State-of-the-Art Calculation of the Decay Rate of Electroweak Vacuum

Takeo Moroi (Tokyo)

Refs:

Endo, TM, Nojiri, Shoji [1703.09304 & 1704.03492]

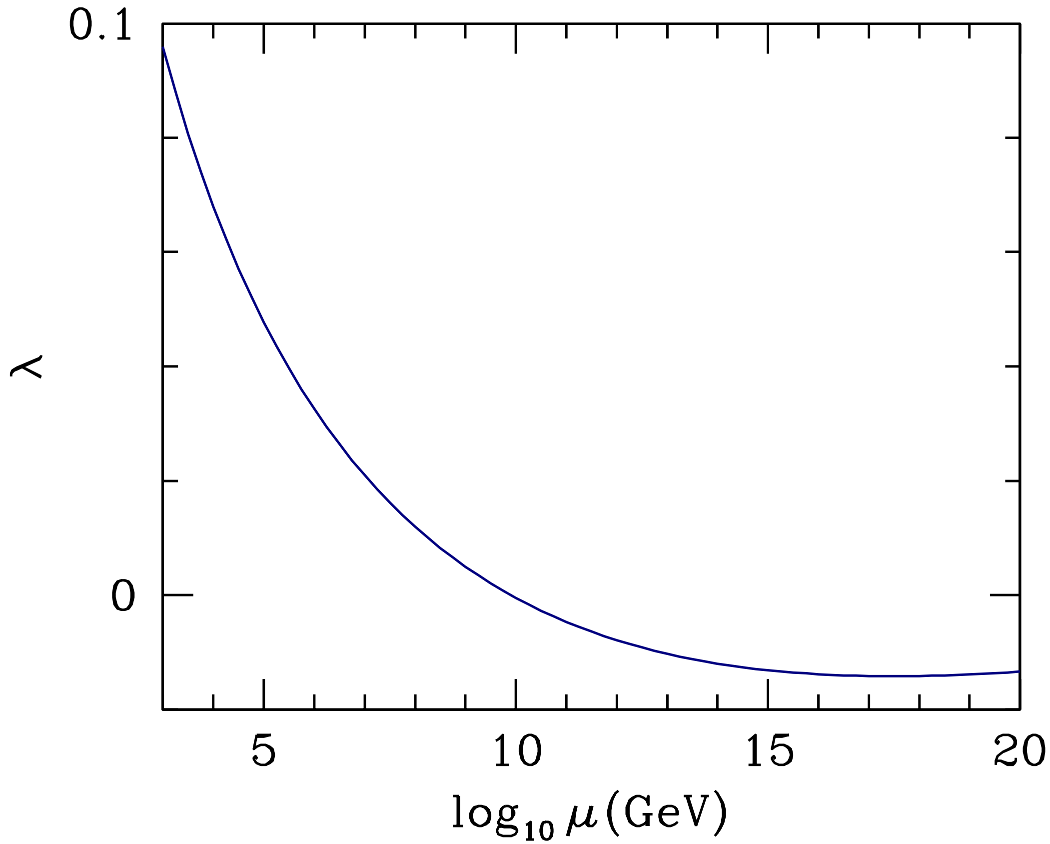
Chigusa, TM, Shoji [1707.09301]

1. Introduction

If the standard model (SM) is valid up to the Planck scale

The electroweak vacuum is (probably) metastable

The Higgs quartic coupling becomes negative at $\mu \gg M_{\text{weak}}$

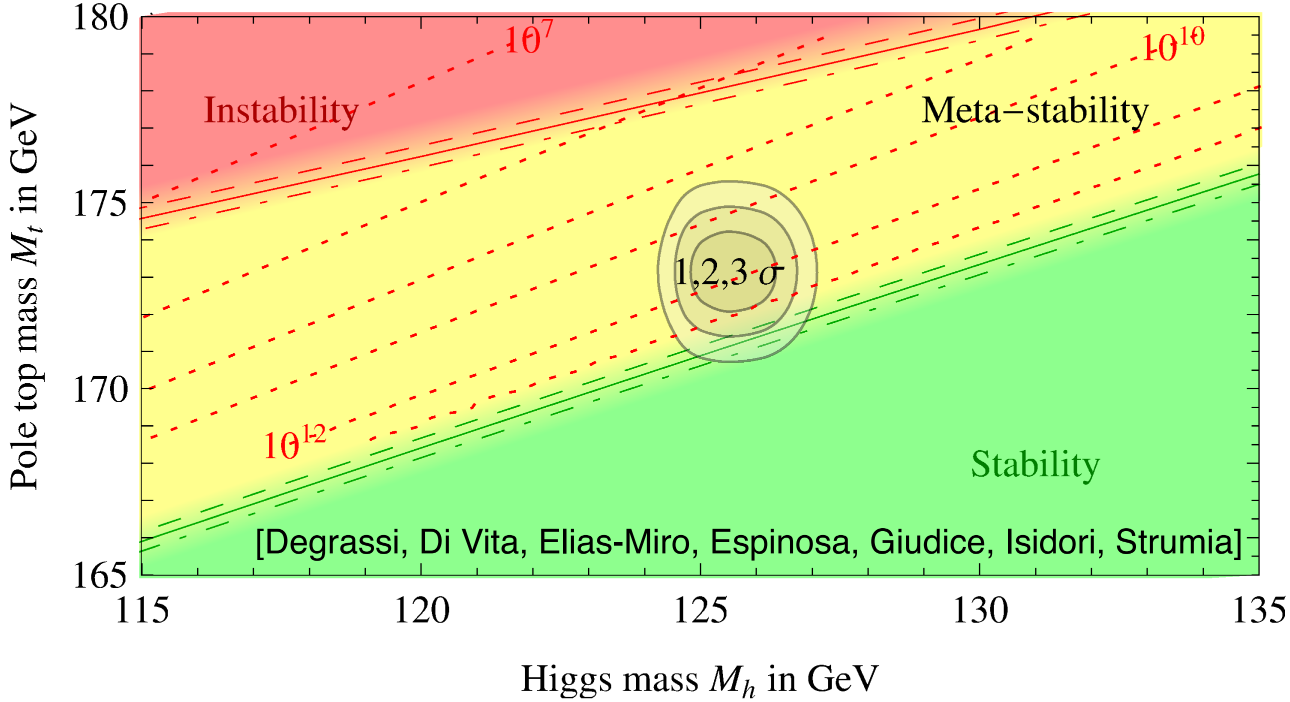


How large is the Decay rate of the EW vacuum?

⇒ Is the decay rate small enough so that $t_{\rm now} \simeq 13.6$ Gyr?

Stability of the EW vacuum was studied in the past

[Isidori, Ridolfi & Strumia; Degrassi et al.]



Problems in the previous calculations

- Effects of zero-modes were not properly taken care of
- The calculations were not simple

I explain how to calculate the decay rate of the EW vacuum

- A calculation without problems in previous studies
- Gauge-invariant expression of the decay rate
- Our result (with best-fit SM parameters)

$$\gamma \equiv \frac{\Gamma}{(\text{Volume})} \simeq 10^{-554}\ \text{Gyr}^{-1}\text{Gpc}^{-3}$$

$$\Leftrightarrow H_0^{-4} \sim 10^{3}\ \text{GyrGpc}^{3}$$

Outline

1. Introduction

2. Coleman's Method

3. Bounce in the SM

4. Effects of Higgs Mode

5. Effects of Gauge and NG Modes (probably, I will skip)

6. Total Decay Rate

7. Numerical Results

8. Summary

2. Coleman's Method

Calculation of the decay rate using "bounce"

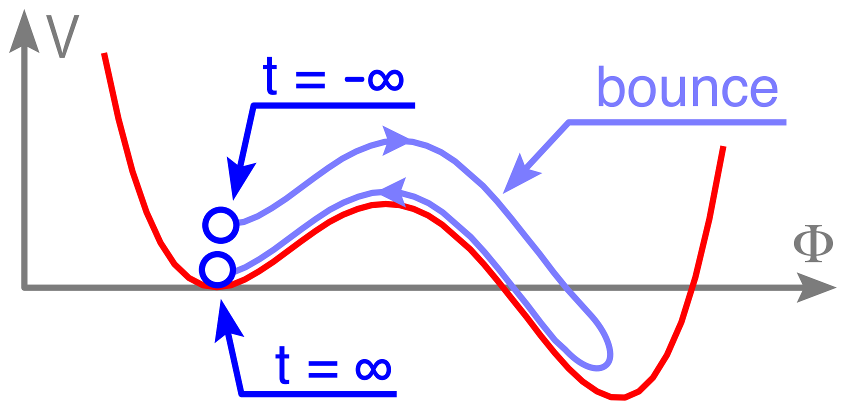
[Coleman; Callan & Coleman]

