# Fisher Information Metric from Holography and its Application to Scrambling

Phys. Rev. Lett 115 (2015) (hep-th/1507.07555)
with Numasawa, Shiba, Takayanagi, Watanabe
"Distance between Quantum States and Gauge-Gravity Duality"

JHEP 1609 (2016) 002 (hep-th/1607.01467)
"Butterflies from Information Metric"

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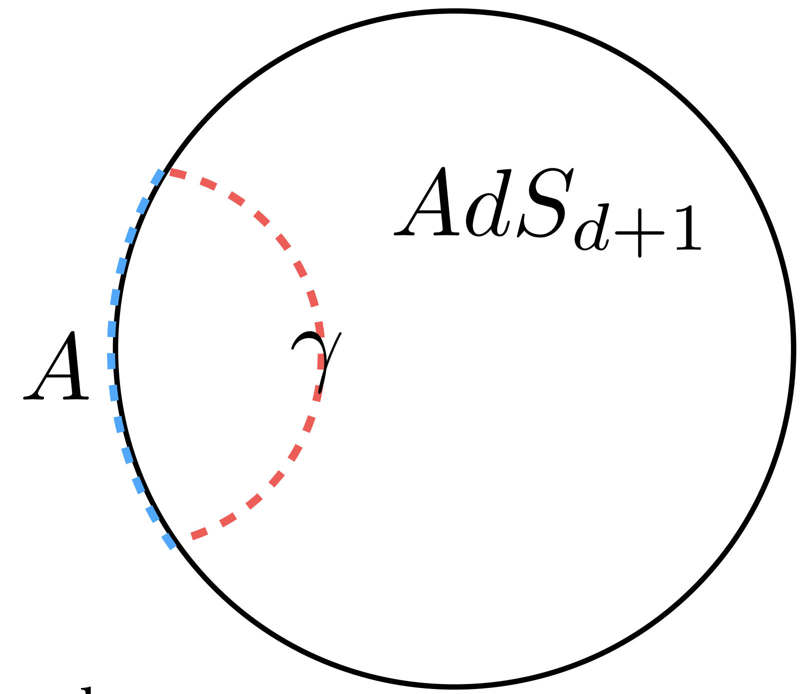
Motivation

## Quantum information and AdS/CFT




- AdS/CFT [Maldacena]

- Ryu-Takayanagi formula [Ryu, Takayanagi]

  The formula relates bulk minimal surface to entanglement entropy of boundary subregion.

  - First law of Entanglement Entropy = Linearized Einstein equation
    [Nozaki, Numasawa, Takayanagi][Lashkari, McDermott, Raamsdonk]
  - Renyi entropy, Relative entropy [Dong][Lashkari, Raamsdonk]
    [Jefferis, Lewkowycz, Maldacena, Suh]
  - Averaged Null energy condition = Monotonicity of relative entropy
    [Lashkari, Rabideau, Sabella-Garnier, Raamsdonk]

It is natural to expect Quantum information theoretic quantities have holographic duals !

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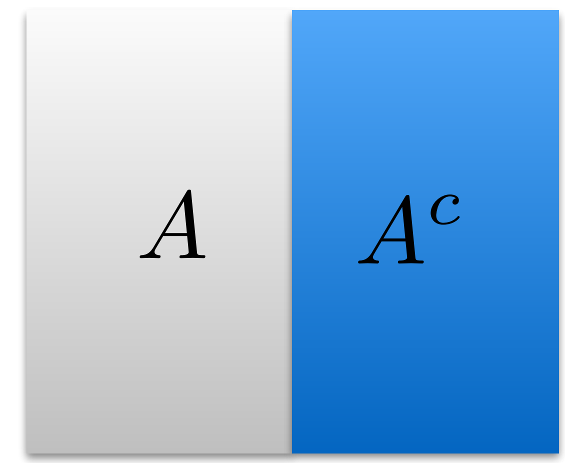
# 1. Fisher Information

# Quantum Information Theory : Entanglement Entropy

## Entanglement entropy

$$S_A := -tr_{H_A}\rho_A log \rho_A$$




Measure of entanglement between subregion and outside.

- Non local quantity.

- Order parameter of quantum phase transitions.

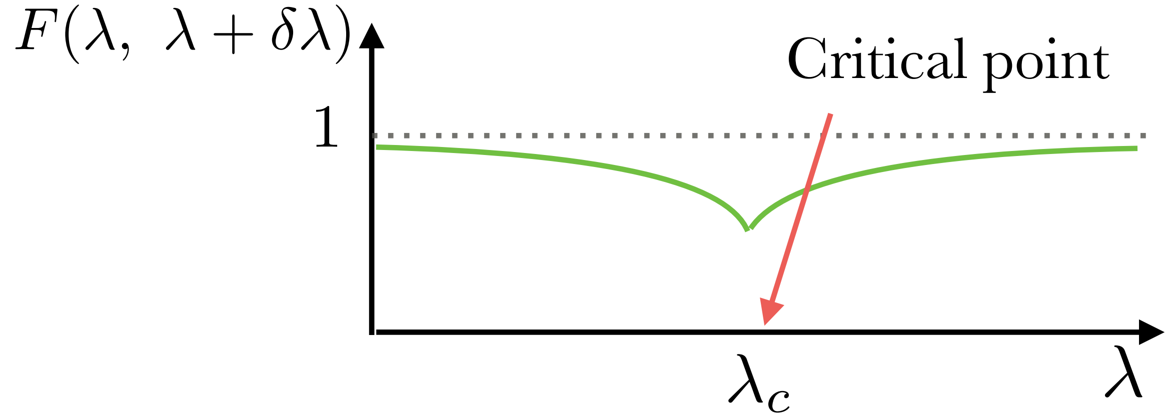
  Topological entanglement entropy characterizes topological phases of quantum states. [Kitaev, Preskill][Levin, Wen]

## Fisher information metric

$$\underbrace{F(\lambda, \lambda + \delta\lambda)}_{\text{Fidelity}} = |\langle \Psi_\lambda | \Psi_{\lambda+\delta\lambda} \rangle|^2 = 1 - 2\underbrace{G_{\lambda\lambda}}_{\text{Fisher Information Metric}} \cdot (\delta\lambda)^2 + \mathcal{O}(\delta\lambda^3)$$

- Order parameter for quantum phase transitions. Fidelity of vacuum states $|\Psi_\lambda\rangle$ of $H_0 + \lambda V$ drops suddenly at critical points.

