トップクォークの $\overline{\text{MS}}$ 質量の精密決定

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First, let us see the case of electron mass.

by far the best precision

$$m_{\mathrm{e}} = 5.485\ 799\ 0932\ (29) \times 10^{-4}\ m_{\mathrm{u}} \qquad m_{\mathrm{u}} \equiv \frac{m_{^{12}\mathrm{C}}}{12}$$

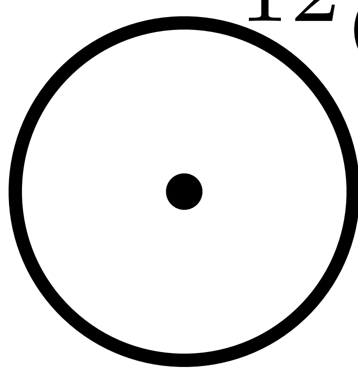
$$= 0.510\ 998\ 928\ (11)\ \mathrm{MeV}$$

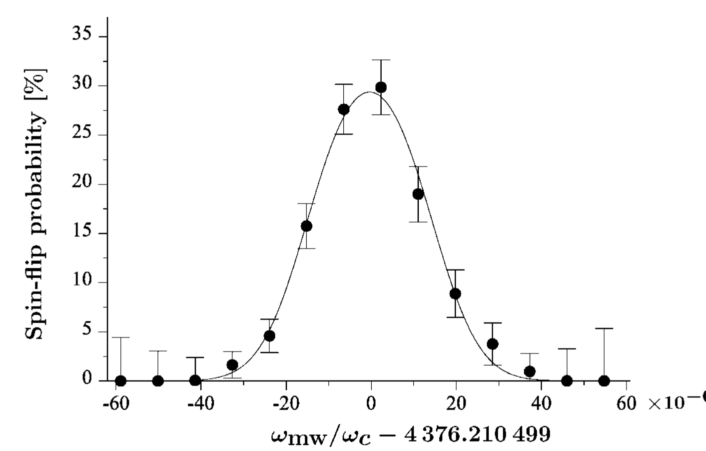
[CODATA2010 (latest version)]

This value is obtained by spectroscopy of hydrogen-like atoms.

[Beier et. al. PRL 88 011603]

$^{12}\mathrm{C}^{5+}$





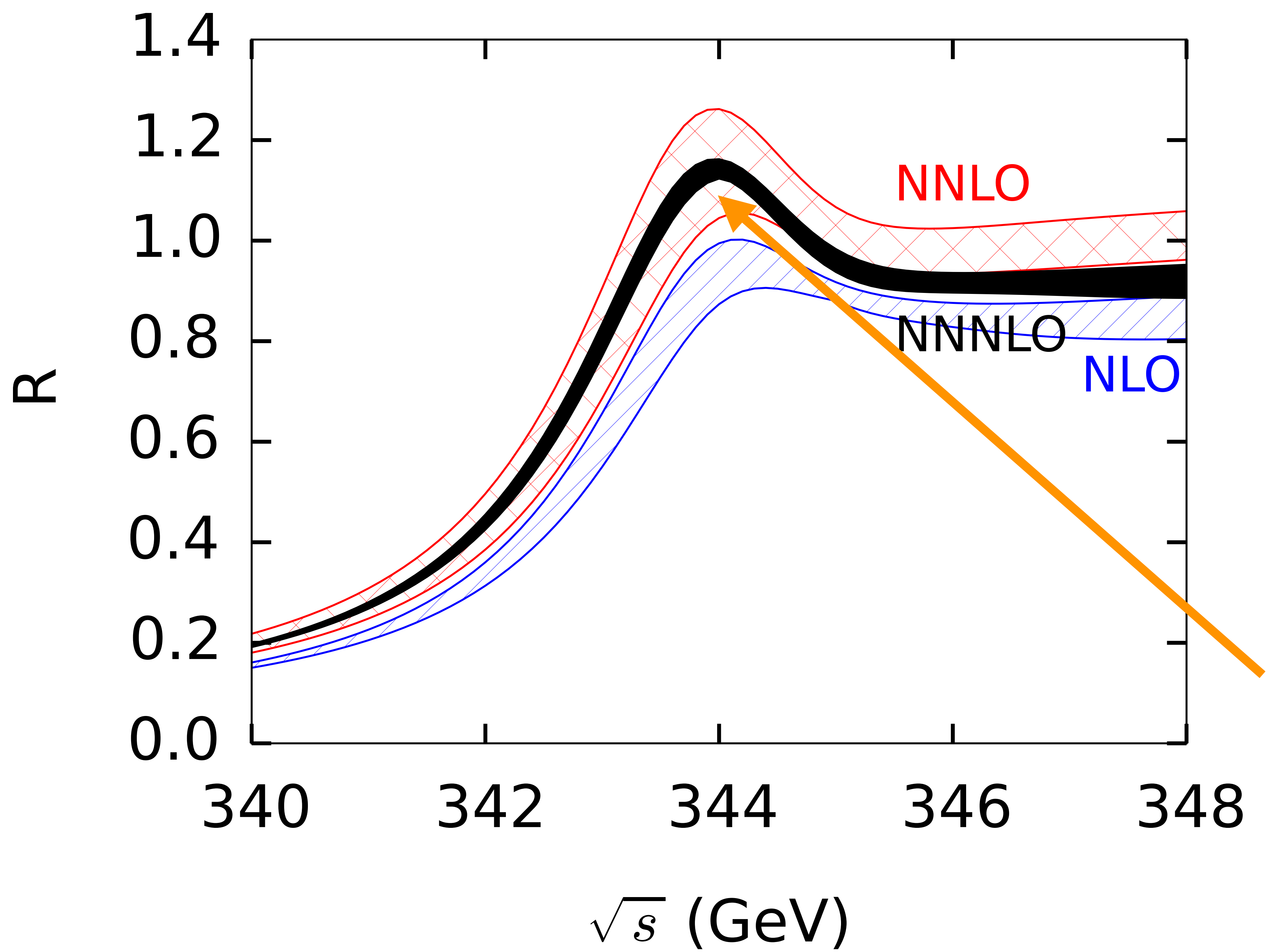
All calculation is based on pole-mass scheme.

Mass of bound state is sum of masses of elements and binding energy.

$$m_{^{12}\mathrm{C}^{5+}} = m_{^{12}\mathrm{C}^{6+}} + m_{\mathrm{e}} + E_{\mathrm{bin}}$$

Let us try similar procedure with top quark.

Top quarks does not form stable bound state, but its signal can be visible at future linear collider.



theoretical prediction

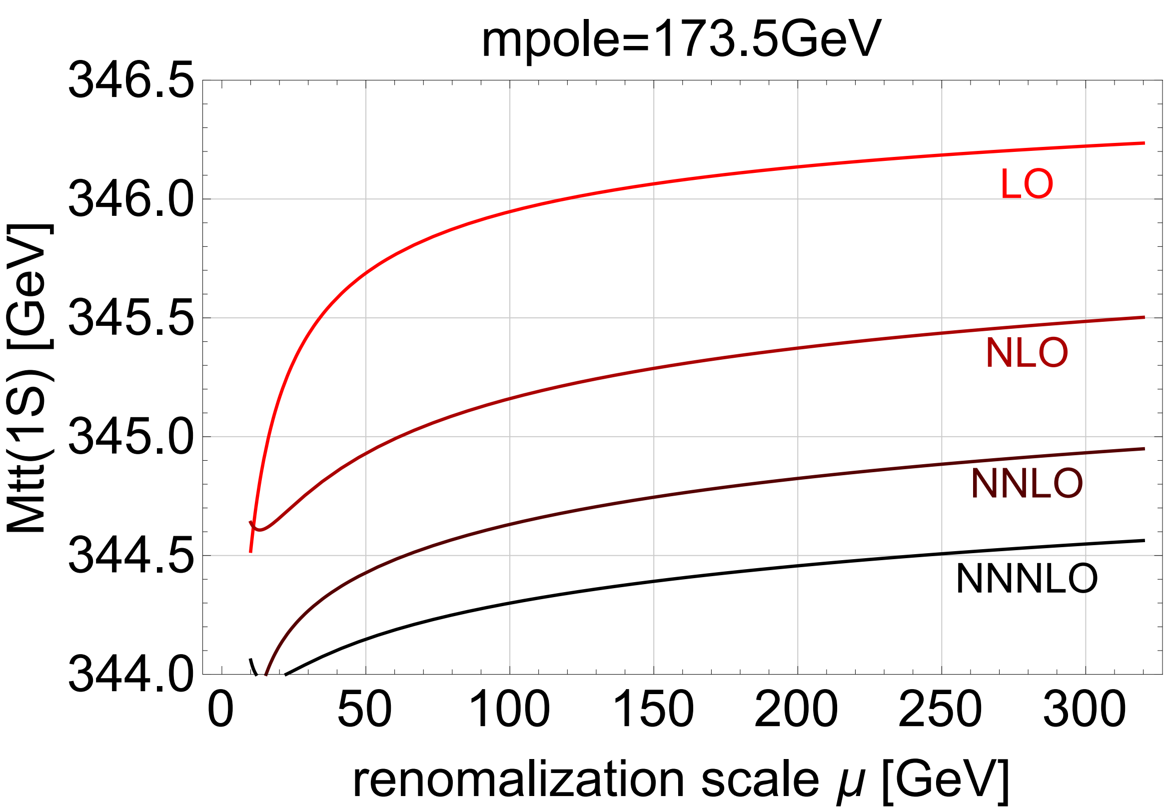
This peak position is not exactly the toponium mass because it consists of bound state + continuum.

[Beneke, Kiyo, Marquard, Penin, Piclum, Steinhauser, 2015]

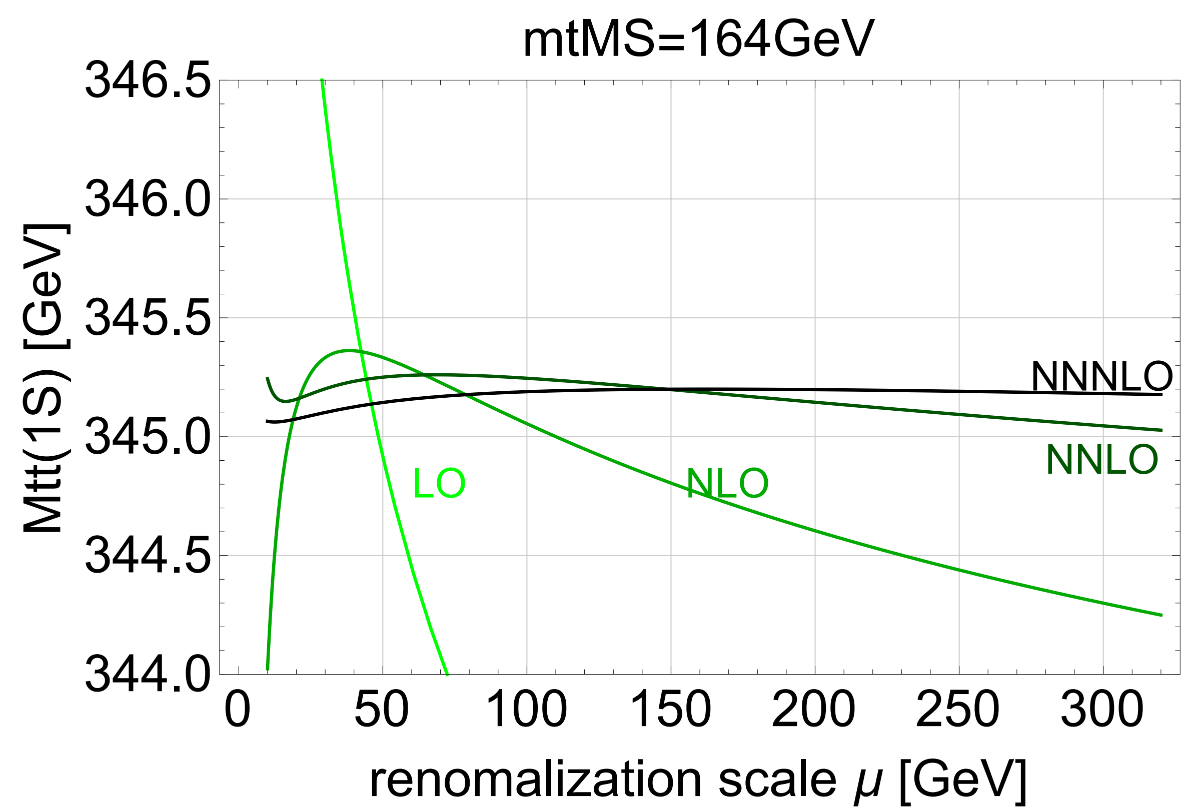
highlight of this talk

- We analyze toponium bound state mass at NNNLO.

- Cancelation of u=1/2 renormalon contribution is crucial.



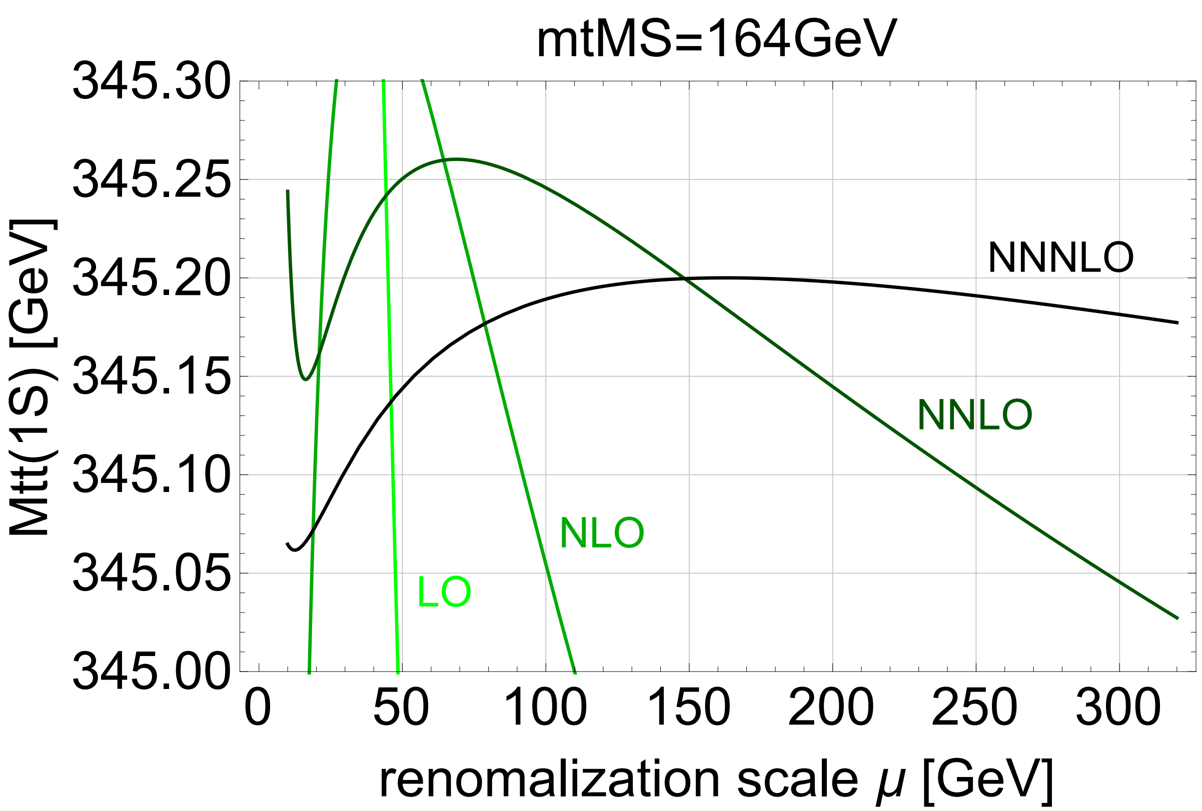
before cancelation

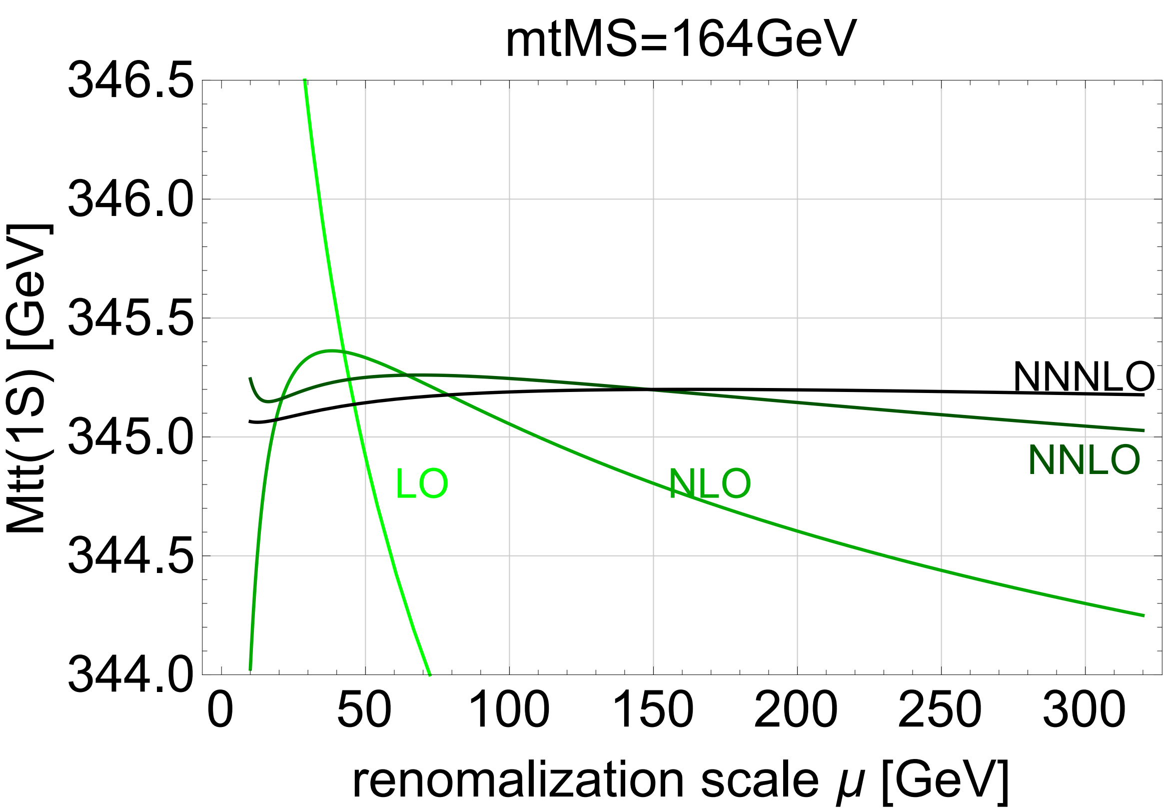


after cancelation

highlight of this talk

- By using proper mass definition, we find that the precision of 30 MeV in the top quark mass is possible in principle.