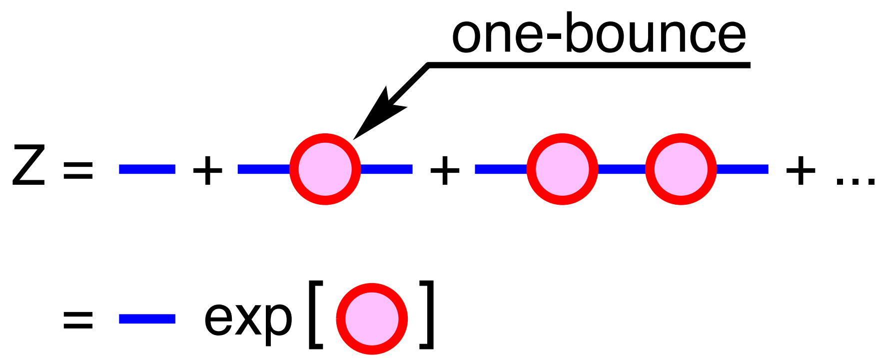
- The decay rate is related to Euclidean partition function

$$Z = \langle \mathsf{FV} | e^{-HT} | \mathsf{FV} \rangle \simeq \int \mathcal{D}\Psi \, e^{-S_{\mathsf{E}}} \propto \exp(i\gamma V T)$$

- The false vacuum decay is dominated by the classical path

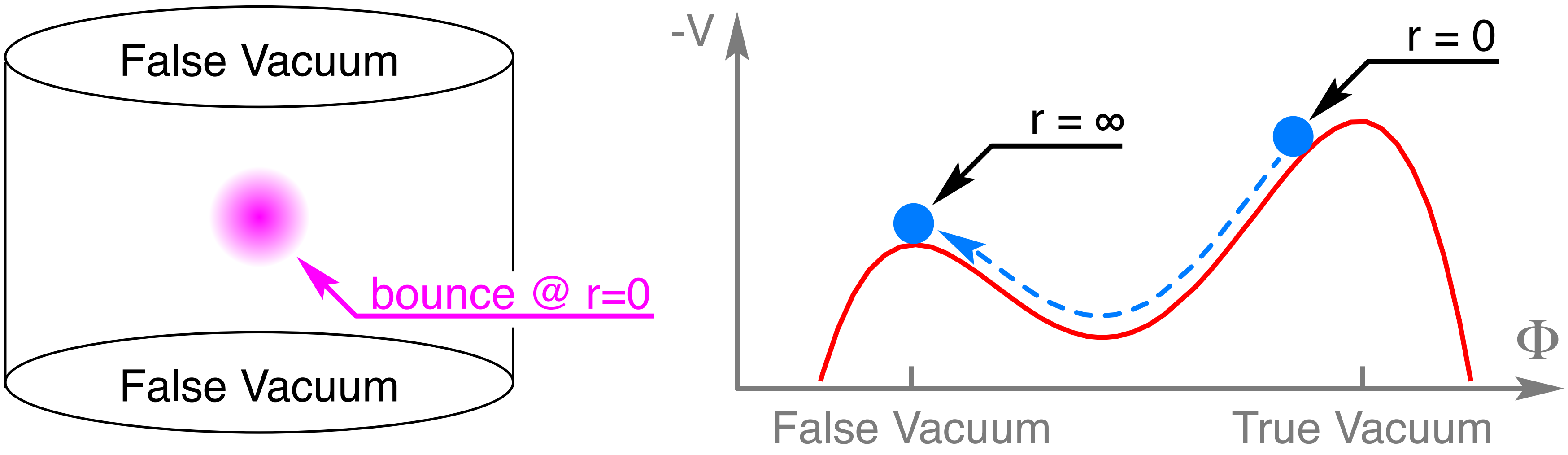
Bounce: saddle-point solution of the EoM





The bounce: $O(4)$ symmetric solution of Euclidean EoM

[Coleman, Glaser & Martin; Blum, Honda, Sato, Takimoto & Tobioka]



$$\left[\partial^2\Phi - \frac{\partial V}{\partial \Phi}\right]_{\Phi\to\bar\phi} = \left[\partial_r^2\Phi + \frac{3}{r}\partial_r\Phi - \frac{\partial V}{\partial \Phi}\right]_{\Phi\to\bar\phi} = 0$$

with $\begin{cases} \bar\phi(r=\infty) = v : \text{ false vacuum} \\ \bar\phi'(0) = \bar\phi'(\infty) = 0 \end{cases}$

The decay rate per unit volume w.r.t. one-bounce action

$$\gamma \simeq \frac{1}{VT} \text{Im} \left[\frac{Z_{\text{1-bounce}}}{Z_{\text{0-bounce}}} \right] \simeq \frac{1}{VT} \text{Im} \left[\frac{\int_{\text{1-bounce}} \mathcal{D}\Psi \, e^{-S_{\text{E}}}}{\int_{\text{0-bounce}} \mathcal{D}\Psi \, e^{-S_{\text{E}}}} \right]$$

We expand the action around the "classical path"

$$S_{\text{E}}[\bar{\phi} + \Psi] = S_{\text{E}}[\bar{\phi}] + \frac{1}{2} \int d^4x \Psi \mathcal{M} \Psi + O(\Psi^3)$$

$$S_{\text{E}}[v + \Psi] = S_{\text{E}}[v] + \frac{1}{2} \int d^4x \Psi \widehat{\mathcal{M}} \Psi + O(\Psi^3)$$

$\bar{\phi}$: Classical solution (bounce)

Ψ: Field fluctuations around the classical solution

$\mathcal{M}$ and $\widehat{\mathcal{M}}$: so-called "fluctuation operators"

Final expression for γ (at the one-loop level)

$$\gamma = \mathcal{A} e^{-\mathcal{B}} \text{ with } \mathcal{B} = S_{\mathsf{E}}[\bar{\phi}] - S_{\mathsf{E}}[v]$$

Prefactor $\mathcal{A}$ (for bosonic contribution)

$$\mathcal{A} \simeq \frac{1}{VT} \left| \frac{\mathrm{Det}\mathcal{M}}{\mathrm{Det}\widehat{\mathcal{M}}} \right|^{-1/2}$$

$\mathcal{M}$: fluctuation operator around the bounce

$\widehat{\mathcal{M}}$: fluctuation operator around the false vacuum

Main subject today:

- Calculation of the prefactor $\mathcal{A} \simeq \mathcal{A}^{(h)} \times \mathcal{A}^{(\mathsf{Gauge})} \times \cdots$
- Naive calculation gives $\mathcal{A} \to \infty$, if $\mathcal{M}$ has zero-eigenvalue

3. Bounce in the SM

Higgs potential: $V = m^2 H^\dagger H + \lambda(H^\dagger H)^2$

- We consider very large Higgs amplitude for which $\lambda < 0$
- It happens when $|H| \gg m$, so we neglect m^2-term

We use the following potetial: $V = -|\lambda|(H^\dagger H)^2$

$\Rightarrow$ This potential does not have local minimum

$\Rightarrow$ It still has "bounce solution"

$$H_{\text{bounce}} = \frac{1}{\sqrt{2}} e^{i\sigma^a\theta^a} \begin{pmatrix} 0 \\ \bar{\phi} \end{pmatrix} \quad \text{with} \quad \partial_r^2 \bar{\phi} + \frac{3}{r}\partial_r \bar{\phi} + 3|\lambda|\bar{\phi}^2 = 0$$

$\Rightarrow$ Expliit form of the bounce:

$$\bar{\phi}(r) = \frac{8\bar{\phi}_C}{8 + |\lambda|\bar{\phi}_C^2 r^2} \quad \Leftarrow \quad \bar{\phi}_C = \bar{\phi}(r=0)\text{: free parameter}$$

Bounce action for the SM

$$\mathcal{B} = \frac{8\pi^2}{3|\lambda|}$$

Possible deformation of the bounce

- $SU(2)$ transformation: change of θ^a
- Scale transformation: change of $\bar{\phi}_C$
- Translation (in 4D space)