[Quan, Song, Liu, Zanardi, Sun] [Zanardi, Paunkovic]



## Estimation theory

Reciprocal of Fisher Information gives lower bound of variance of unbiased estimator of parameter λ. (Cramer-Rao bound)

For any linear operator with $\langle \Psi_\lambda | \hat{\lambda} | \Psi_\lambda \rangle = \lambda$,

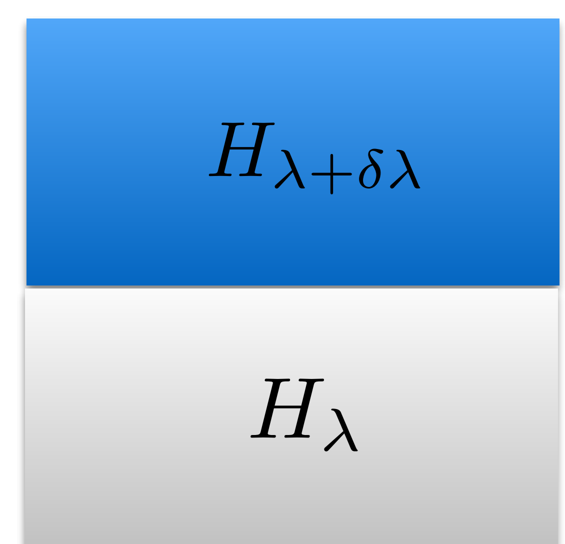
$$\langle \Psi_\lambda | (\hat{\lambda} - \lambda)^2 | \Psi_\lambda \rangle \geq \frac{1}{G_{\lambda\lambda}} \quad \text{holds.}$$

## Vacuum states

Fisher information metric for vacuum states can be obtained from two point functions.

$$H_\lambda = H_0 + \lambda \cdot V$$

$$G_{\lambda\lambda} = \frac{1}{2} \langle \int_\epsilon^\infty dt_1 V(t_1) \int_{-\infty}^{-\epsilon} V(t_2) \rangle_\lambda$$



# 2. Gravity dual of Fisher Information Metric

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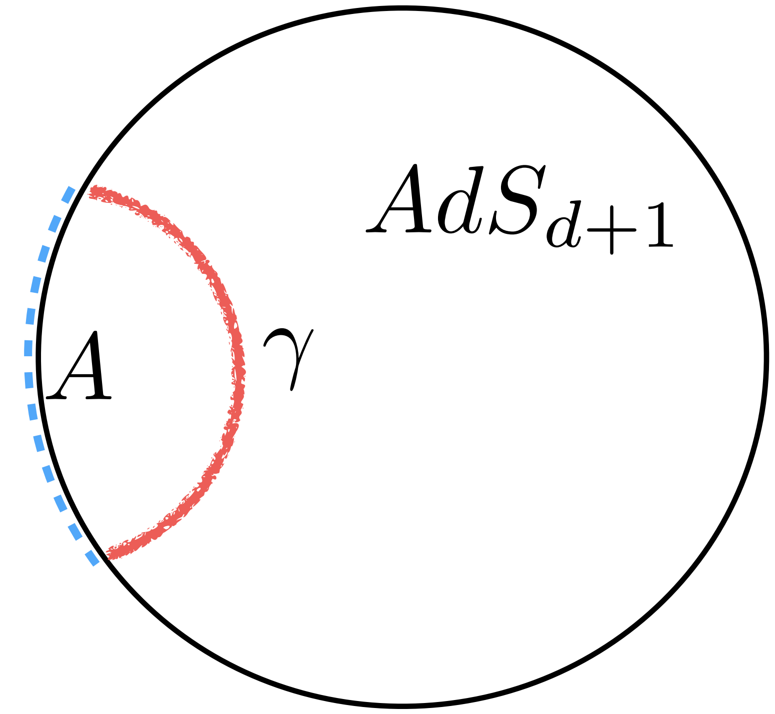
# Gravity dual of Entanglement entropy: Ryu-Takayanagi formula

Entanglement entropy of region A of dual Euclidean CFT of Einstein gravity is given by

$$S_A = \frac{Area(\gamma)}{4G_N}$$




where $\gamma$ is the minimal area surface whose edge coincides with that of A.

[Ryu, Takayanagi]

⇨

- Expression of E.E in terms of geometric object in the bulk.
- How about extremal volume surface in the bulk?
- How about Fisher Information metric?

# Holographic dual of Fisher Information metric

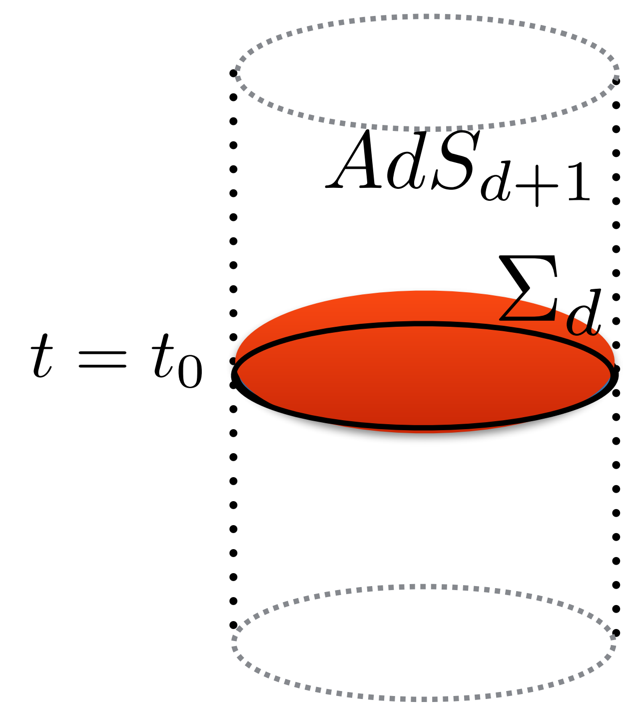
[M.M, Numasawa, Shiba, Takayanagi, Watanabe ]