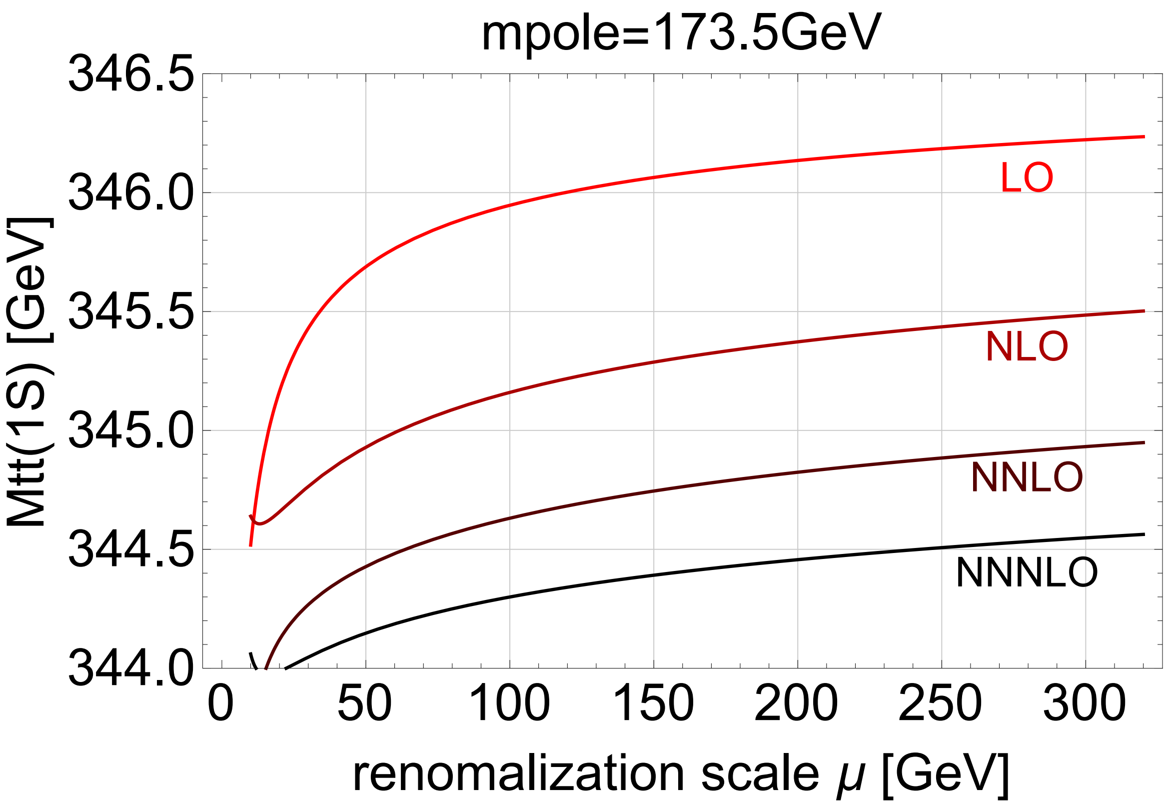




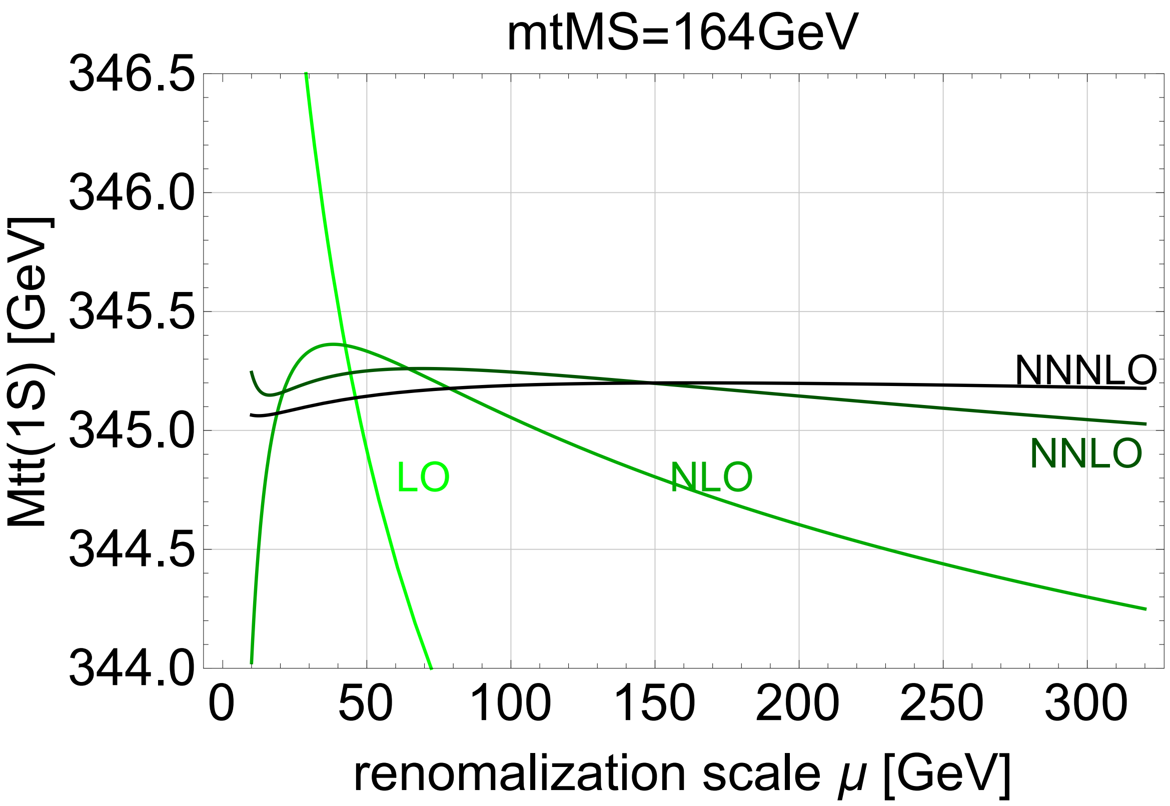
Theoretical uncertainty is estimated by the scale dependence of prediction.

highlight of this talk

- We suggest that the cancelation happens not only in u=1/2 renormalon but also in more general IR contributions.



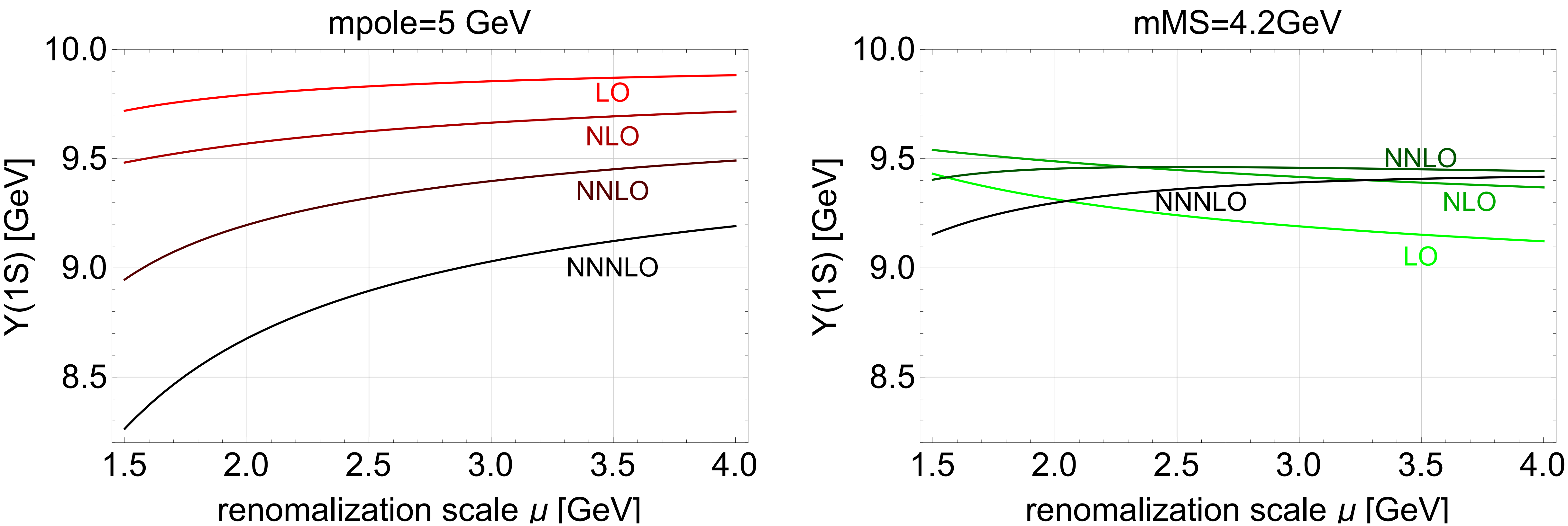
before cancelation



after cancelation

highlight of this talk

- Improvement of perturbative convergence is more drastic in the case of bottomonium, due to rather low energy scale.



before cancelation after cancelation

Plan of my talk

- Current status of perturbative QCD
- Cancelation of u=1/2 renormalon in binding energy and quark self energy.
- Strong IR cancelation in heavy quarkonium
- Summary

Current status of perturbative QCD

calculation of the toponium mass

$$M_{t\bar{t}}(1S) = 2m_t^{\text{pole}} + E_{\text{bin}}$$

The binding energy can be calculated systematically with the use of "potential non-relativistic QCD".

[Pineda, Soto, 1998, Brambilla, Pineda, Soto, Vairo, 2000]

QCD ➔ non-relativistic QCD

"integrating out hard momentum mode"

Higher dimensional terms are determined by matching with QCD.

non-relativistic QCD ➔ potential non-relativistic QCD

"integrating out soft momentum mode"

Dynamical fields are mesonic composite and ultra-soft gluon.

Equation of motion of meson becomes usual Schrodinger equation.

(correction to potential is determined by matching with NRQCD)

All calculation is based on pole-mass scheme.

calculation of the toponium mass

$$M_{t\bar{t}}(1S) = 2m_t^{\text{pole}} + E_{\text{bin}}$$

$$E_{\text{bin}} = -\frac{4}{9} m_t^{\text{pole}} \alpha_S^2 \left(P_0 + P_1 \frac{\alpha_S}{\pi} + P_2 \frac{\alpha_S^2}{\pi^2} + P_3 \frac{\alpha_S^3}{\pi^3} \right) + \mathcal{O}(\alpha_S^6)$$

$$P_0(L_\mu) = 1, \qquad P_1(L_\mu) = \beta_0 \, L_\mu + c_1,$$

$$P_2(L_\mu) = \frac{3}{4}\beta_0^2 \, {L_\mu}^2 + \left(-\frac{1}{2}\beta_0^2 + \frac{1}{4}\beta_1 + \frac{3}{2}\beta_0 c_1 \right) L_\mu + c_2,$$

$$P_3(L_\mu) = \frac{1}{2}\beta_0^3 L_\mu^3 + \left(-\frac{7}{8}\beta_0^3 + \frac{7}{16}\beta_0\beta_1 + \frac{3}{2}\beta_0^2 c_1 \right) L_\mu^2 + \left(\frac{1}{4}\beta_0^3 - \frac{1}{4}\beta_0\beta_1 + \frac{1}{16}\beta_2 - \frac{3}{4}\beta_0^2 c_1 + \frac{3}{8}\beta_1 c_1 + 2\beta_0 c_2 \right) L_\mu + c_3$$

$$L_\mu = \log\left(\mu / C_F \alpha_S m_t^{\text{pole}} \right) + 1$$

[Anzai, Kiyo, Sumino, 2009, Smirnov, Smirnov, Steinhauser, 2009]
[Kiyo, Sumino, 2014]

All calculation is based on pole-mass scheme.

relation between pole mass and $\overline{\text{MS}}$ mass

$$m^{\text{pole}} = \bar{m}\left(d_0 + d_1\frac{\alpha_S}{\pi} + d_2\frac{\alpha_S^2}{\pi^2} + d_3\frac{\alpha_S^3}{\pi^3}\right) + \mathcal{O}(\alpha_S^4)$$

[Marquard, Smirnov, Smirnov, Steinhauser, 2015]

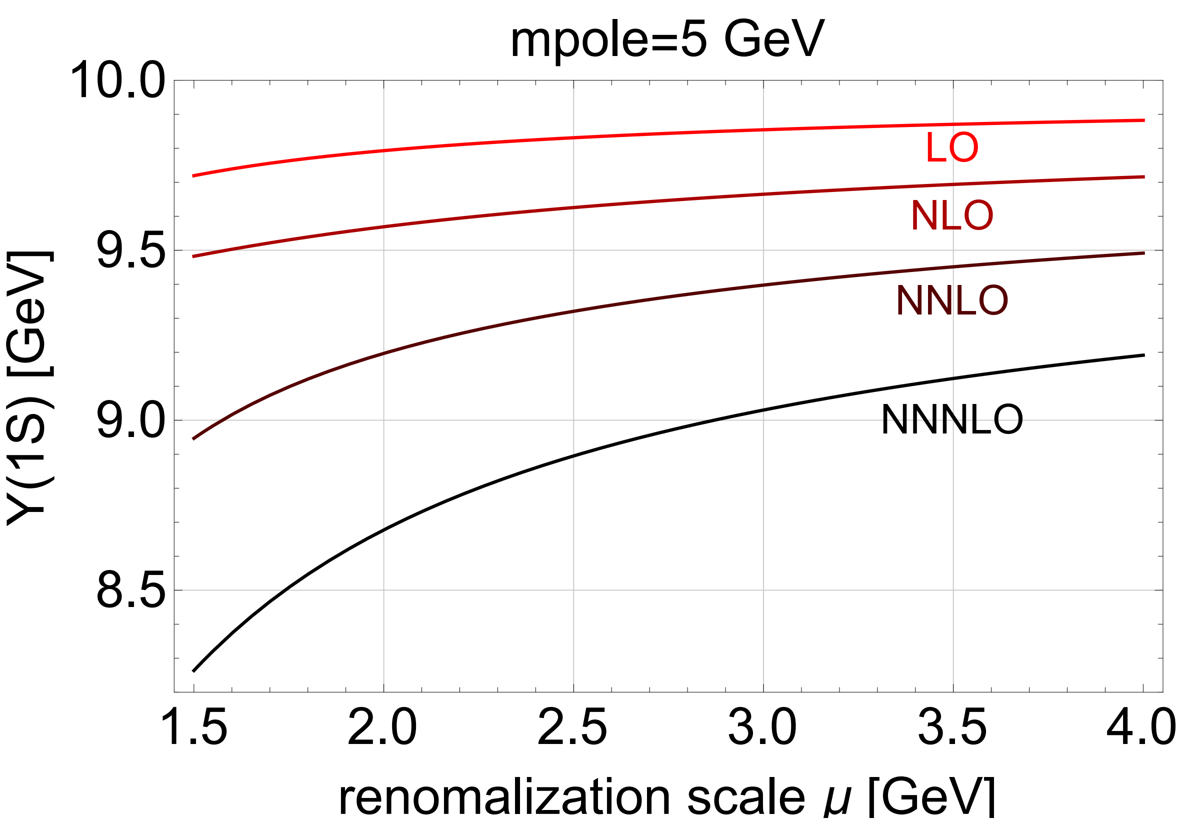
$\overline{\text{MS}}$ mass $\quad m_{\overline{\text{MS}}}(\mu) \quad \bar{m} = m_{\overline{\text{MS}}}(m_{\overline{\text{MS}}})$

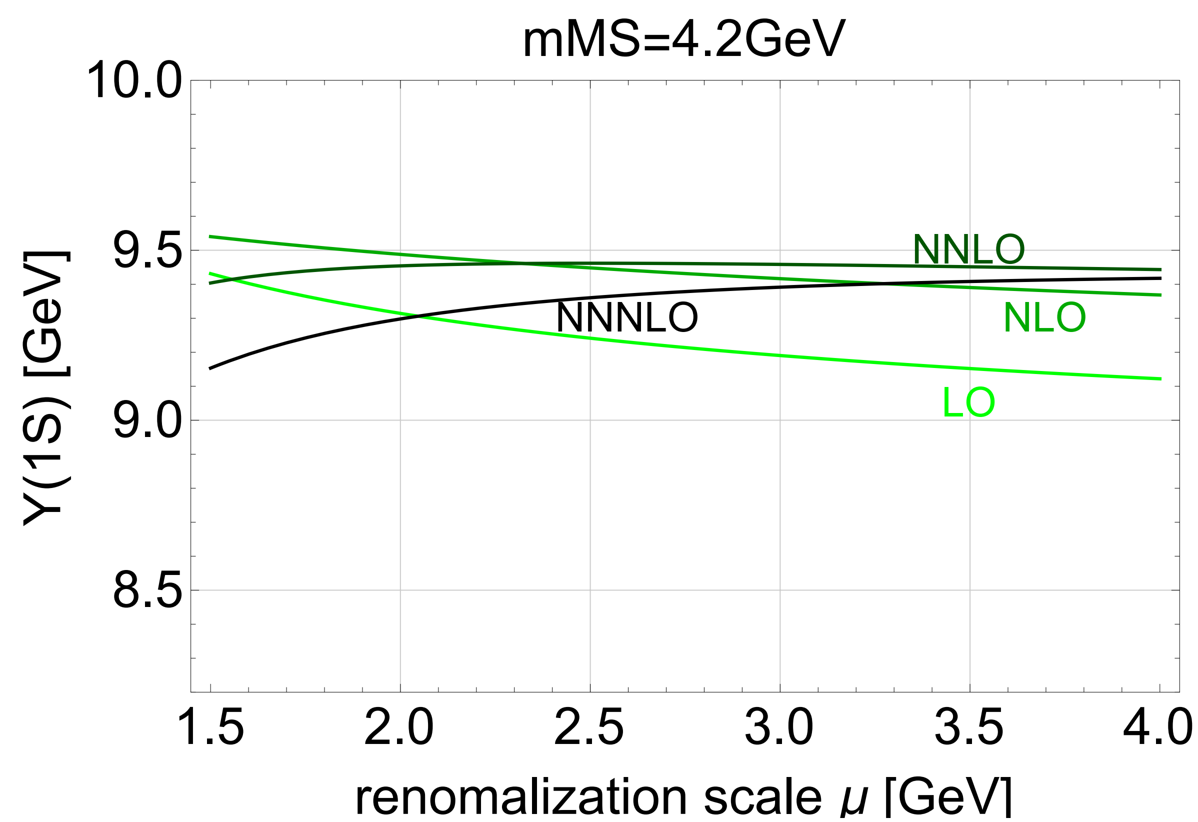
- defined perturbatively by subtracting UV-divergence
- Convenient and widely used choice of the renormalization scale is the mass of quark itself.