Expansion around the bounce:

$$H = \frac{1}{\sqrt{2}} e^{i\sigma^a\theta^a} \begin{pmatrix} \varphi^1 + i\varphi^2 \\ \bar{\phi} + h - i\varphi^3 \end{pmatrix}, \quad W_\mu^a = w_\mu^a, \quad B_\mu = b_\mu$$

4. Effects of the Higgs Mode

Fluctuation operator of the Higgs mode

$$\mathcal{L} \ni \frac{1}{2} h \left(-\partial^2 - 3|\lambda|\bar{\phi}^2\right) h = \frac{1}{2} h \, \mathcal{M}^{(h)} \, h$$

We need to calculate $\mathrm{Det}\mathcal{M}^{(h)}$

$$\mathrm{Det}\mathcal{M}^{(h)} \sim \prod_n \omega_n \qquad \omega_n\text{: Eigenvalue of } \mathcal{M}^{(h)}$$

We expand h using 4D spherical harmonics $\mathcal{Y}_{J,m_A,m_B}$

$$h(x) = \sum_{n,J,m_A,m_B} \alpha_{n,J,m_A,m_B} \rho^{(h)}_{n,J}(r) \mathcal{Y}_{J,m_A,m_B}(\hat{\mathbf{r}})$$

α_{n,J,m_A,m_B}: expansion coefficient (integration variable)

4D Laplacian acting on angular-momentum eigenstate

$$\partial^2 \to \Delta_J \equiv \partial_r^2 + \frac{3}{r}\partial_r - \frac{4J(J+1)}{r^2} \equiv \partial_r^2 + \frac{3}{r}\partial_r - \frac{L^2}{r^2}$$

Radial mode function $\rho_{n,J}^{(h)}(r)$:

- $\mathcal{M}_J^{(h)} \rho_{n,J}^{(h)} \equiv \left[-\partial_r^2 - \frac{3}{r}\partial_r + \frac{4J(J+1)}{r^2} - 3|\lambda|\bar{\phi}^2\right] \rho_{n,J}^{(h)} = \omega_{n,J} \rho_{n,J}^{(h)}$
- $\rho_{n,J}^{(h)}(0) < \infty$ to make S_{E} finite
- $\rho_{n,J}^{(h)}(\infty) = 0$, because $h(\infty) = 0$

We calculate the functional determinant of $\mathcal{M}_J^{(h)}$

$$\Rightarrow \mathrm{Det}\mathcal{M}^{(h)} = \prod_{J=0}^{\infty} \left[\mathrm{Det}\mathcal{M}_J^{(h)}\right]^{(2J+1)^2} = \prod_{J=0}^{\infty} \left[\prod_n \omega_{n,J}\right]^{(2J+1)^2}$$

Regularization with angular-momentum cutoff

Quadratic part of the action (to perform Gaussian integral)

$$
\begin{aligned}
S_{\mathsf{E}} &= \mathcal{B} + \frac{1}{2}\sum_{n,J,m_A,m_B} \alpha^2_{n,J,m_A,m_B} \int dr r^3 \rho^{(h)}_{n,J} \mathcal{M}^{(h)}_J \rho^{(h)}_{n,J} + \cdots \\
&= \mathcal{B} + \frac{1}{2}\sum_{n,J,m_A,m_B} 2\pi\omega_{n,J}\alpha^2_{n,J,m_A,m_B} + \cdots
\end{aligned}
$$

Normalization: $\int dr r^3 \rho^{(h)}_{n,J}\rho^{(h)}_{n',J'} = 2\pi\delta_{JJ'}\delta_{nn'}$

Path integral over h:

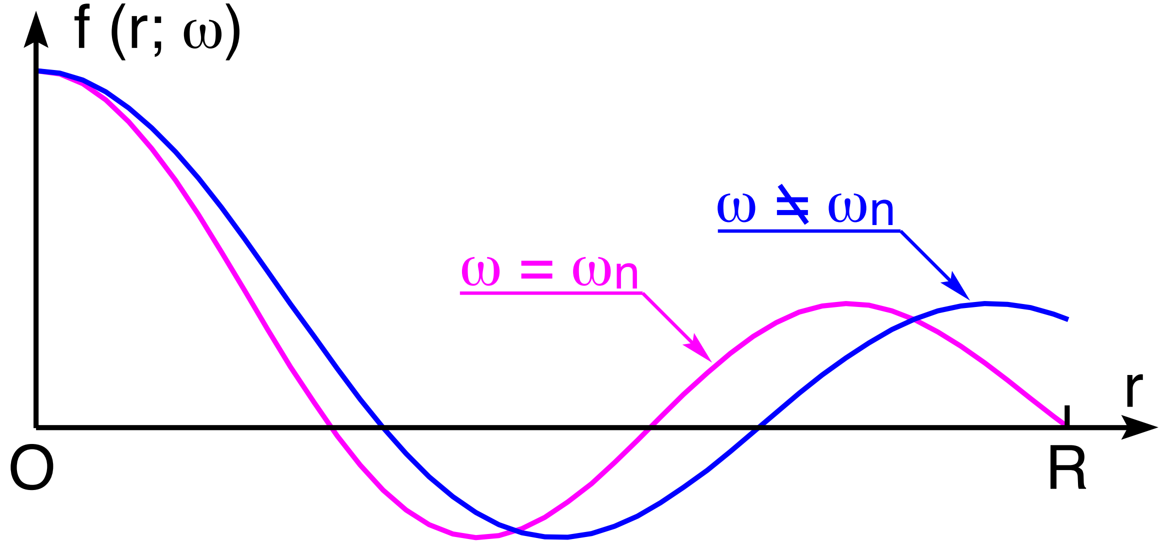
$$
\begin{aligned}
\int \mathcal{D}h e^{-S_{\mathsf{E}}} &\equiv \int \prod_{n,J,m_A,m_B} d\alpha_{n,J,m_A,m_B} e^{-S_{\mathsf{E}}} \\
&\simeq e^{-\mathcal{B}} \prod_J \left[\prod_n \omega_{n,J}\right]^{-(2J+1)^2/2} \\
&= e^{-\mathcal{B}} \left[\mathsf{Det}\mathcal{M}^{(h)}\right]^{-1/2}
\end{aligned}
$$

Functional determinant for operators defined in $0 \le r \le R$

$$\mathrm{Det}\mathcal{M} \simeq \prod_n \omega_n \text{ with } \begin{cases} \mathcal{M}\rho_n = \omega_n \rho_n \text{ with } \mathcal{M} = -\Delta_J + \delta W(r) \\ \rho_n(0) < \infty \\ \rho_n(R) = 0 \end{cases}$$

We introduce a function f which obeys: $(\mathcal{M} - \omega)\, f(r;\omega) = 0$

- $f(r = R;\omega)|_{\omega=\omega_n} = 0$
- $\mathrm{Det}(\mathcal{M} - \omega)|_{\omega=\omega_n} = 0$



We can use "Gelfand-Yaglom theorem"

[Coleman; Dashen, Hasslacher & Neveu; Kirsten & McKane; $\cdots$]

$$\frac{\mathrm{Det}(\mathcal{M}-\omega)}{\mathrm{Det}(\widehat{\mathcal{M}}-\omega)} = \frac{f(r=R;\omega)}{\hat{f}(r=R;\omega)} \quad \text{with} \quad \begin{cases} \mathcal{M} f(r;\omega) = \omega f(r;\omega) \\ \widehat{\mathcal{M}} \hat{f}(r;\omega) = \omega \hat{f}(r;\omega) \\ f(r=0) = \hat{f}(r=0) < \infty \end{cases}$$

$\Rightarrow$ Notice: LHS and RHS have the same analytic behavior

- LHS and RHS have same zeros and infinities
- LHS and RHS becomes equal to 1 when $\omega \to \infty$

We need $f(r;\omega=0)$ and $\hat{f}(r;\omega=0)$, which obey

- $\mathcal{M} f = 0$
- $\widehat{\mathcal{M}} \hat{f} = 0$

Higgs-mode contribution to the prefactor $\mathcal{A}$

$$\mathcal{A}^{(h)} \simeq \lim_{r_\infty\to\infty} \prod_J \left[\frac{f_J^{(h)}(r_\infty)}{r_\infty^{2J}} \right]^{-(2J+1)/2}$$

$$\mathcal{M}_J^{(h)} f_J^{(h)} = \left[-\partial_r^2 - \frac{3}{r}\partial_r + \frac{4J(J+1)}{r^2} - 3|\lambda|\bar{\phi}^2(r) \right] f_J^{(h)} = 0$$

$$f_J^{(h)}(r\to 0) \simeq r^{2J}$$

However, $f_0^{(h)}(r\to\infty) = f_{1/2}^{(h)}(r\to\infty) = 0$

- Conformal zero-mode in $J=0$: $f_0^{(h)} = \dfrac{\partial\bar{\phi}}{\partial\bar{\phi}_C}$
- Translation zero-mode in $J=1/2$: $f_{1/2}^{(h)} = -\dfrac{4}{|\lambda|\bar{\phi}_C^3}\partial_r\bar{\phi}$

