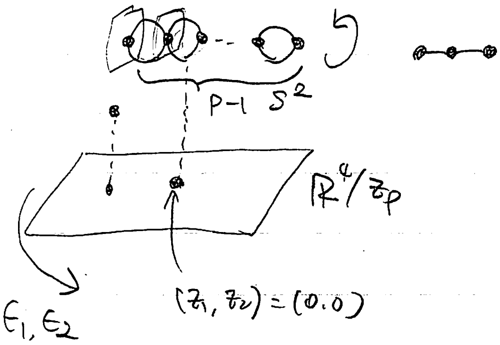
Instanton partition functions on A_{p-1} ALE space



$$\mathbb{R}^4 / \mathbb{Z}_p \xrightarrow{\text{resolve}} A_{p-1} \text{ ALE}$$

$\mathbb{R}^4 \ni (z_1, z_2)$, $\quad \mathbb{Z}_p$: $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to \begin{pmatrix} e^{2\pi i/p} & \\ & e^{\frac{-2\pi i}{p}} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

~~$\mathcal{N}=2$ $U(N)$~~

G : $U(1)^2 \times U(1)^N$

$U(1)^2$: $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to \begin{pmatrix} e^{i\epsilon_1} & \\ & e^{i\epsilon_2} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

$U(1)^N \longrightarrow$ gauge trans $\vec{a}$

- $Z^{A_{p-1}, \text{orbi}}_{\text{inst}} (\vec{a}, \epsilon_{1,2}) \xleftarrow[\mathbb{Z}_p\text{-inv. part}]{\text{choose}} Z^{\mathbb{R}^4}_{\text{inst}}$

- $Z^{A_{p-1}, \text{resolved}} (\vec{a}, \epsilon_{1,2}) = \sum_{\underbrace{C^{(r)}_\alpha}_{\text{flux}}} \prod_{r=0}^{p-1} {}^{\ell}\!\!\diagup\, Z^{\mathbb{R}^4}_{\text{inst}} (\vec{a}^{(r)}, \epsilon^{(r)}_{1,2})$

$$C^{(r)}_\alpha = \frac{1}{2\pi} \int_{r\text{-th } S^2} F_\alpha \in \mathbb{Z} \qquad \alpha = 1, \cdots, N$$

match for
- pure gauge thy.
- adj hyp.

mismatch for $\underbrace{N_f}_{\text{fund. hyp.}} = 2N$

Motivation AGT corresp.

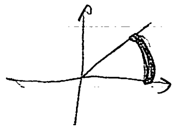
2D

- $Z^{\mathbb{R}^4,\, U(2),\, N_f = 4}_{\text{inst}} \longleftrightarrow$ (Free boson) $\times$ (Liouville)

$\langle V^{fb}_2 V^{fb}_3 \rangle \qquad \langle V_1 V_2 V_3 V_4 \rangle$

$Z^{\mathbb{R}^4/\mathbb{Z}_2}_{\text{inst}} \longleftrightarrow$ (free boson) $\times$ (affine alg) $\times$ (super Liouville)

$\mathrm{Hol} = e^{\oint A}$ $\quad \Theta = \begin{pmatrix} U_1 & & \\ & \ddots & \\ & & U_N \end{pmatrix}$ $\quad U_\alpha{}^p = 1$



origin $\quad c^{(r)} = \frac{1}{2\pi} \mathrm{Tr} \int_{r\text{-th } S^2} F \quad \in \mathbb{Z}$

$p-1$ S^2

$$Z^{A_{p-1}, \Theta}_{\mathrm{inst}, \mathrm{Hol}} = \sum_{\{c^{(r)}\}} Z_{\mathrm{inst}, \mathrm{Hol}, c^{(r)}}$$

$Z^{\mathrm{orbi.}}$

$$Z^{\mathbb{R}^4}_{\mathrm{inst.}} = \int_{\underbrace{M_{k\text{-inst}}}} e^{-S} = \sum_{\text{fixed pts.}} \frac{1}{\det_{TM_k}} \; G/\Theta \longrightarrow Z^{A_{p-1}, \mathrm{orbi.}}$$

k

choose Θ fixed pts., factors in $\det_{TM_k}$

pure gauge thy.

