行列の幾何と コヒーレント状態

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Introduction

- MR is a sequence of linear maps $(T_N)_N \ (N=1,2,\cdots\infty)$

$T_N : C^\infty(\mathcal{M}) \to M_N(C)$ is a linear map satisfying

$$\begin{cases} \lim_{N\to\infty} ||T_N(f)T_N(g) - T_N(fg)|| = 0 & \text{Preserving the product} \\ \lim_{N\to\infty} ||iN[T_N(f), T_N(g)] - T_N(\{f,g\})|| = 0 & \text{Quantizing Poisson alg.} \\ \lim_{N\to\infty} \mathrm{Tr}T_N(f) = \int f & \text{Integral} \Leftrightarrow \text{Trace} \end{cases}$$

[Arnlind-Hoppe-Huisken]

- For any compact Kahler manifold (including any Riemann surfaces), MR can be constructed by the Toeplitz quantization. [Bordemann-Meinrenken-Shlichenmaier]

Introduction

- Matrix models $\Rightarrow$ Nonperturbative formulation of M/string theories. [BFSS, IKKT, DVV]

$$Z = \int \prod_{\mu=1}^{D} DX^{\mu} e^{-S}$$

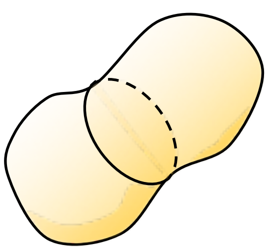
$X^{\mu} : N \times N$ Hermitian matrices

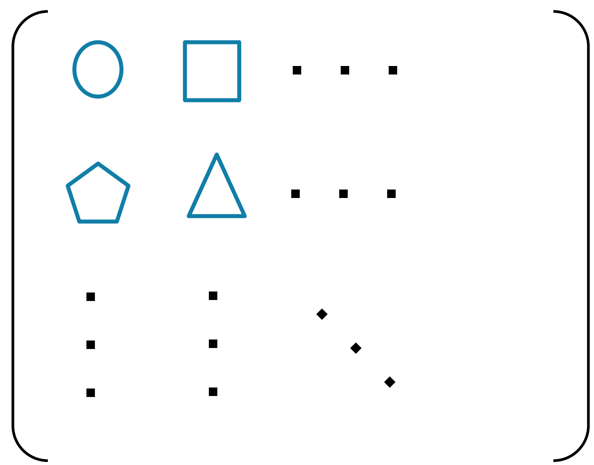
"quantized version" of embedding function

$$y^{\mu} : \mathcal{M} \to R^{D}$$

Fundamental objects $\Rightarrow$ Matrix configurations

- Membranes
- Strings
- D-branes





[Hoppe, de Wit-Hoppe-Nicolai]

Problem

How are matrix configurations related to shapes of strings/membranes ?

Problem 1 : Given geometry $\Rightarrow$ Construct matrices

- Construction of the matrix regularization
- Geometric/Toeplitz quantization

Problem 2 : Given matrices $\Rightarrow$ Find the underlying geometry $\leftarrow$ This talk

- Inverse of the problem 1

[Hanada-Kawai-Kimura] $\Rightarrow$ Geometry from Infinitely large matrices.

Motivation

◇ Understanding matrix model as gravitational theory

◇ Numerical simulations of matrix models,

[Kim-Nishimura-Tsuchiya, Anagnostopoulos-Hanada-Nishimura-Takeuchi, Catterall-Wiseman, Hanada-Hyakutake-Ishiki-Nishimura, Kadoh-Kamata, Filev-O'Connor, Berkovitz-Hanada-Maltz, Asano-Filev-Kovacik-O'connor ....]