Conformal zero-mode: $\mathcal{M}_0^{(h)} \rho_{\mathsf{conf}}(r) = 0$

$$\rho_{\mathsf{conf}}(r)\mathcal{Y}_{0,0,0} = \mathcal{N}_{\mathsf{conf}} \frac{\partial \bar{\phi}}{\partial \bar{\phi}_C} = \mathcal{N}_{\mathsf{conf}} \left(1 - \frac{|\lambda|}{8} \bar{\phi}_C^2 r^2\right) \left(1 + \frac{|\lambda|}{8} \bar{\phi}_C^2 r^2\right)^{-2}$$

Normalization factor

$$\mathcal{N}_{\mathsf{conf}}^{-2} = \frac{1}{2\pi} \int_0^{r_\infty} d^4 r \left(\frac{\partial \bar{\phi}}{\partial \bar{\phi}_C}\right)^2 \simeq \frac{64\pi}{|\lambda|^2 \bar{\phi}_C^4} \ln r_\infty$$

Path integral over conformal zero-mode = integral over $\bar{\phi}_C$

$$H \ni \frac{1}{\sqrt{2}}(\bar{\phi} + h) = \frac{1}{\sqrt{2}} \left[\bar{\phi} + \alpha_{\mathsf{conf}} \mathcal{N}_{\mathsf{conf}} \frac{\partial \bar{\phi}}{\partial \bar{\phi}_C} + \cdots \right]$$

$$\Rightarrow \int \mathcal{D} h^{(\mathsf{conf})} \equiv \int d\alpha_{\mathsf{conf}} \to \int \frac{d\bar{\phi}_C}{\mathcal{N}_{\mathsf{conf}}}$$

Functional determinant

$$\int \prod_n d\alpha_{n,J=0}\, e^{-S_{\mathsf{E}}^{(J=0)}} \simeq \left[\mathsf{Det}\mathcal{M}_0^{(h)}\right]^{-1/2} \to \int \frac{d\bar{\phi}_C}{\mathcal{N}_{\mathsf{conf}}} \left[\mathsf{Det}'\mathcal{M}_0^{(h)}\right]^{-1/2}$$

"Prime": zero-eigenvalue is omitted from the Det

Functional determinant with zero-eigenvalue omitted

$$\frac{\mathsf{Det}'\mathcal{M}_0^{(h)}}{\mathsf{Det}\widehat{\mathcal{M}}_0^{(h)}} = \lim_{\nu\to 0} \nu^{-1} \frac{\mathsf{Det}(\mathcal{M}_0^{(h)} + \nu)}{\mathsf{Det}\widehat{\mathcal{M}}_0^{(h)}} = \lim_{\nu\to 0} \frac{f_0^{(h)}(r_\infty) + \nu \check{f}_0^{(h)}(r_\infty)}{\nu}$$

$$(\mathcal{M}_0^{(h)} + \nu)(f_0^{(h)} + \nu \check{f}_0^{(h)}) = O(\nu^2) \;\Rightarrow\; \check{f}_0^{(h)} = -\left[\mathcal{M}_0^{(h)}\right]^{-1} f_0^{(h)}$$

The function $\check{f}_0^{(h)}$

$$\check{f}_0^{(h)}(r_\infty) = \int_0^{r_\infty} dr_1 r_1^{-3} \int_0^{r_1} dr_2 r_2^3 \frac{\partial\bar{\phi}}{\partial\bar{\phi}_C} \simeq -\frac{4}{|\lambda|\bar{\phi}_C^2} \ln r_\infty \simeq -\frac{|\lambda|\bar{\phi}_C^2}{16\pi\mathcal{N}_{\mathsf{conf}}^2}$$

$J = 0$ contribution: $\ln r_\infty$ disappears

$$\left|\frac{\text{Det}\mathcal{M}_0^{(h)}}{\text{Det}\widehat{\mathcal{M}}_0^{(h)}}\right|^{-1/2} \to \int \frac{d\bar{\phi}_C}{\mathcal{N}_{\text{conf}}} \left|\check{f}_0^{(h)}(r_\infty)\right|^{-1/2} = \int \frac{d\bar{\phi}_C}{\bar{\phi}_C} \left(\frac{16\pi}{|\lambda|}\right)^{1/2}$$

We can also take care of the translation zero-mode:

[Callan & Coleman]

$$\frac{\mathcal{A}^{(h)}}{VT} \to \int \frac{d\bar{\phi}_C}{\bar{\phi}_C} \frac{\mathcal{B}^2}{4\pi^2} \left(\frac{16\pi}{|\lambda|}\right)^{1/2} \left[\frac{\check{f}_{1/2}^{(h)}(r_\infty)}{r_\infty}\right]^{-2} \prod_{J\geq 1} \left[\frac{f_J^{(h)}(r_\infty)}{r_\infty^{2J}}\right]^{-(2J+1)^2/2}$$

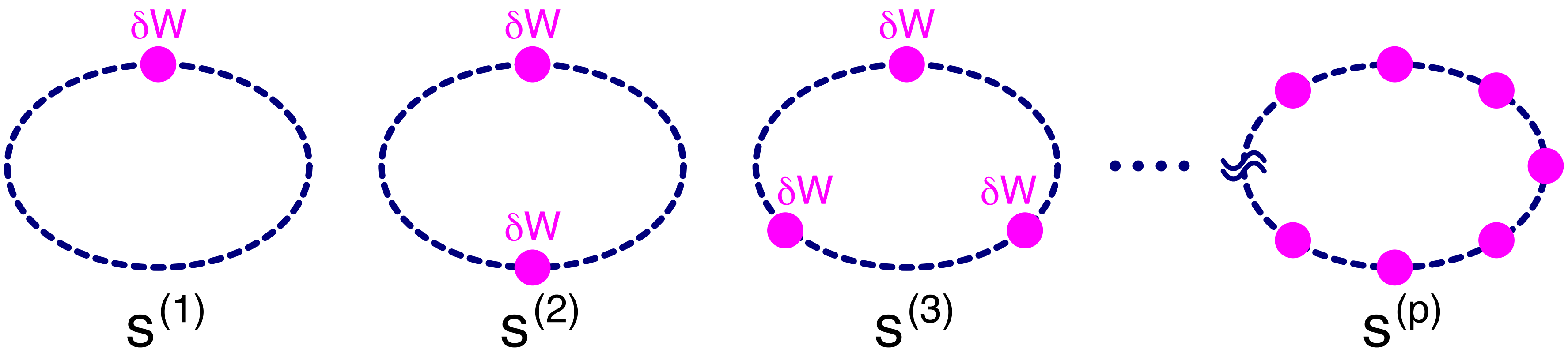
$$\mathcal{M}_{1/2}^{(h)} \check{f}_{1/2}^{(h)} = -r \left(1 + \frac{|\lambda|}{8} \bar{\phi}_C^2 r^2\right)^{-2} \Rightarrow \check{f}_{1/2}^{(h)}(r_\infty) \propto \frac{r_\infty}{\bar{\phi}_C^2}$$

We are calculating one-loop effective action

$\Rightarrow$ Renormalization is necessary

$$\mathcal{A} = \frac{1}{VT} \left| \frac{\text{Det } (-\partial^2 + \delta W)}{\text{Det } (-\partial^2)} \right|^{-1/2} e^{-\mathcal{S}_{\text{C.T.}}} \quad \text{with} \quad \delta W = -3|\lambda|\bar{\phi}^2$$

$$\Rightarrow \ln \mathcal{A} \sim -\frac{1}{2} \text{Tr} \ln \left[1 - \frac{\delta W(r)}{\partial^2} \right] + \cdots$$



$\Rightarrow$ We calculate the divergent part (i.e., $s^{(1)} + s^{(2)}$) in two ways

