Novel construction and the monodromy relation for 3pt-point functions @ weak coupling

Takuya Nishimura

Institute of physics, University of Tokyo, Komaba

July, 1st, 2015@Kyoto University.

Based on

- Y. Kazama, S. Komatsu and T.N, [arXiv:1410.8533]
- Y. Kazama, S. Komatsu and T.N, [arXiv:1506.03203]

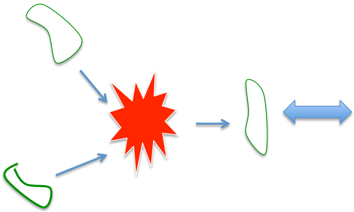
Introduction

In this talk, I will talk about 3pt. functions in $\mathrm{AdS}_5/\mathrm{CFT}_4$ at weak coupling.

Why 3pt functions in $\mathcal{N} = 4$ SYM?

- They are fundamental building blocks of the theory together with the 2pt. functions.
- They encode the dynamics of the string theory on the AdS background.

We need to study these fundamental observables in detail to reveal the underlying mechanism of AdS/CFT.

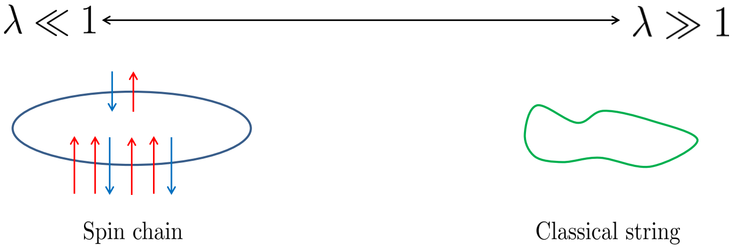


$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle$$

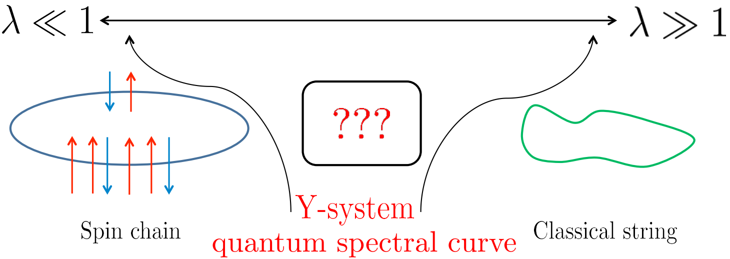
Integrability based approach

In the planar limit, integrability turned out to be a powerful tool.

1. At $\lambda \ll 1$ or perturbative $\mathcal{N} = 4$ SYM,
   (1-loop dilatation operator)= Integrable spin chain Hamiltonian
2. At $\lambda \gg 1$ or classical string,
   A large class of classical solutions is constructed from algebraic curves. $\Rightarrow$ Semiclassical string spectrum



Assuming the integrability, all-loop results are obtained!



Fortunately, we have solved the spectrum problem by using integrability.

- Spectrum problem has been studied using various integrability techniques.
  $\Rightarrow$It means the underlying $1+1$ dim system (light-cone gauge fixed sigma model, spin chain.) is integrable.
- However, "integrability" is not oblivious beyond the spectrum problem a priori.

Is there any notion of "integrability" beyond the spectrum problem? If there is, what is the precise meaning of "integrability"?

Monodromy relation @strong coupling

At the strong coupling regime, the so-called monodromy relation plays an quite important role ['11 Janik,Wereszczynski],['11,'12,'13 Kazama,Komatsu].

$$(\text{e.o.m}) \Leftrightarrow (d+A(u))^2 = 0\ ,\ \ \Omega_i(u) = \mathrm{P}\exp\left[\int_{C_i} A(u)\right]$$

$$\Omega_1(u)\Omega_2(u)\Omega_3(u) = 1$$

- The monodromy relation provide a global information even without knowing the exact form of the vertex operators and the saddle pt. configuration.
- Combined with the analyticity, it determines the semi-classical three-point functions completely.

What is the weak coupling counter part of this relation? How is it useful to constrain three-point functions?

Motivations for our works

1. We would like to build weak coupling correlators respecting the symmetry.
2. We wish to find the weak coupling analogue of the monodormy relation.

Results

1. We develop a new formalism in which the symmetry is manifest.
   $\Rightarrow$ We can simplify the analysis exploiting the symmetry.
2. We also derive the monodromy relations for correlator at weak coupling.

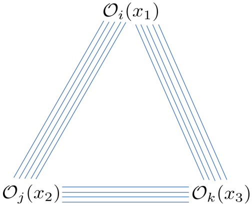
Plan of talk

1. Introduction
2. Construction of 3pt. functions @weak coupling
3. Monodromy relations
4. Summary and prospects

Construction of 3pt. functions @weak coupling

Tree-level three-point functions

Tree-level three-point fuctions are calculated by taking all possible Wick contractions. In particular, only planar graphs contribute in the large N_c limit.



- At tree-level, the dimensions of operators are highly degenerate.
- According to the usual degenerate perturbation theory, $\mathcal{O}_i(x_1), \mathcal{O}_j(x_2), \mathcal{O}_k(x_3)$ must be eigenstates of the 1-loop dilatation operator.

$\Rightarrow$ Combinatorics of Wick contractions.

Two-point functions and dilatation operator

$\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\rangle = \frac{\delta_{ij}}{|x-y|^{2\Delta_i}}$. $\mathcal{O}_i(x)$: with conformal dimension Δ_i

We must take into acount the operator mixing!

$$\mathcal{O}_i^{\text{ren}} = Z_i^j \mathcal{O}_j^{\text{bare}},$$
$$\frac{d}{d\ln\Lambda}\mathcal{O}^{\text{ren}} = \Gamma\mathcal{O}^{\text{ren}}, \quad \Gamma := \frac{dZ}{d\ln\Lambda}Z^{-1}$$

To obtain the conformal dimension, we need to

1. determine Γ,
2. and diagonalize $\Gamma \Longrightarrow \Delta_i = \Delta_i^{(0)} + \gamma_i \quad \gamma_i$: eigenvalues.

'02 Minahan, Zarembo

In the planar limit $N_c \to \infty$ of $\mathcal{N} = 4$ SYM,

Diagonalization of Γ = Diagonalization of spin chain Hamiltonian

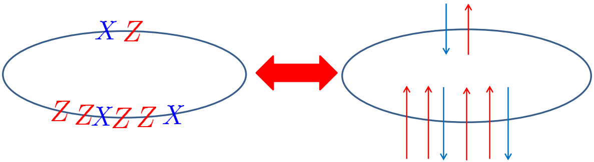
SU(2) sector

The simplest example is $SU(2)$ subsector where all single trace operators are made up of two types of complex scalar fields:

$$\mathcal{O}(x) = \mathrm{Tr}[ZZZ\cdots ZXZ\cdots], \quad Z = \phi_1 + i\phi_2, \quad X = \phi_3 + i\phi_4$$

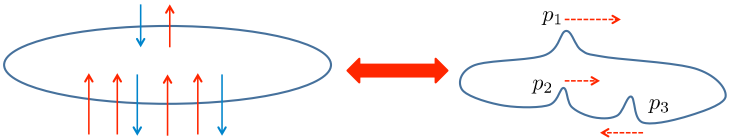
For this sector, the 1-loop dilatation operator becomes the Hamiltonian of the Heisenberg spin chain!!

