

# Soft theorem and large gauge transformation

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# Soft theorem

- Pion(axion), 60's

$$\lim_{\omega \rightarrow 0} \langle \text{out} | a_{\omega \hat{q}}^{(\pi)} \mathcal{S} | \text{in} \rangle = J^{(1)}(q) \langle \text{out} | \mathcal{S} | \text{in} \rangle$$

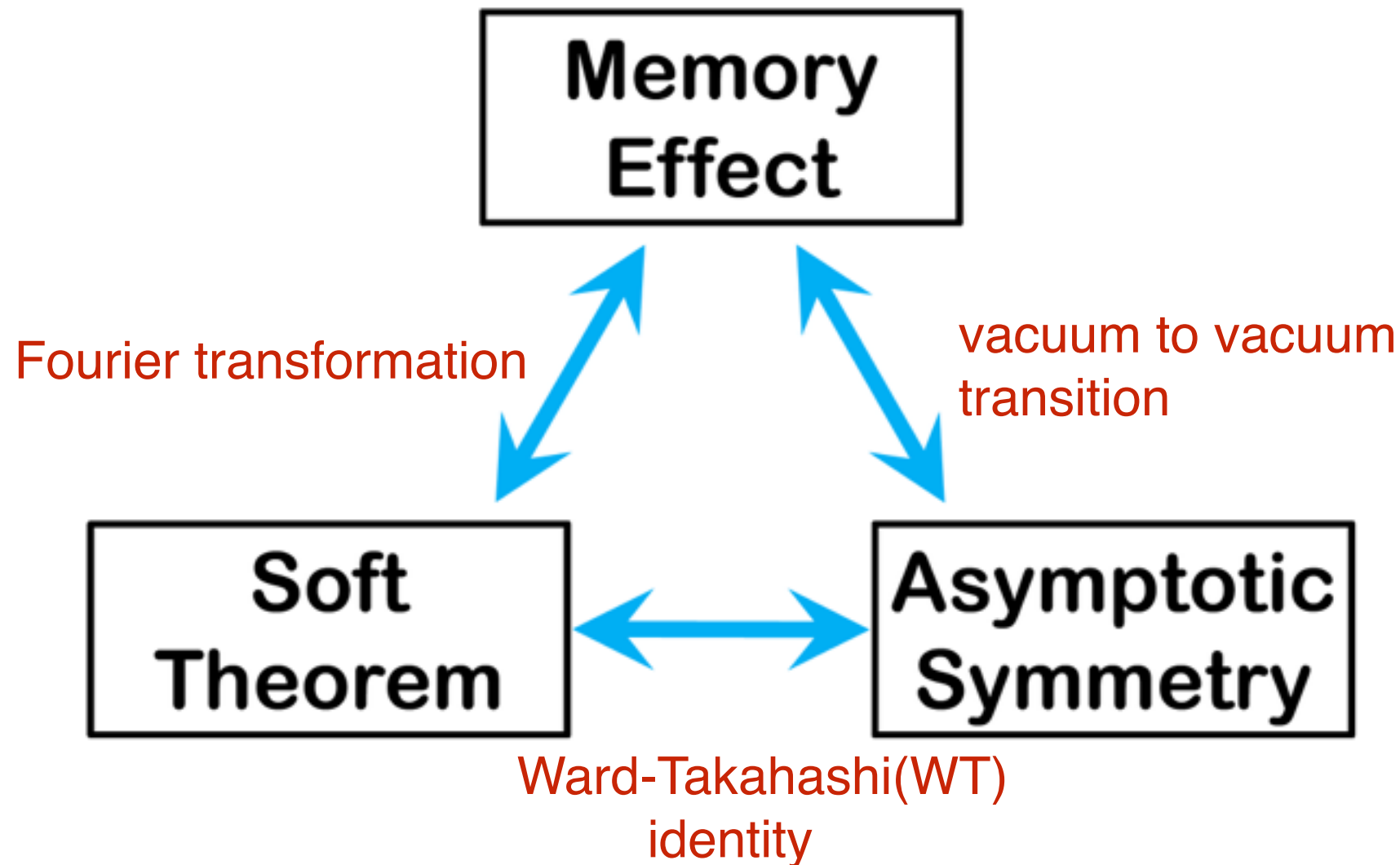
$$J^{(1)}(q) = \sum_k \frac{-iy \eta_k}{2m p_k \cdot q} \epsilon_{\mu\nu\rho\sigma} q^\mu p_k^\nu J_k^{\rho\sigma}$$

- Cosmology [Maldacena '02]

$$\lim_{\vec{q} \rightarrow 0} \frac{1}{P_\zeta(q)} \langle \zeta_{\vec{q}} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle' = -\vec{k}_1 \cdot \partial_{\vec{k}_1} \langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle'$$

# Triangular relation

[Strominger '13, ...]