## Proposal

Fisher information metric of states with Hamiltonian $H_{CFT} + \lambda \cdot V$

$$|\langle \Psi_{\lambda+\delta\lambda}(t_0)|\Psi_\lambda(t_0)\rangle| = 1 - G_{\lambda\lambda}(\delta\lambda)^2 + \mathcal{O}((\delta\lambda)^3)$$

for marginal deformation V is given by the volume

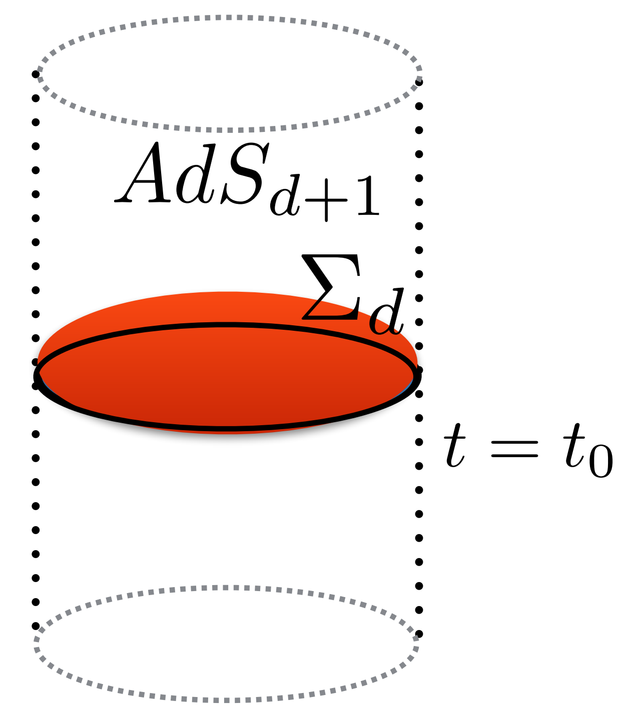



$$\boxed{G_{\lambda\lambda} = \frac{n_d}{L^d_{AdS_{d+1}}} \int_{\Sigma_d} \sqrt{g}}$$

$\Sigma_d$: Codimension 1 extremal volume surface with boundary at $t = t_0$.

$n_d$ : Unspecified, positive, dimensionless $\mathcal{O}(1)$ constant.

- We can compute nontrivial time dependent Fisher information.




- Consistent with nontrivial time dependent example.

$$\boxed{G_{\lambda\lambda} = \frac{n_d}{L^d_{AdS_{d+1}}} \int_{\Sigma_d} \sqrt{g}}$$

- Different from the distance between vacuum and excited states. [Lashkari,Raamsdonk][Lashkari, Lin, Ooguri, Stoica, Raamsdonk]

# Motivation from AdS/Tensor Network proposal

Tensor network: Contraction of tensors can express complicated states.

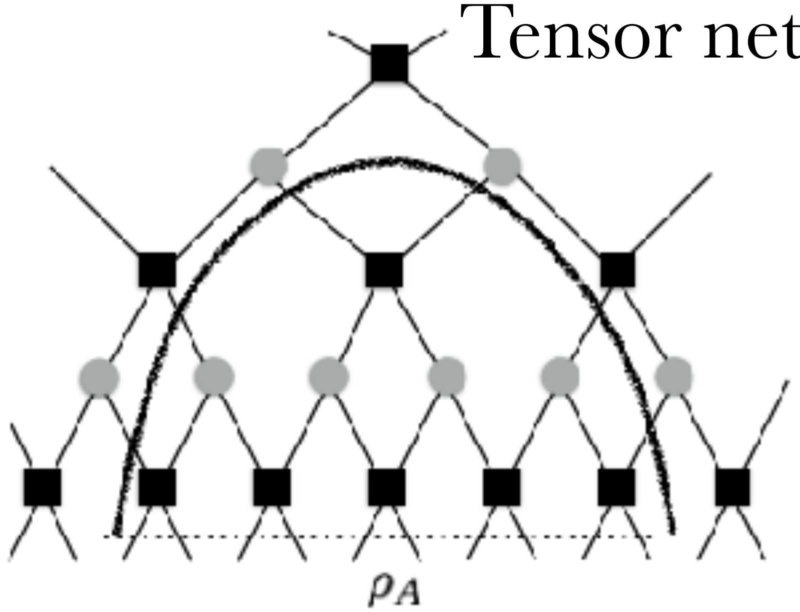
[White][Vidal]

- Geometrical expression of entanglement of state.
- Particular structure of network gives particular class of states.

Entanglement entropy of MERA tensor network state is given by area of minimal surface in tensor network, same as RT formula.

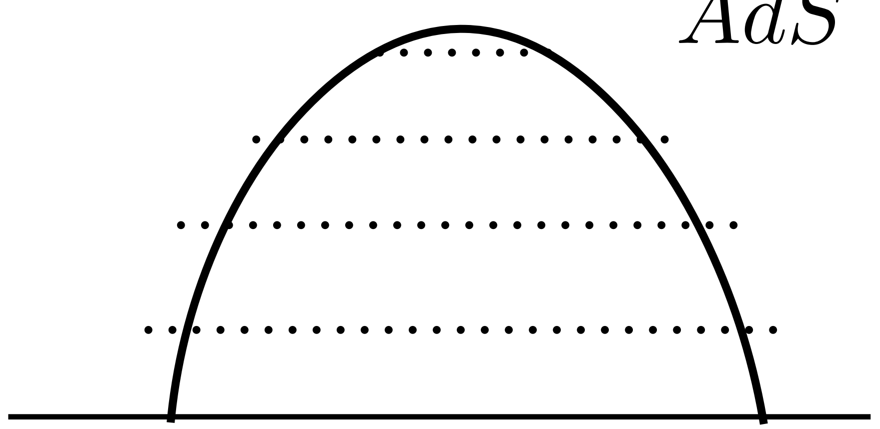
⇨ $\boxed{\text{MERA tensor network = Bulk in AdS/CFT ?}}$ [Swingle]

Tensor network




?