$$\begin{aligned} \Gamma|_{SU(2)} &\longleftrightarrow H_{XXX} \\ Z, X &\longleftrightarrow |\uparrow\rangle, |\downarrow\rangle \\ \mathrm{Tr}[ZZ\ldots Z] &\longleftrightarrow |\mathrm{Vac}\rangle = |\uparrow\uparrow\ldots\uparrow\rangle \\ \mathcal{O}_i, \Delta_i = E_i &\longleftrightarrow |E_i\rangle, E_i \end{aligned}$$



Bethe ansatz

1. Define the Bethe states:Fock states of the pseudo-particles (Magnon).
   $|\boldsymbol{u}\rangle = \prod_{i=1}^{M} B(u_i)|\uparrow^{\ell}\rangle, \quad p = i\ln\frac{u+i/2}{u-i/2}$
2. The momenta (rapidities) of magnons are quantized due to the periodicity condition. $\Longrightarrow$ Bethe ansatz equation!
3. We can read off the spectrum from the disppersion realation.
   $E = \sum_{i=1}^{M} \varepsilon(u_i), \quad \varepsilon(u) = \frac{\lambda}{8\pi^2}\frac{1}{u^2+1/4}$



$$\mathcal{O}(x) \leftrightarrow |\boldsymbol{u}\rangle = \prod_{i=1}^{M} B(u_i)|\uparrow^{\ell}\rangle \ , \ \Omega(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

$$\Omega(u) = L_1(u) \cdots L_\ell(u) \ \ , \ L_n(u) = u + i\vec{S}_n \cdot \vec{\sigma} = \begin{pmatrix} u + iS_n^3 & iS_n^- \\ iS_n^+ & u - iS_n^3 \end{pmatrix}$$

They are eigenstates of the Hamiltonian if and only if they satisfy the Bethe ansatz equations:

$$T(u)|\boldsymbol{u}\rangle = t_{\boldsymbol{u}}(u)|\boldsymbol{u}\rangle \ , \ T(u) = \mathrm{tr}\Omega(u) = A(u) + D(u) \ ,$$

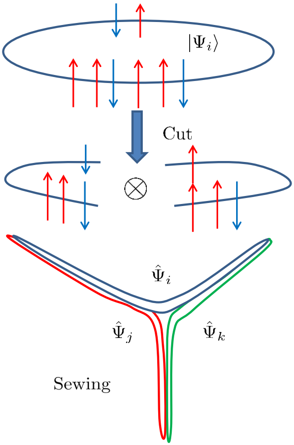
$$\prod_{k=1}^{\ell} \left(\frac{u_k + i/2}{u_k - i/2} \right)^{\ell} = \prod_{j \neq k} \frac{u_k - u_j + i}{u_k - u_j - i}$$

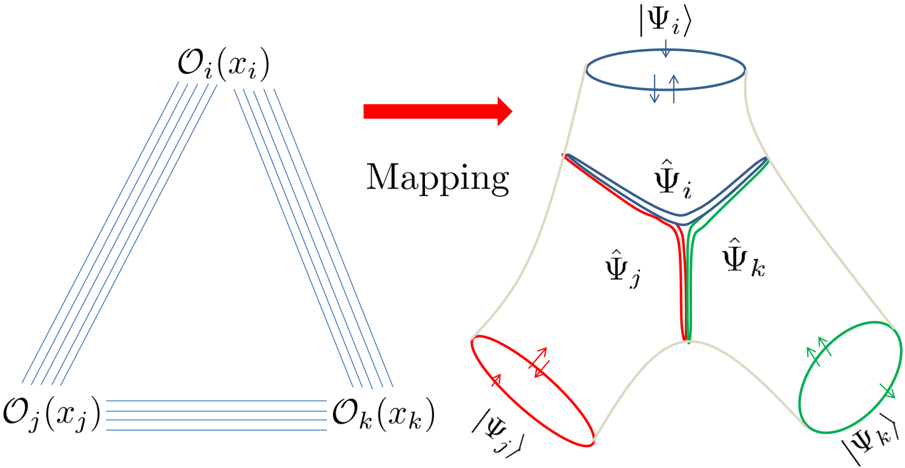
- The transfer matrix $T(u)$ generates a family of conserved charges including the Hamiltonian.
- $|\boldsymbol{u}\rangle$ is called on-shell when their rapidities satisfy the BAEs.
- On-shell Bethe states are highest weight states of SU(2).

Tailoring for spin chain

The tailoring
is an efficient method to calculate relevant contractions ['11 Escobedo,Gromov,Sever,Vieira].

1. Cutting spin chains into the subchains: $|\Psi_i\rangle = \sum |\Psi_i\rangle_l \otimes |\Psi_i\rangle_r$.
2. Flipping the half of the states: $|\Psi_i\rangle \to \hat{\Psi}_i = \sum |\Psi_i\rangle_l \otimes {}_r\overleftarrow{\langle\Psi_i|}$
3. Sewing the states: $C_{ijk} \propto \sum\sum\sum \overleftarrow{\langle\Psi_i}|\Psi_j\rangle \overleftarrow{\langle\Psi_j}|\Psi_k\rangle \overleftarrow{\langle\Psi_k}|\Psi_i\rangle$





Tree-level three-point functions are mapped to the overlaps of the spin chain wave functions.

A special class of 3pt. functions in SU(2) sector

$$\mathcal{O}_1 = \sum \mathrm{Tr}[ZX\cdots] \leftrightarrow \sum |\uparrow\downarrow\cdots\rangle$$
$$\mathcal{O}_2 = \sum \mathrm{Tr}[\bar{Z}\bar{X}\cdots] \leftrightarrow \sum |\uparrow\downarrow\cdots\rangle$$
$$\mathcal{O}_3 = \sum \mathrm{Tr}[Z\bar{X}\cdots] \leftrightarrow \sum |\uparrow\downarrow\cdots\rangle$$

The tailoring gives the structure constant in terms of scalar product of Bethe states.

Remarks

- In this special case, the result only involves $\langle \mathrm{off-shell}|\mathrm{on-shell}\rangle$.
  $\Rightarrow$ Determinant formulas.
- The mapping to a single SU(2) spin chain is not natural from a symmetry point of view.

Basic ideas

In the literatures, there are some unsatisfactory points:

- The symmetries are not manifest.
- A very limited class of 3pt. functions has been treated.