# Talk Plan

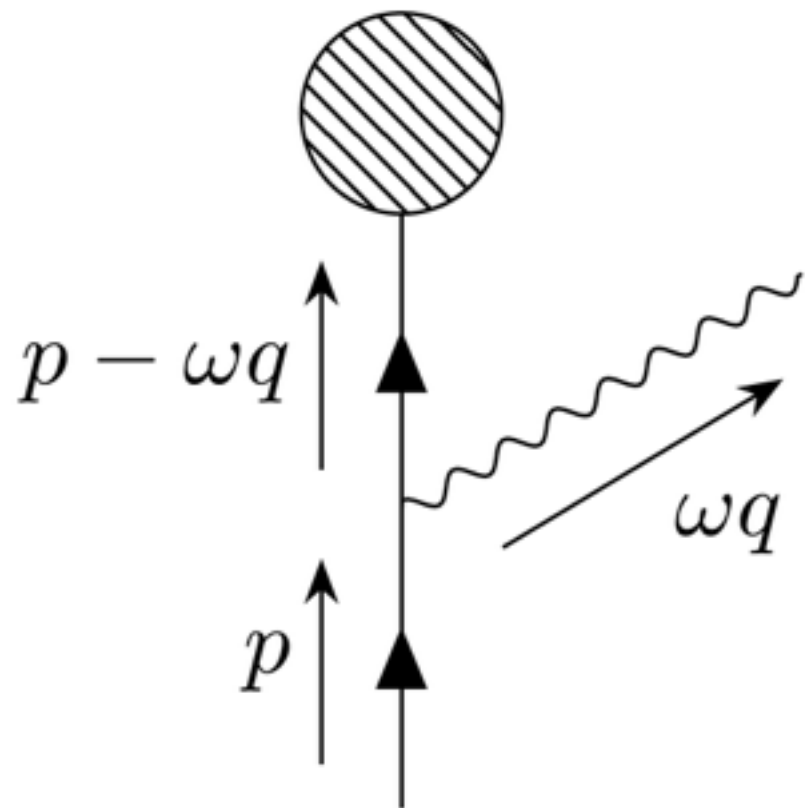
1. Triangular relation (review)
2. Large gauge transformation and soft theorem [YH, Shiu '18]
3. pion/axion memory effect [YH, Sugishita '17]
4. Memory effect in curved background [YH, Seo, Shiu '17]

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# Diagram w/ soft photon

- massless QED,  $\omega \rightarrow 0$ .



$$\begin{aligned}
 & \epsilon_\nu \frac{i}{p_\mu \gamma^\mu - \omega q_\mu \gamma^\mu} \gamma^\nu u(p) \\
 & \sim i\epsilon_\nu \frac{p_\mu \gamma^\mu}{-2\omega p \cdot q} \gamma^\nu u(p) \\
 & = -i\epsilon_\nu \frac{p^\nu}{\omega p \cdot q} u(p)
 \end{aligned}$$

**Soft factor**

Leading contribution comes from diagram where soft photon attaches to external line.

# Soft photon theorem

[Low '58]

- leading and subleading

$$\lim_{\omega \rightarrow 0} \langle \text{out} | a_{\omega \hat{q}}^{(\gamma)} \mathcal{S} | \text{in} \rangle = (J^{(1)} + J^{(2)}) \langle \text{out} | \mathcal{S} | \text{in} \rangle$$

$$J^{(1)} = e \sum_k Q_k \eta_k \frac{p_k \cdot \epsilon}{p_k \cdot q}$$

Leading

$$J^{(2)} = -ie \sum_k Q_k \eta_k \frac{q_\mu \epsilon_\nu J_k^{\mu\nu}}{p_k \cdot q}$$

Subleading

For spin 1/2

$$J_{\mu\nu} := -i \left( p_\mu \frac{\partial}{\partial p^\nu} - p_\nu \frac{\partial}{\partial p^\mu} \right) - \frac{i}{2} \gamma_{\mu\nu}$$

# Soft graviton theorem

- leading, subleading and subsubleading

$$\lim_{\omega \rightarrow 0} \langle \text{out} | a_{\omega \hat{q}}^{(\text{grav})} \mathcal{S} | \text{in} \rangle = (J^{(1)} + J^{(2)} + J^{(3)}) \langle \text{out} | \mathcal{S} | \text{in} \rangle$$

$$J^{(1)} = \sum_k \eta_k \frac{e_{\mu\nu} p_k^\mu p_k^\nu}{p_k \cdot q} \quad [\text{Weinberg '65}] \quad \text{Leading}$$

$$J^{(2)} = -i \sum_k \eta_k \frac{e_{\mu\nu} p_k^\mu q_\rho J_k^{\rho\nu}}{p_k \cdot q} \quad \text{Subleading}$$