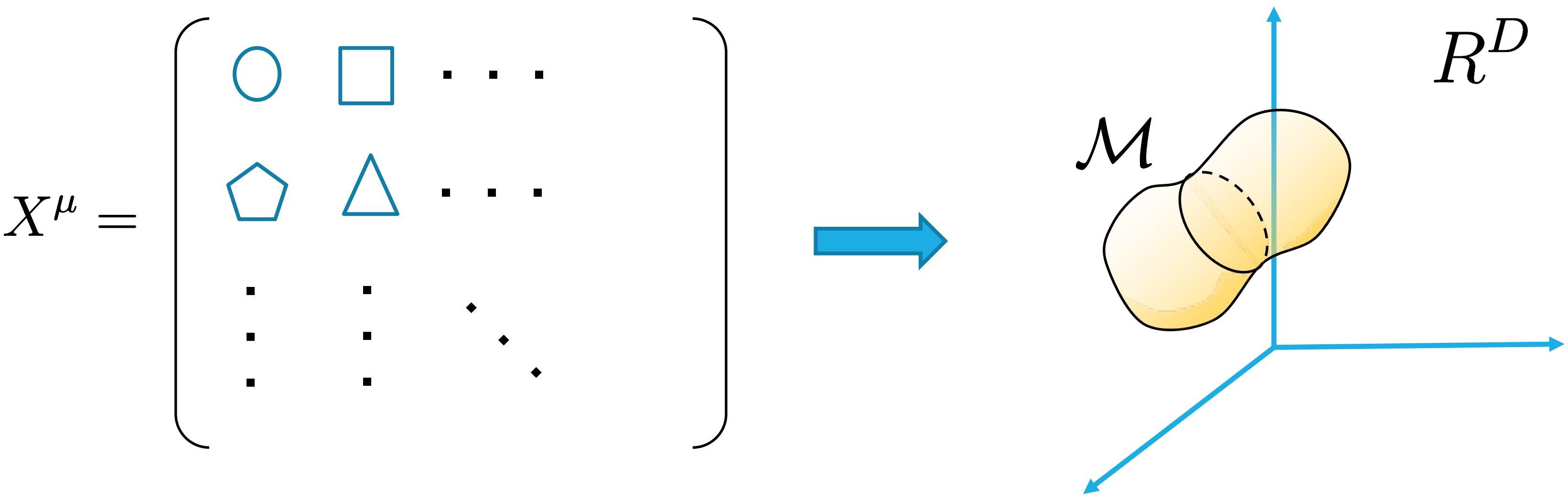
Path-integrals of matrices will numerically generate matrix configurations like

$$X^1_{11} = 2.53..., X^1_{12} = 1.62... + i3.24..., \cdots$$

From these numbers, how can we recover geometry of strings/membranes/D-branes?

Matrices $\Rightarrow$ Geometry ?

◇ For given matrices X^μ, how can we associate a “classical space” $\mathcal{M} \subset R^D$?



- Dirac operator [Berenstein-Dzienkowski, Asakawa-Sugimoto-Terashima]
- Coherent states [Ishiki, Schneiderbauer-Steinaker] ← This talk
- Morse theory [Shimada, …]

$\vdots$

Contents

2. MATRIX GEOMETRY AND COHERENT STATES

How can we recover R² from QM

We know that classical geometry of 1D quantum mechanics (NC plane) is R²

$[X_1, X_2] = i\hbar$ ⟹ plane (x_1, x_2)

But how can we find this?

$\Rightarrow$ COHERENT STATES

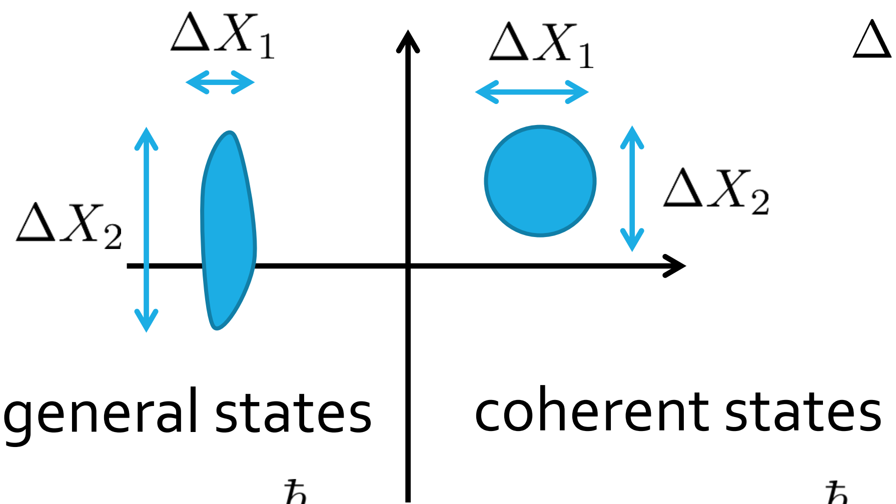
$$[X_1, X_2] = i\hbar \Longrightarrow \{\text{Canonical coherent states}\} \sim R^2$$