To improve these points, we develop a new formalism:

- Mapping to a double spin chain.
- Novel interpretation for the Wick contractions.

$$\times \quad \mathcal{O}_1\mathcal{O}_2 = \langle \mathcal{O}_1 | \mathcal{O}_2 \rangle$$

$$\bigcirc \quad \langle \mathcal{O}_1\mathcal{O}_2 \rangle = \langle V_{12} | (|\mathcal{O}_1\rangle \otimes |\mathcal{O}_2\rangle)$$

As we will see, *the symmetries (Ward identities) are realized as the invariance for the vertex*

$$\langle V_{12} | (J_1 + J_2) = 0 \Leftrightarrow \langle (J\mathcal{O}_1)\mathcal{O}_2 \rangle + \langle \mathcal{O}_1(J\mathcal{O}_2) \rangle = 0$$

$\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ double spin chain formalism

From a symmetry point of view, it is natural to introduce a double spin chain ['15 Kazama,Komatsu,T.N].

$$\Phi_{a\tilde{a}} = \begin{pmatrix} Z & X \\ -\bar{X} & \bar{Z} \end{pmatrix}_{a\tilde{a}} \leftrightarrow \begin{pmatrix} |\uparrow\rangle_L\otimes|\uparrow\rangle_R & |\uparrow\rangle_L\otimes|\downarrow\rangle_R \\ |\downarrow\rangle_L\otimes|\uparrow\rangle_R & |\downarrow\rangle_L\otimes|\downarrow\rangle_R \end{pmatrix}_{a\tilde{a}}$$

- SO(4)≅$\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ transformation is $\Phi\to g_L\Phi g_R$.
- A general SO(4) scalar is labeled by bi-spinor $P^{a\tilde{a}}$: $P\cdot\Phi = P^{a\tilde{a}}\Phi_{a\tilde{a}}$.

$$\begin{aligned} P\cdot\Phi \leftrightarrow\; & P^{1\tilde{1}}|\uparrow\rangle_L\otimes|\uparrow\rangle_R + P^{1\tilde{2}}|\uparrow\rangle_L\otimes|\downarrow\rangle_R \\ & + P^{2\tilde{1}}|\downarrow\rangle_L\otimes|\uparrow\rangle_R + P^{2\tilde{2}}|\downarrow\rangle_L\otimes|\downarrow\rangle_R \end{aligned}$$

General rotated vacua

To treat generic operators in the SU(2) sector, we first consider rotated vacua on which non-BPS operators are build.

- General BPS operators are obtained by SO(4) rotations $\mathrm{Tr}[Z^{\ell}] \to \mathrm{Tr}[(P\cdot\Phi)^{\ell}]$:

$$Z = \Phi_{1\tilde{1}} \to (g_L \Phi g_R)_{1\tilde{1}} = (g_L)_1{}^a \Phi_{a\tilde{a}} (g_R)^{\tilde{a}}{}_{\tilde{1}} \ ,$$
$$P^{a\tilde{a}} = \mathfrak{n}^a \tilde{\mathfrak{n}}^{\tilde{a}} \ , \mathfrak{n}^a = (g_L)_1{}^a \ , \tilde{\mathfrak{n}}^{\tilde{a}} = (g_R)^{\tilde{a}}{}_{\tilde{1}}$$

- Hence, the vacua are assigned two polarization spinors:

$$\mathrm{Tr}[(P\cdot\Phi)^{\ell}] \leftrightarrow |\mathfrak{n}^{\ell}\rangle_L \otimes |\tilde{\mathfrak{n}}^{\ell}\rangle_R \ ,$$
$$|\mathfrak{n}^{\ell}\rangle_L = |\mathfrak{n}\rangle_L \otimes \cdots \otimes |\mathfrak{n}\rangle_L \ , \ |\tilde{\mathfrak{n}}^{\ell}\rangle_R = |\tilde{\mathfrak{n}}\rangle_R \otimes \cdots \otimes |\tilde{\mathfrak{n}}\rangle_R \ ,$$
$$|\mathfrak{n}\rangle_L = \mathfrak{n}^1 |\uparrow\rangle_L + \mathfrak{n}^2 |\downarrow\rangle_L \ , \ |\tilde{\mathfrak{n}}\rangle_R = \tilde{\mathfrak{n}}^{\tilde{1}} |\uparrow\rangle_R + \tilde{\mathfrak{n}}^{\tilde{2}} |\downarrow\rangle_R \ .$$

- For convenience, we normalize polarization spinors as

$$\mathfrak{n}^a \bar{\mathfrak{n}}_a = \tilde{\mathfrak{n}}^{\tilde{a}} \bar{\tilde{\mathfrak{n}}}_{\tilde{a}} = 1 \ , \ \bar{\mathfrak{n}}_a = (\mathfrak{n}^a)^* \ , \ \bar{\tilde{\mathfrak{n}}}_{\tilde{a}} = (\tilde{\mathfrak{n}}^{\tilde{a}})^*$$

General non-BPS operators in SU(2) sector

Non-BPS operators in the SU(2) sector can be obtained by adding magnons either on the SU(2)$_L$ or SU(2)$_R$ sector:

$$\text{Type I}: \ |\boldsymbol{u};\mathfrak{n}^{\ell}\rangle_L \otimes |\tilde{\mathfrak{n}}^{\ell}\rangle_R \ , \ \text{Type II}: \ |\mathfrak{n}^{\ell}\rangle_L \otimes |\tilde{\boldsymbol{u}};\tilde{\mathfrak{n}}^{\ell}\rangle_R \ .$$

Such states are related to $|\boldsymbol{u};\uparrow^{\ell}\rangle_L = \prod_{i=1}^{M} B_L(u_i)|\uparrow^{\ell}\rangle_L$. by SU(2)$_L\times$SU(2)$_R$ transformation:

$$|\boldsymbol{u};\mathfrak{n}^{\ell}\rangle_L = \mathrm{g}_L|\boldsymbol{u};\uparrow^{\ell}\rangle_L \ , \ |\mathfrak{n}^{\ell}\rangle = \mathrm{g}_L|\uparrow^{\ell}\rangle \ , \ \mathrm{g}_L \in \mathrm{SU}(2)/\mathrm{U}(1) \ ,$$

It is convenient to parametrize g_L as

$$\mathrm{g}_L = e^{\zeta S_- - \bar{\zeta} S_+} = e^{z S_-} e^{-\ln(1+|z|^2) S_3} e^{-\bar{z} S_+}$$

where $z = \frac{\zeta}{|\zeta|}\tan|\zeta|$. This is the so-called coherent state representation and z is the projective coordinate for the coset SU(2)/U(1)$\equiv S^2$.

With this parametrization $\mathrm{g}_L = e^{zS_-}e^{-\ln(1+|z|^2)S_3}e^{-\bar{z}S_+}$, we find

$$|\mathfrak{n}\rangle_L = \mathrm{g}_L|\uparrow\rangle_L \ , \mathfrak{n} = \mathrm{g}_L \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} 1 \\ z \end{pmatrix} \ ,$$

Furthermore, the Bethe states on the rotated vacuum are expressed as follows

$$|\boldsymbol{u};\mathfrak{n}^\ell\rangle_L = \mathrm{g}_L|\boldsymbol{u};\uparrow^\ell\rangle_L = \left(\frac{1}{1+|z|^2}\right)^{\ell/2-M} e^{zS_-}|\boldsymbol{u};\uparrow^\ell\rangle_L \ .$$

Notice that the Bethe state is a highest weight state of SU(2)$_L$, i.e. $S_+|\boldsymbol{u};\uparrow^\ell\rangle_L = 0$. Similarly, we have

$$|\boldsymbol{u};\tilde{\mathfrak{n}}^\ell\rangle_R = \mathrm{g}_R|\tilde{\boldsymbol{u}};\uparrow^\ell\rangle_R = \left(\frac{1}{1+|\tilde{z}|^2}\right)^{\ell/2-\tilde{M}} e^{\tilde{z}\tilde{S}_-}|\tilde{\boldsymbol{u}};\uparrow^\ell\rangle_R \ .$$