⇔



# Motivation from AdS/Tensor network proposal

Each tensors for MERA of CFT vacuum state are identical.

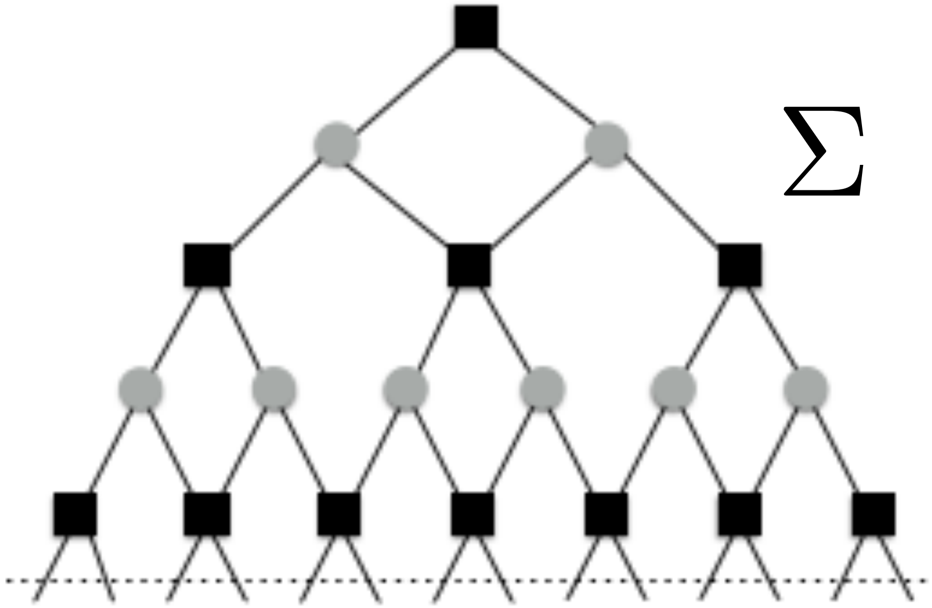
Marginal deformation

$$H_{CFT} + \lambda \cdot V \longrightarrow H_{CFT} + (\lambda + \delta\lambda) \cdot V$$

modifies those tensors uniformly.

One can assume those tensors contribute to Fisher information metric uniformly.

$$\Longrightarrow \boxed{G_{\lambda\lambda} \propto \frac{\mathrm{Vol}(\Sigma)}{L^d_{AdS_{d+1}}}}$$



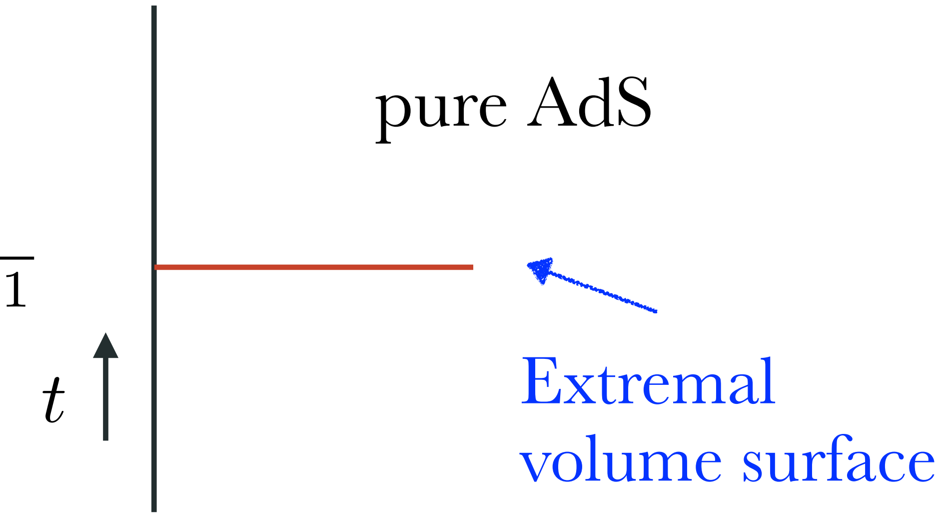

Extremal volume condition is imposed to ensure covariance.

## Examples

<u>AdS</u> Dual to vacuum states $|\Omega_\lambda\rangle$ of

The surface is flat and $G_{\lambda\lambda} = \dfrac{n_d V_{d-1}}{(d-1)\epsilon^{d-1}}$