$$m_t^{\mathrm{pole}} = \bar{m}_t \left(d_0 + d_1 \frac{\alpha_S}{\pi} + d_2 \frac{\alpha_S^2}{\pi^2} + d_3 \frac{\alpha_S^3}{\pi^3} \right)$$

$$M_{t\bar{t}}(1S) = 2m_t^{\mathrm{pole}} + E_{\mathrm{bin}}$$

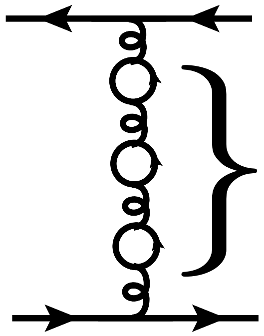
$$E_{\mathrm{bin}} = -\frac{4}{9} m_t^{\mathrm{pole}} \alpha_S^2 \left(P_0 + P_1 \frac{\alpha_S}{\pi} + P_2 \frac{\alpha_S^2}{\pi^2} + P_3 \frac{\alpha_S^3}{\pi^3} \right)$$





Cancelation of u=1/2 renormalon in binding energy and quark self energy

Bad convergence behavior is reproduced by the leading log resummation.



$$V^{\mathrm{LL}}(\mathbf{q^2}) = -\frac{4\pi C_F \alpha_S}{\mathbf{q^2}} \frac{1}{1+\frac{\alpha_S \beta_0}{4\pi}\log\frac{\mathbf{q^2}}{\mu^2}} = -\frac{4\pi C_F \alpha_S}{\mathbf{q^2}} \sum_{n=0}^{\infty} \left(\frac{\alpha_S \beta_0}{4\pi} \log \frac{\mu^2}{\mathbf{q^2}} \right)^n$$

$C_F = 4/3$

α_S : strong coupling constant

μ : renormalization scale

$$V^{\mathrm{LL}}(r) = \int_{\mathbf{q}} V^{\mathrm{LL}}(\mathbf{q^2}) \mathrm{e}^{\mathrm{i}\mathbf{qr}}$$

There are factorially growing contributions.

$$V^{\mathrm{LL}}\Big|^{u=1/2}_{\mathrm{renormalon}} = -\frac{2C_F \alpha_S \mu}{\pi} \sum_{n=0}^{N} \left(\frac{\alpha_S \beta_0}{2\pi} \right)^n n!$$

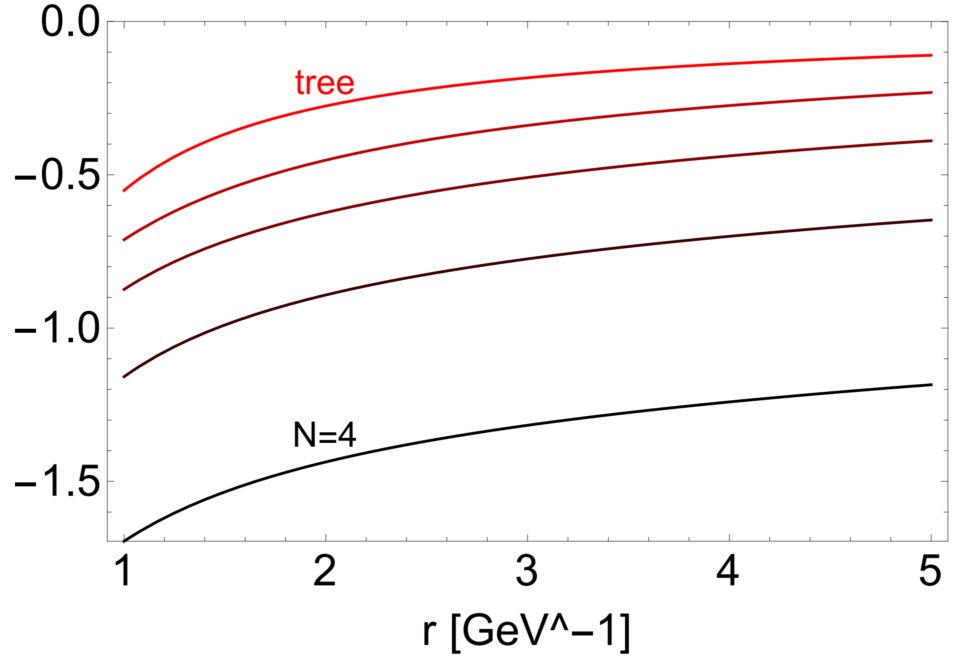
← just a constant (r-independent)

$$\delta E^{\mathrm{LL}}_{\mathrm{bin}}\Big|^{u=1/2}_{\mathrm{renormalon}} = -\frac{(C_F \alpha_S)^2 M}{\pi} \left(\frac{\mu}{C_F \alpha_S M} \right) \sum_{n=0}^{\infty} 2^{n+1} \left(\frac{\alpha_S \beta_0}{4\pi} \right)^n n!$$

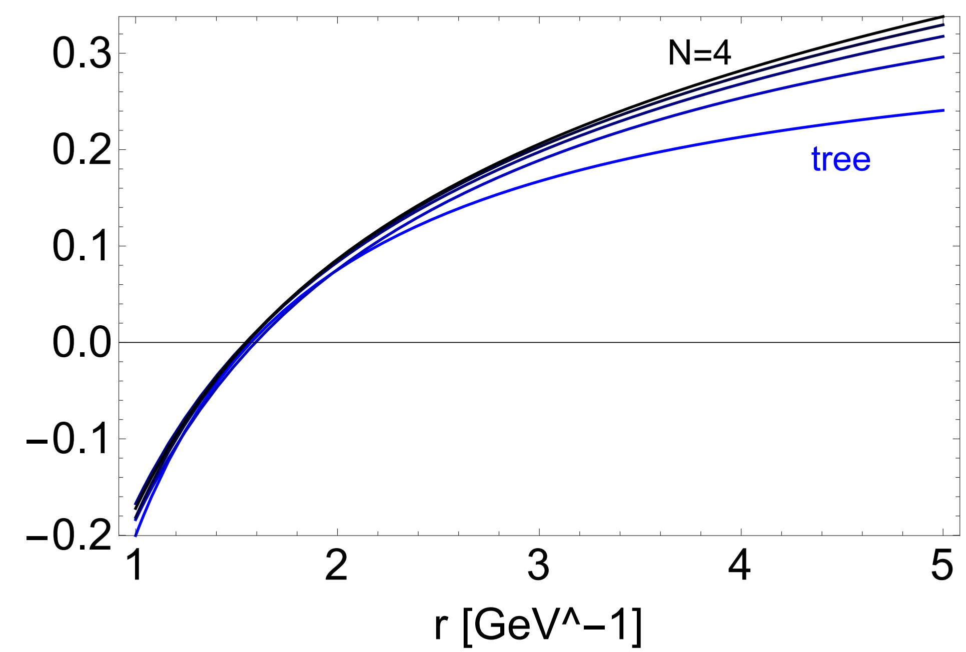
Renormalon

Source of the bad convergence is u=1/2 renormalon.

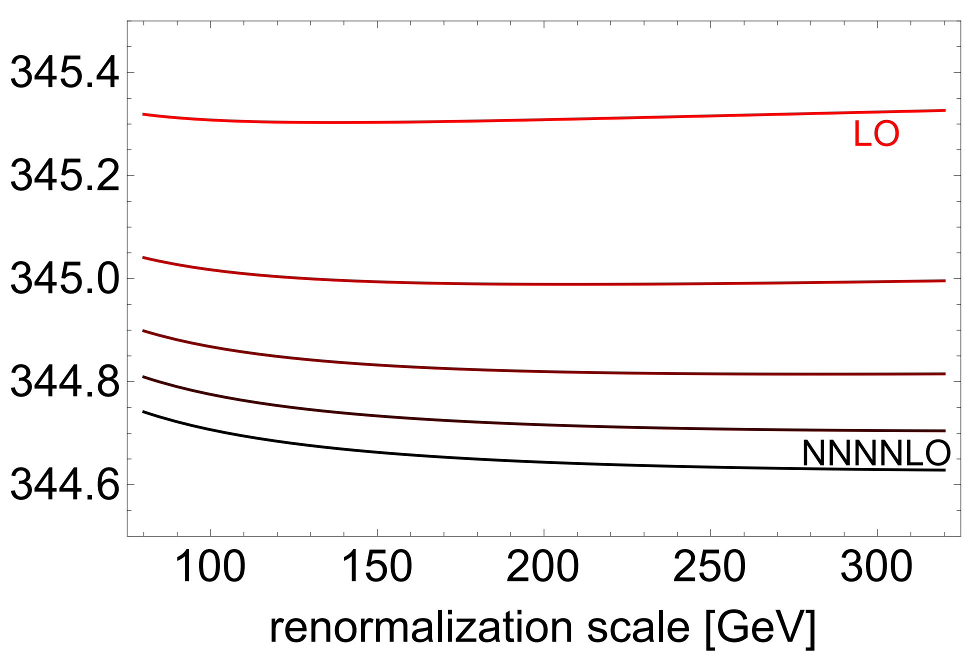
$$V^{\rm LL}(r)\ [\rm GeV]$$



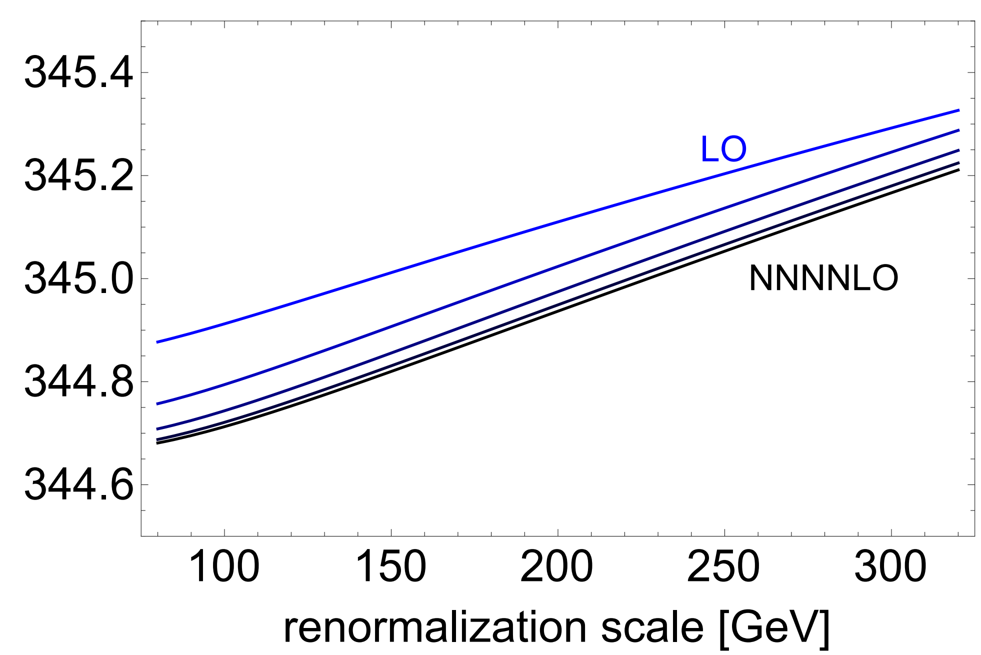
$$V^{\rm LL}(r) - V^{\rm LL}\Big|^{u=1/2}_{\rm renormalon}\ [\rm GeV]$$



$$2m_t^{\rm pole} + E_{\rm bin}^{\rm LL}\ [\rm GeV]$$



$$2m_t^{\rm pole} + E_{\rm bin}^{\rm LL} - E_{\rm bin}^{\rm LL}\Big|^{u=1/2}_{\rm renormalon}\ [\rm GeV]$$



“renormalization” of renormalon

$$E_{\text{tot}}(r) = 2m_t^{\text{pole}} + V(r)$$
$$= 2(m_t^{\text{pole}} - \delta^r m_t) + \left[V(r) - V|_{\text{renormalon}}^{u=1/2}\right]$$
$$= 2m_t^r + \left[V(r) - V|_{\text{renormalon}}^{u=1/2}\right]$$

We define $\delta^r m_t$ so that $2\delta^r m_t + V|_{\text{renormalon}}^{u=1/2} = 0$

Note that $V|_{\text{renormalon}}^{u=1/2}$ is a constant and thus $V|_{\text{renormalon}}^{u=1/2} = E_{\text{bin}}|_{\text{renormalon}}^{u=1/2}$

$$m_{t\bar{t}}(1S) = 2m_t^{\text{pole}} + E_{\text{bin}}$$
$$= 2(m_t^{\text{pole}} - \delta^r m_t) + \left[E_{\text{bin}} - E_{\text{bin}}|_{\text{renormalon}}^{u=1/2}\right]$$
$$= 2m_t^r + \left[E_{\text{bin}} - E_{\text{bin}}|_{\text{renormalon}}^{u=1/2}\right]$$

Actually m_{pole} has u=1/2 renormalon.

The pole mass is IR sensitive quantity.

$$m_{\text{pole}} = m_{\overline{\text{MS}}}(\mu)(1+\Delta_m)$$

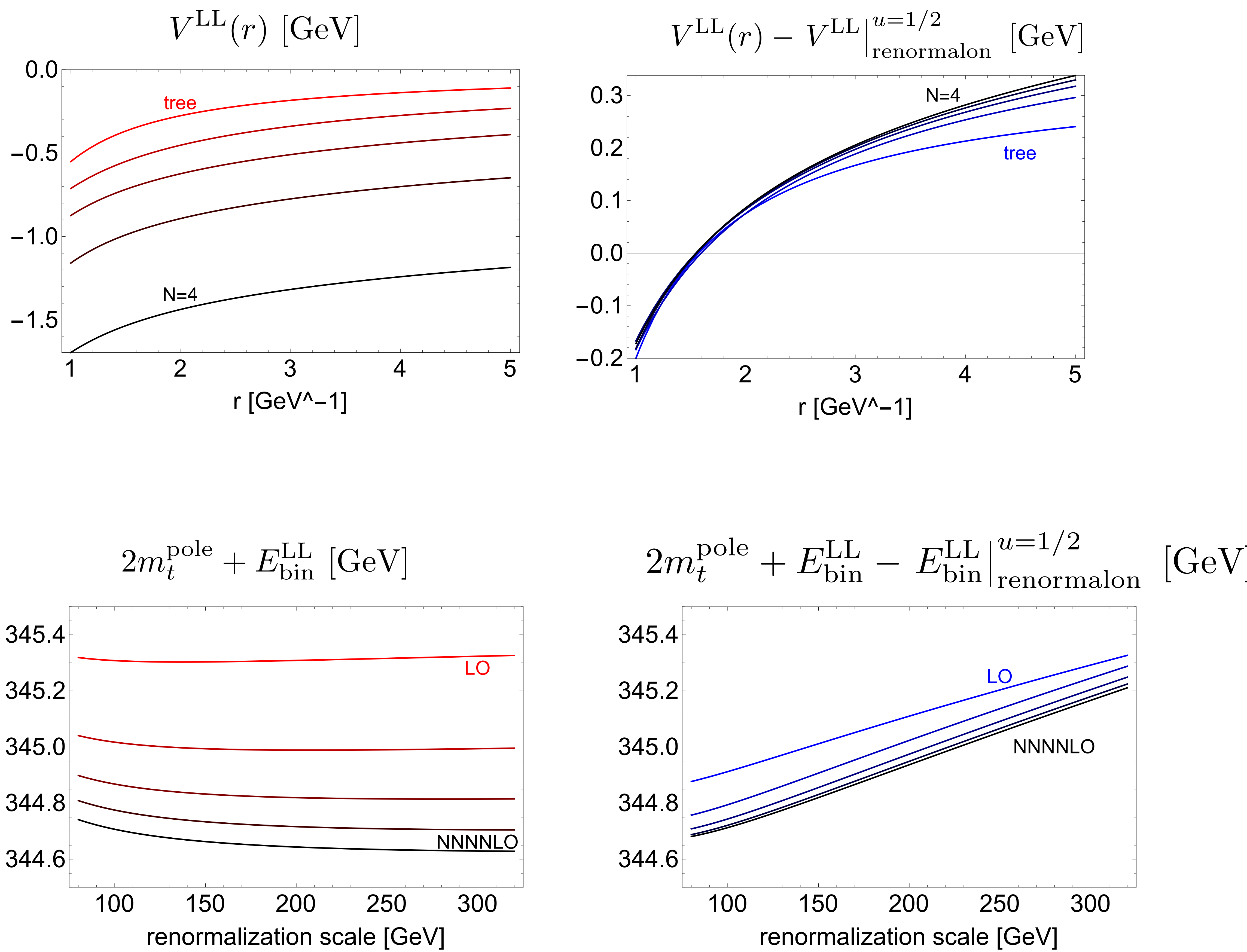
$$B[\Delta_m^{\text{LL}}] = \frac{C_F \alpha_S}{2\pi}\left(\frac{\mu}{m_{\overline{\text{MS}}}}\right)^{2u} 3(1-u)\frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)}$$