Wick contraction as singlet projection

The Wick contractions for Z, X and their conjugate are summarized as

$$\overbracket{\Phi_{a\tilde{a}}\Phi_{b\tilde{b}}} = \epsilon_{ab}\epsilon_{\tilde{a}\tilde{b}} \ , \ \ \Phi_{a\tilde{a}} = \begin{pmatrix} Z & X \\ -\bar{X} & \bar{Z} \end{pmatrix}_{a\tilde{a}}$$

To implement the above contractions in a spin chain language, we introduce a singlet projector

$$\langle \mathbf{1}| = \epsilon_{ab}\langle a| \otimes \langle b| \ ,$$
$$|1\rangle = |\uparrow\rangle \ , \ \ |2\rangle = |\downarrow\rangle \ , \ \ \langle a|b\rangle = \delta_{ab} \ .$$

With the singlet projector, we find

$$\overbracket{\Phi_{a\tilde{a}}\Phi_{b\tilde{b}}} = {}_L\langle \mathbf{1}|(|a\rangle_L \otimes |b\rangle_L) \cdot {}_R\langle \mathbf{1}|(|a\rangle_R \otimes |b\rangle_R)$$

For $P_1 \cdot \Phi = \mathfrak{n}_1^a \tilde{\mathfrak{n}}_1^{\tilde{a}} \Phi_{a\tilde{a}}$ and $P_2 \cdot \Phi = \mathfrak{n}_2^b \tilde{\mathfrak{n}}_2^{\tilde{b}} \Phi_{b\tilde{b}}$,

$$\begin{aligned} P_1 \overbracket{\cdot \Phi P_2} \cdot \Phi &= {}_L\langle \mathbf{1}|(|\mathfrak{n}_1\rangle_L \otimes |\mathfrak{n}_2\rangle_L) \cdot {}_R\langle \mathbf{1}|(|\tilde{\mathfrak{n}_1}\rangle_R \otimes |\tilde{\mathfrak{n}_2}\rangle_R) \\ &= \langle \mathfrak{n}_1, \mathfrak{n}_2\rangle\langle \tilde{\mathfrak{n}}_1, \tilde{\mathfrak{n}}_2\rangle = (\epsilon_{ab}\mathfrak{n}_1^a \mathfrak{n}_2^b)(\epsilon_{\tilde{a}\tilde{b}}\tilde{\mathfrak{n}}_1^{\tilde{a}}\tilde{\mathfrak{n}}_2^{\tilde{b}}) \end{aligned}$$

Composite operators in the SU(2) sector are schematically written as

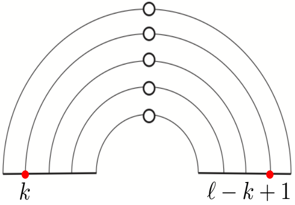
$$\mathcal{O}_i \mapsto |\mathcal{O}_i\rangle_L \otimes |\tilde{\mathcal{O}}_i\rangle_R \ .$$

The Wick contractions for the composite operators are given by

$$\overbrace{\mathcal{O}_1\mathcal{O}_2} = \langle|\mathcal{O}_1\rangle_L, |\mathcal{O}_2\rangle_L\rangle \cdot \langle|\tilde{\mathcal{O}}_1\rangle_R, |\tilde{\mathcal{O}}_2\rangle_R\rangle$$

$$\langle|\Psi_1\rangle, |\Psi_2\rangle\rangle = \langle V_{12}|(|\Psi_1\rangle \otimes |\Psi_2\rangle) \ , \langle V_{12}| = \prod_{k=1}^{\ell} \langle \mathbf{1}_{k,\ell-k+1}|$$

$$\langle V_{12}|(S_k^{(1)} + S_{\ell-k+1}^{(2)}) = 0$$



The contributions from the left and right sector are completely factorized.

Cutting and sewing in our formalism

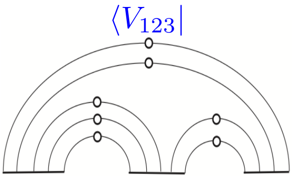
Let us concentrate on the SU$(2)_L$ part.

1. Cutting:

$$|\mathcal{O}_i\rangle_L \to \sum_a |\mathcal{O}_{i_a}\rangle^l \otimes |\mathcal{O}_{i_a}\rangle^r \ (i = 1, 2, 3)$$

2. Sewing:

$$\begin{aligned}&\langle |\mathcal{O}_1\rangle_L, |\mathcal{O}_2\rangle_L, |\mathcal{O}_3\rangle_L\rangle \\ &= \sum_{a,b,c} \left\langle |\mathcal{O}_{1_a}\rangle^r, |\mathcal{O}_{2_b}\rangle^l \right\rangle \left\langle |\mathcal{O}_{2_b}\rangle^r, |\mathcal{O}_{3_c}\rangle^l \right\rangle \left\langle |\mathcal{O}_{3_c}\rangle^r, |\mathcal{O}_{1_a}\rangle^l \right\rangle \\ &= \langle V_{123}|(|\mathcal{O}_1\rangle_L \otimes |\mathcal{O}_2\rangle_L \otimes |\mathcal{O}_3\rangle_L)\end{aligned}$$



z dependence from the Ward identities

Three-point functions must satisfy the Ward identity for SU$(2)_L$:

$$0 = \langle S^* |\mathcal{O}_1\rangle_L, |\mathcal{O}_2\rangle_L, |\mathcal{O}_3\rangle_L\rangle + \langle |\mathcal{O}_1\rangle_L, S^* |\mathcal{O}_2\rangle_L, |\mathcal{O}_3\rangle_L\rangle$$
$$+ \langle |\mathcal{O}_1\rangle_L, |\mathcal{O}_2\rangle_L, S^* |\mathcal{O}_3\rangle_L\rangle \ , \ S^* \in \mathfrak{su}(2)_L \ .$$

Putting $|\hat{\mathcal{O}}_i\rangle_L = e^{z_i S_-} |\boldsymbol{u_i}; \uparrow^\ell\rangle$ into the above, we obtain the following differential representation for the Ward identities:

$$\sum_{i=1}^{3} \rho_{z_i}(S^*) \left\langle |\hat{\mathcal{O}}_1\rangle_L, |\hat{\mathcal{O}}_2\rangle_L, |\hat{\mathcal{O}}_3\rangle_L \right\rangle = 0 \ ,$$

$$\rho_{z_i}(S^-) = \frac{d}{dz_i} \ , \ \rho_{z_i}(S^3) = L_i - z_i \frac{d}{dz_i} \ , \ \rho_{z_i}(S^+) = L_i z_i - \frac{z_i^2}{2} \frac{d}{dz_i} \ .$$

L_i: $\mathfrak{su}(2)_L$ charge. These equations completely fix the z_i dependence:

$$\left\langle |\hat{\mathcal{O}}_1\rangle_L, |\hat{\mathcal{O}}_2\rangle_L, |\hat{\mathcal{O}}_3\rangle_L \right\rangle = z_{21}^{L_1+L_2-L_3} z_{32}^{L_2+L_3-L_1} z_{13}^{L_3+L_1-L_2} \mathcal{G}$$