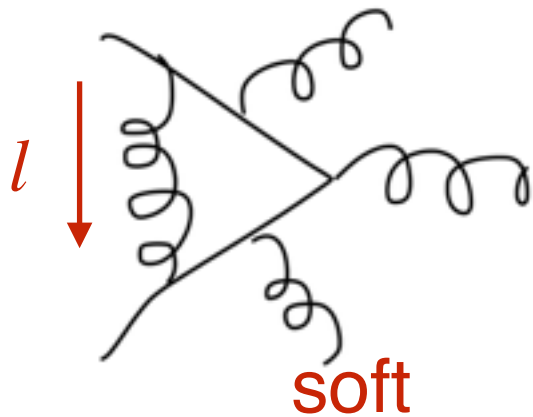
[Cachazo, Strominger '14]

$$J^{(3)} = -\frac{1}{2} \sum_k \eta_k \frac{e_{\mu\nu} q_\rho J_k^{\rho\mu} q_\sigma J_k^{\sigma\mu}}{p_k \cdot q} \quad \text{Subsubleading}$$



# Loop correction?

- Beyond leading order, there is **loop correction** to the soft theorem. related to IR div.



- Subleading theorem suffers from correction only from 1-loop diagram.

[Bern, Davies, Nohle '14, He, Huang, Wen '14]

- Subsubleading, 2-loop.

# Asymptotic sym.

- Asymptotically flat

$$\begin{aligned} ds^2 = & -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ & + \mathcal{O}\left(\frac{1}{r}\right) du^2 + \mathcal{O}\left(\frac{1}{r^2}\right) dudr + \mathcal{O}(1) dudz + \mathcal{O}(1) dud\bar{z} \\ & + \mathcal{O}(r) dz^2 + \mathcal{O}(r) d\bar{z}^2 + \mathcal{O}(r) dzd\bar{z}, \end{aligned}$$

- gauge choice (Bondi gauge)

$$g_{rA} = g_{rr} = 0, \qquad \det\left(\frac{g_{AB}}{r^2}\right) = (\gamma_{z\bar{z}})^2$$

# Supertranslation/rotation

- The transformation which preserves gauge&boundary condition.

**f: real function**

$$u \rightarrow u + \epsilon_u(u, r, z, \bar{z}),$$

$$r \rightarrow r + \epsilon_r(u, r, z, \bar{z})$$

$$z \rightarrow z + \epsilon_z(u, r, z, \bar{z}),$$

$$\bar{z} \rightarrow \bar{z} + \epsilon_{\bar{z}}(u, r, z, \bar{z})$$

$$\epsilon_u = -f(z, \bar{z}) + \int du \left[ \partial^{\bar{z}} \left( \frac{a_z(z)}{(1 + |z|^2)^2} \right) + \partial^z \left( \frac{a_{\bar{z}}(\bar{z})}{(1 + |z|^2)^2} \right) \right],$$

$$\epsilon_r = D^z D_z \epsilon_u - r \left[ \partial^{\bar{z}} \left( \frac{a_z}{(1 + |z|^2)^2} \right) + \partial^z \left( \frac{a_{\bar{z}}}{(1 + |z|^2)^2} \right) \right],$$

$$\epsilon_z = -\frac{1}{r} \partial^z \epsilon_u + a_z(z),$$

$$\epsilon_{\bar{z}} = -\frac{1}{r} \partial^{\bar{z}} \epsilon_u + a_{\bar{z}}(\bar{z}).$$

# Asymptotic charge

- For gravity, Only leading and subleading theorems can be understood as WT identity of asymptotic transformations.
- Leading soft theorem  $\longleftrightarrow$  Supertranslation
- Subleading soft theorem  $\longleftrightarrow$  Superrotation
- Subsubleading soft theorem  $\longleftrightarrow$  ? ? ?
- For U(1), only leading.

# Memory effect

- Solving classical equation of motion with fixed trajectory of charged particle.

$$y_n^\mu(\tau) = \frac{p_n^\mu}{m_n} \tau + x_0^\mu \quad (\tau < 0)$$

$$y_{n'}^\mu(\tau) = \frac{p_{n'}^\mu}{m_{n'}} \tau + x_0^\mu \quad (\tau > 0)$$

kick of the charged particle.

# Leading, subleading,...

$$u = t - r$$

permanent change of  $O(r^{-1})$

$$A^\mu(x) = \Theta(u) \sum_{n'} \frac{e_{n'} p_{n'}^\mu}{4\pi p_{n',r}^u} + \Theta(-u) \sum_n \frac{e_n p_n^\mu}{4\pi p_n^u r} - \delta(u) \left[ \sum_{n'} \frac{e_{n'} J_{n'}^{u\mu}}{4\pi p_{n',r}^u} - \sum_n \frac{e_n J_n^{u\mu}}{4\pi p_n^u r} \right]$$