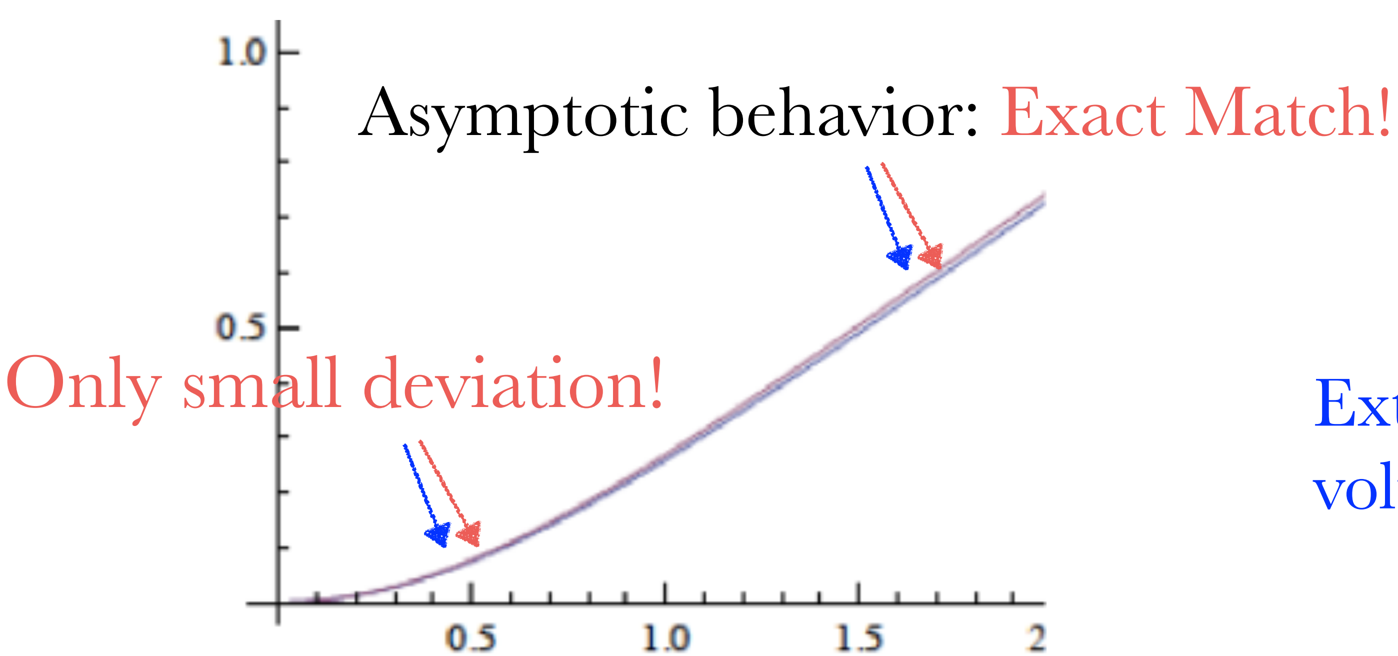
consistent with field theory result.



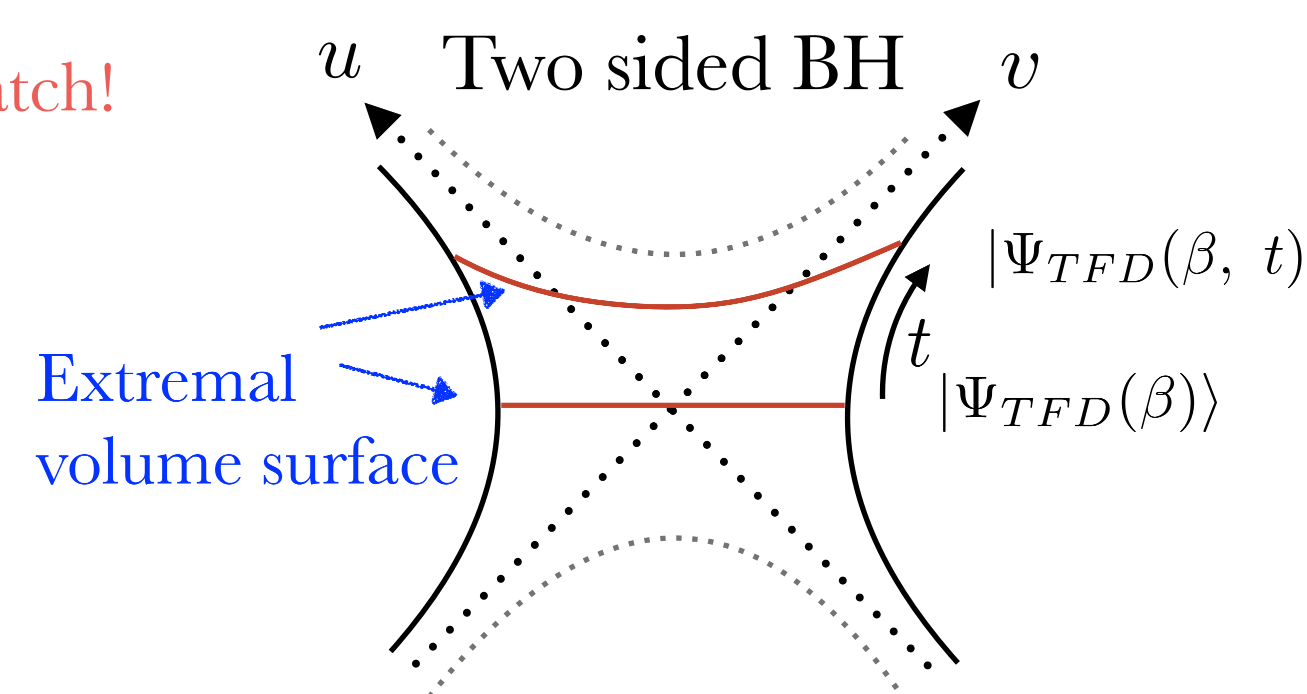

<u>Two sided BH</u>

Dual to TFD states of CFT $|\Psi_{TFD}(\beta, \lambda, t)\rangle$

Almost identical time evolution was confirmed in 2d.




$d = 2$



# 3. Application to scrambling

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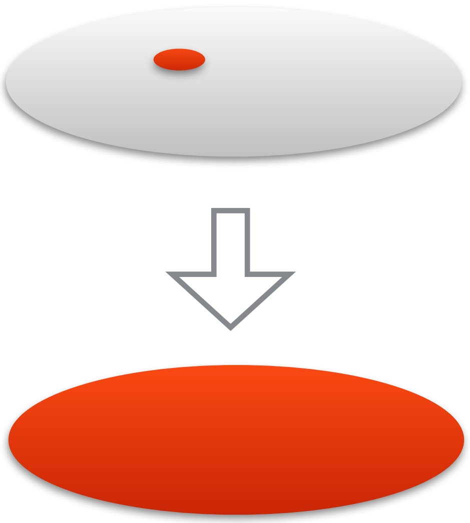
## Scrambling



Local excitation W is added to thermal equilibrium.