Representation in terms of pDWPF

The cutting for the Bethe state $|\boldsymbol{u};\uparrow^{\ell}\rangle$ is determined in EGSV:

$$|\boldsymbol{u};\uparrow^{\ell}\rangle \to \sum_{\alpha_l \cup \alpha_r = \boldsymbol{u}} H_\ell(\alpha_l,\alpha_r)|\alpha_l;\uparrow^{\ell_l}\rangle \otimes |\alpha_r;\uparrow^{\ell_r}\rangle \ ,$$

$$H_\ell(\alpha_l,\alpha_r) = \prod_{u\in\alpha_l}\prod_{v\in\alpha_r}\left(u-\frac{i}{2}\right)^{\ell_r}\left(v+\frac{i}{2}\right)^{\ell_l}\left(\frac{u-v+i}{u-v}\right)$$

From $|\boldsymbol{u_i};\mathfrak{n}_i^{\ell_i}\rangle \propto e^{z_i S_-}|\boldsymbol{u_i};\uparrow^{\ell_i}\rangle$, we find

$$|\boldsymbol{u_i};\mathfrak{n}_i^{\ell_i}\rangle \to \sum_{\alpha_l \cup \alpha_r = \boldsymbol{u}} H_\ell(\alpha_l,\alpha_r)e^{zS_-^l}|\alpha_l;\uparrow^{\ell_l}\rangle \otimes e^{zS_-^r}|\alpha_r;\uparrow^{\ell_r}\rangle \ .$$

The sewing procedure produces the following building blocks:

$$\left\langle e^{z_i S_-}|\boldsymbol{x};\uparrow^{\ell}\rangle, e^{z_j S_-}|\boldsymbol{y};\uparrow^{\ell}\rangle\right\rangle$$

1. Using the defining property of the singlet $\langle \mathbf{1}_{12}|(S^{(1)}+S^{(2)})=0$,

$$\left\langle e^{z_i S_-}|\boldsymbol{x};\uparrow^\ell\rangle, e^{z_j S_-}|\boldsymbol{y};\uparrow^\ell\rangle\right\rangle = \left\langle |\boldsymbol{x};\uparrow^\ell\rangle, e^{(z_j-z_i)S_-}|\boldsymbol{y};\uparrow^\ell\rangle\right\rangle \ .$$

2. With the relation $\langle \mathbf{1}_{12}|B^{(1)}(u) = -\langle \mathbf{1}_{12}|B^{(2)}(u)$, we can gather all the excitations to the one side:

$$(-1)^{M_i}\left\langle |\uparrow^\ell\rangle, e^{(z_j-z_i)S_-}|\boldsymbol{x}\cup\boldsymbol{y};\uparrow^\ell\rangle\right\rangle$$

3. From $\langle \mathbf{1}_{12}|(|\uparrow^\ell\rangle\otimes|\Psi_2\rangle) = \langle\downarrow^\ell|\Psi_2\rangle$, one finds

$$(-1)^{M_i}\langle\downarrow^\ell|e^{(z_j-z_i)S_-}|\boldsymbol{x}\cup\boldsymbol{y};\uparrow^\ell\rangle = (-1)^{M_i}(z_j-z_i)^{\ell-M_i-M_j}Z_\ell(\boldsymbol{x}\cup\boldsymbol{y})$$

where $Z_\ell(\boldsymbol{u})$ is the so-called partial Domain Wall Partition Function (pDWPF):

$$Z_\ell(\boldsymbol{u}) = \frac{1}{(\ell-M)!}\langle\downarrow^\ell|(S_-)^{\ell-M}\prod_{i=1}^{M}B(u_i)|\uparrow^\ell\rangle \ .$$

Finally, we have

$$\langle |\mathcal{O}_1\rangle_L, |\mathcal{O}_2\rangle_L, |\mathcal{O}_3\rangle_L\rangle$$
$$= \left(\frac{1}{1+|z_1|^2}\right)^{\ell_1/2-M_1} \left(\frac{1}{1+|z_2|^2}\right)^{\ell_2/2-M_2} \left(\frac{1}{1+|z_3|^2}\right)^{\ell_3/2-M_3}$$
$$\times \sum_{\alpha_l^{(k)}\cup\alpha_r^{(k)}=\boldsymbol{u_k}} z_{21}^{\ell_{12}-|\alpha_r^{(1)}|-|\alpha_l^{(2)}|} z_{32}^{\ell_{23}-|\alpha_r^{(2)}|-|\alpha_l^{(3)}|} z_{13}^{\ell_{31}-|\alpha_r^{(3)}|-|\alpha_l^{(1)}|} \mathcal{D}_{\{\alpha_{l,r}^{(1)},\alpha_{l,r}^{(2)},\alpha_{l,r}^{(3)}\}}$$

$$\mathcal{D}_{\{\alpha_{l,r}^{(1)},\alpha_{l,r}^{(2)},\alpha_{l,r}^{(3)}\}}$$
$$= (-1)^{|\alpha_r^{(1)}|+|\alpha_r^{(2)}|+|\alpha_r^{(3)}|} \prod_{k=1}^{3} H_{\ell_k}(\alpha_l^{(k)},\alpha_r^{(k)}) Z_{\ell_{kk+1}}(\alpha_r^{(k)}\cup\alpha_l^{(k+1)})$$

where $z_{ij} := z_i - z_j$, $|\alpha|$ is the number of elements in α.

- pDWPF has a determinant expression.
- The summation should be simplified so that it reproduces the correct kinematical dependence.

Kinematical dependence and further simplification

In particular, if we consider $|\mathcal{O}_1\rangle_L = |\boldsymbol{u};\mathfrak{n}_1\rangle$, $|\mathcal{O}_2\rangle_L = |\boldsymbol{v};\mathfrak{n}_2\rangle$, $|\mathcal{O}_3\rangle_L = |\mathfrak{n}_3\rangle$,

$$\left\langle |\hat{\mathcal{O}}_1\rangle_L, |\hat{\mathcal{O}}_2\rangle_L, |\hat{\mathcal{O}}_3\rangle_L \right\rangle = z_{12}^{\ell_{21}-M_1-M_2} z_{32}^{\ell_{23}-M_2+M_1} z_{13}^{\ell_{31}-M_1+M_2} \mathcal{G} \ ,$$

which yields the following highest order term in z_3:

$$(-1)^{\ell_{31}-M_1+M_2} z_3^{\ell_3} z_{21}^{\ell_{12}-M_1-M_2} \mathcal{G} \ .$$

On the other hands, the higher order term in the summation we have derived is given by

$$(-1)^{\ell_{31}} z_3^{\ell_3} z_{21}^{\ell_{12}-M_1-M_2} \ \mathcal{D}_{\{\alpha_{l,r}^{(1)},\alpha_{l,r}^{(2)},\emptyset\}}\Big|_{\alpha_l^{(1)}=\alpha_r^{(2)}=\emptyset}$$

By comparing the z_3 dependence,

$$\mathcal{G} = (-1)^{M_1+M_2} \ \mathcal{D}_{\{\alpha_{l,r}^{(1)},\alpha_{l,r}^{(2)},\emptyset\}}\Big|_{\alpha_l^{(1)}=\alpha_r^{(2)}=\emptyset} \propto H(\emptyset,\boldsymbol{u})H(\boldsymbol{v},\emptyset)Z_{\ell_{12}}(\boldsymbol{u}\cup\boldsymbol{v})$$