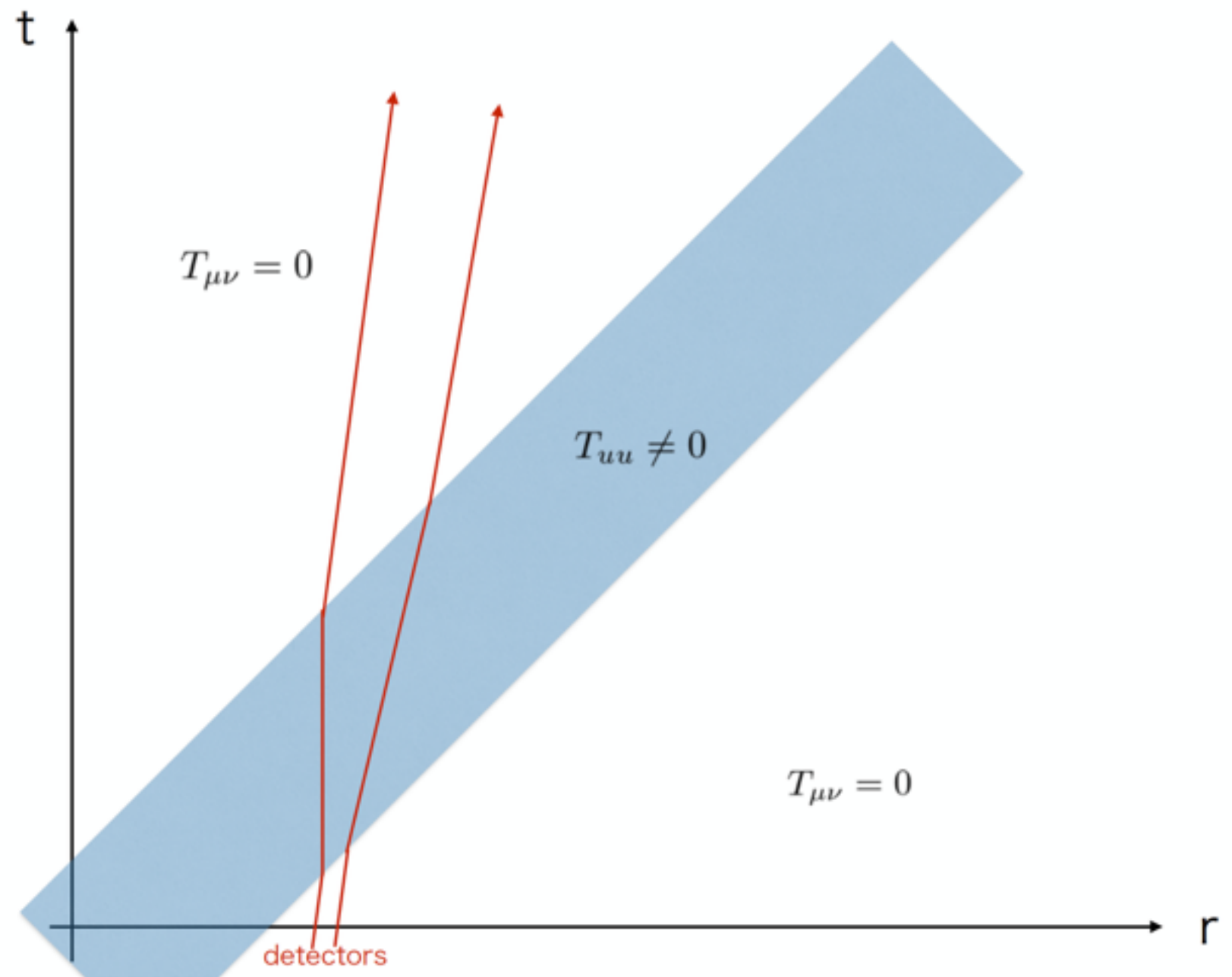
Leading memory effect

subleading memory effect

$$\begin{aligned} \frac{1}{16\pi G_N} \bar{h}^{\mu\nu}(x) = & \Theta(u) \sum_{n'} \frac{p_{n'}^\mu p_{n'}^\nu}{4\pi p_{n',r}^u} + \Theta(-u) \sum_n \frac{p_n^\mu p_n^\nu}{4\pi p_n^u r} - \delta(u) \left[ \sum_{n'} \frac{p_{n'}^\nu J_{n'}^{u\mu}}{4\pi p_{n',r}^u} - \sum_n \frac{p_n^\nu J_n^{u\mu}}{4\pi p_n^u r} \right] \\ & + \frac{1}{2} \delta'(u) \left[ \sum_{n'} \frac{J_{n'}^{u\mu} J_{n'}^{u\nu}}{4\pi p_{n',r}^u} - \sum_n \frac{J_n^{u\mu} J_n^{u\nu}}{4\pi p_n^u r} \right] \end{aligned}$$

# Gravitational memory effect

The proper distance between 2 particles changes.



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2. Large gauge transformation and soft theorem [YH, Shiu '18]
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# Our work

- We present different point of view.
- We consider the WT identity of  $O(x^n)$  gauge parameter, and show that this corresponds to  $O(q^n)$  soft photon/graviton theorem.
- We focus on the **tree level** formula.