[Cf. H. Grosse and P. Presnajder]

Canonical Coherent states

$$[X_1, X_2] = i\hbar$$

$$\Delta\mathcal{O} = \sqrt{\langle\mathcal{O}^2\rangle - \langle\mathcal{O}\rangle^2}$$



$$\Delta X_1 \Delta X_2 > \frac{\hbar}{2}$$

$$\Delta X_1 \Delta X_2 = \frac{\hbar}{2}$$

$$\Delta X_1 = \Delta X_2 = \sqrt{\hbar/2} \to 0 \quad (\hbar \to 0)$$

◇ Canonical coherent states have minimal wave packets, which shrink to a point in the classical limit.

◇ There exists a coherent state at every point on the phase space.

Coherent states ⇔ points on classical space

Coherent States $\Rightarrow$ Geometry

Given matrices

$$X^\mu$$

Construct states satisfying

$$\Delta X^\mu \to 0$$

{ A set of all such states } $\sim$ classical space ?

◇ This works for
- Fuzzy plane (canonical coherent states)
- Fuzzy sphere (Bloch coherent states)

◇ Let us generalize this for more general X^μ

Common feature of coherent states

◇ Coherent states can be defined as the ground states of $H(y)=\frac{1}{2}(X^\mu-y^\mu)^2$

- Canonical coherent states $[X_1,X_2]=i\hbar$

$$H(q,p)=\frac{1}{2}(X_1-q)^2+\frac{1}{2}(X_2-p)^2 \qquad (q,p)\in R^2$$

- Bloch coherent states $X_i=\frac{2}{N}L_i$ N-dim irrep of SU(2) generators

$$H(y)=\frac{1}{2}(X_1-y_1)^2+\frac{1}{2}(X_2-y_2)^2+\frac{1}{2}(X_3-y_3)^2 \qquad (y_1,y_2,y_3)\in R^3$$

This Hamiltonian is useful to find the shape of the classical space

Classical space = loci of zeros of $E_0(y)$ in the classical limit

$(\hbar\to 0,\ N\to\infty)$

Classical Geometry as Zeros of E_0

$$E_0(y) = \langle H(y) \rangle = \frac{1}{2}\langle (X^\mu - y^\mu)^2 \rangle = \frac{1}{2}\langle X_\mu^2 \rangle - \langle X^\mu \rangle y_\mu + \frac{1}{2} y_\mu^2$$

$$= \frac{1}{2}\langle X_\mu^2 \rangle \underline{- \frac{1}{2}\langle X_\mu \rangle^2 + \frac{1}{2}\langle X_\mu \rangle^2} - \langle X^\mu \rangle y_\mu + \frac{1}{2} y_\mu^2$$

$$= \frac{1}{2}(\Delta X^\mu)^2 + \frac{1}{2}(y^\mu - \langle X^\mu \rangle)^2 \to 0 \quad \Longleftrightarrow \quad \begin{cases} \langle X^\mu \rangle \to y^\mu \\ \Delta X^\mu \to 0 \end{cases}$$

◇ Iff ground state energy vanishes, there exist minimal wave packets

$\Leftrightarrow$ points on the classical space

$$X^\mu \Longrightarrow H(y) = \frac{1}{2}(X^\mu - y^\mu)^2 \Longrightarrow E_0(y) \Longrightarrow \mathcal{M}$$

Our definition of classical space

- We assume we are given large-N family of D Hermitian matrices

$$\{(X_1^{(N)}, X_2^{(N)}, \cdots, X_D^{(N)}) \mid N = 1, 2, 3, \cdots\}$$

$N \times N$ Hermitian

- We consider Hamiltonian $H(y) = \frac{1}{2}(X^\mu - y^\mu)^2 \quad y \in R^D$

- Classical space := zeros of $E_0(y)$ in the large-N limit

$$\mathcal{M} = \{y \in R^D | f(y) = 0\} \qquad f(y) = \lim_{N\to\infty} E_0(y)$$