The result is

$$\langle |\mathcal{O}_1\rangle_L, |\mathcal{O}_2\rangle_L, |\mathcal{O}_3\rangle_L\rangle = z_{12}^{\ell_{21}-M_1-M_2} z_{32}^{\ell_{23}-M_2+M_1} z_{13}^{\ell_{31}-M_1+M_2}$$
$$\times \left(\frac{1}{1+|z_1|^2}\right)^{\ell_1/2-M_1} \left(\frac{1}{1+|z_2|^2}\right)^{\ell_2/2-M_2} \left(\frac{1}{1+|z_3|^2}\right)^{\ell_3/2-M_3}$$
$$\times \prod_{k=1}^{M_1}\left(u_k+\frac{i}{2}\right)^{\ell_{31}} \prod_{l=1}^{M_2}\left(v_l-\frac{i}{2}\right)^{\ell_{23}} Z_{\ell_{12}}(\boldsymbol{u}\cup\boldsymbol{v})$$

Comments

- We have treated correlators for two type I operators and one type II operator: $|\boldsymbol{u_i};\mathfrak{n}_i\rangle_L \otimes |\tilde{\mathfrak{n}}_i\rangle_R$ $(i=1,2)$, $|\mathfrak{n}_3\rangle_L \otimes |\tilde{\boldsymbol{u}}_3;\tilde{\mathfrak{n}}_3\rangle_R$.
  $\Rightarrow$ A special class of 3pt. functions in EGSV is contained in this class.
- The correlators for three type I operators, namely, $|\boldsymbol{u_i};\mathfrak{n}_i\rangle_L \otimes |\tilde{\mathfrak{n}}_i\rangle_R$ $(i=1,2,3)$ *involves a sum over partitions, even after the simplification*.
- pDWPF has a determinant expression and the semi-classical limit is obtained ['12 Kostov].

Determinant formulas and semi-classical limit

pDWPF has various (determinant) expressions:['11 Foda,Wheeler], ['12 Kostov,Matsuo], ['13 Kazama,Komatsu,T.N]

$$
\begin{aligned}
Z_\ell(\boldsymbol{u}) &= \frac{1}{(\ell-M)!}\langle \downarrow^\ell |(S^-)^{\ell-M}\prod_{i=1}^{M} B(u_i)|\uparrow^\ell\rangle \\
&= \frac{\prod_{i=1}^{M}(u_i-i/2)^\ell}{\prod_{i<j}(u_i-u_j)}\det\left(u_k^{l-1}-\left(\frac{u_k-+i/2}{u_k-i/2}\right)^\ell (u_k-i)^{l-1}\right)_{1\le k,l\le M} \\
&= \prod_{i=1}^{L-1}\oint_C \frac{dx_i}{2\pi i}\prod_{j<k}(x_j-x_k)(e^{2\pi x_j}-e^{2\pi x_k})\prod_{l=1}^{L-1}\frac{Q_u(x_l)Q_v(x_l)}{(x_l^2+1/4)^L}e^{2\pi x_l}
\end{aligned}
$$

where $Q_u(x) := \prod_{i=1}^{M}(x-u_i)$. Semi-classical limit ($\ell\to\infty$, M/ℓ : fixed) for pDWPF is obtained in ['12 Kostov]

$$
\ln Z_\ell(\boldsymbol{u}) \sim \oint_{C_u}\frac{dx}{2\pi}\mathrm{Li}_2(e^{ip_{\boldsymbol{u}}(x)})\ ,\quad p_{\boldsymbol{u}}(x)=\sum_{i=1}^{M}\frac{1}{x-u_i}-\frac{\ell}{2x}\ .
$$

Lessons

1. The Wick contractions are efficiently performed using the singlet projector $\langle \mathbf{1}_{12}|$.
2. The defining property of the singlet ensures the Ward identities of the symmetry.
3. In this formalism, it is possible to deal a more general class of 3pt. functions and the structure constants can be expressed in terms of pDWPFs, rather than the scalar products.
4. The Ward identities and the kinematical dependence greatly simplify the result.

Monodromy relations for correlators @weak coupling

Monodromy relation for basic 2pt. functions

It is sufficient to consider the 2pt. point functions. We would like to show the following form of monodormy relation

$$\langle V_{12}|\Omega(u-i/2)\Omega(u+i/2) \propto \langle V_{12}|\mathbf{1} \ ,$$
$$\Omega(u) = L_1(u)\cdots L_\ell(u) \ , \ L_n(u) = u + i\vec{S}_n\cdot\vec{\sigma}$$

To prove this, we use two important equations:

$$\text{Crossing}: \quad \langle V_{12}|L_n^{(1)}(u) = -\langle V_{12}|L^{(2)}_{\ell-n+1}(-u)$$
$$\text{Inversion}: \quad L_n(-u+i/2)L_n(u+i/2) = -(u^2+1)\mathbf{1}$$

Crossing relation

With the definition $L_n^{(k)}(u) = u + i\vec{S}_n^{(k)} \cdot \vec{\sigma}$ and the property of the vertex $\langle V_{12}|(S_n^{(1)} + S_{\ell-n+1}^{(2)})$,

$$\begin{aligned}\langle V_{12}|L_n^{(1)}(u) &= \langle V_{12}|(u + i\vec{S}_n^{(1)} \cdot \vec{\sigma}) \\ &= \langle V_{12}|(u - i\vec{S}_{\ell-n+1}^{(2)} \cdot \vec{\sigma}) \\ &= -\langle V_{12}|L_{\ell-n+1}^{(2)}(-u) = \sigma^2 L_{\ell-n+1}^{(2)T}(u)\sigma^2\end{aligned}$$

With the crossing relation for the Lax operators, we have

$$\langle V_{12}|\Omega^{(1)}(u) = (-1)^\ell \langle V_{12}|\overleftarrow{\Omega}^{(2)}(-u) = \langle V_{12}|\sigma^2\Omega^{(2)T}(u)\sigma^2 \ ,$$

$$\overleftarrow{\Omega}(u) := L_\ell(u)\cdots L_1(u) \ , \quad \sigma^2\Omega^T(u)\sigma^2 = \begin{pmatrix} D(u) & -B(u) \\ -C(u) & A(u) \end{pmatrix}$$

In particular, one finds $\langle V_{12}|B^{(1)}(u) = -\langle V_{12}|B^{(2)}(u)$.

Inversion relation

By the explicit calculation, we can show

$$L(v)L(u) = (vu - 3/4) + i(v + u - i)\vec{S}\cdot\vec{\sigma}$$

Hence,

$$\begin{aligned}
&\langle V_{12}|\Omega^{(1)}(u - i/2)\Omega^{(2)}(u + i/2) \\
&= (-1)^{\ell}\langle V_{12}|\overleftarrow{\Omega}^{(2)}(-u + i/2)\Omega^{(2)}(u + i/2) \\
&= (-1)^{\ell}\langle V_{12}|\cdots L_1(-u + i/2)L_1(u + i/2)\cdots \\
&= (u^2 + 1)^{\ell}\langle V_{12}|
\end{aligned}$$

- To invert the Lax operator, the shift for the spectral parameter is necessary.
- These results are obtained by the reduction from monodormy relation of $\mathfrak{psu}(2,2|4)$ sector ['14 Jiang,Kostov,Petrovskii,Serban],['15 Kazama,Komatsu,T.N].