# Large gauge $\text{tr.}(U(1))$

- After Lorenz gauge fixing, residual gauge tr. is

$$\partial^2 \chi = 0$$

- Explicitly,

$$\chi = \sum_{M=1}^{\infty} \eta_{i_1 i_2 \dots i_M} x^{i_1} x^{i_2} \dots x^{i_M}, \quad \eta_{i i i_3 \dots i_n} = 0,$$

- Large gauge tr. which does not fall off at infinity.

# WT identity

- Starting from

$$\lim_{R \rightarrow \infty} \langle 0 | (Q_{R,\alpha} E_1 B - B E_1 Q_{R,\alpha}) | 0 \rangle = \lim_{R \rightarrow \infty} \langle 0 | [Q_{R,\alpha}, B] | 0 \rangle$$

$E_1$ : projection on the zero-mass one-particle state

$Q_{R,\alpha}$ : charge of large gauge transformation

$B$ : arbitrary operator

- LSZ reduction will be performed to obtain relation in terms of amplitude.

$$Q_{R,\alpha} := \int d^3x f_R(\vec{x}) J_0 \quad f_R(x)=1, 0 \text{ for } |x| < R \text{ and } |x| > R$$

# LHS

$$f(\vec{x}) = \begin{cases} 1 & \text{for } |\vec{x}| < 1, \\ 0 & \text{for } |\vec{x}| > 1. \end{cases}$$

$$f_R(\vec{x}) := f(\vec{x}/R)$$

$$\begin{aligned} \lim_{R \rightarrow \infty} \int d^3x \, f_R(\vec{x}) j_0(\vec{x}) &= \lim_{R \rightarrow \infty} \int d^3k \, \tilde{f}_R(-\vec{k}) \tilde{j}_0(\vec{k}) = \lim_{R \rightarrow \infty} \int d^3k \, R^3 \tilde{f}(R\vec{k}) \tilde{j}_0(\vec{k}) \\ &= \lim_{R \rightarrow \infty} \int d^3k \, \tilde{f}(\vec{k}) \tilde{j}_0(\vec{k}/R) = \int d^3k \, \tilde{f}(\vec{k}) \lim_{\vec{q} \rightarrow 0} \tilde{j}_0(\vec{q}) \\ &= \lim_{\vec{q} \rightarrow 0} \tilde{j}_0(\vec{q}) \end{aligned}$$

# Soft photon insertion

- Second term survives after spatial integration and EoM

$$J_{0,U(1)} = \partial_i(A_0 \partial^i \chi) - \dot{A}_i \partial^i \chi + \chi \partial^2 A_0$$

- LHS becomes

$$\text{LHS} = \lim_{q \rightarrow 0} (\partial^i \chi) \big|_{x_j = i \partial_{qj}} \langle \vec{q}, i | B | 0 \rangle$$

# RHS

$$\lim_{q \rightarrow 0} \partial^i \chi \big|_{x_j = i \partial_{qj}} \langle \vec{q}, i | B | 0 \rangle = \langle 0 | \delta B | 0 \rangle$$

- U(1) field w/ charged scalar

$$A_\mu \rightarrow A_\mu + \partial_\mu \chi, \quad \psi \rightarrow e^{-ie\chi} \psi.$$

$$\chi = \sum_{M=1}^{\infty} \eta_{i_1 i_2 \dots i_M} x^{i_1} x^{i_2} \dots x^{i_M}, \quad \eta_{i i i_3 \dots i_n} = 0,$$

- $x \rightarrow -i \partial_p$  in momentum space.

$\delta A_\mu$  does not contribute to amplitude.

# Soft photon theorem

$$\begin{aligned}
 & \lim_{q \rightarrow 0} \eta_{i i_2 \dots i_M} \partial_{q_2} \dots \partial_{q_M} \mathcal{M}_{n+2m+1}^i(q; p, p', k) \\
 &= \lim_{q \rightarrow 0} \sum_{j=1}^m \eta_{i i_2 \dots i_M} \partial_{q_{i_2}} \dots \partial_{q_{i_M}} \quad \text{U(1) w/ charged scalar} \\
 & \times \left[ \Gamma^i(-p_j - q, p_j, q) S(p_j + q) \mathcal{M}_{n+2m}(p_j + q) \right. \\
 & \left. + \mathcal{M}_{n+2m}(p'_j + q) S(-p'_j - q) \Gamma^i(-p'_j - q, p'_j, q) \right] \\
 & + \frac{Q\phi}{M} \eta_{i_1 \dots i_M} [\partial_{p_{i_1}} \dots \partial_{p_{i_M}} \mathcal{M}_{n+2m} + (-1)^{M-1} (p \leftrightarrow p')]
 \end{aligned}$$

Note

$$\chi = \sum_{M=1}^{\infty} \eta_{i_1 i_2 \dots i_M} x^{i_1} x^{i_2} \dots x^{i_M}, \quad \eta_{i i i_3 \dots i_n} = 0,$$

# Note

- $M=1$  corresponds to leading and subleading soft theorem.
- $M\geq 2$  corresponds to the constraint on the subsub...leading soft behavior of amplitude.