⇩

Perturbation is scrambled.

= Any local measurement cannot recover the original excitation.

<u>Motivation from BH physics</u>

The minimal time for excitations to be mixed with BH is argued to be

$$t_s \gtrsim \frac{1}{T_H} \log S_{BH}$$

in order to avoid cloning. [Hayden, Preskill]

<u>Fast scrambling conjecture</u>

[Sekino, Susskind] conjectured this bound holds for any system.

[Maldacena, Shenker, Stanford] gave convenient definition of scrambling and proved this conjecture for large N theories.

## Definition of scrambling

Scrambling of local excitation W is characterized by the growth of commutator

$$\langle [W(t),\ V][W(t),\ V]^\dagger \rangle_\beta$$

for all local Vs.

Assuming V and W commute at first and $\langle V^2\rangle_\beta \langle W^2\rangle_\beta = 1$

## Onset of scrambling

For large N CFT,

Generically λ can depend on both V and W.

$$\langle [W(t),\ V][W(t),\ V]^\dagger \rangle_\beta \propto \frac{1}{N^2} e^{\lambda t}$$

with "Lyapunov exponent" $\lambda \leq \frac{2\pi}{\beta}$

Saturated by holographic CFT

[Maldacena, Shenker, Stanford]

and scrambling time $t_s = \frac{1}{\lambda} \log N^2$

## Relation to classical chaos

For classically chaotic theory, semiclassically

$$\langle [q(t),\ p(0)][q(t),\ p(0)]^\dagger \rangle \ \sim \hbar^2 \left(\frac{\delta q(t)}{\delta q(0)}\right)^2 \ \sim \hbar^2 e^{2\lambda t}$$

[Larkin, Ovchinnikov]

⇨ Quantum mechanical definition of "Lyapunov exponent"?

But this "Lyapunov exponent" is huge: $\dfrac{2\pi k_B T}{\hbar} \sim 10^{11}\mathrm{s}^{-1}$ (For 1K)

- Why is this "Lyapunov exponent" so huge?
- What is the relation to conventional quantum chaos? (such as level spacing statistics)
- Can we understand this "Lyapunov exponent" quantum mechanically?

# Butterfly Effect

Perturbation to initial state:

Inner product between two states with identical Hamiltonian will be conserved.

Perturbation to system:

Inner product between two states with different Hamiltonians $H^{\lambda}$ and $H^{\lambda+\delta\lambda}$ decays rapidly in chaotic systems.

[Peres, Jalabert, Pastawski, Jacquod, Silvestrov, Beenakker, Cerruti, Wisniacki, Cucchietti, Gorin, Prosen, Zurek, Seligman, etc…]

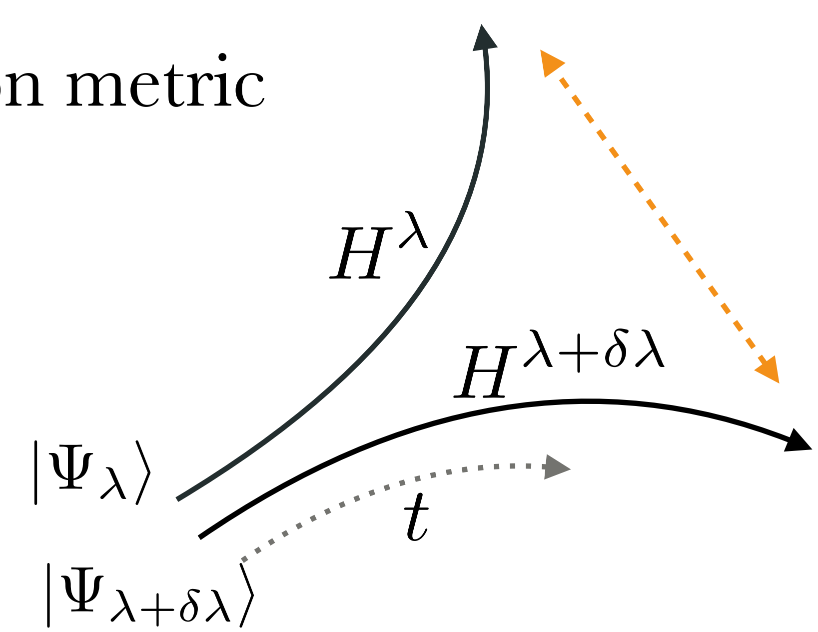
$$|\langle\Psi_{\lambda+\delta\lambda}|e^{iH^{\lambda+\delta\lambda}t}e^{-iH^{\lambda}t}|\Psi_{\lambda}\rangle|$$

$$= 1 - G_{\lambda\lambda}(\delta\lambda)^2 + \mathcal{O}((\delta\lambda)^3)$$

Fisher information metric

- Exponential dump can be observed.

$$|\langle\Psi|e^{iH^{\lambda+\delta\lambda}t}e^{-iH^{\lambda}t}|\Psi\rangle|^2 \propto e^{-\lambda_{Lyapnov}t}$$



## Fisher Information metric of TFD state

Fisher information metric $G^{W}_{\lambda\lambda} = G^{(0)}_{\lambda\lambda} + G^{W:c}_{\lambda\lambda}$ for

$$|\Psi_{TFD}(\beta,\lambda,t_w)\rangle_W = e^{-iH^\lambda t_w} W e^{iH^\lambda t_w} |\Psi_{TFD}(\beta,\lambda)\rangle$$