$$m_{\overline{\text{MS}}}\Delta_m^{\text{LL}}\Big|_{\text{renormalon}}^{u=1/2} = \frac{C_F \alpha_S \mu}{\pi}\sum_{n=0}^{N}\left(\frac{\alpha_S \beta_0}{2\pi}\right)^n n!$$

$$= -\frac{V^{\text{LL}}\Big|_{\text{renormalon}}^{u=1/2}}{2}$$

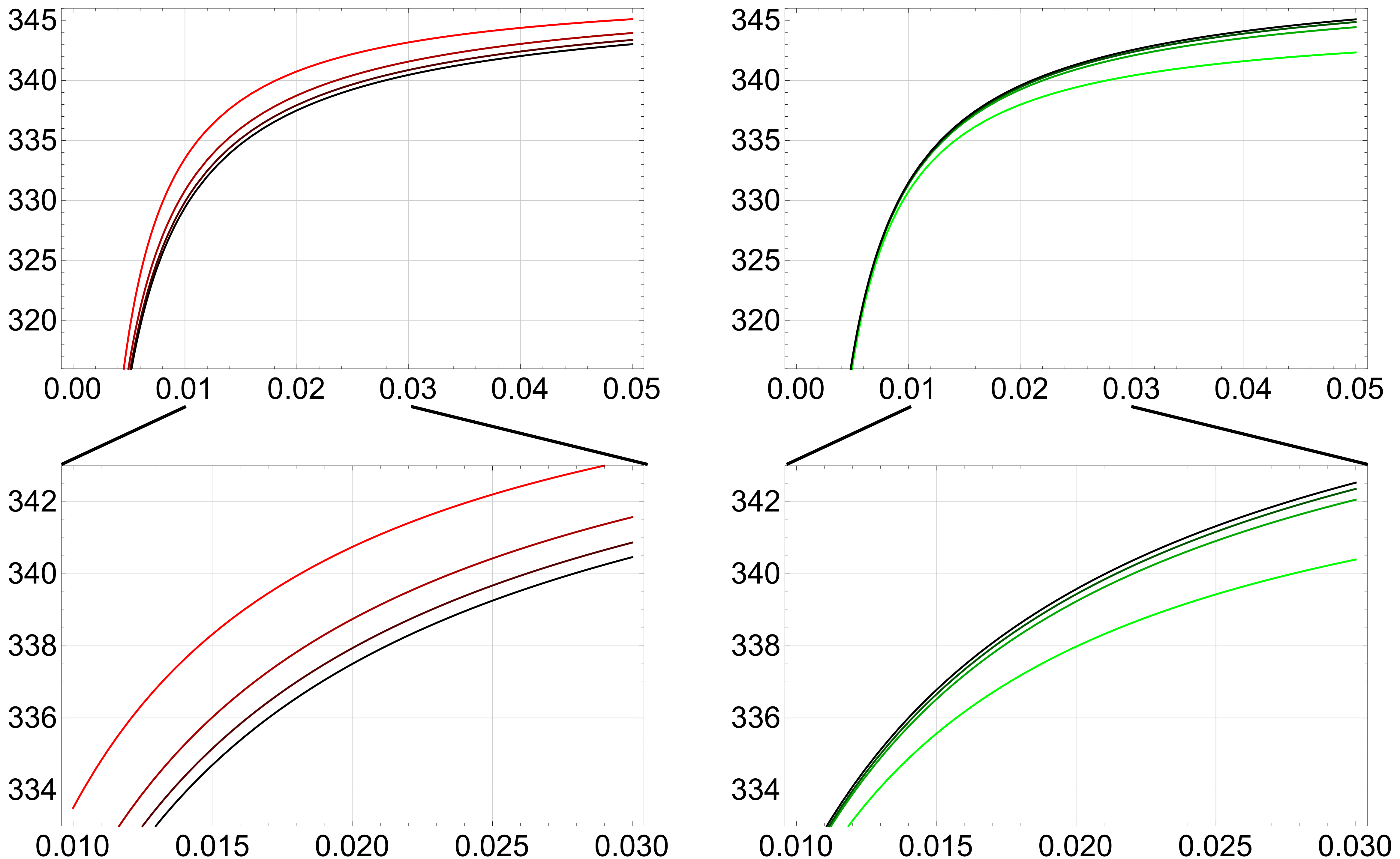
If we express the pole mass in IR insensitive mass, we can extract u=1/2 renormalon.

Source of the bad convergence is u=1/2 renormalon.



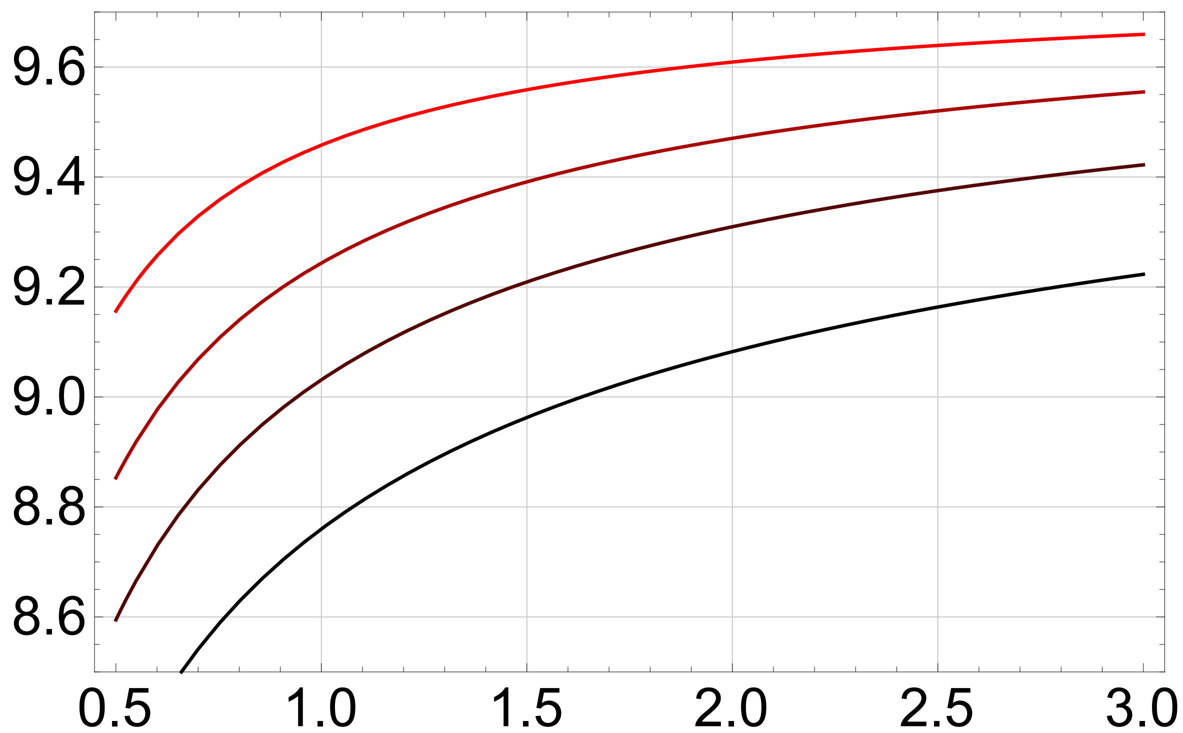
realistic case

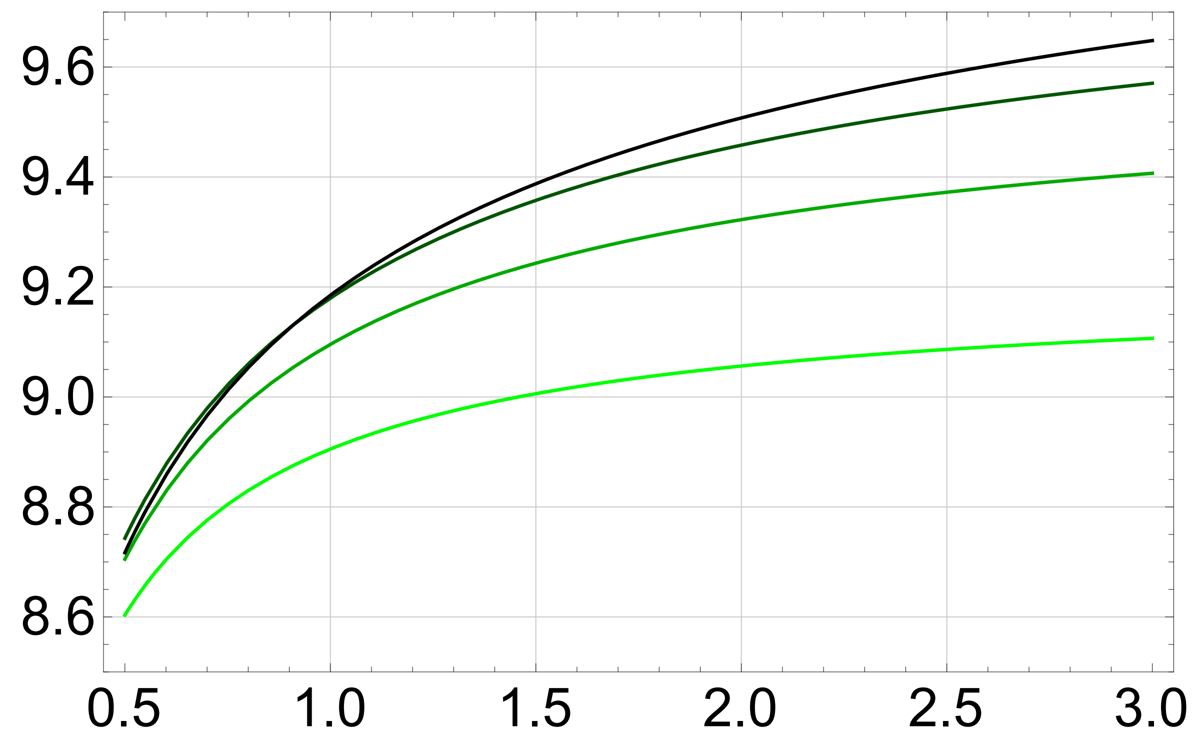
All plots are $E_{\rm tot}(r) = 2m_t^{\rm pole} + V(r)$ [GeV] of r [GeV^{-1}]



realistic case

All plots are $E_{\rm tot}(r) = 2m_b^{\rm pole} + V(r)$ [GeV] of r [GeV^{-1}]



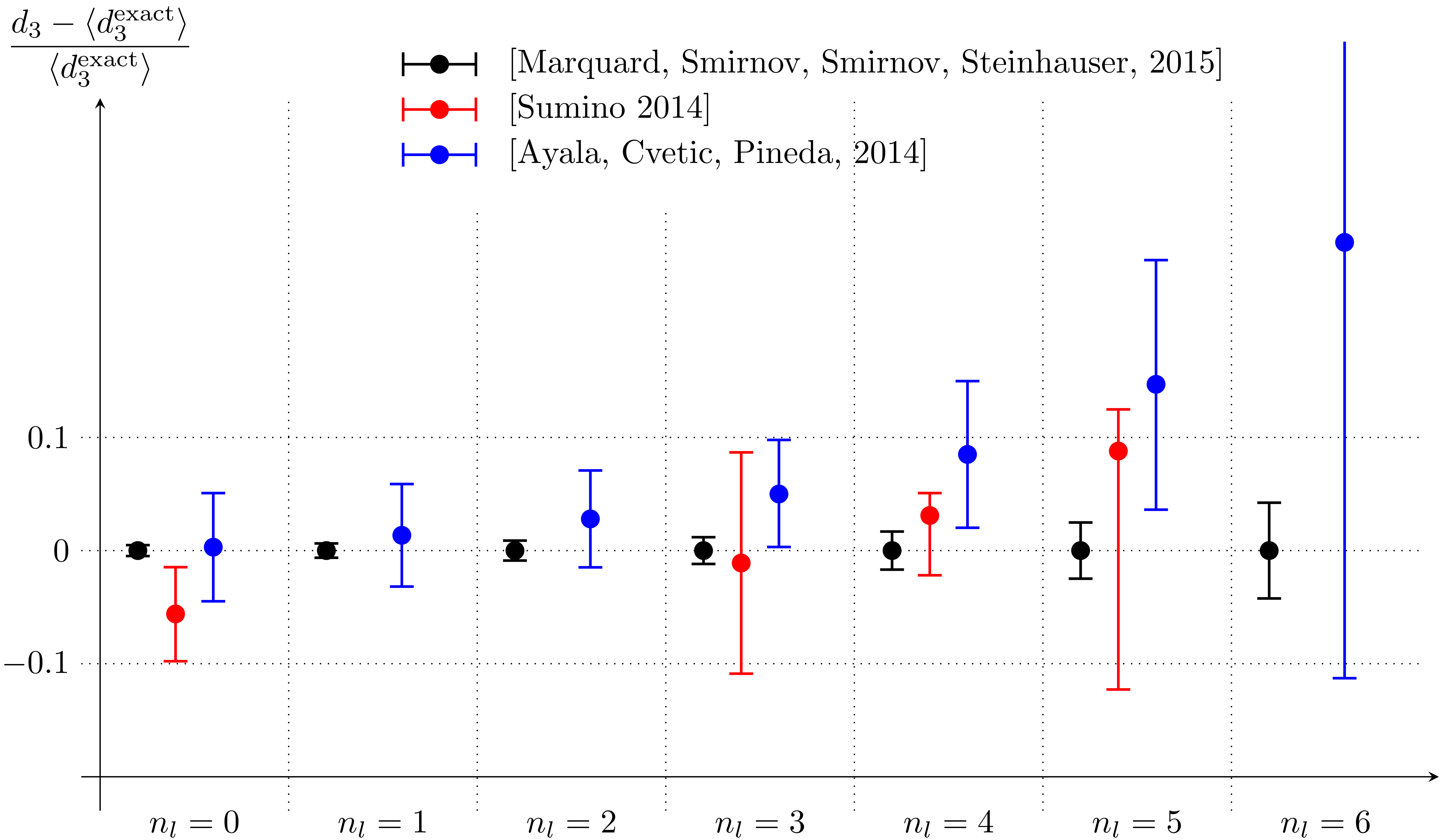


As mentioned before, the improvement of convergence is more visible in the case of bottomonium.

Strong IR cancelation in heavy quarkonium

What can we learn from the value of d3?

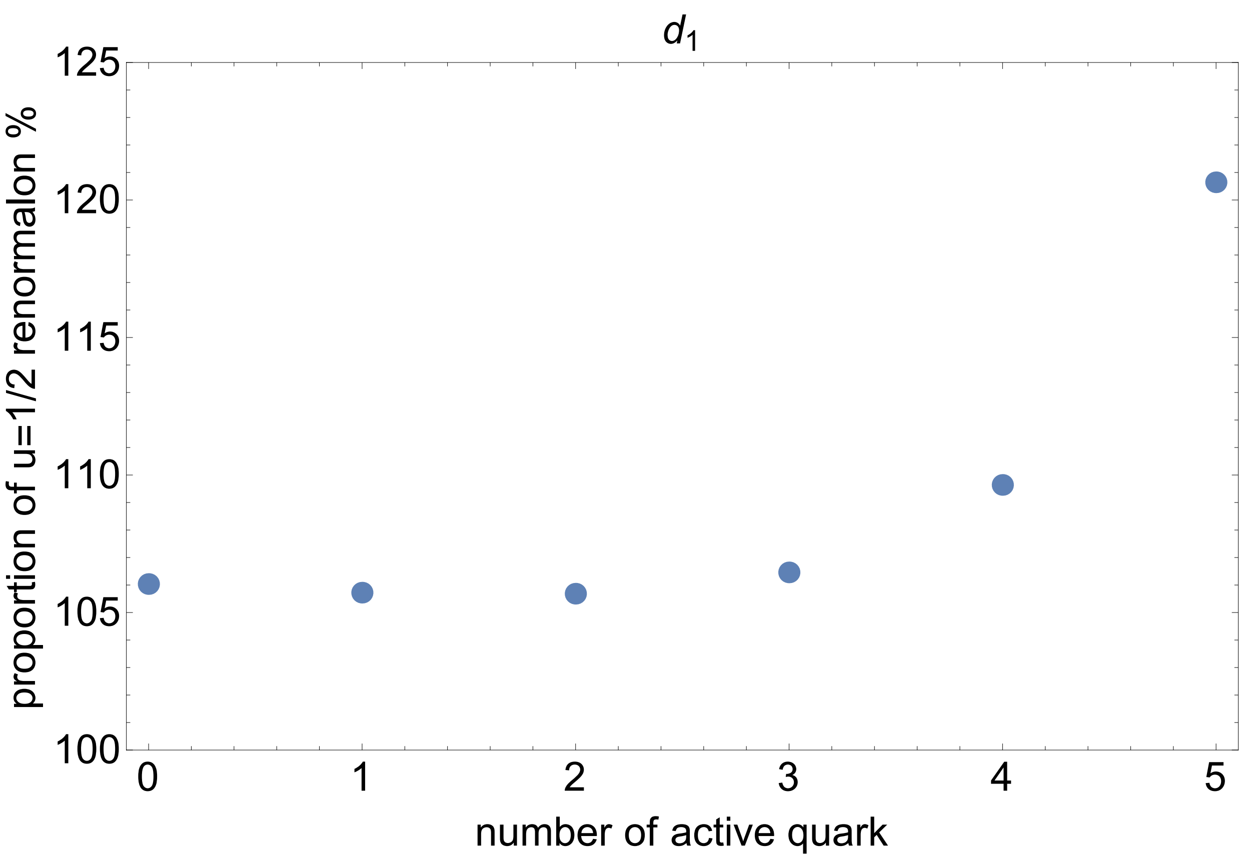
$$m^{\text{pole}} = \bar{m}\left(d_0 + d_1\frac{\alpha_S}{\pi} + d_2\frac{\alpha_S^2}{\pi^2} + d_3\frac{\alpha_S^3}{\pi^3}\right) + \mathcal{O}(\alpha_S^4)$$