1. Gelfand-Yaglom theorem

$$s^{(1)} + s^{(2)} + s^{(3)} + \cdots = \sum_J \frac{(2J+1)^2}{2} \ln \left[\frac{f_J^{(h)}(r_\infty)}{r_\infty^{2J}} \right]$$

- We expand $f_J^{(h)}$ w.r.t. δW: $(-\Delta_J + \delta W) f_J^{(h)} = 0$

$$f_J^{(h)}(r) = r^{2J} + \sum_{p=1}^{\infty} F_p(r) \quad \text{with} \quad \Delta_J F_p = \delta W(r) F_{p-1}$$

- We calculate $f_J^{(h)}$ up to $O(\delta W^2)$

$$\begin{aligned} s^{(1)} + s^{(2)} &= \sum_J \frac{(2J+1)^2}{2} \left[\frac{F_1(r_\infty)}{r_\infty^{2J}} + \frac{F_2(r_\infty)}{r_\infty^{2J}} - \frac{1}{2} \left(\frac{F_1(r_\infty)}{r_\infty^{2J}} \right)^2 \right] \\ &\equiv \sum_J s_J \end{aligned}$$

2. Dimensional regularization (with $\overline{\text{MS}}$ subtraction)

$$s^{(1)}_{\overline{\text{MS}}} = 0$$

$$s^{(2)}_{\overline{\text{MS}}} = \frac{1}{128\pi^4} \int dk k^3 \widetilde{\delta W}(k) \widetilde{\delta W}(-k) \left[\frac{1}{\bar{\epsilon}} + \ln \frac{k^2}{\mu^2} + \cdots \right] - (\text{C.T.})$$

$$\widetilde{\delta W}(k) \equiv F.T.[\delta W(x)]$$

- In $\overline{\text{MS}}$ scheme, $\bar{\epsilon}^{-1}$ is subtracted by counter terms
- The μ-dependent part

$$s_{\overline{\text{MS}}} \equiv s^{(1)}_{\overline{\text{MS}}} + s^{(2)}_{\overline{\text{MS}}} = - \int d^4x \left[\frac{1}{2} \gamma_{\text{wf}} \, \bar{\phi} \partial^2 \bar{\phi} + \frac{1}{4} \delta_{\text{vtx}} \, \bar{\phi}^4 \right] \ln \mu + \cdots$$

γ_{wf} : wave-function correction

δ_{vtx} : vertex correction

Counter term (for the Higgs mode)

$$\mathcal{S}_{\text{C.T.}} = s_{\overline{\text{MS}}} - \sum_J s_J$$

Prefactor $\mathcal{A}^{(h)}$:

$$\frac{\mathcal{A}^{(h)}}{VT} e^{-\mathcal{S}_{\text{C.T.}}} \rightarrow \int \frac{d\bar{\phi}_C}{\bar{\phi}_C} e^{-s_{\overline{\text{MS}}}} e^{s_0 + s_{1/2}} \frac{\mathcal{B}^2}{4\pi^2} \left(\frac{16\pi}{|\lambda|}\right)^{1/2} \left[\frac{\check{f}^{(h)}_{1/2}(r_\infty)}{r_\infty}\right]^{-2}$$
$$\prod_{J\geq 1} e^{s_J} \left[\frac{f_J^{(h)}(r_\infty)}{r_\infty^{2J}}\right]^{-(2J+1)^2/2}$$

e^{s_J} subtracts the divertent part from $\text{Det}\mathcal{M}_J^{(h)}$

$$e^{s_J} \left[\frac{f_J^{(h)}(r_\infty)}{r_\infty^{2J}}\right]^{-(2J+1)^2/2} \rightarrow 1 \quad \text{as} \quad J \rightarrow \infty$$

μ-dependence of $\gamma = \mathcal{A} e^{-\mathcal{B}} \sim \exp\left[-(\mathcal{B} + s_{\overline{\mathsf{MS}}} + \cdots)\right]$

$$\mathcal{B}(\mu) = \int d^4x \left[-\frac{1}{2}\bar{\phi}\partial^2\bar{\phi} + \frac{1}{4}\lambda(\mu)\bar{\phi}^4\right]$$

Running of the quartic coupling

$$\lambda(\mu) = \lambda(\mu_0) + (2\,\gamma_{\mathsf{wf}}\,\lambda + \delta_{\mathsf{vtx}})\ln(\mu/\mu_0) + \cdots$$

μ-dependence of the decay rate

$$\mathcal{B} + s_{\overline{\mathsf{MS}}} = \cdots + \frac{1}{2}\int d^4x\ \gamma_{\mathsf{wf}}\left[\bar{\phi}\left(\frac{\partial V}{\partial\bar{\phi}} - \partial^2\bar{\phi}\right)\right]\ln\mu + \cdots$$

At the leading order, μ-dependence vanishes due to EoM

[Endo, TM, Nojiri, Shoji]

5. Gauge and NG Contributions

Next subject: effects of gauge and NG fields

$\Rightarrow$ For simplicity, let us consider $U(1)$ gauge symmetry

H: scalar field with charge $+1$

Our choice of gauge-fixing function: $\mathcal{F} = \partial_\mu A_\mu$

[Such a choice was also suggested by Kusenko, Lee & Weinberg]

$$\mathcal{L} = \cdots + \frac{1}{2\xi}(\partial_\mu A_\mu)^2 + \bar{c}\partial^2 c$$

With this choice of gauge-fixing function:

- Ghosts do not couple to the bounce
- Gauge-fixing terms do not affect the EoM of the bounce

$$\Rightarrow H_{\text{bounce}} = \frac{1}{\sqrt{2}} e^{i\theta} \bar{\phi}$$

Comment: People used to adopt R_ξ gauge

$$\mathcal{F}^{(R_\xi)} = \partial_\mu A_\mu - 2\xi g(\mathrm{Re}H)(\mathrm{Im}H)$$

EoM of the bounce is affected by the gauge-fixing terms

$$\Rightarrow (H, A_\mu)_{\text{bounce}} = \left(\frac{1}{\sqrt{2}} e^{i\Theta(r)} \bar{\phi}, \ \frac{1}{g} \partial_\mu \Theta(r) \right)$$

$$\partial_r^2 \Theta + \frac{3}{r} \partial_r \Theta - \frac{1}{2} \xi g^2 \bar{\phi}^2 \sin 2\Theta = 0.$$

$\Theta(r)$ is determined by $\Theta(0)$, because $\Theta'(0) = 0$

Fluctuation operator depends on the choice of Θ

$\Rightarrow$ I could not understand how we take into account the effects of all the possible bounce configuration

Fluctuation operator for A_μ and φ

$$\mathcal{M}^{(A_\mu,\varphi)} \equiv \begin{pmatrix} -\partial^2\delta_{\mu\nu} + \left(1 - \frac{1}{\xi}\right)\partial_\mu\partial_\nu + g^2\bar{\phi}^2 & g(\partial_\nu\bar{\phi}) - g\bar{\phi}\partial_\nu \\ 2g(\partial_\mu\bar{\phi}) + g\bar{\phi}\partial_\mu & -\partial^2 - |\lambda|\bar{\phi}^2 \end{pmatrix}$$

We expand fluctuations using $\mathcal{Y}_{J,m_A,m_B}$

$$A_\mu(x) \ni \rho^{(S)}_{J,m_A,m_B}(r)\frac{x_\mu}{r}\mathcal{Y}_{J,m_A,m_B} + \rho^{(L)}_{J,m_A,m_B}(r)\frac{r}{L}\partial_\mu\mathcal{Y}_{J,m_A,m_B}$$

$$+ \sum_{a=1,2}\rho^{(T_a)}_{J,m_A,m_B}(r)\, i\epsilon_{\mu\nu\rho\sigma}V^{(a)}_\nu(x_\rho\partial_\sigma - x_\sigma\partial_\rho)\mathcal{Y}_{J,m_A,m_B}$$

$$\varphi(x) \ni \rho^{(\varphi)}_{J,m_A,m_B}(r)\mathcal{Y}_{J,m_A,m_B}$$

$V^{(a)}_\nu$: arbitrary constant 4D vector

Fluctuation operator after angular-momentum decomposition

$$
\begin{aligned}
S_{\mathsf{E}} &\simeq S_{\mathsf{E}}[\bar{\phi}] + \frac{1}{2}\int d^4x\,(a_\mu,\varphi)\,\mathcal{M}^{(A_\mu,\varphi)}\,(a_\mu,\varphi)^T \\
&\simeq S_{\mathsf{E}}[\bar{\phi}] \\
&\quad + \frac{1}{2}\sum_{J,m_A,m_B}\int dr r^3 \left(\rho^{(S)},\rho^{(L)},\rho^{(\varphi)}\right)\mathcal{M}_J^{(S,L,\varphi)}\left(\rho^{(S)},\rho^{(L)},\rho^{(\varphi)}\right)^T \\
&\quad + \frac{1}{2}\sum_{J,m_A,m_B}\sum_{a=1,2}\int dr r^3 \rho^{(T_a)}\mathcal{M}_J^{(T)}\rho^{(T_a)}
\end{aligned}
$$