# Graviton

- Similarly, the formula for soft graviton is

$$\begin{aligned}
 & \lim_{q \rightarrow 0} (\eta_{ij\alpha_2 \dots \alpha_M} + \eta_{ji\alpha_2 \dots \alpha_M}) \partial_{q_{\alpha_2}} \dots \partial_{q_{\alpha_M}} \mathcal{M}_{n+2m+1}^{ij} \\
 &= \lim_{q \rightarrow 0} \sum_l (\eta_{ij\alpha_2 \dots \alpha_M} + \eta_{ji\alpha_2 \dots \alpha_M}) \partial_{q_{\alpha_2}} \dots \partial_{q_{\alpha_M}} \\
 &\times \left[ \Gamma_{\text{matt}}^{ij}(q, p_l, -p_l - q) S(p_l + q) \mathcal{M}_{n+2m}(p_l + q) \right. \\
 &+ \mathcal{M}_{n+2m}(p'_l + q) S(-p'_l - q) \Gamma_{\text{matt}}^{ij}(q, p'_l, -p'_l - q) \\
 &+ \left. \Gamma^{(ij, i'_l j'_l, i_l j_l)}(q, k_l, -k_l - q) D^{(i'_l j'_l, i''_l j''_l)}(k_l + q) \mathcal{M}_{n+2m}(k_l + q)_{i''_l j''_l} \right] \\
 &+ (\text{All derivatives act on } \mathcal{M}_{n+2m}).
 \end{aligned}$$

# Note

- M=1 corresponds to leading and subleading soft theorem.

- For M=2, starting from ansatz,

$$\mathcal{M}_{n+1} \sim \{ (\text{leading}) + (\text{subleading}) \\ + A(\epsilon_{\mu\nu} p^\mu p^\nu)(q \cdot p) + B(\epsilon_{\mu\nu} p^\mu p^\nu)(q \cdot \partial_p) + C(\epsilon_{\mu\nu} \partial_p^\mu \partial_p^\nu)(q \cdot p) \\ + D(\epsilon_{\mu\nu} \partial_p^\mu \partial_p^\nu)(q \cdot \partial_p) + E(\epsilon_{\mu\nu} p_p^\mu \partial_p^\nu)(q \cdot p) + F(\epsilon_{\mu\nu} p_p^\mu \partial_p^\nu)(q \cdot \partial_p) \} \mathcal{M}_n$$

we can fix all coefficients.

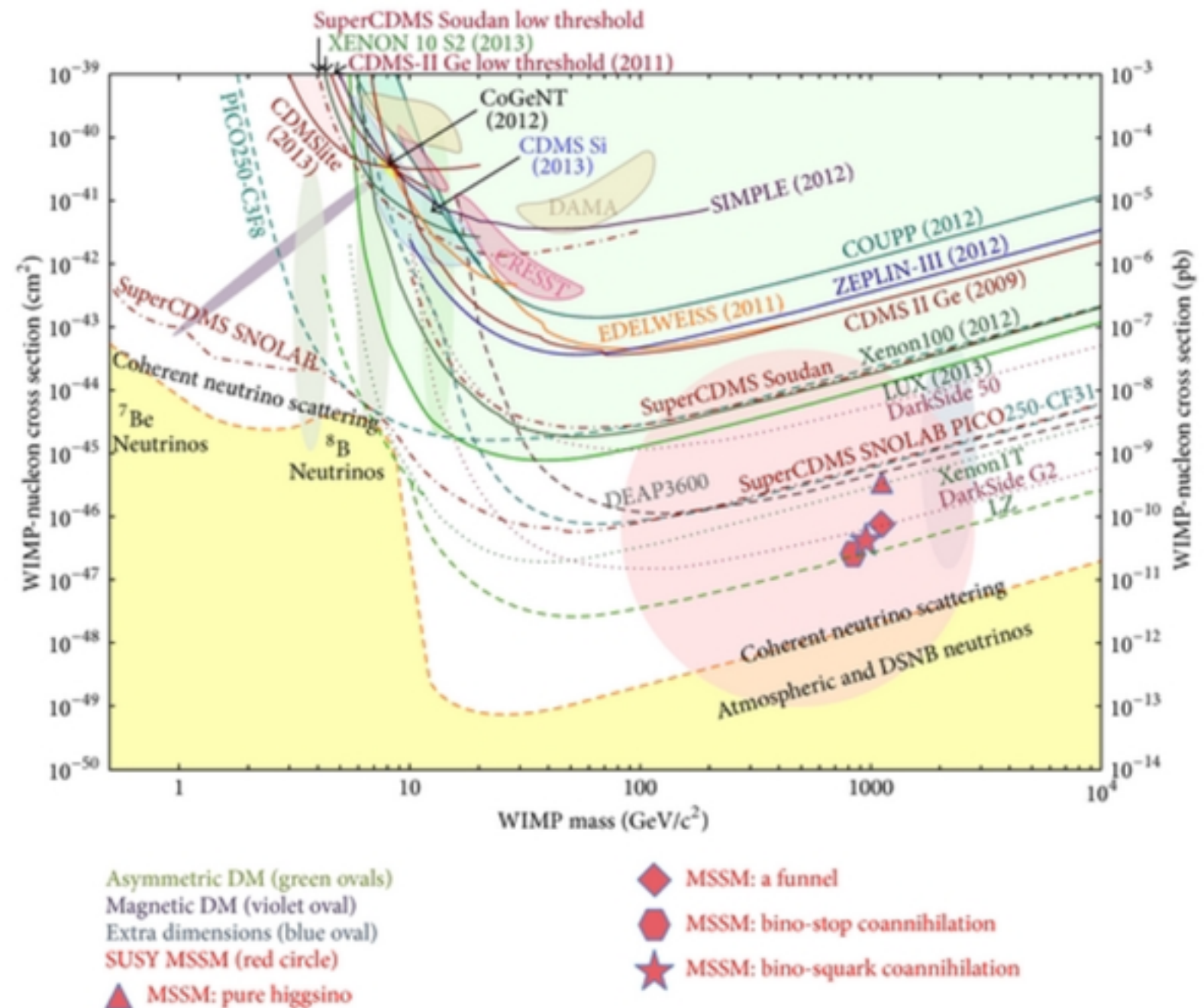
- M $\geq$ 3: Constraints on higher order terms.