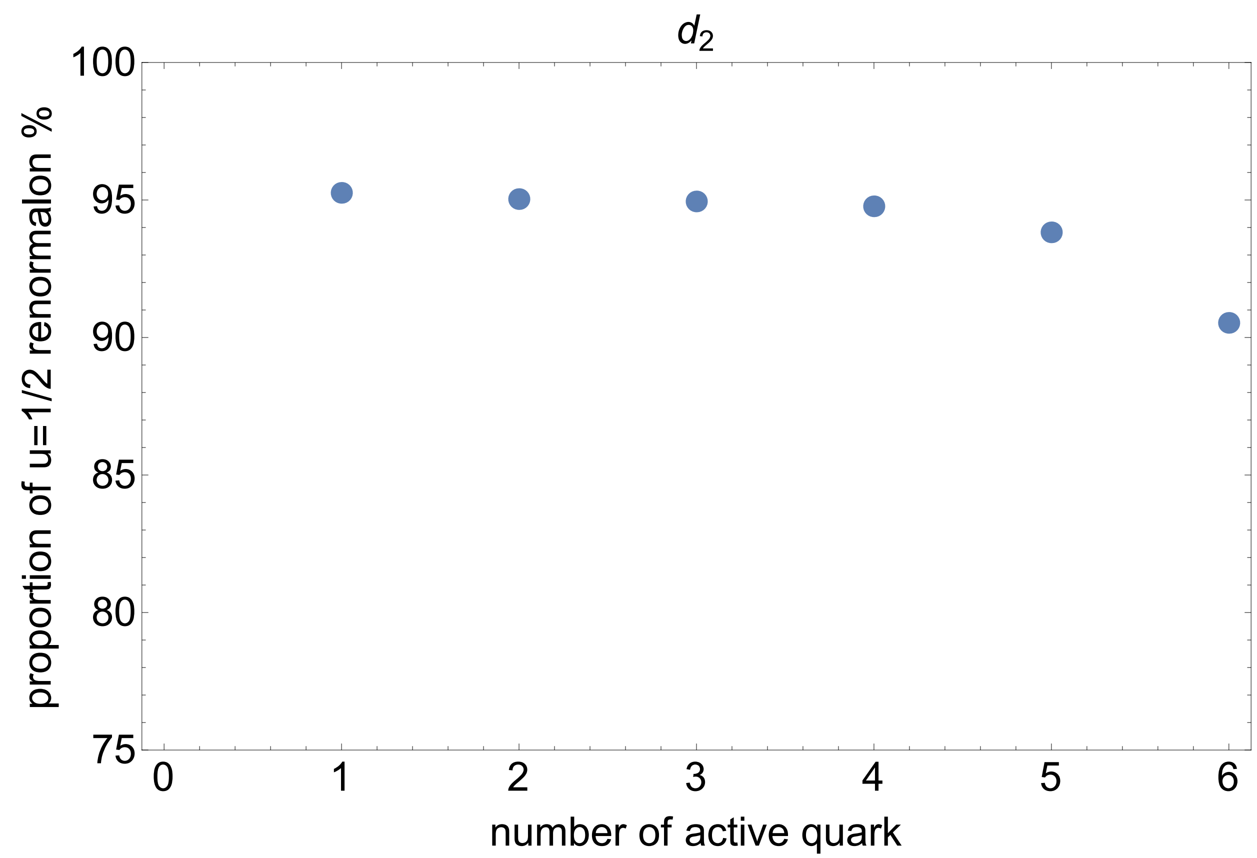


d3 may be dominated by u=1/2 renormalon.

[Ayala, Cvetic, Pineda, 2014]

$$d_N \simeq \pi N_m \left(\frac{\beta_0}{2}\right)^N \frac{\Gamma(\nu+N+1)}{\Gamma(\nu+1)}$$
$$\times \left[1+\frac{\nu\tilde{c}_1}{N+\nu}+\frac{\nu(\nu-1)\tilde{c}_2}{(N+\nu)(N+\nu-1)}+\frac{\nu(\nu-1)(\nu-2)\tilde{c}_3}{(N+\nu)(N+\nu-1)(N+\nu-2)}+\mathcal{O}(\frac{1}{N^4})\right]$$



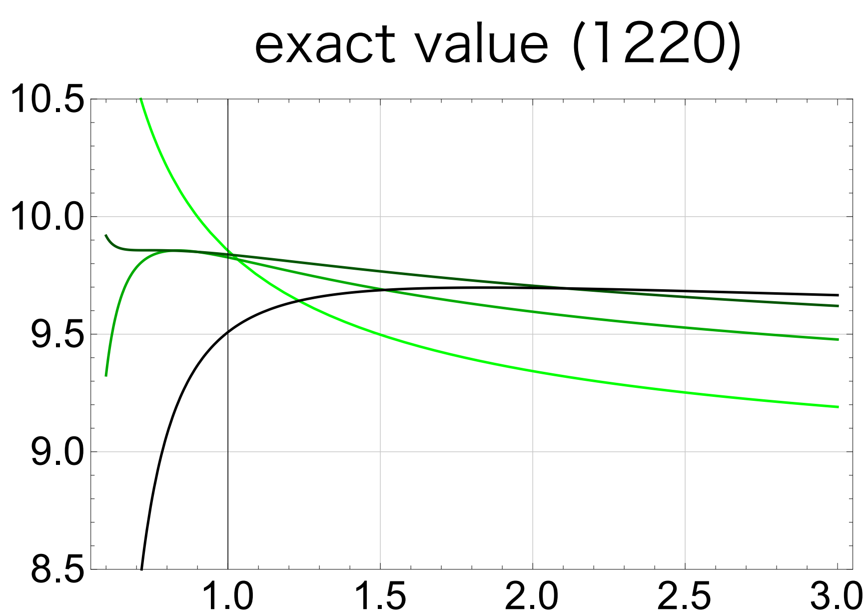


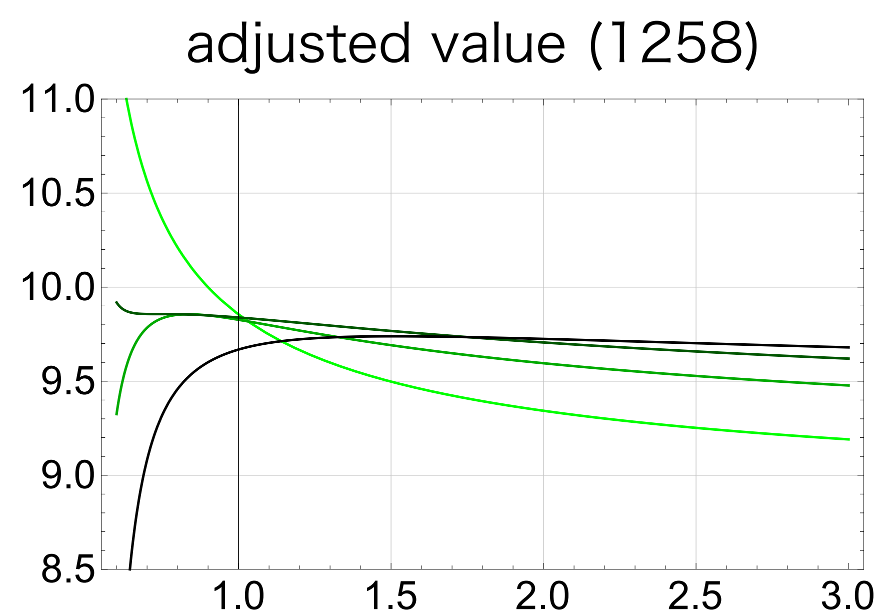
Each points has about 10% uncertainty.

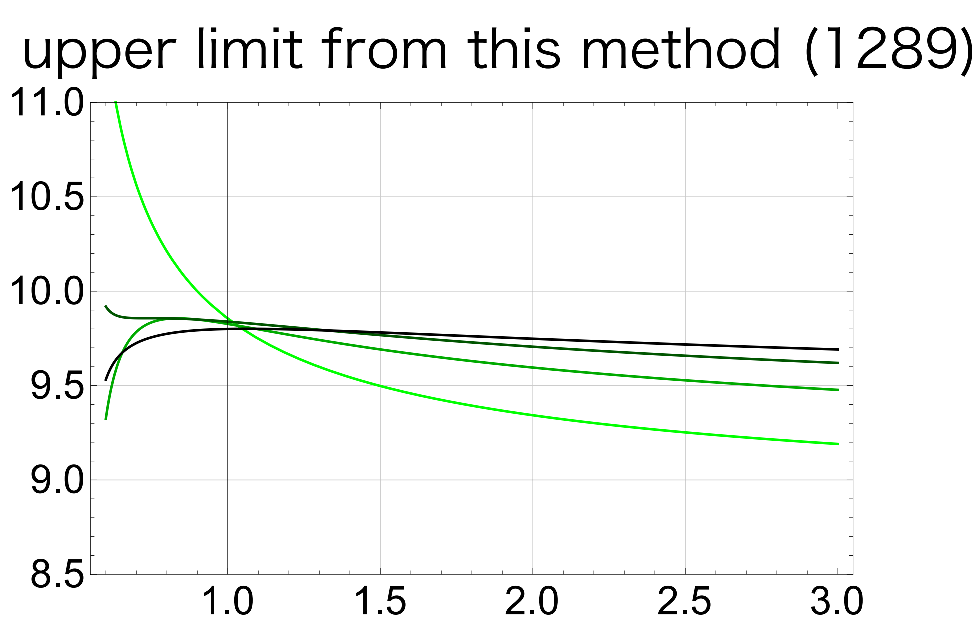
Stability of potential is sensitive to d3.

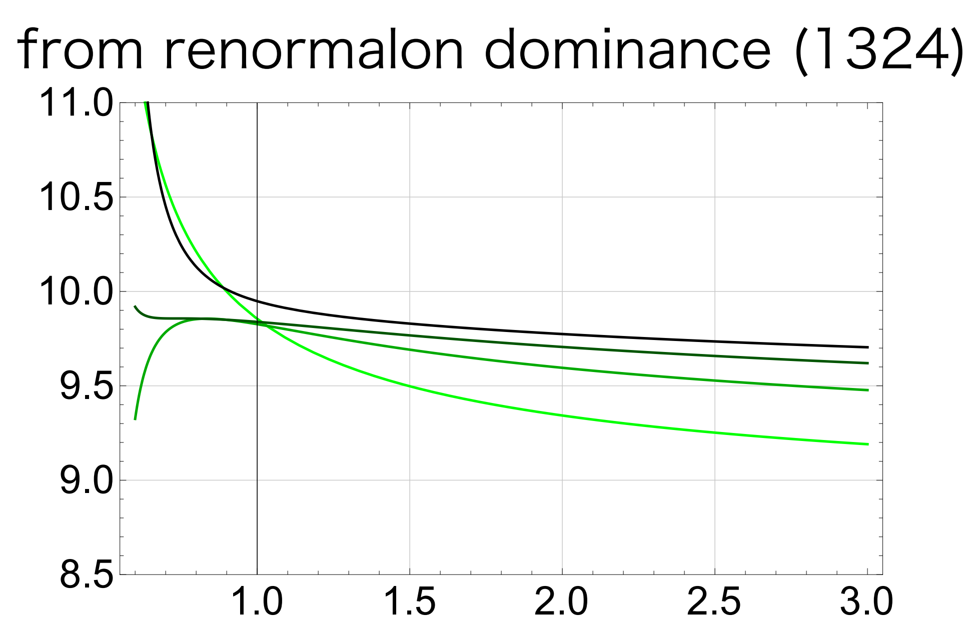
[Sumino, 2014]

All plots shows renormalization scale dependence of $E_{\text{tot}}(r) = 2m_b^{\text{pole}} + V(r)\Big|_{r=2.8\ \text{GeV}^{-1}}$



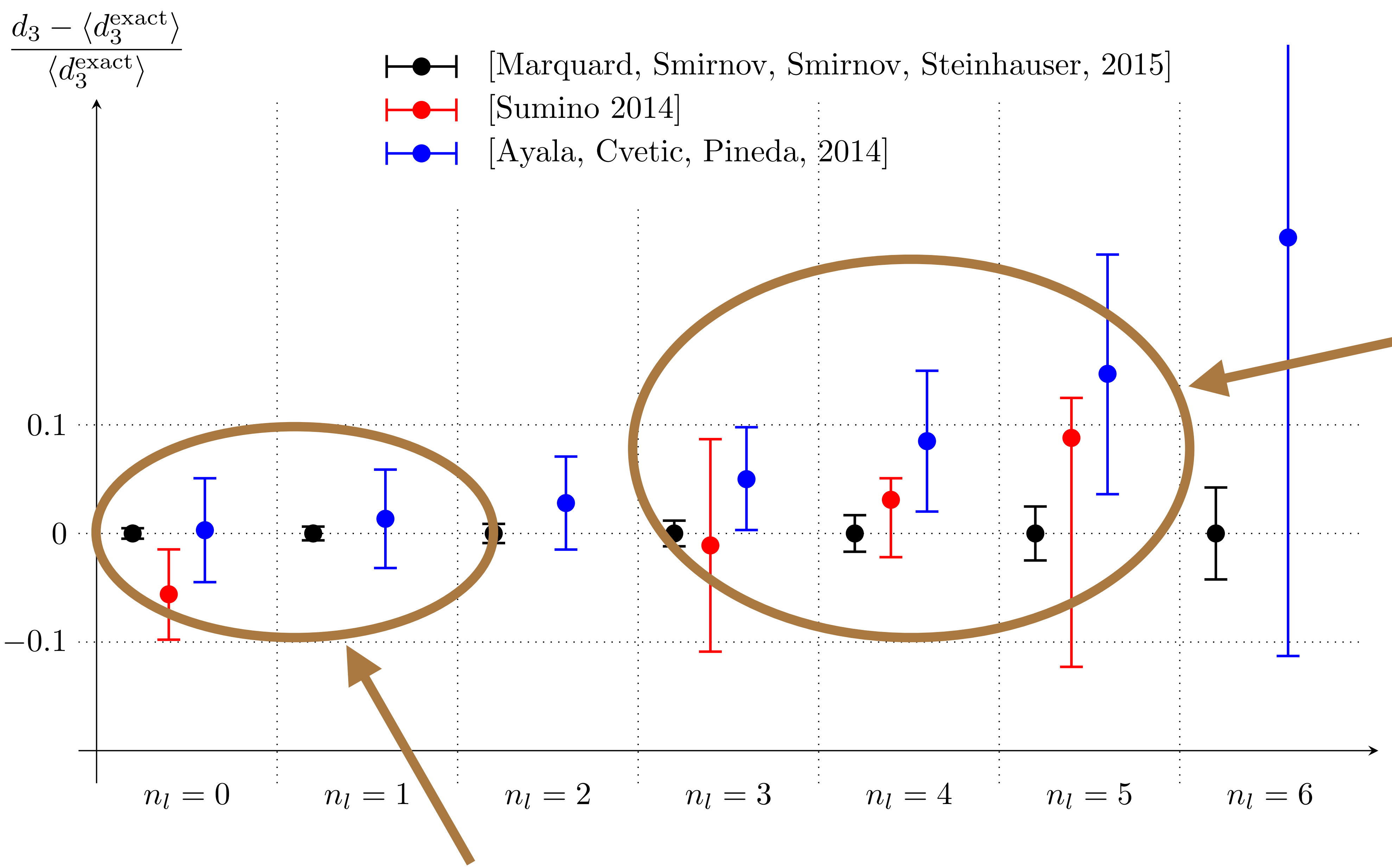






minimal-sensitivity scale disappears

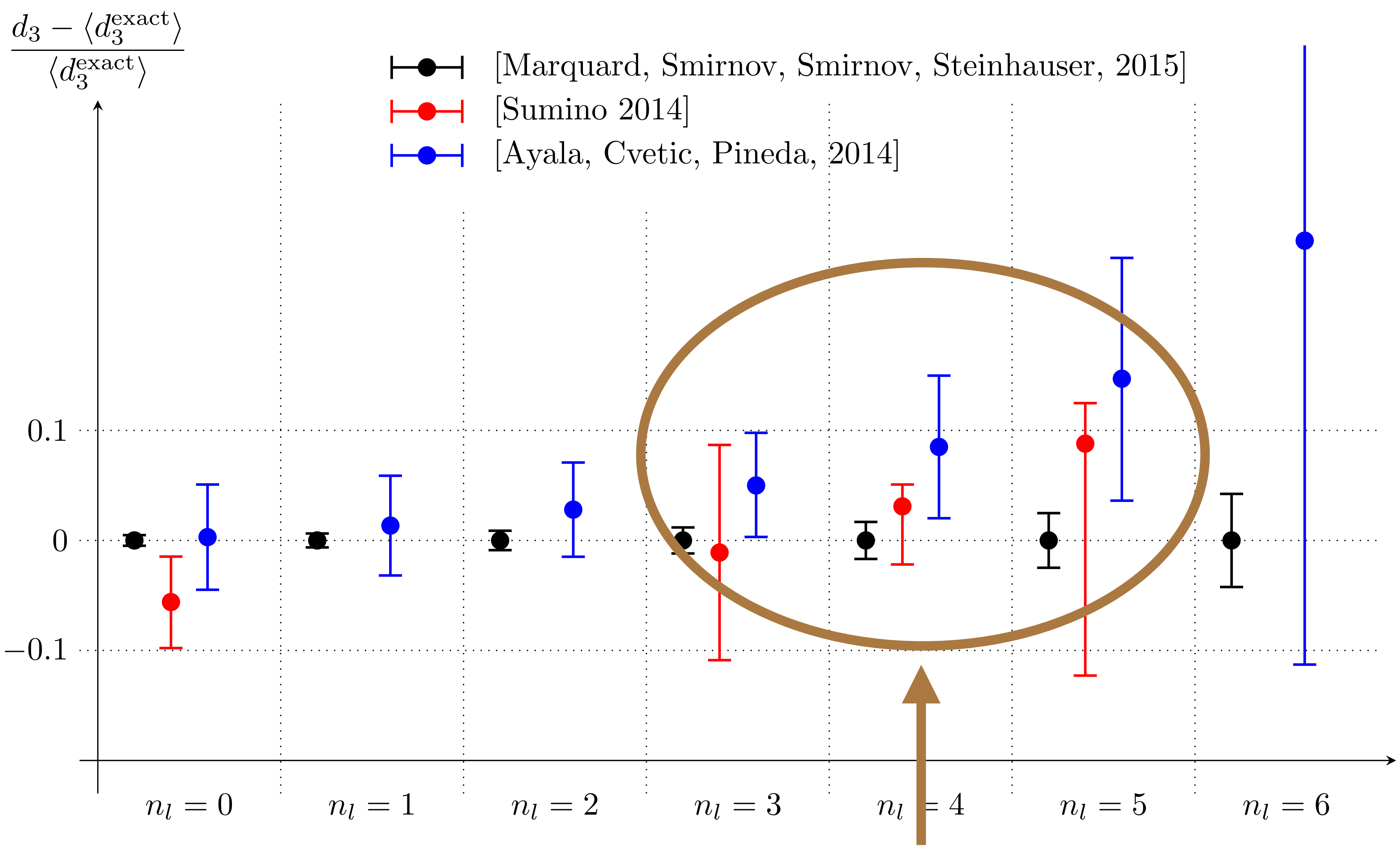
What can we learn from the value of d3?