$U(2)$, $k=1$

$\mathbb{C}^2/\mathbb{Z}_2 \to A_1\text{-ALE}$

$c = 3$, $\mathrm{Hol} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$$Z^{\mathbb{R}^4}_{1\text{-inst}} = Z_{(\square, \cdot)} + \cancel{Z_{(\cdot, \square)}} \longrightarrow$$

$$Z_{(\square, \cdot)} = \frac{1}{\cancel{(a_1 - a_2)\epsilon_1 \epsilon_2 (a_1 - a_2 + \epsilon_1 + \epsilon_2)}} \longrightarrow 1$$

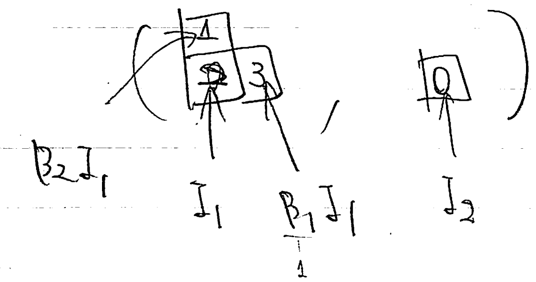
$$\mathcal{M}^{U(N)}_{k\text{-inst}} = \{ B_1, B_2, I, J \mid \mu = 0 \} / \underset{e^{i\phi}}{\underset{\Psi}{U(k)}}$$

B_1, B_2: $k\times k$; I: $k\times N$

$\cancel{U(2)}\ \ k=4$

ex) $U(2)$, $k=4$ $\qquad G\cdot(B_1, B_2, I, J) = e^{i\phi}\cdot(B_1, B_2, I, J)$ $\quad \exists \phi$

$\mathcal{M} \ni$



$I = \underbrace{\begin{pmatrix} I_1 & \cdots & I_N \end{pmatrix}}_{N}$ k

I_α, $\alpha = 1, \ldots, N$

	B_1	B_2	I_1	I_2
$\mathbb{Z}_\Phi$	1	-1	2	0

$\mathbb{Z}_1$ → B_1, $\mathbb{Z}_2$ → B_2

$I_\alpha \xrightarrow{\mathbb{Z}_\Phi} U_\alpha I_\alpha$

$I_1 \to$

I_2

$\mathbb{R}^4/\mathbb{Z}_4$, $\quad Hol = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$

$$= \begin{pmatrix} e^{\frac{2\pi i}{4}} & \\ & e^{\frac{0\cdot\pi i}{4}} \end{pmatrix} \Big\} N$$

# of box $\qquad$ # of box with $\mathbb{Z}_4 = 0$

$$k = k_0 + k_1 + k_2 + k_3 \qquad (k_0, k_1, k_2, k_3) = (1, 1, 1, 1)$$

$$N = N_0 + N_1 + N_2 + N_3 \qquad (N_0, N_1, N_2, N_3) = (1, 0, 1, 0)$$

For given $\{c^{(r)}\}_{r=1,2,3}$, $\qquad c^{(r)} = N_r - 2k_r + k_{r-1} + k_{r+1}$

choose fixed pts that satisfy

$V = \mathbb{C}^k$, $W = \mathbb{C}^N$ $\qquad B_{1,2} : V \to V$

$$\underset{X_0}{\boxed{V}} \xrightarrow{\alpha} \underset{X_1}{\boxed{\begin{matrix} Q\otimes V \\ \oplus \\ W \end{matrix}}} \xrightarrow{\beta} \underset{X_2}{\boxed{V}}$$

$Q = R_1 + R_{-1}$

$Q\otimes V = \sum_r k_r (R_{r+1} + R_{r-1})$

$W = \sum_r N_r R_r$

$V = \sum_r k_r R_r$

$$E = \frac{\ker B}{\operatorname{Im} \alpha} \qquad c(E) = c_1(X_1) - c_1(X_2) - c_1(X_0)$$

choose factors in det L

	ϵ_1	ϵ_2	a_1	a_2
$\mathbb{Z}_p$	1	-1	2	0

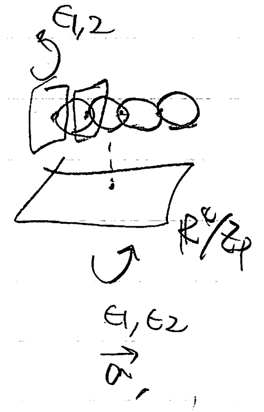
$$Z^{\mathbb{R}^4}_{(\square\!\square,\square)} = \frac{1}{\cancel{(-a_1+a_2)}\,(a_1-a_2+2\epsilon_1)\cdots}$$

$-2+0$ $2-0+2=4\equiv 0$

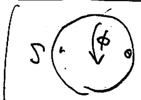
[Kronheimer–Nakajima '90]
[Fucito et al '04]

- $Z^{\text{resolved}}_{A_{p-1}}$ $p=2$ [Fucito et al '09] $p\geq 2$ [Bonelli et al '12]

$$\underline{Z^{A_{p-1},\,\text{resolved}}_{\text{full}\,(Hol)}} = \sum_{\substack{c_\alpha \\ (Hol)}} \prod_{r=0}^{p-1} \underline{Z^{\mathbb{R}^4}_{\text{full}}(\vec{a}^{(r)}, \epsilon^{(r)}_{1,2})}$$