$\rho^{(X)}$ have indices J, m_A, and m_B

We calculate the contribution of each J

$$
\mathsf{det}\mathcal{M}^{(A_\mu,\varphi)} \simeq \prod_J \left[\mathsf{det}\mathcal{M}_J^{(S,L,\varphi)}\left(\mathsf{det}\mathcal{M}_J^{(T)}\right)^2\right]^{(2J+1)^2}
$$

Fluctuation operators around the bounce

$$\mathcal{M}_J^{(S,L,\varphi)} \equiv \begin{pmatrix} -\Delta_J + \frac{3}{r^2} + g^2\bar{\phi}^2 & -\frac{2L}{r^2} & g\bar{\phi}' - g\bar{\phi}\partial_r \\ -\frac{2L}{r^2} & -\Delta_J - \frac{1}{r^2} + g^2\bar{\phi}^2 & -\frac{L}{r}g\bar{\phi} \\ 2g\bar{\phi}' + g\bar{\phi}\partial_r + \frac{3}{r}g\bar{\phi} & -\frac{L}{r}g\bar{\phi} & -\Delta_J - |\lambda|\bar{\phi}^2 \end{pmatrix}$$

$$+\left(1-\frac{1}{\xi}\right)\begin{pmatrix} \partial_r^2 + \frac{3}{r}\partial_r - \frac{3}{r^2} & -L\left(\frac{1}{r}\partial_r - \frac{1}{r^2}\right) & 0 \\ L\left(\frac{1}{r}\partial_r + \frac{3}{r^2}\right) & -\frac{L^2}{r^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{M}_J^{(T)} \equiv -\Delta_J + g^2\bar{\phi}^2 \;\Leftarrow\; \xi\text{-independent}$$

$$\Delta_J \equiv \partial_r^2 + \frac{3}{r}\partial_r - \frac{L^2}{r^2} \quad \text{with} \quad L^2 = 4J(J+1)$$

S-, L-, and NG-mode contributions to the prefactor $\mathcal{A}$

$$\mathcal{A}^{(S,L,\varphi)} = \prod_J \left[\frac{\mathrm{Det}\mathcal{M}_J^{(S,L,\varphi)}}{\mathrm{Det}\widehat{\mathcal{M}}_J^{(S,L,\varphi)}} \right]^{-(2J+1)^2/2}$$

$$= \prod_J \left[\left(\frac{\mathcal{D}_J^{(S,L,\varphi)}(r \to 0)}{\widehat{\mathcal{D}}_J^{(S,L,\varphi)}(r \to 0)} \right)^{-1} \frac{\mathcal{D}_J^{(S,L,\varphi)}(r \to \infty)}{\widehat{\mathcal{D}}_J^{(S,L,\varphi)}(r \to \infty)} \right]^{-(2J+1)^2/2}$$

$$\mathcal{D}^{(S,L,\varphi)}(r) \equiv \mathrm{det}(\Psi_1(r)\ \Psi_2(r)\ \Psi_3(r))$$

$$\widehat{\mathcal{D}}^{(S,L,\varphi)}(r) \equiv \mathrm{det}(\widehat{\Psi}_1(r)\ \widehat{\Psi}_2(r)\ \widehat{\Psi}_3(r))$$

We need solutions of 2nd order differential equation

$$\mathcal{M}_J^{(S,L,\varphi)} \Psi_I(r) = 0$$

$\Rightarrow$ We need three independent solutions: Ψ_1, Ψ_2, Ψ_3

Let us consider 3×3 functions $\mathcal{G}_J(r)$ and $\hat{\mathcal{G}}_J(r)$, obeying

- $\mathcal{M}_J^{(S,L,\varphi)} \mathcal{G}_J(r) = 0$
- $\widehat{\mathcal{M}}_J^{(S,L,\varphi)} \hat{\mathcal{G}}_J(r) = 0$

We may choose:

- $\hat{\mathcal{G}}_J(r) = \begin{pmatrix} 2J r^{2J-1} & L\left[(J+1)\xi - J\right] r^{2J+1} & 0 \\ L r^{2J-1} & 2J\left[(J+1)\xi - (J+2)\right] r^{2J+1} & 0 \\ 0 & 0 & r^{2J} \end{pmatrix}$
- $\mathcal{G}_J(r \to 0) \simeq \hat{\mathcal{G}}_J(r \to 0)$

Asymptotic behavior at $r \to \infty$

- $\mathcal{G}_J$ and $\widehat{\mathcal{G}}_J$ obey (almost) the same equation at $r \to \infty$
- Columns of $\mathcal{G}_J$ are linear combinations of those of $\widehat{\mathcal{G}}_J$

$$\Rightarrow \mathcal{G}_J(r \to \infty) \simeq \widehat{\mathcal{G}}_J(r \to \infty)\, \mathcal{T}_{3\times 3}(\xi)$$

$\mathcal{T}_{3\times 3}$: 3×3 "transfer matrix"

Is $\det\mathcal{T}_{3\times 3}$ ξ-independent?

$$\left[\frac{\mathrm{Det}\mathcal{M}_J^{(S,L,\varphi)}}{\mathrm{Det}\widehat{\mathcal{M}}_J^{(S,L,\varphi)}}\right] = \left[\frac{\det\mathcal{G}_J(r \to \infty)}{\det\widehat{\mathcal{G}}_J(r \to \infty)}\right] = \det\mathcal{T}_{3\times 3}$$

Comment:

Calculation of $\mathcal{T}_{3\times 3}$ is numerically challenging

We found that $\mathcal{M}_J^{(S,L,\varphi)}\Psi(r) = 0$ holds for Ψ, where:

$$\Psi \equiv \begin{pmatrix} \partial_r \chi \\ \frac{L}{r}\chi \\ g\bar{\phi}\chi \end{pmatrix} + \begin{pmatrix} \frac{1}{rg^2\bar{\phi}^2}\eta \\ \frac{1}{Lr^2g^2\bar{\phi}^2}\partial_r(r^2\eta) \\ 0 \end{pmatrix} + \begin{pmatrix} -2\frac{\bar{\phi}'}{g^2\bar{\phi}^3}\zeta \\ 0 \\ \frac{1}{g\bar{\phi}}\zeta \end{pmatrix}$$

Requirements on the three functions: $\chi(r)$, $\eta(r)$, $\zeta(r)$

$$\Delta_J \chi = \frac{2\bar{\phi}'}{rg^2\bar{\phi}^3}\eta + \frac{2}{r^3}\partial_r\left(\frac{r^3\bar{\phi}'}{g^2\bar{\phi}^3}\zeta\right) - \xi\zeta$$

$$(\Delta_J - g^2\bar{\phi}^2)\eta - \frac{2\bar{\phi}'}{r^2\bar{\phi}}\partial_r\left(r^2\eta\right) = -\frac{2L^2\bar{\phi}'}{r\bar{\phi}}\zeta$$

$$\Delta_J \zeta = 0$$

We can find three independent solutions:

1. $\eta = \zeta = 0$

   $\Rightarrow \chi = r^{2J}$

2. $\zeta = 0$

   $\Rightarrow \eta_2 \equiv f_J^{(\eta)}$ with $(\Delta_J - g^2\bar{\phi}^2)f_J^{(\eta)} - \frac{2\bar{\phi}'}{r^2\bar{\phi}}\partial_r\left(r^2 f_J^{(\eta)}\right) = 0$