It does not work in these cases.

Renormalon dominance hypothesis holds when number of active quark is small.

What can we learn from the value of d3?



Estimation from potential stability predicts better values.

This suggests potential is stabilized by stronger cancelation than u=1/2 renormalon.

Summary

- We analyze toponium bound state mass at NNNLO in perturbative QCD.
- Cancelation between the quark self-energy and the binding energy is crucial to meaningful predictions.
- By using proper mass definition, we find that the precision of 20 - 30 MeV in the top quark mass is possible in principle.
- We suggest that the cancelation happens not only in u=1/2 renormalon but also in more general IR contributions.

Introduction to Borel transformation

leading-log resumed potential

$$V^{\mathrm{LL}}(\mathbf{q^2}) = -\frac{4\pi C_F \alpha_S}{\mathbf{q^2}} \sum_{n=0}^{\infty} \left(\frac{\alpha_S \beta_0}{4\pi}\right)^n \left(\log \frac{\mu^2}{\mathbf{q^2}}\right)^n$$

Borel transformation is defined as

$$B[V^{\mathrm{LL}}(\mathbf{q^2})](u) \equiv -\frac{4\pi C_F \alpha_S}{\mathbf{q^2}} \sum_{n=0}^{\infty} \frac{u^n}{n!} \left(\log \frac{\mu^2}{\mathbf{q^2}}\right)^n$$

$$= -\frac{4\pi C_F \alpha_S}{\mathbf{q^2}} \left(\frac{\mu^2}{\mathbf{q^2}}\right)^u$$

so that the original function is obtained by

$$V^{\mathrm{LL}}(\mathbf{q^2}) = \int_0^{\infty} B[V^{\mathrm{LL}}(\mathbf{q^2})](\tfrac{\alpha_S \beta_0}{4\pi} u) \mathrm{e}^{-u} \mathrm{d}u$$

$B[E_{\rm bin}]$ shows singularities.

$$\delta E_{\rm bin} = \langle 1S|\ \delta V\ |1S\rangle$$

$$\mathbf{q} = \mathbf{p} - \mathbf{p}'$$

$$\lambda = C_F \alpha_S M/2$$

$$B[\delta E^{\rm LL}_{\rm bin}] = \langle 1S|\ B[\delta V^{\rm LL}]\ |1S\rangle$$

$$= \int_{\mathbf{p},\mathbf{p}'} \langle 1S|\mathbf{p}\rangle\ B[\delta V^{\rm LL}(\mathbf{q^2})]\ \langle \mathbf{p}'|1S\rangle$$

$$= \int_{\mathbf{p},\mathbf{p}'} \frac{\sqrt{\pi\lambda^5}}{4(\mathbf{p}^2+\lambda^2)^2} \left[-\frac{4\pi C_F\alpha_S}{\mathbf{q^2}}\left(\frac{\mu^2}{\mathbf{q^2}}\right)^{u}\right] \frac{\sqrt{\pi\lambda^5}}{4(\mathbf{p}'^2+\lambda^2)^2}$$

$$= -\frac{(C_F\alpha_S)^2 M}{\pi}\left(\frac{\mu}{C_F\alpha_S M}\right)^{2u} \Gamma(1/2-u)\,\Gamma(3/2+u)$$

$$u = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$$

singularities correspond to IR div. in the original integral.

IR renormalon

$$u = -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \cdots$$

singularities correspond to UV div. in the original integral.

UV renormalon

Renormalon contribution shows n! growth.

$$B[\delta E^{\rm LL}_{\rm bin}] = -\frac{(C_F\alpha_S)^2 M}{\pi}\left(\frac{\mu}{C_F\alpha_S M}\right)^{2u}\Gamma(1/2-u)\ \Gamma(3/2+u)$$

near $u=1/2$

$$\sim -\frac{(C_F\alpha_S)^2 M}{\pi}\left(\frac{\mu}{C_F\alpha_S M}\right)\frac{1}{1/2-u}$$

$$= -\frac{(C_F\alpha_S)^2 M}{\pi}\left(\frac{\mu}{C_F\alpha_S M}\right)\sum_{n=0}^{\infty}2^{n+1}u^n$$

inverse Borel transformation

$$\delta E^{\rm LL}_{\rm bin}\Big|^{u=1/2}_{\rm renormalon} = -\frac{(C_F\alpha_S)^2 M}{\pi}\left(\frac{\mu}{C_F\alpha_S M}\right)\sum_{n=0}^{\infty}2^{n+1}\left(\frac{\alpha_S\beta_0}{4\pi}\right)^n n!$$

asymptotic series

Renormalon contribution shows n! growth.

$$B[\delta E_{\rm bin}^{\rm LL}] = -\frac{(C_F\alpha_S)^2 M}{\pi}\left(\frac{\mu}{C_F\alpha_S M}\right)^{2u}\Gamma(1/2-u)\,\Gamma(3/2+u)$$

near $u=k+1/2$

$$\delta E_{\rm bin}^{\rm LL}\Big|_{\rm renormalon}^{u=k+1/2} = -\frac{(C_F\alpha_S)^2 M}{\pi}$$

$$\times\left(\frac{\mu}{C_F\alpha_S M}\right)^{2k+1}(k+1)!\sum_{n=0}^{\infty}\frac{2^{n+1}}{(2k+1)^{n+1}}\left(\frac{\alpha_S\beta_0}{4\pi}\right)^n n!$$

Speed of growth becomes milder as k increases.

And we will see the mass of heavy quark is a strong suppression factor.

Renormalon contribution shows n! growth.

$$B[\delta E^{\rm LL}_{\rm bin}] = -\frac{(C_F\alpha_S)^2 M}{\pi}\left(\frac{\mu}{C_F\alpha_S M}\right)^{2u}\Gamma(1/2-u)\;\Gamma(3/2+u)$$

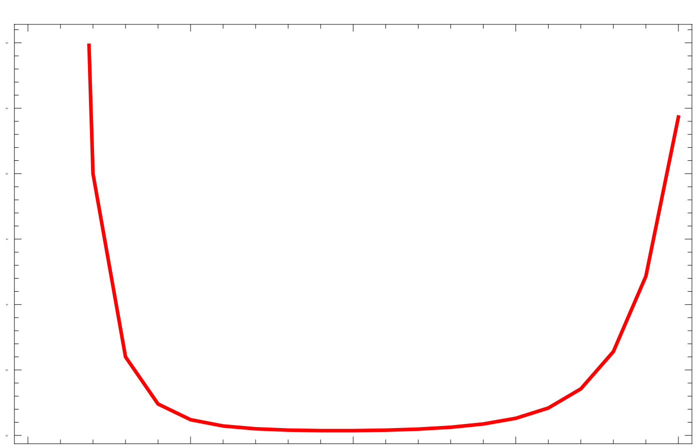
near $u=-3/2$

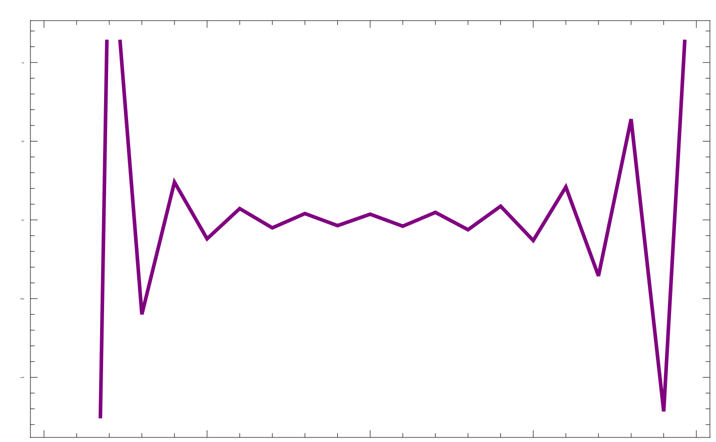
$$\delta E^{\rm LL}_{\rm bin}\Big|^{u=-3/2}_{\rm renormalon} = -\frac{(C_F\alpha_S)^2 M}{\pi}\left(\frac{C_F\alpha_S M}{\mu}\right)^{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^{n+1}\left(-\frac{\alpha_S\beta_0}{4\pi}\right)^{n} n!$$

sign-alternating asymptotic series

(better convergence)

IR renormalon





UV renormalon is controlable.

for example

$$\delta E_{\rm bin}^{\rm LL}\Big|_{\rm renormalon}^{u=-3/2} \propto \sum_{n=0}^{\infty}(-a)^n n! \qquad a = \frac{\alpha_S \beta_0}{6\pi}$$

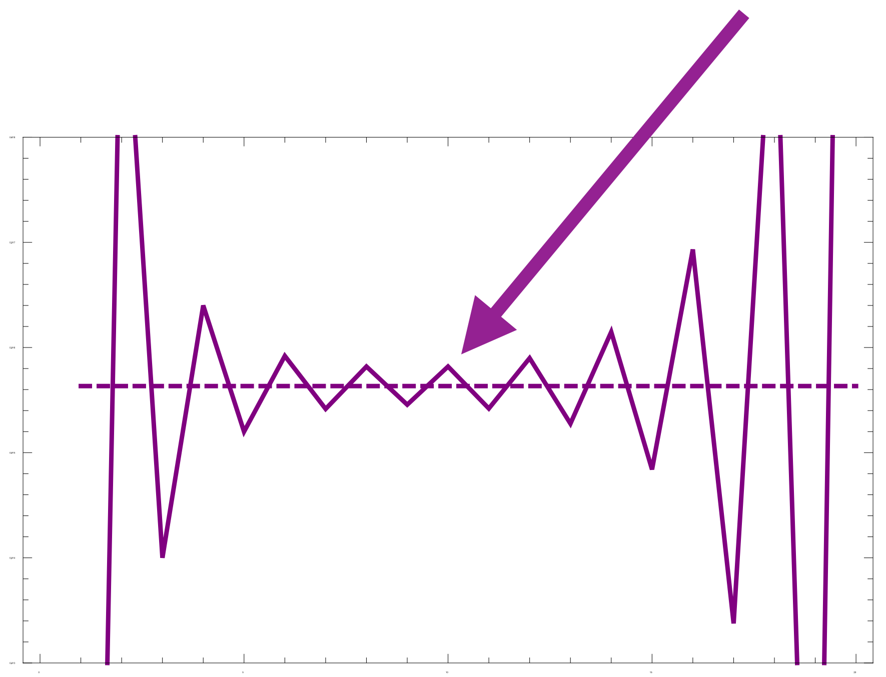
$$\sum_{n=0}^{\infty}(-a)^n n! = \sum_{n=0}^{\infty}(-a)^n \int_0^{\infty} s^n \mathrm{e}^{-s}\mathrm{d}s = \underbrace{\int_0^{\infty}\frac{\mathrm{e}^{-s}}{1+as}\mathrm{d}s}_{I_{u=-3/2}(a)}$$

consider this to be "true value"

Uncertainty of truncated value is less than the term of next order.

$$\left| I_{u=-3/2}(a) - \sum_{n=0}^{N}(-a)^n n! \right| < a^{N+1}(N+1)!$$

The best truncation is easy to find.



There is unavoidable ambiguity in IR renormalon.

for example

$$\delta E^{\rm LL}_{\rm bin}\Big|^{u=1/2}_{\rm renormalon} \propto \sum_{n=0}^{\infty} a^n n! \qquad a = \frac{\alpha_S \beta_0}{2\pi}$$

$$\sum_{n=0}^{\infty} a^n n! = \sum_{n=0}^{\infty} a^n \int_0^{\infty} s^n \mathrm{e}^{-s} \mathrm{d}s = \int_0^{\infty} \frac{\mathrm{e}^{-s}}{1-as} \mathrm{d}s$$

ambiguity of how to regularize this

The best truncation may be

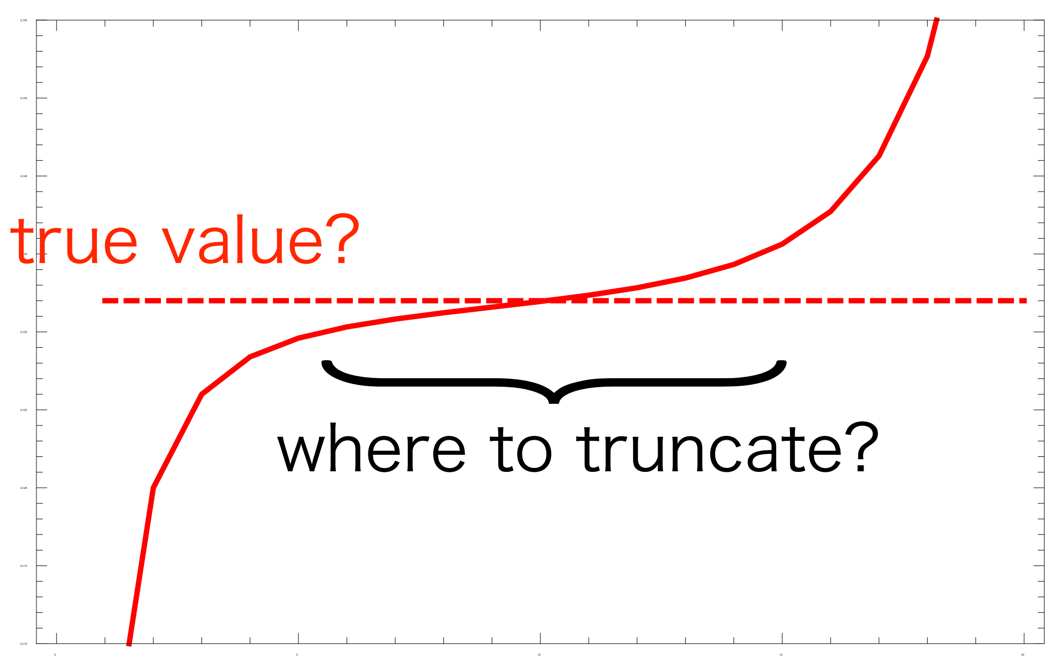
$$a^{n_*} n_*! \simeq a^{n_*+1}(n_*+1)! \rightarrow n_* = 1/a$$

Uncertainty form truncation ambiguity is estimated to be

$$\sum_{n_*-\sqrt{n_*}}^{n_*+\sqrt{n_*}} a^n n! \simeq 2\sqrt{n_*} a^{n_*} n_*!$$

$$\simeq 2\sqrt{2\pi} n_*^{n_*+1} a^{n_*} \mathrm{e}^{-n_*}$$

$$\simeq \frac{2\sqrt{2\pi}}{a} \mathrm{e}^{-1/a}$$



There is unavoidable ambiguity in IR renormalon.

Apply ambiguity estimation of previous slide.

$$\delta E_{\rm bin}^{\rm LL}\Big|_{\rm renormalon}^{u=1/2} = -\frac{2C_F\alpha_S\mu}{\pi}\underbrace{\sum_{n=0}^{\infty} a^n n!} \qquad a = \frac{\alpha_S\beta_0}{2\pi}$$

Estimated ambiguity is $\dfrac{2\sqrt{2\pi}}{a}\,\mathrm{e}^{-1/a}$

Therefore ambiguity form u=1/2 renormalon becomes

$$\frac{2\sqrt{2\pi}4\pi C_F}{\beta_0}\mu\,\mathrm{e}^{-\frac{2\pi}{\alpha_S\beta_0}} = \frac{2\sqrt{2\pi}4\pi C_F}{\beta_0}\Lambda_{\rm QCD}$$

This result seems to be reasonable, isn't it?