If we consider the limit $u \to \infty$ and expand the monodormy relation in power of $1/u$, we obtain the Ward identities of the form

$$\langle S^* | \mathcal{O}_1 \rangle, |\mathcal{O}_2\rangle\rangle + \langle |\mathcal{O}_1\rangle, S^* | \mathcal{O}_2\rangle\rangle = 0 \ .$$

- The expansion of the monodormy matrix in power of $1/u$ generates the Yangian generators.
  $\Rightarrow$ *The monodormy relations include a kind of the Ward identities of them.*
- This in turn suggests the Yangian invariance of the vertex:

$$\langle V_{123} | \Omega_1 \Omega_2 \Omega_3 \propto \langle V_{123} |$$

Summary and prospects

Summary

- We have devolved a new formalism in which the Wick contractions are expressed as singlet projections.
  $\Rightarrow$ *The Ward identities automatically follows*.
- For three-point functions in the SU(2) sector, the structure constants are given in terms of the pDWPFs.
- The knowledge of the kinematical dependence greatly helps the analysis.
- The monodromy relations at weak coupling are derived using the crossing and inversion relation, both for the fundamental and the harmonic R-matrix.
- *The* $1/u$ *expansion for them generates non-trivial identities for correlators, including the usual Ward identities*.

Prospects

- More on the monodromy relations. *How they constrain the three-point functions?*
  $\Rightarrow$ Semi-classical three-point functions from Landau-Lifshitz model.
  `[Kazama,Komatsu,T.N, in progress]`
- 1-loop corrections
  $\Rightarrow$ *Is it possible to constrain the 1-loop corrections using the symmetries?*

$$\langle V_{123}(g)|(J_1(g)+J_2(g)+J_3(g))=0$$

$$(\langle V_{123}^{(0)}|+g\langle V_{123}^{(1)}|+\cdots)\sum_{i=1}^{3}(J_i^{(0)}+gJ_i^{(1)}+\cdots)=0$$

$$\Leftrightarrow \sum_{i=1}^{3}\langle V_{123}^{(0)}|J_i^{(1)}+\sum_{i=1}^{3}\langle V_{123}^{(1)}|J_i^{(0)}=0$$

- Relation to recent non-perturbative results, SFT vertex (form factor) `['15 Bajnok,Janik]`, hexagon form factor `['15 Basso,Komatsu,Vieira]` .

Back up

D-scheme oscillator represenattion

To construct the representations, It is convenient to introduce the oscillators satisfying

$$[\mu^\alpha, \lambda_\beta] = \delta^\alpha{}_\beta \ , \ [\tilde{\mu}^{\dot\alpha}, \tilde{\lambda}_{\dot\beta}] = \delta^{\dot\alpha}{}_{\dot\beta} \ , \ \{\xi^a, \bar{\xi}_b\} = \delta^a{}_b \ ,$$

where α, β, $\underline{\dot\alpha, \dot\beta}$ are the spinor indices of the Lorentz group $SL(2,\mathbb{C}) \times \overline{SL(2,\mathbb{C})}$ and a, b are the SU(4) R-symmetry indices.

$$J^A{}_B = \bar{\zeta}^A \zeta_B \ , \quad \bar{\zeta}^A = \begin{pmatrix} \lambda_\alpha \\ i\tilde{\mu}^{\dot\alpha} \\ \bar{\xi}_a \end{pmatrix}^A \ , \quad \zeta_A = \begin{pmatrix} \mu^\alpha \\ i\tilde{\lambda}_{\dot\alpha} \\ \xi^a \end{pmatrix}_A \ ,$$

$$[J^A{}_B, J^C{}_D] = \delta^C{}_B J^A{}_D - (-1)^{(|A|+|B|)(|C|+|D|)} \delta^A{}_D J^C{}_D \ ,$$

$$C := \mathrm{tr} J = J^A{}_A \ , \quad B := \mathrm{str} J = (-1)^{|A|} J^A{}_A$$

$J^A{}_B$: generators of $\mathfrak{u}(2,2|4)$, C: central charge, B: hypercharge.

$$\mathfrak{u}(2,2|4) \underbrace{\longrightarrow}_{\text{remove } B} \mathfrak{su}(2,2|4) \underbrace{\longrightarrow}_{\text{remove } C} \mathfrak{psu}(2,2|4)$$

Fundamental fields as Fock states

The fundamental fields of $\mathcal{N}=4$ SYM are represented as follows:

$$
\begin{aligned}
\mu^\alpha|0\rangle = \tilde{\mu}^{\dot\alpha}|0\rangle &= \xi^a|0\rangle = 0 \\
F_{\alpha\beta}(0) &\leftrightarrow \lambda_\alpha\lambda_\beta|0\rangle \ , \\
\psi_{\alpha a}(0) &\leftrightarrow \lambda_\alpha\bar{\xi}_a|0\rangle \ , \\
\phi_{ab}(0) &\leftrightarrow \bar{\xi}_a\bar{\xi}_b|0\rangle \ , \\
\bar{\psi}^a_{\dot\alpha}(0) &\leftrightarrow \frac{1}{3!}\epsilon^{abcd}\tilde{\lambda}_{\dot\alpha}\bar{\xi}_b\bar{\xi}_c\bar{\xi}_d|0\rangle \ , \\
\bar{F}_{\dot\alpha\dot\beta}(0) &\leftrightarrow \frac{1}{4!}\epsilon^{abcd}\tilde{\lambda}_{\dot\alpha}\tilde{\lambda}_{\dot\beta}\bar{\xi}_a\bar{\xi}_b\bar{\xi}_c\bar{\xi}_d|0\rangle \ ,
\end{aligned}
$$

with the vanishing central charge $C=0$. The operator at the general position x is obtained by

$$
|\mathcal{O}(0)\rangle \to |\mathcal{O}(x)\rangle := e^{iP\cdot x}|\mathcal{O}(0)\rangle \ .
$$

Thus, the derivatives $-i\frac{\partial}{\partial x^{\dot\beta\alpha}}$ are expressed as the action of $P_{\alpha\dot\beta} = \lambda_\alpha\tilde{\lambda}_{\dot\beta}$.

Singlet state for $\mathfrak{psu}(2,2|4)$

We wish to construct a singlet state under $\mathfrak{psu}(2,2|4)$. For this purpose, it is convenient to prepare new vacua

$$|Z\rangle = |0\rangle_B \otimes \bar{\xi}_3\bar{\xi}_4|0\rangle_F \ , \quad \overline{|\bar{Z}\rangle} = |\bar{0}\rangle_B \otimes \bar{\xi}_1\bar{\xi}_2|0\rangle_F \ ,$$
$$\lambda_\alpha|\bar{0}\rangle_B = \tilde{\lambda}_{\dot{\alpha}}|\bar{0}\rangle_B = 0 \ ,$$

$$|\mathbf{1}_{12}\rangle = \sum_{\boldsymbol{n},\boldsymbol{m}\geq 0} f(\boldsymbol{n},\boldsymbol{m})|\boldsymbol{n}\rangle \otimes \overline{|\boldsymbol{m}\rangle} \ ,$$
$$f(\boldsymbol{n},\boldsymbol{m}) = \delta_{\boldsymbol{n},\boldsymbol{m}}(-1)^{n_{\tilde{\lambda}_{\dot{1}}}+n_{\tilde{\lambda}_{\dot{2}}}+n_{c_1}+n_{c_2}} f(C)$$

where $f(C)$ is an arbitrary function of the central charge. In particular, if we take $f(C)=1$,

$$|\mathbf{1}_{12}\rangle = \exp\left(\lambda_\alpha \otimes \mu^\alpha - \tilde{\lambda}_{\dot{\alpha}} \otimes \tilde{\mu}^{\dot{\alpha}} + \bar{c}_i \otimes c^i - \bar{d}_j \otimes d^j\right)|Z\rangle \otimes \overline{|\bar{Z}\rangle}$$

which is used in `['14 Jiang,Kostov,Petrovskii,Serban]`.