   $\Rightarrow \chi = a_2 r^{2J} + \delta\chi_2$

   $\delta\chi_2$: non-homogeneous terms generated by $f_J^{(\eta)}$

3. $\zeta = r^{2J}$

   $\Rightarrow \eta_3 = b_3 f_J^{(\eta)} + \delta\eta_3$

   $\Rightarrow \chi = a_3 r^{2J} + b_3\delta\chi_2 + \delta\chi_3$

Around the bounce

$$\mathcal{D}(r\to 0)\simeq \mathrm{det}\begin{pmatrix} 2Jr^{2J-1} & \frac{1}{8(J+1)}r^{2J+1} & -\frac{1}{4}\xi r^{2J+1} \\ Lr^{2J-1} & \frac{J(J+2)}{L^3}r^{2J+1} & -\frac{J}{2L}\xi r^{2J+1} \\ g\bar{\phi}_C r^{2J} & -\frac{1}{2Jg\bar{\phi}_C}r^{2J} & \frac{1}{g\bar{\phi}_C}r^{2J} \end{pmatrix} \sim O(r^{6J})$$

$$\mathcal{D}(r\to \infty)\simeq \mathrm{det}\begin{pmatrix} 2Jr^{2J-1} & -\frac{r}{L^2}f_J^{(\eta)} & -\frac{(J+1)\xi - J}{4(J+1)}r^{2J+1} \\ Lr^{2J-1} & -\frac{2r}{L^3}f_J^{(\eta)} & -\frac{2J[(J+1)\xi-(J+2)]}{4L(J+1)}r^{2J+1} \\ g\bar{\phi}r^{2J} & -\frac{1}{2m_\phi g\bar{\phi}r}f_J^{(\eta)} & O(\bar{\phi}r^{2J+2}) \end{pmatrix}$$

$\mathcal{D}(r\to 0)$ and $\mathcal{D}(r\to\infty)$ are both proportional to $(J+1)\xi+J$

$\Rightarrow$ The ξ-dependence cancels out

Functional determinant of the fluctuation operator

$$\frac{\text{Det}\mathcal{M}_J^{(S,L,\varphi)}}{\text{Det}\widehat{\mathcal{M}}_J^{(S,L,\varphi)}} = \frac{|\lambda| J \bar{\phi}_C^2 f_J^{(\eta)}(r_\infty)}{8(J+1) r_\infty^{2J-2}}$$

$$(\Delta_J - g^2\bar{\phi}^2) f_J^{(\eta)} - \frac{2\bar{\phi}'}{r^2\bar{\phi}} \partial_r \left(r^2 f_J^{(\eta)} \right) = 0$$

A special care is needed for $J = 0$

- There is no L nor T mode
- Zero-mode exists in association with gauge symmetry

$$\varphi^{(\text{gauge})} = \mathcal{N}_{\text{gauge}} \bar{\phi} \quad \text{with} \quad \mathcal{N}_{\text{gauge}}^{-2} = \frac{1}{2\pi} \int d^4x \bar{\phi}^2$$

$$H_{\text{bounce}} = \frac{1}{\sqrt{2}} e^{i\theta} \bar{\phi} \Rightarrow \int \mathcal{D}\varphi^{(\text{gauge})} \to \int \frac{d\theta}{\mathcal{N}_{\text{gauge}}} = \frac{\mathcal{V}_{U(1)}}{\mathcal{N}_{\text{gauge}}}$$

Fluctuation operator for $J = 0$ (only S- and NG modes)

$$\mathcal{M}_{J=0}^{(S,\varphi)} = \begin{pmatrix} -\Delta_0 + \frac{3}{r^2} + g^2\bar{\phi}^2 & g\bar{\phi}' - g\bar{\phi}\partial_r \\ 2g\bar{\phi}' + g\bar{\phi}\partial_r + \frac{3}{r}g\bar{\phi} & -\Delta_0 - |\lambda|\bar{\phi}^2 \end{pmatrix} + \left(1 - \frac{1}{\xi}\right)\begin{pmatrix} \Delta_0 - \frac{3}{r^2} & 0 \\ 0 & 0 \end{pmatrix}$$

Two independent solutions of $\mathcal{M}_{J=0}^{(S,\varphi)}\Psi = 0$

- $\Psi_1(r) = \begin{pmatrix} 0 \\ g\bar{\phi} \end{pmatrix}$ $\Leftarrow$ Zero-mode
- $\Psi_2(r) = \begin{pmatrix} -\frac{1}{4}\xi r \\ -\frac{1}{8}\xi r^2 g\bar{\phi} \end{pmatrix}$

Functional determinant for $J=0$

$$\left[\frac{\text{Det}\mathcal{M}_0^{(S,\varphi)}}{\text{Det}\widehat{\mathcal{M}}_0^{(S,\varphi)}}\right]^{-1/2} \to \frac{\mathcal{V}_{U(1)}}{\mathcal{N}_{\text{gauge}}}\left[\frac{\text{Det}'\mathcal{M}_0^{(S,\varphi)}}{\text{Det}\widehat{\mathcal{M}}_0^{(S,\varphi)}}\right]^{-1/2}$$

Calculation of $\text{Det}'\mathcal{M}_0^{(S,\varphi)}$ (rough sketch)

1. Find a solution of $\left[\text{Det}\mathcal{M}_0^{(S,\varphi)} + \text{diag}(\nu,\nu)\right]\Psi^{(\nu)} = 0$

$$\begin{aligned}\Psi^{(\nu)}(r_\infty) &= \Psi_1(r_\infty) + \nu\left[c\Psi_2 + \begin{pmatrix} 0 \\ \dfrac{1}{2\pi\mathcal{N}_{\text{gauge}}^2\bar{\phi}r^2}\end{pmatrix}\right]_{r_\infty} + O(\nu^2) \\ &\equiv \nu\check{\Psi}_1(r_\infty) + O(\nu^2)\end{aligned}$$

2. Use the Gelfand-Yaglom theorem

$$\text{Det}'\mathcal{M}_0^{(S,\varphi)} \sim \lim_{\nu\to 0}\frac{\text{det}(\Psi^{(\nu)}\ \ \Psi_2)_{r_\infty}}{\nu} = \text{det}(\check{\Psi}_1\ \ \Psi_2)_{r_\infty}$$

Functional determinant for $J = 0$

$$\left[\frac{\mathrm{Det}\mathcal{M}_0^{(S,\varphi)}}{\mathrm{Det}\widehat{\mathcal{M}}_0^{(S,\varphi)}}\right]^{-1/2} \to \frac{\mathcal{V}_{U(1)}}{\mathcal{N}_{\mathrm{gauge}}}\left[\frac{\mathrm{Det}'\mathcal{M}_0^{(S,\varphi)}}{\mathrm{Det}\widehat{\mathcal{M}}_0^{(S,\varphi)}}\right]^{-1/2} = \mathcal{V}_{U(1)}\left(\frac{16\pi}{|\lambda|}\right)^{1/2}$$

Gauge and NG contribution to the Prefactor

$$\mathcal{A}^{(\mathrm{Gauge,NG})} = \mathcal{V}_{U(1)}\left(\frac{16\pi}{|\lambda|}\right)^{1/2}\prod_{J\geq 1/2} \left[\frac{|\lambda| J\bar{\phi}_C^2 f_J^{(\eta)}(r_\infty)}{8(J+1)r_\infty^{2J-2}}\right]^{-(2J+1)^2/2}\left[\frac{f_J^{(T)}(r_\infty)}{r_\infty^{2J}}\right]^{-(2J+1)^2}$$

$$(\Delta_J - g^2\bar{\phi}^2)f_J^{(\eta)} - \frac{2\bar{\phi}'}{r^2\bar{\phi}}\partial_r\left(r^2 f_J^{(\eta)}\right) = 0$$

$$(\Delta_J - g^2\bar{\phi}^2)f_J^{(T)} = 0$$

6. Total Decay Rate

Decay rate:

$$\gamma = \int d \ln \bar{\phi}_C \left[I^{(h)} I^{(W,Z,\mathsf{NG})} I^{(t)} e^{-\mathcal{S}_{\mathsf{C.T.}}} e^{-\mathcal{B}} \right]$$

Contributions of various fields

$$I^{(h)} = \frac{\mathcal{B}^2}{4\pi^2} \left(\frac{16\pi}{|\lambda|} \right)^{1/2} \left[\frac{\check{f}^{(h)}_{1/2}(r_\infty)}{r_\infty} \right]^{-2} \prod_{J \geq 1} \left[\frac{f^{(h)}_J(r_\infty)}{r_\infty^{2J}} \right]^{-(2J+1)^2/2}$$

$$I^{(V,Z,\mathsf{NG})} = \mathcal{V}_{SU(2)} \left(\frac{16\pi}{|\lambda|} \right)^{3/2} \prod_{V=W,Z} \prod_{J \geq 1/2} \left[\frac{|\lambda| J \bar{\phi}_C^2 f_J^{(\eta^V)}(r_\infty)}{8(J+1) r_\infty^{2J-2}} \right]^{-(2J+1)^2/2} \left[\frac{f_J^{(T^V)}(r_\infty)}{r_\infty^{2J}} \right]^{-(2J+1)^2}$$