Current knowledge of $\delta E^{\rm LL}_{\rm bin}\Big|_{\rm renormalon}$

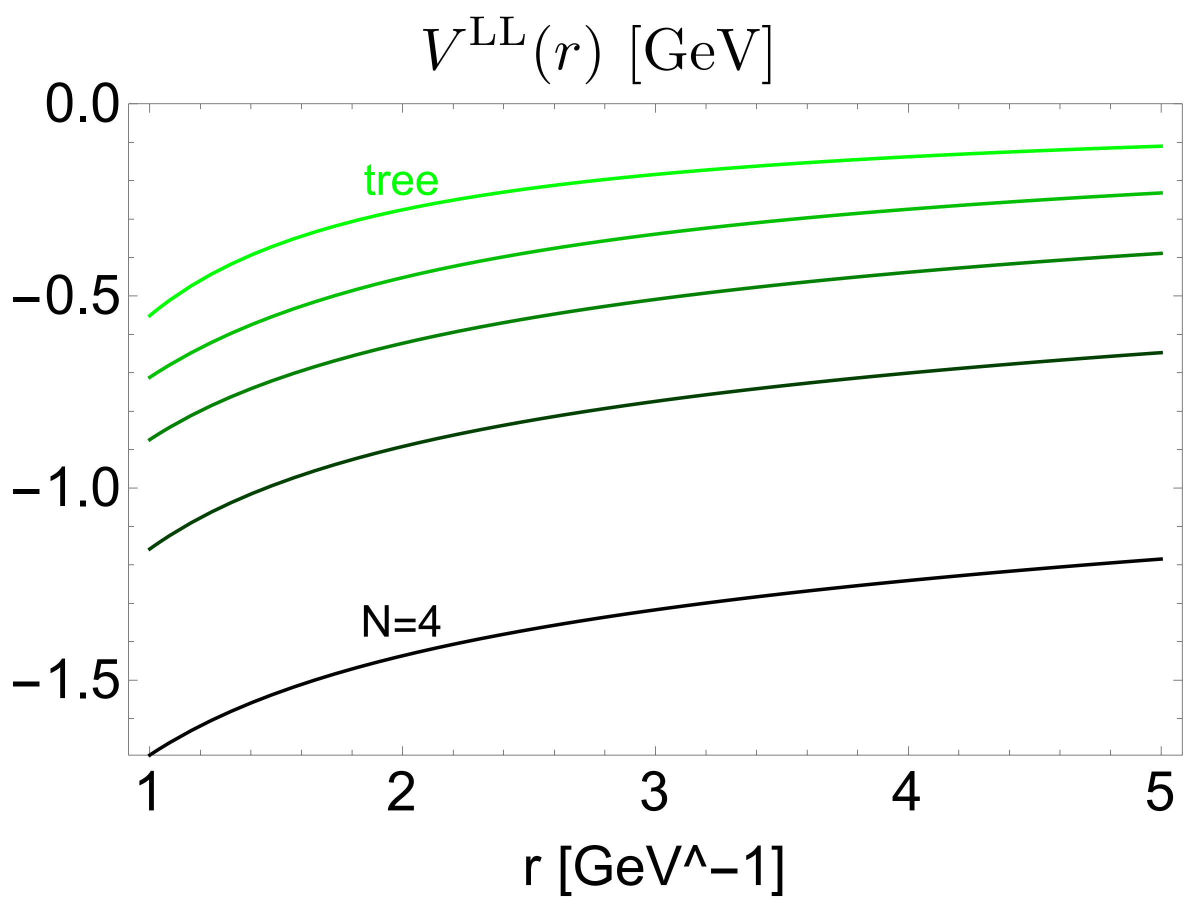
	UV renormalon	IR renormalon
origin	large momentum	small momentum
position	$u=-3/2,-5/2,-7/2,...$	$u=1/2,3/2,5/2,...$
sign of series	alternative sign	same sign
Borel sum	summable	not summable
Uncertainty estimation	$\left\lvert I_{u=-3/2}(a)-\sum_{n=0}^{N}(-a)^n n!\right\rvert < a^{N+1}(N+1)!$	$\frac{2\sqrt{2\pi}4\pi C_F}{\beta_0}\mu\ \mathrm{e}^{-\frac{2\pi}{\alpha_S\beta_0}} = \frac{2\sqrt{2\pi}4\pi C_F}{\beta_0}\Lambda_{\rm QCD}$

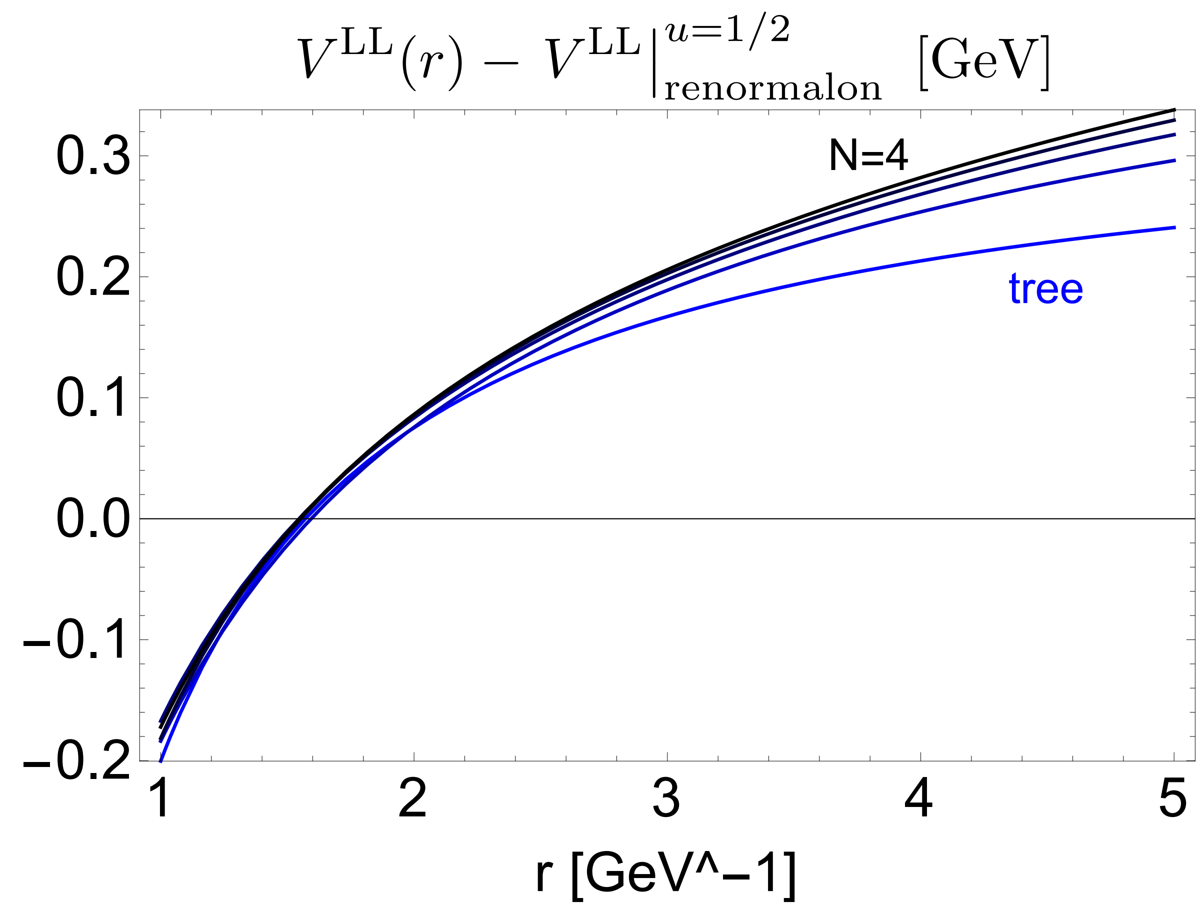
Let us see another aspect, potential.

$$V^{\mathrm{LL}}(r) = \int_{\mathbf{q}} V^{\mathrm{LL}}(\mathbf{q^2}) e^{\mathbf{iqr}} \qquad B[V^{\mathrm{LL}}(r)](u) = -\frac{C_F \alpha_S}{\sqrt{\pi} r} \left(\frac{\mu r}{2}\right)^{2u} \frac{\Gamma(1/2-u)}{\Gamma(1+u)}$$

$$V^{\mathrm{LL}}\Big|^{u=1/2}_{\mathrm{renormalon}} = -\frac{2 C_F \alpha_S \mu}{\pi} \sum_{n=0}^{N} \left(\frac{\alpha_S \beta_0}{2\pi}\right)^n n!$$

just a constant (r-independent)

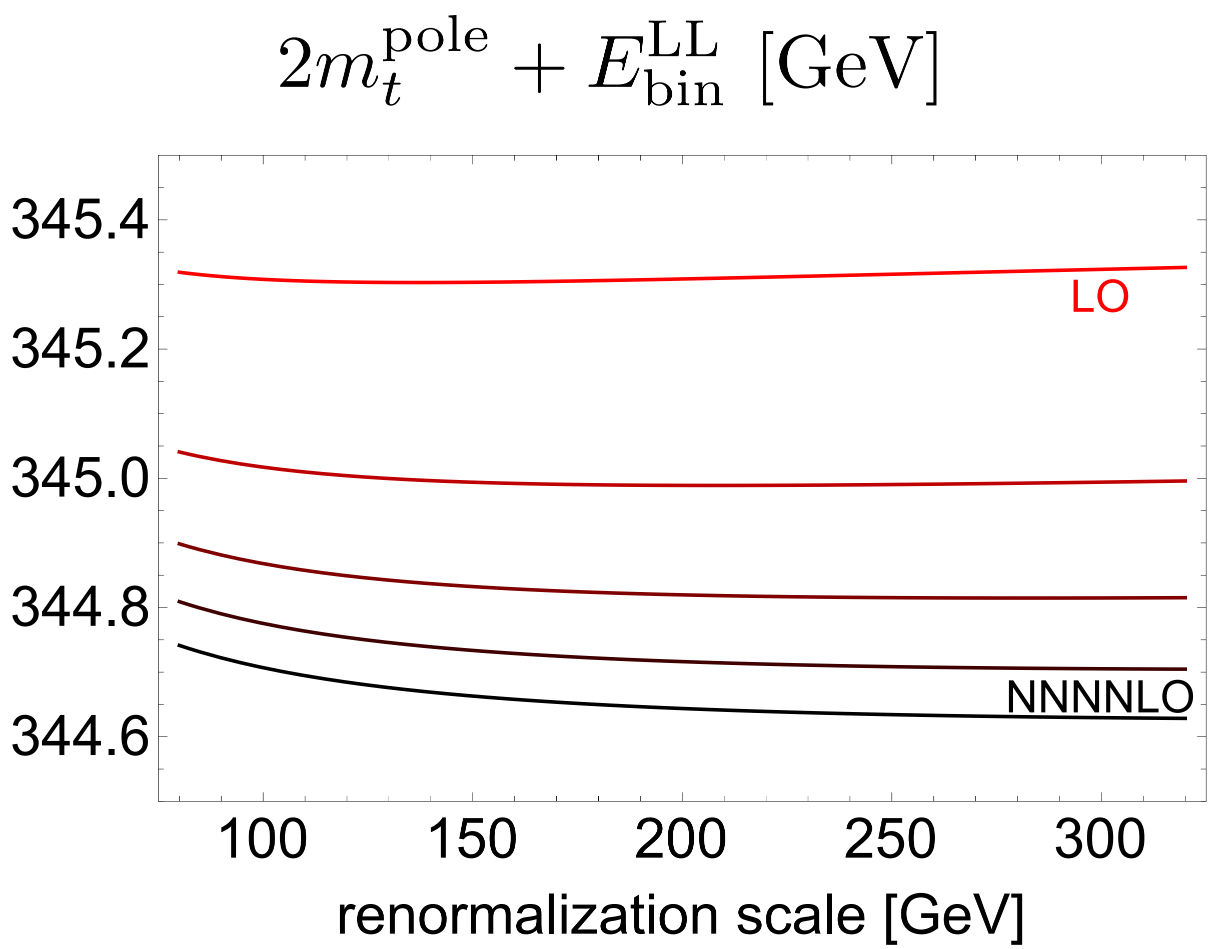


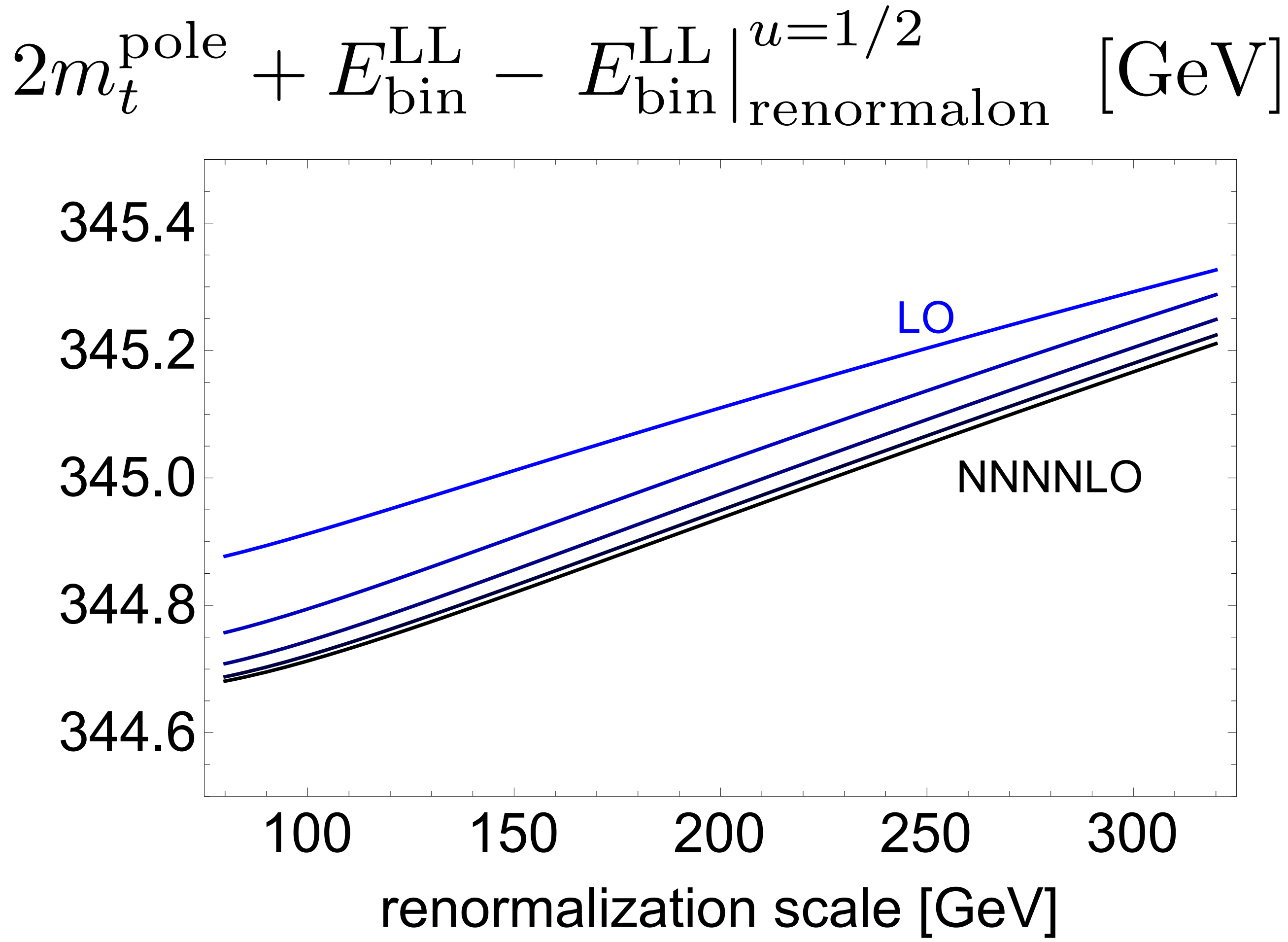


Source of the bad convergence is u=1/2 renormalon.

(constant shift of the potential)

Convergence of $E_{\rm bin}^{\rm LL}$ is also improved.





This may be reasonable once the convergence of potential is confirmed.

※ For the sake of easier comparison,
I adjust the value of $m_t^{\rm pole}$
as 173 GeV and 172.6 GeV respectively.

$\overline{\text{MS}}$ mass $m_{\overline{\text{MS}}}(\mu)$ $\bar{m} = m_{\overline{\text{MS}}}(m_{\overline{\text{MS}}})$

- defined perturbatively by subtracting UV-divergence
- Convenient and widely used choice of the renormalization scale is the mass of quark itself.