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3. **pion/axion memory effect** [YH, Sugishita '17]
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# Dark matter

- WIMP dark matter has not been discovered yet.
- Alternative candidate is axion, whose coherent oscillation accounts for DM density.



# Kick of charged particle

- Axion memory

$$\begin{aligned} a(x) &= \epsilon_{\mu\nu\rho\sigma} \frac{1}{r} p^\mu \left( \Theta(u) \frac{p'^\nu J_{p'}^{\rho\sigma}}{\omega_{p'} (1 - \vec{v}' \cdot \hat{x})} - \Theta(-u) \frac{p^\nu J_p^{\rho\sigma}}{\omega_p (1 - \vec{v} \cdot \hat{x})} \right) + \mathcal{O}(r^{-2}) \\ &= -\epsilon_{\nu\rho\sigma}^u \frac{1}{r} \delta(u) \left( \frac{p'^\nu J_{p'}^{\rho\sigma}}{p'_r} + \frac{p^\nu J_p^{\rho\sigma}}{p_r} \right) + \mathcal{O}(r^{-2}) \\ &=: \frac{1}{r} \delta(u) S^{(1)} + \frac{1}{r^2} \Theta(u) S^{(2)}. \end{aligned}$$

- There is no particle which has monopole charge of pion/axion. (assumption: no explicit breaking)

# Detection of axion memory effect

- Non-relativistic Hamiltonian

$$H_{\text{NR}} = -\frac{p^2}{2M} - \kappa(\vec{\sigma} \cdot \vec{\nabla} a)$$

- Heisenberg eq.

$$\dot{s}_z = -\kappa(\vec{\nabla} \times \vec{\sigma})_z a.$$

New way of axion detection?

# Detection by spin

- Similarly, we can consider the detection of the GW memory effect by the spin direction.

$$H_{\text{NR, spin}} = -\frac{1}{4m}\epsilon^{kil}\sigma^l\{m(\partial_k h^{0i}) - p_j(\partial_k h^{ij})\}$$

$$\dot{s}^n = \frac{i}{4m}(\delta^{kn}\sigma^i - \delta^{in}\sigma^k)\{m(\partial_k h^{0i}) - p_j(\partial_k h^{ij})\}$$

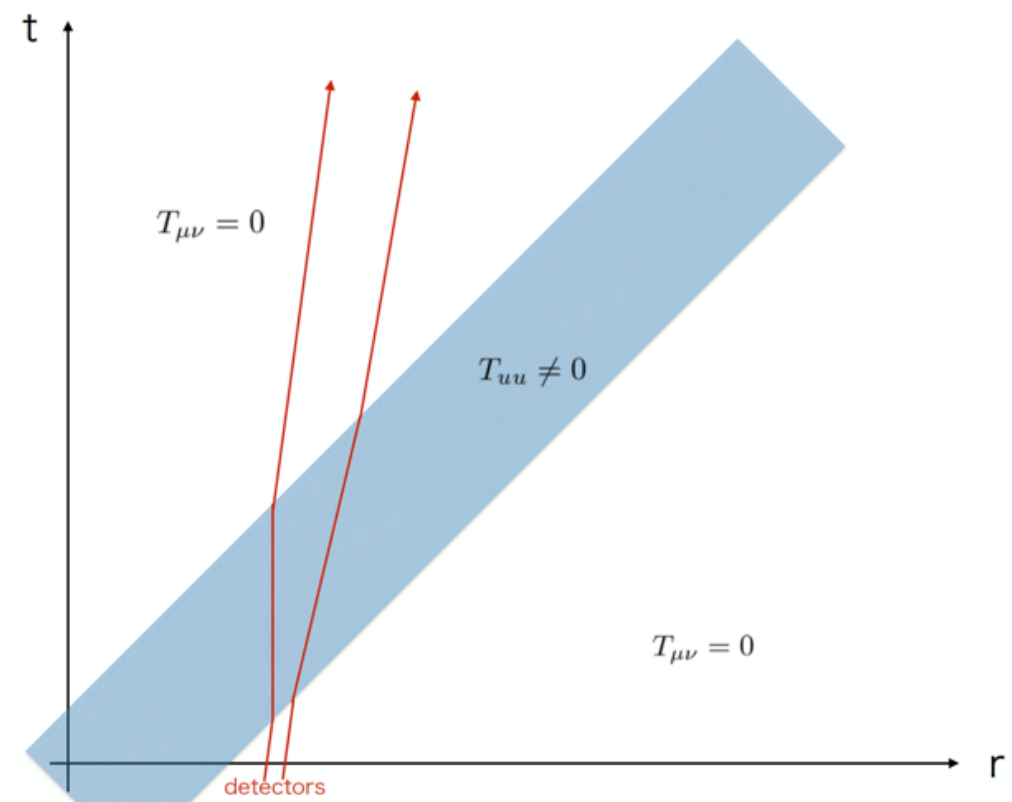
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# Memory in dS