Example 1: Fuzzy sphere

$$X^\mu = \frac{2}{\sqrt{N^2-1}} L^\mu, \quad (\mu = 1,2,3) \qquad \sum_{\mu=1}^{3} (X^\mu)^2 = 1$$

N dim irrep of SU(2) generators

$$H(y) = \frac{1}{2}(X_1 - y_1)^2 + \frac{1}{2}(X_2 - y_2)^2 + \frac{1}{2}(X_3 - y_3)^2 \qquad (y_1, y_2, y_3) \in R^3$$

$$f(y) = \lim_{N\to\infty} E_0(y) = \frac{1}{2}(1 - |y|)^2$$

➡ Classical space is $\mathcal{M} = \{y \in R^3 | f(y) = 0\} = S^2$

Example 2: Fuzzy torus

$$VU = e^{i\theta} UV \quad (\theta = 2\pi/N)$$ Represented by the clock-shift matrices

$$U = X^1 + iX^2, \quad V = X^3 + iX^4$$ "Fuzzy Clifford Torus"

$$H(y) = \frac{1}{2}\sum_{\mu=1}^{4}(X^\mu - y^\mu)^2$$

$$f(y) = \lim_{N\to\infty} E_0(y) = \frac{1}{2}\left(1 - \sqrt{y_1^2 + y_2^2}\right)^2 + \frac{1}{2}\left(1 - \sqrt{y_3^2 + y_4^2}\right)^2$$

⇒ Classical space is $\mathcal{M} = \{y \in R^4 | f(y) = 0\} = T^2$

3. KAHLER STRUCTURE

Natural geometric structures?

◇ In general, one can consider some geometric structures on a given manifold

Riemannian, symplectic, Poisson, complex,....

◇ Now, we saw that from matrices X^μ, we can define a classical space as

$$\mathcal{M} = \{y \in R^D | f(y) = 0\} \qquad f(y) = \lim_{N\to\infty} E_0(y)$$

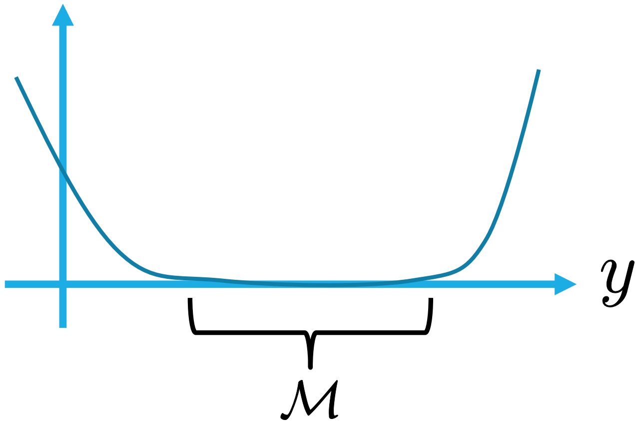
Do matrices X^μ also contain information of geometric structures on $\mathcal{M}$?

◇ When X^μ satisfy some conditions, there exists a natural Kahler structure

$$X^\mu \Longrightarrow (g, \omega, J)$$

Assumptions

$$f(y) = \lim_{N\to\infty} E_0(y)$$



◇ "Smoothness of $\mathcal{M}$"

$E_0(y)$ is a smooth function near $\mathcal{M}$

◇ "Compactness of $\mathcal{M}$"

$$||X^\mu|| < \infty \quad (N \to \infty)$$

◇ "Existence of a good commutative limit"

$$[X^\mu, X^\nu] = \frac{1}{C_N} F^{\mu\nu}(X) + \mathcal{O}(1/C_N^2) \qquad C_N \to \infty \ \ (N \to \infty)$$

Polynomial of X^μ s.t. degree and coefficients are independent of N

◇ Just for simplicity,

$E_0(y)$ is nondegenerate for $y \in \mathcal{M}$ $\qquad$ ($\Leftrightarrow$ A single brane)

