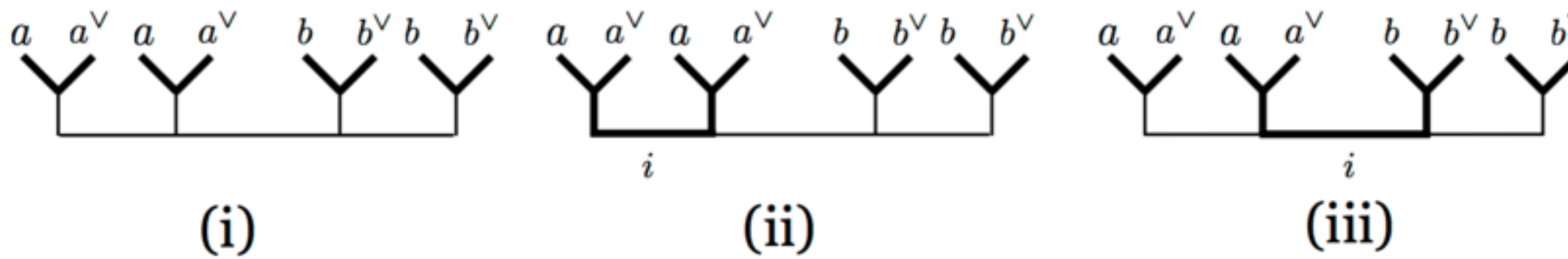
In the same manner, we can also take the fusion rules separately for arbitrary n-th Renyi entropy. Therefore

$$\Delta S_A^{(n)} = \log d_a + \log d_b$$

This can also be generalized to arbitrary number of excitations.

Non rational CFTs

Consider 2nd Renyi Entropy and the following conformal blocks:



(i)Dominant one in RCFTs

Coefficient $F_{00}[a] = 1/d_a$ generically vanishes

(cf: Holographic CFTs)

(ii)(iii) Ignored (subleading of ϵ) in RCFTs

(iii)contains interactions effect between $\mathcal{O}_a$'s and $\mathcal{O}_b$'s

and can contribute to the time evolution of (Renyi) EE

→ In this case, we cannot take fusion rules separately

(5)Conclusion

- We study the time evolution of E.E after the local operator excitation.
- Time evolution after single excitation characterizes the Rational CFTs and non-rational CFTs .
- Time evolution (or final value) of entanglement entropy after the multiple excitation can also be written as the summation of quantum dimensions in RCFTs
- In general CFTs, we expect that the scattering effect appears after multiple excitations

Future Works

- In 2d RCFTs, REE is Written in terms of quantum dimension.

On the other hand, Topological Entanglement Entropy with anyon excitation labeled by a is written in the same form:

$$\Delta S_{topo} = \log d_a$$ [Kitaev-preskill 05]

➡ There is explicit relation?

- Multiple excitations in Holographic CFTs

[with Caputa and Osorio, work in progress]

(5)Conclusion

In this talk, we studied the REE for local operator excited states in the case of free fields in various dim. and 2d RCFTs.

・These results suggest that the late time EE for local operator excited states can detect the "degree of freedom" of local operators.

cf) EE for grand states can detect degree of freedom of theory.
(for example central charge)

・The late time is no changed under the smooth deformation of subsystem A :

➡ "Topological" quantity !

Future problems

- Holographic viewpoint ?

strong interaction → No quasi-particle interpretation

- In 2d RCFTs, REE is Written in terms of quantum dimension.

On the other hand, Topological Entanglement Entropy with anyon excitation labeled by a is written in the same form:

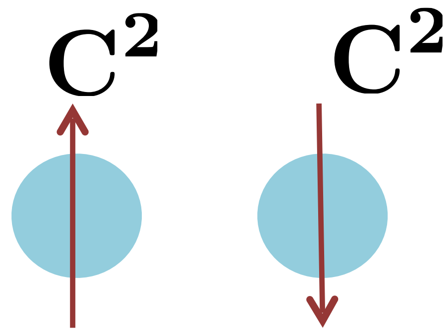
$$\Delta S_{topo} = \log d_a$$ [Kitaev-preskill 05]

➡ There is explicit relation?

(1)Introduction

What is Entanglement Entropy ?

First , we consider the system with two spin. Its Hilbert space is $\mathbf{C}^2 \otimes \mathbf{C}^2$.



First, we cosider the following state:

$$|\Psi\rangle = |\uparrow\rangle \otimes |\uparrow\rangle$$

this is not entangled .