Crossing for individual oscillators and Wick contractions

The Wick contractions for the fundamental fields are simply expressed in terms of

$$I(x,y) := \langle \mathbf{1}_{12}|(e^{iP\cdot x}|0\rangle \otimes e^{iP\cdot y}|0\rangle)$$

Due to the singlet property $\langle \mathbf{1}_{12}|(J^{(1)} + J^{(2)}) = 0$ for all the (super) conformal generators, it satisfies, for example,

$$0 = \langle \mathbf{1}_{12}|(P_\mu^{(1)} + P_\mu^{(2)})(e^{iP\cdot x}|0\rangle \otimes e^{iP\cdot y}|0\rangle) = -i(\partial_\mu^x + \partial_\mu^y) I(x,y)$$

Then, $I(x,y) = I(x-y)$. Similarly, the singlet condition of the dilatation D yields

$$0 = (x^\mu \partial_\mu^x + y^\mu \partial_\mu^y + 2) I(x-y)$$

Thus, we have $I(x,y) = \frac{1}{|x-y|^2}$.

Using the exponential form, one can easily find that the crossing for individual oscillators are given by

$$
\begin{aligned}
\langle \mathbf{1}_{12}|\zeta_A \otimes 1 &= -\langle \mathbf{1}_{12}|1 \otimes \zeta_A \ , \\
\langle \mathbf{1}_{12}|\bar{\zeta}^A \otimes 1 &= \langle \mathbf{1}_{12}|1 \otimes \bar{\zeta}^A \ .
\end{aligned}
$$

Then, all the Wick contractions are expressed in terms of $I(x-y)$

$$
\begin{aligned}
\phi_{ab}(x)\phi_{cd}(y) &= \langle \mathbf{1}_{12}|(\bar{\xi}_a\bar{\xi}_b e^{iP\cdot x}|0\rangle \otimes \bar{\xi}_c\bar{\xi}_d e^{iP\cdot y}|0\rangle) \\
&= \langle \mathbf{1}_{12}|(e^{iP\cdot x}|0\rangle \otimes \bar{\xi}_a\bar{\xi}_b\bar{\xi}_c\bar{\xi}_d e^{iP\cdot y}|0\rangle) = \epsilon_{abcd} I(x-y) \ , \\
\psi_{\alpha a}(x)\bar{\psi}^b_{\dot{\alpha}}(y) &= \delta_a{}^b \langle \mathbf{1}_{12}|(\lambda_\alpha e^{iP\cdot x}|0\rangle \otimes e^{iP\cdot y}\tilde{\lambda}_{\dot{\alpha}}|0\rangle) \\
&= \delta_a{}^b \langle \mathbf{1}_{12}|(e^{iP\cdot x}|0\rangle \otimes e^{iP\cdot y}\lambda_\alpha\tilde{\lambda}_{\dot{\alpha}}|0\rangle) = -i\delta_a{}^b \frac{\partial}{\partial y^{\dot{\alpha}\alpha}} I(x-y) \ , \\
F_{\alpha\beta}(x)\bar{F}_{\dot{\alpha}\dot{\beta}}(y) &= \langle \mathbf{1}_{12}|(e^{iP\cdot x}\lambda_\alpha\lambda_\beta|0\rangle \otimes e^{iP\cdot y}\tilde{\lambda}_{\dot{\alpha}}\tilde{\lambda}_{\dot{\beta}}|0\rangle) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x^{\dot{\alpha}\alpha}}\frac{\partial}{\partial y^{\dot{\beta}\beta}} + \frac{\partial}{\partial x^{\dot{\beta}\alpha}}\frac{\partial}{\partial y^{\dot{\alpha}\beta}}\right) I(x-y) \ .
\end{aligned}
$$

Monodromy relation for fundamental R-matrix

The $\mathfrak{gl}(4|4)$ Lax operator is given by

$$L(u) = u + \eta(-1)^{|B|} E^A{}_B \otimes J^A{}_B$$

where $E^A{}_B$ is the fundamental representation of $\mathfrak{gl}(4|4)$. The crossing and inversion are respectively

$$\text{Crossing}: \quad \langle V_{12}| L_n^{(1)}(u) = -\langle V_{12}| L^{(2)}_{\ell-n+1}(-u+\eta)$$
$$\text{Inversion}: \quad L_n(u) L_n(\eta - u) = u(\eta - u)\mathbf{1} \ ,$$

The non-trivial shift for the spectral parameter under the crossing arises from $\langle \mathbf{1}_{12}| J^A{}_B \otimes 1 = -\langle \mathbf{1}_{12}|(1 \otimes J^A{}_B + (-1)^{|A|}\delta^A{}_B)$.
The monodromy relation becomes['14 Jiang,Kostov,Petrovskii,Serban],['15 Kazama,Komatsu,T.N]

$$\langle V_{12}|\Omega^{(1)}(u)\Omega^{(2)}(u) = (u(u-\eta))^{\ell}\langle V_{12}|$$
$$\Omega(u) := L_1(u)\cdots L_\ell(u)$$

Monodromy relation for harmonic R-matrix

We can also derive the monodromy relation for the R-matrix whose auxiliary space is the singleton representation ['15 Kazama,Komatsu,T.N].

$$\mathbf{R}_{ij}(u) = (-1)^{\mathbb{J}} \frac{\Gamma(\mathbb{J}+u+1)}{\Gamma(\mathbb{J}-u+1)} \in \mathrm{End}(\mathcal{V}_i \otimes \mathcal{V}_j) \ , \quad \mathcal{C}_2 = \mathbb{J}(\mathbb{J}+1)$$

$$\langle V_{12}|\Omega^{(1)}(u)\Omega^{(2)}(u) = \langle V_{12}| \ , \ \Omega^{(i)}(u) := \mathbf{R}^{(i)}_{a1}(u)\cdots\mathbf{R}^{(i)}_{a\ell}(u) \ .$$

It is of particular importance to note the following relation ['04,Beisert,Staudacher].

$$\mathbf{H}_{12} = \frac{d}{du} \ln \mathbf{R}_{12}(u)|_{u=0}$$

- The 1-loop dilatation operator, namely, the spin chain Hamiltonian is easily obtained.
- The harmonic R-matrix is used to construct building blocks for the scattering amplitude as Yangian invariant ['13,Ferro ,Lukowski,Meneghelli,Plefka,Staudacher],['13,Chicherin,Kirschner],['14,Broedel Leeuw,Rosso].