- Current universe is close to dS<sub>4</sub>.
- The effect of the curved background could be important for the detection of the memory effect.



# Strategy

- dS + perturbation

$$\begin{aligned}
 ds^2 = & (-1 + r^2 H^2) du^2 - 2dudr + 2r^2 \gamma_{z\bar{z}} dz d\bar{z} && \text{Static patch} \\
 & + \left\{ \frac{m_B(u, z, \bar{z})}{r} - r H^2 h_{uu}(u, z, \bar{z}) \right\} du^2 + r C_{zz}(u, z, \bar{z}) dz^2 + r C_{\bar{z}\bar{z}}(u, z, \bar{z}) d\bar{z}^2 \\
 & + \{ U_{z1}(u, z, \bar{z}) + H^2 r^2 U_{z2}(u, z, \bar{z}) \} dudz \\
 & + \{ U_{\bar{z}1}(u, z, \bar{z}) + H^2 r^2 U_{\bar{z}2}(u, z, \bar{z}) \} dud\bar{z} + \frac{1}{r^2} (D_{ur1} + H^2 r^2 D_{ur2}) dudr \\
 & + \frac{1}{r} \left( \frac{4}{3} N_z + \frac{4}{3} u \partial_z m_B - \frac{1}{4} \partial_z (C_{zz} C^{zz}) \right) dudz + \frac{(1 + |z|^2)^2}{4} C_{zz} C_{\bar{z}\bar{z}} dz d\bar{z} + \dots
 \end{aligned}$$

- See the consequence of vacuum Einstein eq.

# Result

- From vacuum EoM

$$h_{uu} = -\frac{1}{2} (\partial^z U_{z2} + \partial^{\bar{z}} U_{\bar{z}2}), \quad C_{zz} = -D_z U_{z2}, \quad U_{z1} = D^z C_{zz}$$

$$C_{zz} = 2D_z^2 f \quad f=f(z,\bar{z}) \text{ is arbitrary real function.}$$

- Vacuum solution is parameterized by f.
- The change of f is given by

$$f_{\text{fin}} = - \int d^2 z' \gamma_{z' \bar{z}'} G(z, \bar{z}; z', \bar{z}') \left( m_B \Big|_{u=u_i}^{u=u_f} + \int_{u_i}^{u_f} du T_{uu} \right)$$

$$G(z, \bar{z}; z', \bar{z}') := -\frac{1}{\pi} \sin^2 \frac{\Theta}{2} \log \sin^2 \frac{\Theta}{2}, \quad \sin^2 \frac{\Theta}{2} = \frac{|z - z'|^2}{(1 + z' \bar{z}')(1 + |z|^2)}$$

# Comment

- Memory effect in dS is **same as the flat one** if the proper distance between the source and the detector is the same at the time of detection.

- Change of metric can be understood as BMS-like supertranslation.

$$\begin{aligned} u &\rightarrow u - f, \\ z &\rightarrow z + \frac{1}{r} D^z f, \end{aligned}$$

$$\begin{aligned} r &\rightarrow r - D^{\bar{z}} D_{\bar{z}} f, \\ \bar{z} &\rightarrow \bar{z} + \frac{1}{r} D^{\bar{z}} f. \end{aligned}$$

- Compared with previous results ([Yau et. al. '16, Wald et. al. '16]), **we can treat general situation** (e.g. extended source).

# Summary

- We consider the WT identity of the large gauge transformation.
- The  $O(q^n)$  soft photon/graviton theorem corresponds to WT identity of  $O(x^n)$  gauge parameter.
- The pion/axion memory effect is proposed.
- The memory effect in dS is considered.