Next, we consider the following state:

$$|\Psi\rangle = \frac{1}{2}(|\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle) \quad \text{(EPR state)}$$

This state is not represented as a tensor product state !

Difinition of entangled state

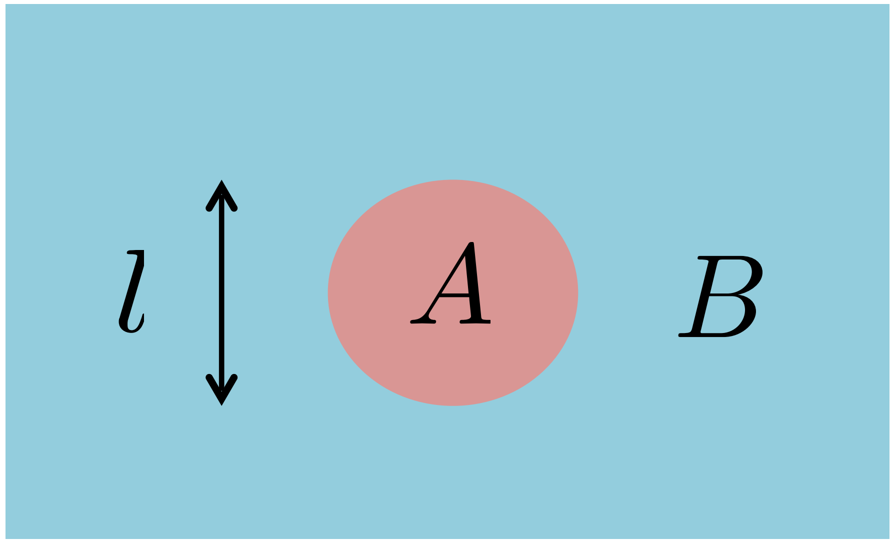
How can we quantify entanglement?

Entanglement Entropy !

Size of subsystem

(1)small size limit $l \to 0$.

In this limit , we find the first law for EE , analogy to the first law of thermodynamics:



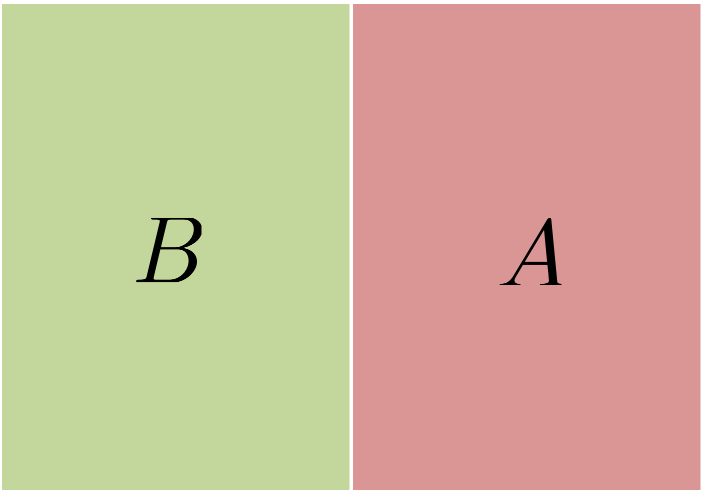
$$\Delta S_A \propto E_A$$

[Bhattacharya-Nozaki-Ugajin-Takayanagi 12]
[Blanco-Casini-Hung-Myers 13]

(2)large size limit $l \to \infty$.

Quite non-local limit .

The main theme of this talk !



In this talk , we consider the subsystem is half plane.

Relation to Lorentian 4-pt functions

In 1+1d CFTs, position dependence of 4-pt functions

is putted in the cross ratio $z = z_{12}z_{34}/z_{13}z_{24}$ ($\bar{z} = \bar{z}_{12}\bar{z}_{34}/\bar{z}_{13}\bar{z}_{24}$)

$$\langle O(z_1, \bar{z}_1) O(z_2, \bar{z}_3) O(z_3, \bar{z}_3) O(z_4, \bar{z}_4) \rangle = |z_{12}|^{-2\Delta} |z_{34}|^{-2\Delta} G(z, \bar{z})$$

Full expression of $G(z, \bar{z})$ contains the info. of CFT data

($\{\Delta_i\}$ and $\{C_{ijk}\}$)

もっと一般に、任意のRCFT、任意のレプリカ数nで

$$\Delta S_A^{(n)} = \log d_a + \log d_b$$

が示せる　[TN, 16]

$n \to 1$ として、von Neumann エントロピーも

$$\Delta S_A = \log d_a + \log d_b$$

→RCFTでは散乱の前後でエントロピーの変化なし