$I^{(t)}$: top quark contribution

So far, we have calculated γ at one-loop level

$\Rightarrow$ Leading $\ln\mu$ depencence cancels out

Scales in the calculation: μ and $\bar{\phi}_C$

$\Rightarrow$ Higher-loop effects should introduce terms proportional to $\ln^p(\bar{\phi}_C/\mu)$, which are not included in our result

$\Rightarrow$ We choose $\mu \sim \bar{\phi}_C$ to minimize such contributions

Decay rate:

$$\gamma = \int d\ln\bar{\phi}_C \left[I^{(h)} I^{(W,Z,\mathsf{NG})} I^{(t)} e^{-\mathcal{S}_{\mathsf{C.T.}}} e^{-\mathcal{B}} \right]_{\mu\sim\bar{\phi}_C}$$

Proper choice of μ is important

- If we use a fixed μ, $I^{(h)}$ is approximately proportional to $\bar{\phi}_C^4$ and $\bar{\phi}_C$ integration does not converge

Due to the RG effect, λ is minimized for $\mu \sim O(10^{17})$ GeV

- $\mathcal{B}$ becomes enhanced for $\mu \gg O(10^{17})$ GeV
- The integrand is significantly suppressed

We use 3-loop RGE for λ to calculate the RG running

- We checked that γ is insensitive to the upper bound of the integration, if $\bar{\phi}_C^{(\mathrm{max})} \gg O(10^{18})$ GeV

7. Numerical Results

Inputs (from PDG):

- $m_h = 125.09 \pm 0.24$ GeV
- $m_t^{(\text{pole})} = 173.5 \pm 1.1$ GeV
- $\alpha_s(m_Z) = 0.1181 \pm 0.0011$

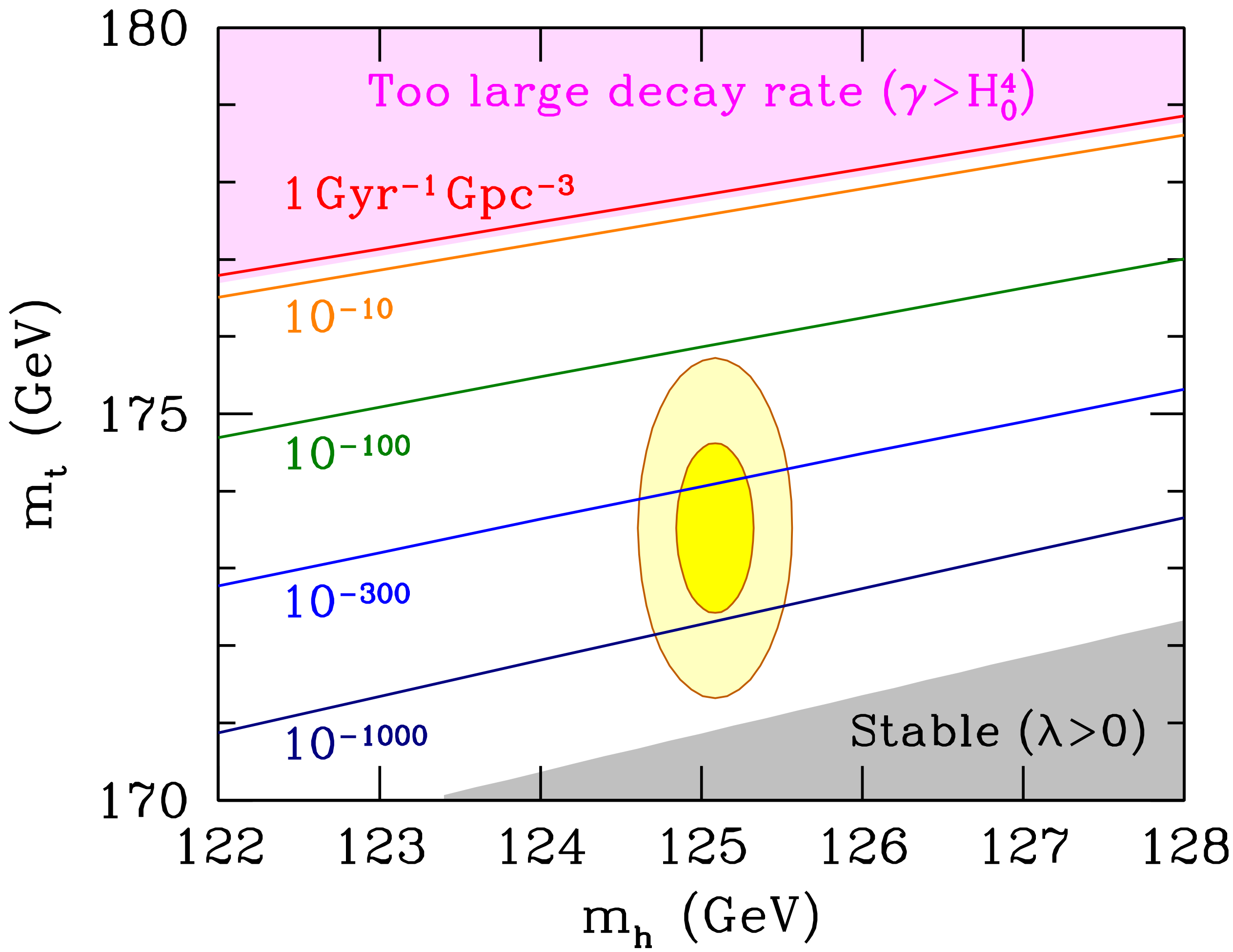
Decay rate of the EW vacuum (taking $\mu = \bar{\phi}_C$)

- $\log_{10}[\,\gamma\ (\text{Gyr}^{-1}\text{Gpc}^{-3})\,] \simeq -554^{+38+270+137}_{-41-817-204}$

For the present universe:

- Cosmic age: $t_0 \simeq 13.6$ Gyr
- Horizon scale: $H_0^{-1} \simeq 4.5$ Gpc

Decay rate per unit volume as a function of m_h and m_t



8. Summary

I have discussed the decay rate of the EW vacuum

- Effects of gauge, Higgs (including NG), and top quark are taken into account
- Path integrals over the conformal and gague zero-modes are properly performed

Numerical result

$$\log_{10}[\,\gamma\ (\mathsf{Gyr}^{-1}\mathsf{Gpc}^{-3})\,] \simeq -554^{+38+270+137}_{-41-817-204}$$

$\Rightarrow$ Uncertainty due to m_t (and α_s) is quite large

The decay rate is extremely small: $\gamma \ll H_0^4$

$\Rightarrow$ We can safely live in the EW vacuum

Back Ups

4D spherical harmonics

$$\partial_\mu \partial_\mu = \partial_r^2 + \frac{3}{r}\partial_r - \frac{L_{\mu\nu}L_{\mu\nu}}{r^2} \text{ with } L_{\mu\nu} = \frac{-i}{\sqrt{2}}(x_\mu \partial_\nu - x_\nu \partial_\mu)$$

Rotation group of 4D Euclidean space: $SU(2)_A \times SU(2)_B$

⇒ States are labeled by the eigenvalues of A^2, B^2, A_3, B_3

$$A_i = \frac{1}{\sqrt{2}}\left(\frac{1}{2}\epsilon_{ijk}L_{ij} - L_{0i}\right) \qquad B_i = \frac{1}{\sqrt{2}}\left(\frac{1}{2}\epsilon_{ijk}L_{ij} + L_{0i}\right)$$

⇒ $A^2 = B^2$ for the 4D spherical harmonics

4D spherical harmonics: eigenstate of $A^2 = B^2$, A_3, B_3

$\mathcal{Y}_{J,m_A,m_B}(\hat{\mathbf{r}}) = \langle \hat{\mathbf{r}} | (J, m_A, m_B) \rangle$, with $J = 0$, $\frac{1}{2}$, 1, $\cdots$

⇒ $L_{\mu\nu}L_{\mu\nu}\mathcal{Y}_{J,m_A,m_B} = 4J(J+1)\mathcal{Y}_{J,m_A,m_B} \equiv L^2 \mathcal{Y}_{J,m_A,m_B}$