Operator on the left Hilbert space

This is bounded from below by the Fisher information of

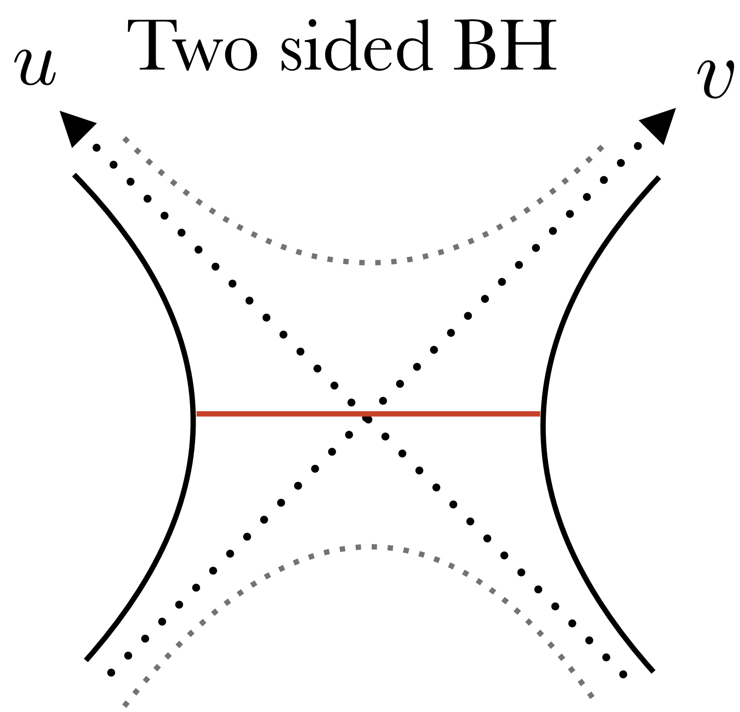
$$e^{-iH^\lambda t_w} W \rho(\beta,\lambda) W e^{iH^\lambda t_w}$$

Explicitly,

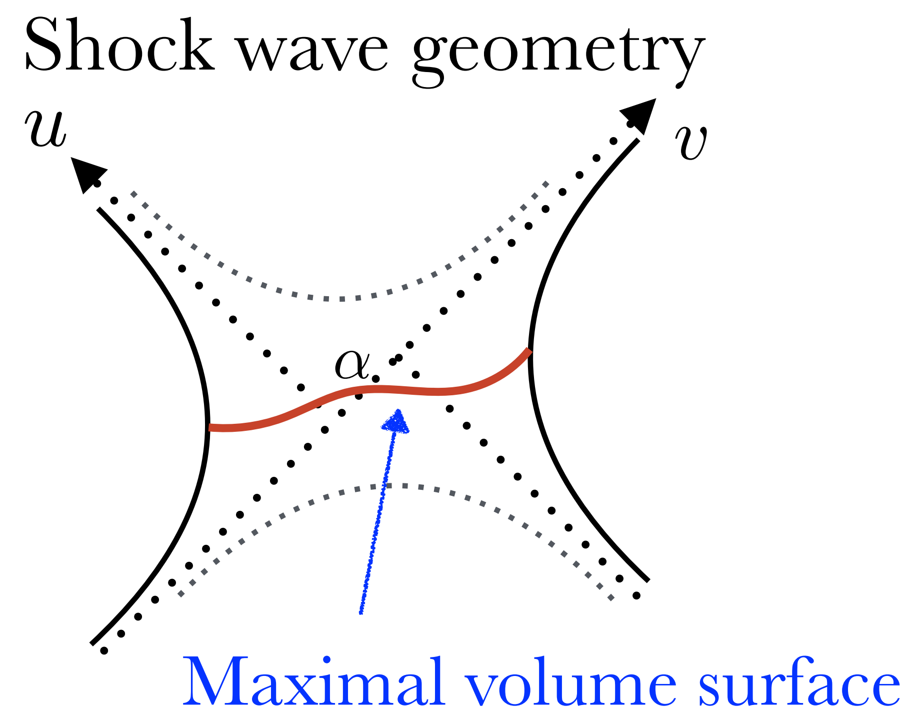
$$\begin{aligned} G^{W:c}_{\lambda\lambda} = \; & Re\Big[\, \frac{1}{2}\int_0^{t_w} dt^1 dt^2 \mathrm{Tr}\Big[\frac{e^{-\beta H}}{Z_W(\beta,\lambda)}\Big[W(-t_w),\; V(-t^1)\Big]\cdot\Big[W(-t_w),\; V(-t^2)\Big]^\dagger\Big] \\ & + \frac{i}{2}\int_0^{\frac{\beta}{2}} dt_E \int_0^{t_w} dt \mathrm{Tr}\Big[\frac{e^{-\beta H}}{Z_W(\beta,\lambda)} e^{Ht_E} V e^{-Ht_E}\Big[\,\Big[W(-t_w),\; V(-t)\Big],\; W(-t_w)\Big]\Big]\,\Big] \\ & - \frac{1}{2}\Big(\int_0^{t_w} dt \mathrm{Tr}\Big[\frac{e^{-\beta H}}{Z_W(\beta,\lambda)}\Big[W(-t_w),\; V(-t)\Big]\; W(-t_w)\Big]\Big)^2 \end{aligned}$$

Exponential growth of Fisher information for all perturbation $\Longleftrightarrow$ Onset of scrambling