PS mass (potential-subtracted mass) $m_{\text{PS}}(\mu_f)$

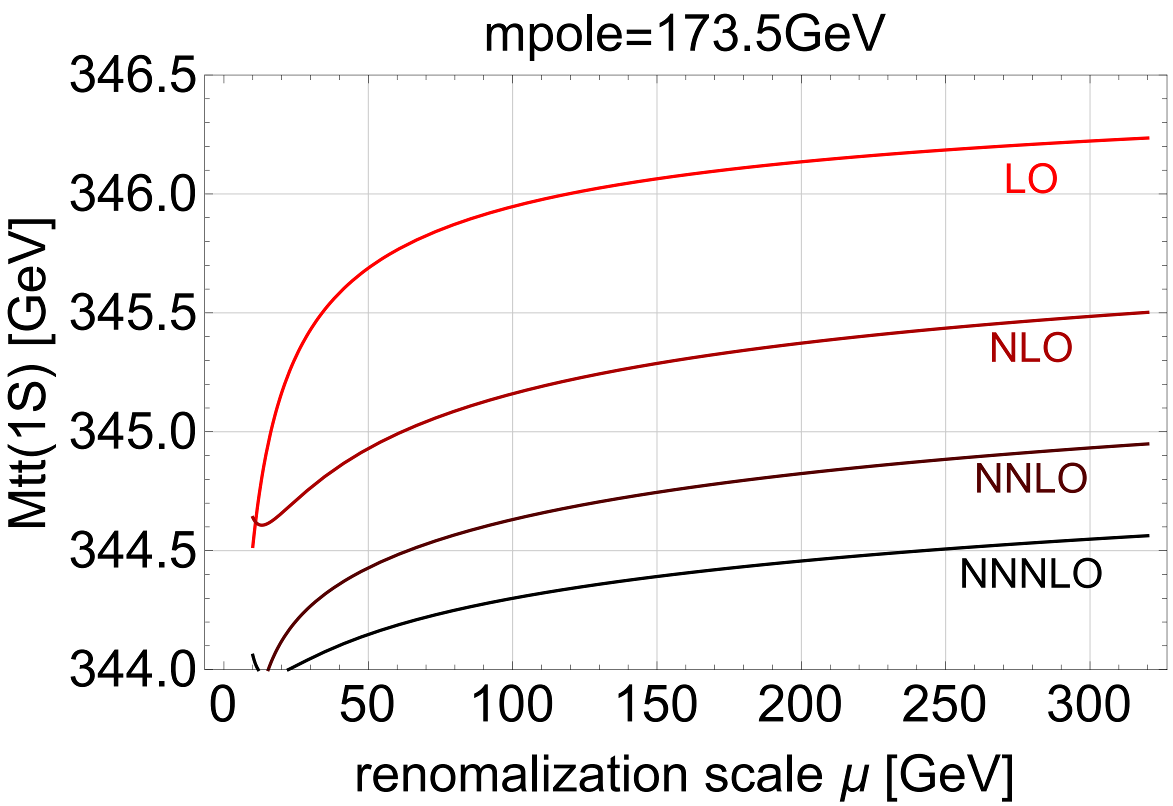
IR contribution of potential is subtracted.

$$m_{\text{PS}}(\mu_f) \equiv m_{\text{pole}} + \frac{1}{2}\int_{q \le \mu_f} V(\mathbf{q})$$

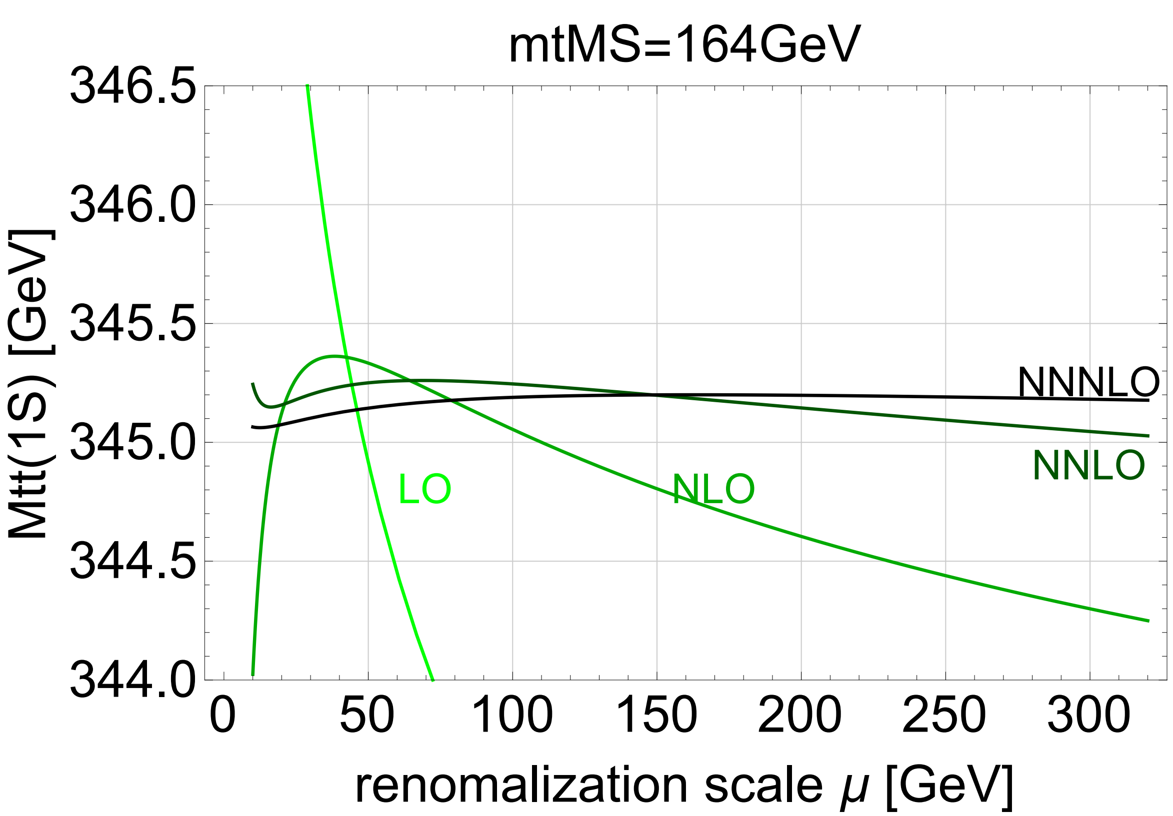
$$\mu_f \simeq \alpha_S m_{\text{pole}}$$

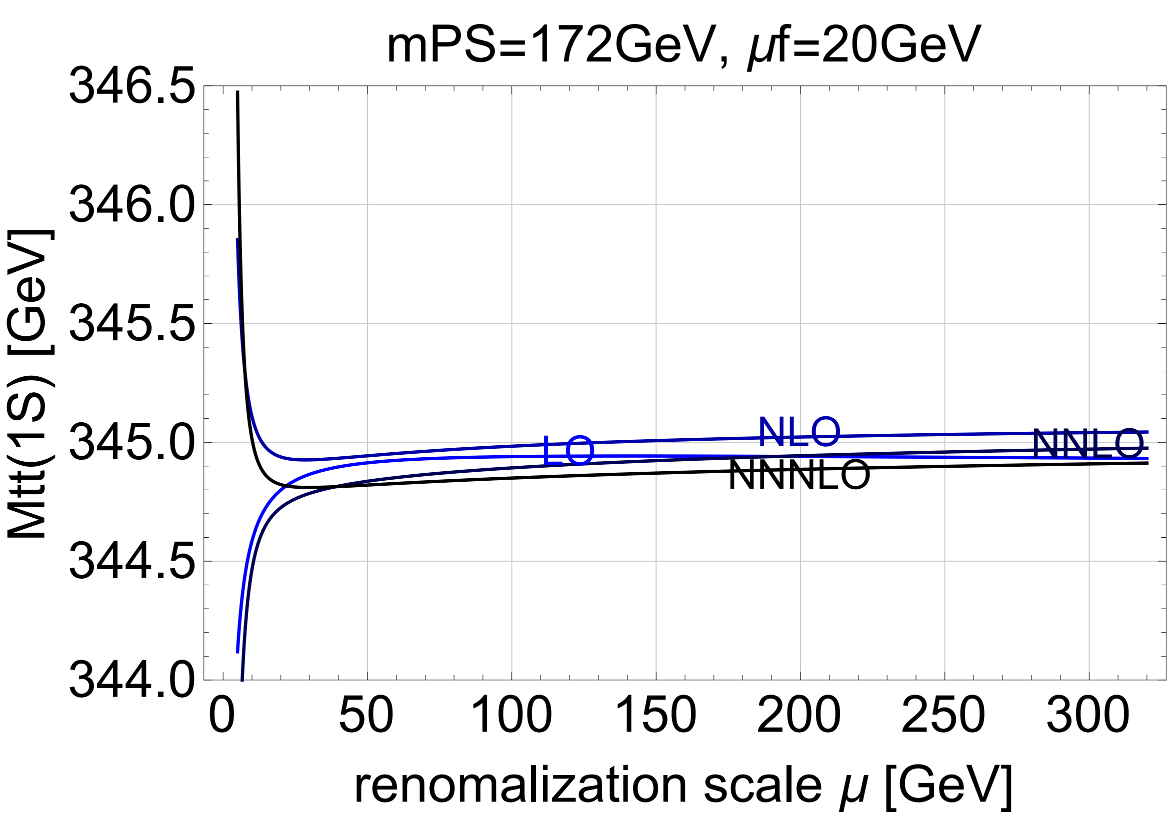
$$E_{\text{tot}}(r) = 2m_{\text{pole}} + V(r)$$
$$= 2m_{\text{PS}} + V(r) - \int_{q \le \mu_f} V(\mathbf{q})$$

Good convergence is achieved.



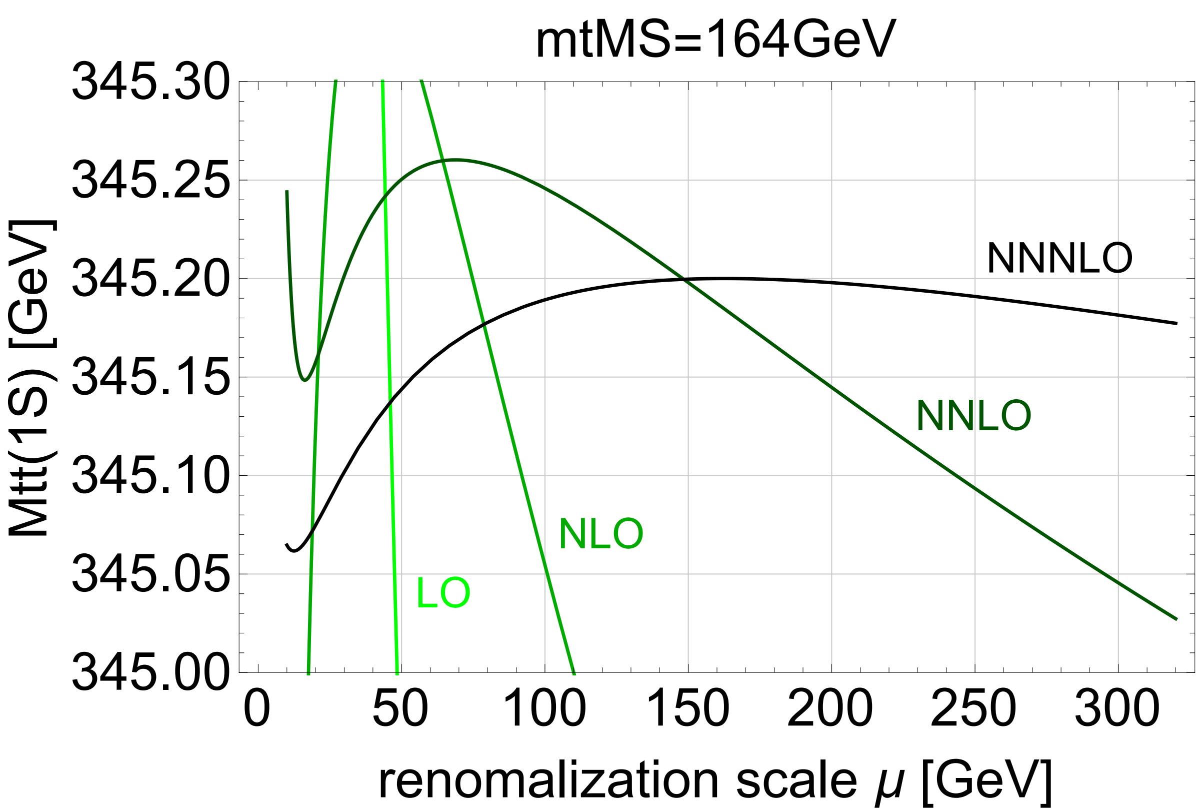
We will have closer look of
two bottom plots in the next slid.

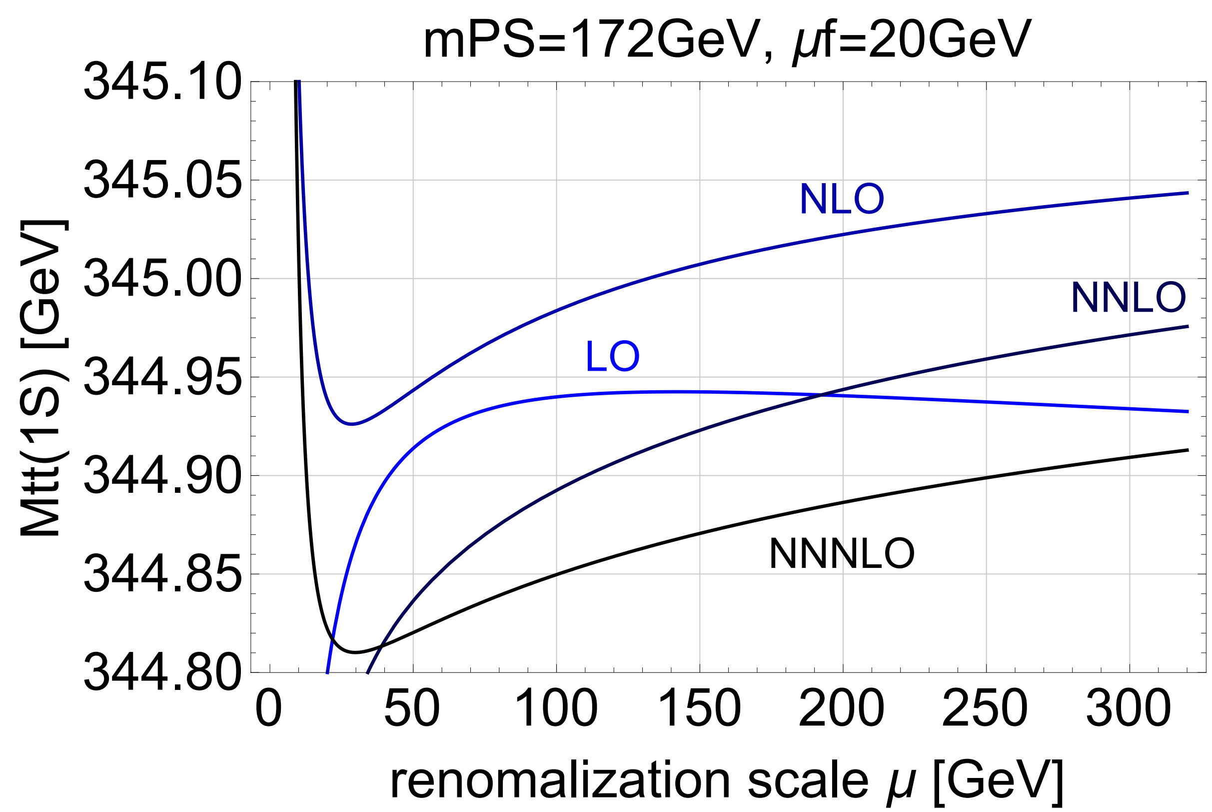




Prediction becomes extremely accurate.

Uncertainty from higher order correction can be estimated by scale dependence, because formally all-order calculation gives scale independent result.





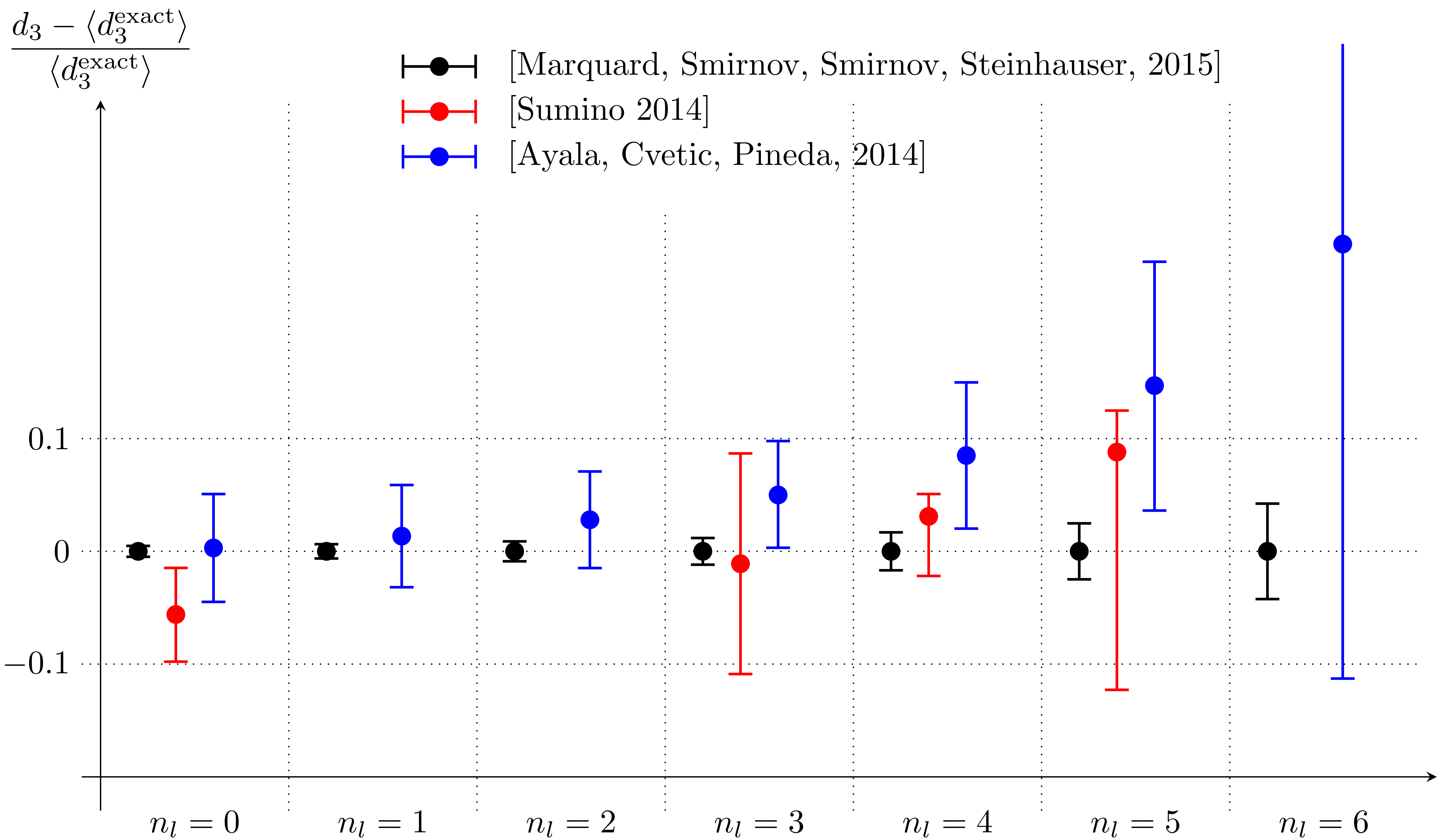
- There are minimal-sensitivity scales.
- NNLO-NNNLO ~ 60 MeV.

- Leading order prediction is stable.
- NNLO-NNNLO ~ 60 MeV

Uncertainty of top mass is about half of that of toponium.➔30 MeV

values of d3

n_l	0	1	2	3	4	5	6
$d_3^{\rm est}[3]$	3351(152)	—	—	1668(167)	1258^{+26}_{-66}	897^{+31}_{-175}	—
$d_3^{\rm est}[4]$	3562(173)	2887(133)	2291(98)	1772(82)	1324(81)	945(92)	629(191)
$d_3^{\rm exact}[1]$	3551.1(21.5)	2848.4(21.5)	2228.4(21.5)	1687.1(21.5)	1220.3(21.5)	824.1(21.5)	494.3(21.5)



Uncertainty from other sources