$Z^{\mathbb{R}^4}_{\text{class}}\, Z^{1\text{-}l}_{\cancel{\text{class}}}\, Z^{\text{inst.}}_{\text{class}}$



$(A_N)_\alpha = (A_S)_\alpha + \partial_\alpha \Psi(\phi)$

$\cancel{W^{(1)}} = e^{i\Psi(\phi)} W^{(0)}$

$\frac{1}{2\pi}\int_{S^2} F_\alpha = c_\alpha \in \mathbb{Z}$

$\Psi_\alpha(\phi+2\pi) - \Psi_\alpha(\phi) = -2\pi c_\alpha$

$a^{(1)}_\alpha = a^{(0)}_\alpha - \epsilon^{(1)}_1 c^{(1)}_\alpha$

- $Z^{A_{p-1}}_{\text{Class}}\, Z^{A_{p-1}}_{1\text{-}l} \sum_c Z^{A_{p-1}}_{\text{inst},\,c}$

$$Z^{A_{p-1}}_{\text{inst},\,c} = \sum_{\substack{c_\alpha \\ Hol \\ c\,|_{\text{fixed}}}} \prod_{r=0}^{p-1} \frac{Z^{\mathbb{R}^4}_{1\text{-}l}(\vec{a}^{(r)}, \epsilon^{(r)})}{Z^{A_{p-1}}_{1\text{-}l}} \cdot Z_{\text{inst.}}(\vec{a}^{(r)}, \epsilon^{(r)})$$

choose $\mathbb{Z}_p$-inv. factor

$Z^{\mathbb{R}^4}_{1\text{-}l}$

$a^{(r)}_\alpha$ depends on a_α, $\epsilon^{(r)}_{1,2}$, $c^{(r)}_\alpha$ $c^{(r)} = \sum_{\alpha=1}^{N} c^{(r)}_\alpha$

$a^{(r)}_\alpha \xrightarrow{\mathbb{Z}_p} a^{(r)}_\alpha$

require

$a_\alpha \xrightarrow{\mathbb{Z}_p} a_\alpha + \frac{2\pi}{p}\cdot$ [illegible] Holonomy

$U(N)$, $N_f = 4$

$$Z^{A_p,\,\mathrm{resolved}}_{\mathrm{inst},\,\mathrm{Hol},\,c^{(r)}}(\vec{a}, m_i, \epsilon_{1,2}) = \begin{cases} Z^{A_p,\,\mathrm{orb}} & c^{(r)} > 0 \quad \forall r \\ Z^{A_p,\,\mathrm{orb}}(-\vec{a}, \epsilon_1+\epsilon_2-m_i, \epsilon) \times (1-(-1)^N q)^{u_{p,N}} & c^{(r)} < 0 \quad \forall r \end{cases}$$

$$u_{p,N} = \frac{(\epsilon_1+\epsilon_2)\left(2\sum a_\alpha + \sum m_i\right)}{p\,\epsilon_1\epsilon_2}$$

AGT for A_1-ALE

2D CFT

$U(2)$, $N_f=4$ $\quad Z^{A_1\ \mathrm{resolved}}_{\mathrm{inst}} = (\text{Free boson}) \times (\text{affine alg}) \times (\text{super Liouville})$

$\langle V^{fb}_2(1) V^{fb}_3(q)\rangle \sim (1-q)^{\#}$ $\qquad \langle V^{sL}_1(\infty) V^{sL}_2(1) V^{sL}_3(q) V^{sL}_4(0)\rangle$

$V^{fb}_2 = e^{\alpha_2\phi}$

$$u_{p=1,\,N=2} = \frac{\langle V^{fb}_2 V^{fb}_3\rangle}{\langle V^{fb}_2 V^{fb}_3\rangle\big|_{m\to\epsilon_1+\epsilon_2-m}}$$

	resolved	orb
$c>0$	$V^{fb}_\alpha V^{rest}_\alpha$	$V^{fb}_\alpha V^{rest}_\alpha$
$c<0$	$V^{fb}_\alpha V^{rest}_{\epsilon_+-\alpha}$	$V^{fb}_\alpha V^{rest}_\alpha$

$$V^{fb}_\alpha V^{rest}_{\epsilon_+-\alpha} = \underbrace{V^{fb}_{\epsilon_+-\alpha} V^{rest}_{\epsilon_+-\alpha}}_{= Z^{orb}(\epsilon_+-m)} \times \frac{V^{fb}_\alpha}{V^{fb}_{\epsilon_+-\alpha}}$$