# Holographic computation

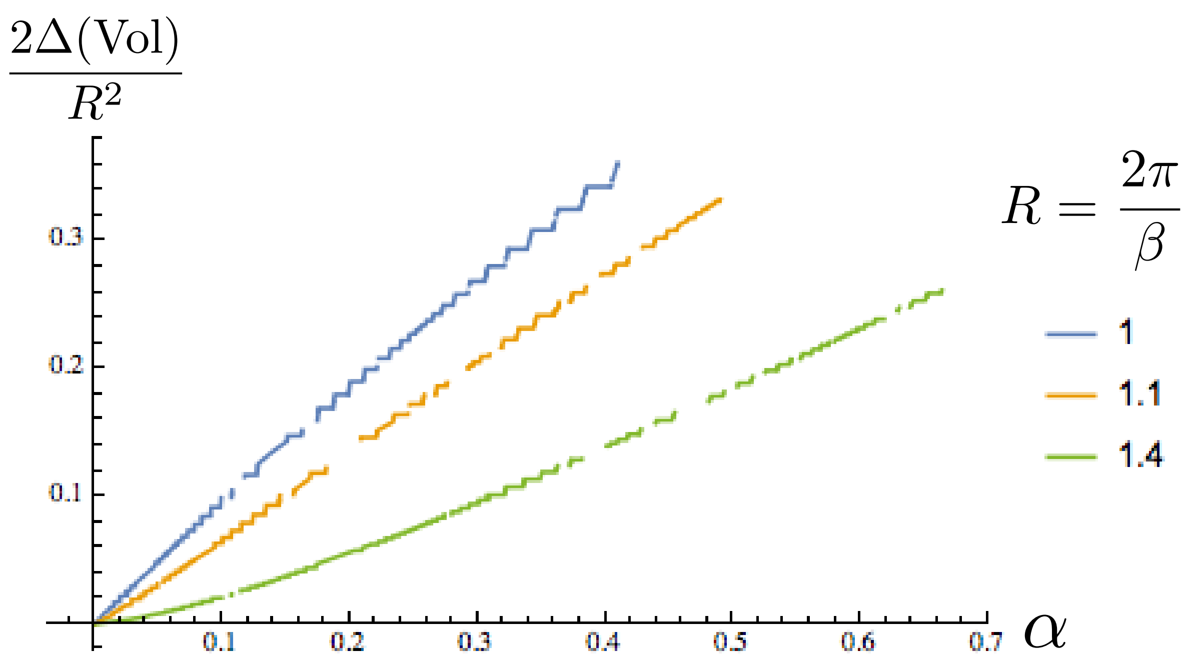



$$G^{(0)}_{\lambda\lambda} = n_d \mathrm{Vol}$$




$$G^{(W:c)}_{\lambda\lambda} = n_d \Delta \mathrm{Vol}$$

Numerical result shows $G^{W:c}_{\lambda\lambda}$ grows proportional to $\alpha = \dfrac{E}{4M} e^{\frac{2\pi}{\beta} t_w}$




$$\boxed{G^{W:c}_{\lambda\lambda} = \frac{f_0}{N^2} e^{\frac{2\pi}{\beta} t_w} + \mathcal{O}(\alpha^2) \qquad f_0 > 0 \quad f_0 \sim \mathcal{O}(N^0)}$$

Exponential growth is confirmed.

# Conclusion

- We proposed volume of extremal volume surface as gravity dual of Fisher information metric.

- We checked our proposal in time dependent Thermofield double state, and confirmed qualitative match.

- Exponential growth of commutator is equivalent to exponential growth of Fisher information, which can be understood as butterfly effect in QM sense.

- Using the gravity dual of Fisher information metric, we confirmed the exponential growth of the information metric for holographic CFT.

# Discussions

Until now, most of the works on fisher information (or overlap), except a few, concluded polynomial growth.

Classical chaoticity & GUE level spacing statistic seems to be insufficient to guarantee the exponential growth of Fisher information.

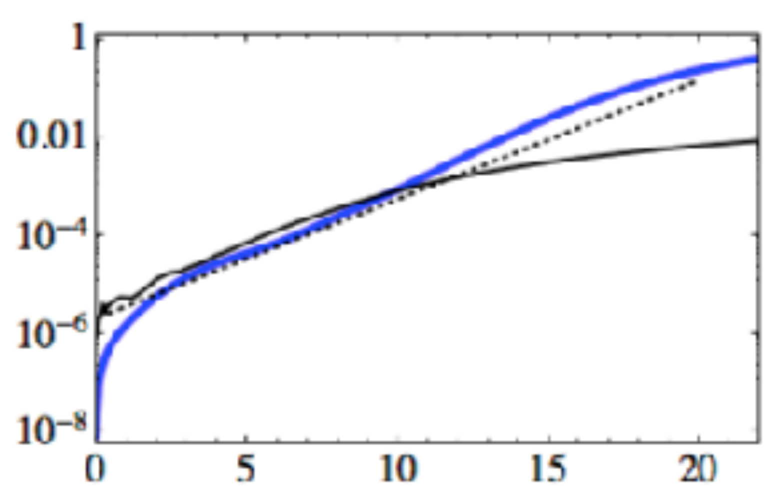
[Fine, Elsayed, Kropf, Wijn(2013)]

Large internal degrees of freedom seems to be neccesary as well.

[Elsayed, Fine(2014)]

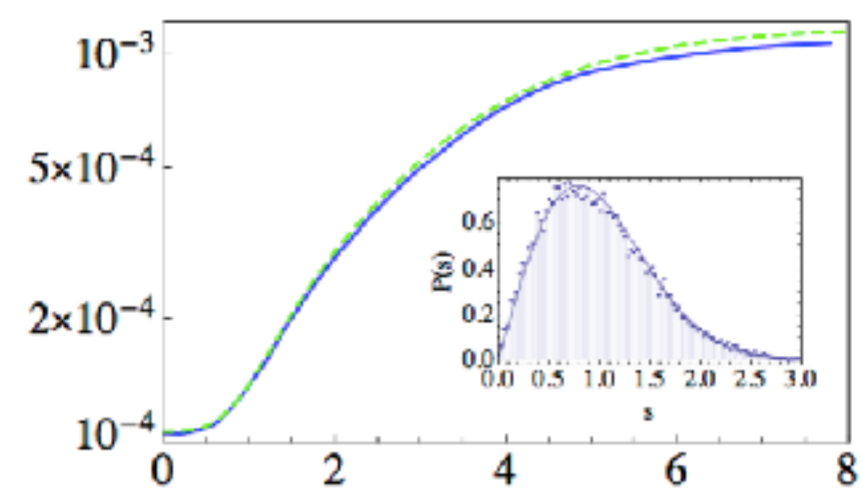
- Precise dependence of "Lyapunov exponent" on internal degrees of freedom should be understood.
- Can we consider more convenient quantity applicable to small N than "Lyapunov exponent"?
- Random matrix theory computation?

Spin 15/2 (Quantum, 6 spins)




[Elsayed, Fine(2014)]

Spin 1/2 (Quantum, 5×5)




[Fine, Elsayed, Kropf, Wijn(2013)]