Differential geometry of coherent states

$$H(y)|n,y\rangle = E_n(y)|n,y\rangle \qquad 0 \leq E_0 \leq E_1 \leq \cdots$$

$$E_0(y+\epsilon) = E_0(y) + \epsilon\cdot\partial E_0(y) + \frac{1}{2}(\epsilon\cdot\partial)^2 E_0(y) + \cdots$$

$$|0,y+\epsilon\rangle = |0,y\rangle + \epsilon\cdot\partial|0,y\rangle + \frac{1}{2}(\epsilon\cdot\partial)^2|0,y\rangle + \cdots$$

◇ This Taylor series contains geometric information of $\mathcal{M}$

◇ Perturbation theory relates each term with matrix elements of X^μ

$$H(y+\epsilon) = H(y) - \epsilon\cdot(X-y) + \frac{1}{2}|\epsilon|^2$$

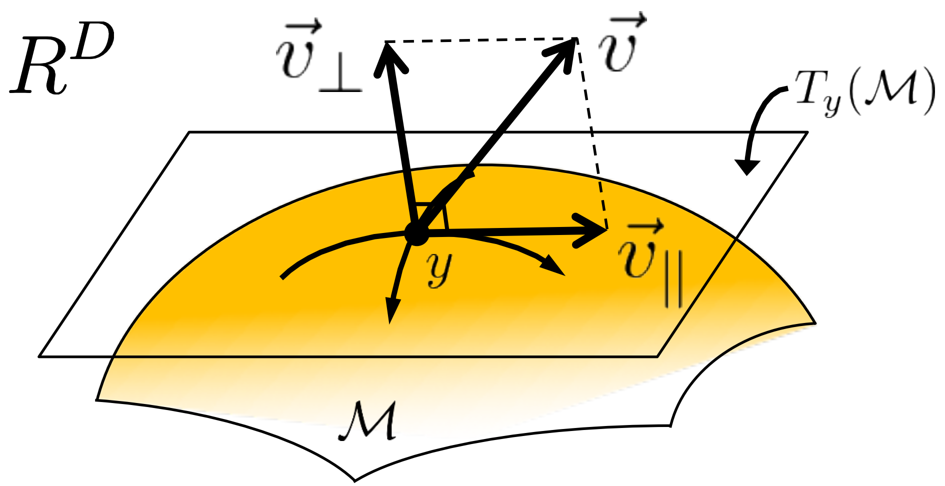
$$\epsilon\cdot\partial E_0(y) = -\langle 0,y|\epsilon\cdot(X-y)|0,y\rangle \quad \text{etc.}$$

$\boxed{\text{Geometric information} \Leftrightarrow \text{matrix elements of } X^\mu}$

Tangent space of M

◇ For example, the following object gives a projection onto tangent vectors on $\mathcal{M}$

$$P^{\mu}{}_{\nu}(y)=\delta^{\mu}_{\nu}-\lim_{N\to\infty}\partial^{\mu}\partial_{\nu}E_0(y)=2\lim_{N\to\infty}\sum_{n=1}^{N-1}\mathrm{Re}\frac{\langle 0,y|X^{\mu}|n,y\rangle\langle n,y|X_{\nu}|0,y\rangle}{E_n(y)-E_0(y)}$$



$$P^{\mu}{}_{\nu}v^{\nu}=v^{\mu}_{||}$$

Riemannian structure

Apart from the induced metric defined by $P^{\mu}{}_{\nu}$, one can define the information metric.

For state vectors $|\lambda\rangle$ labeled by parameters λ^a $(a = 1, 2, \cdots)$,
the information metric (Bures distance) is defined by

$$d(|\lambda\rangle, |\lambda'\rangle) = 1 - |\langle\lambda|\lambda'\rangle|^2$$

In our case,

$$d(|0, y\rangle, |0, y + dy\rangle) = g_{\mu\nu} dy^{\mu} dy^{\nu} + \mathcal{O}(dy^3)$$

$$g_{\mu\nu}(y) = \sum_{n\neq 0} \mathrm{Re} \frac{\langle 0, y|X_{\mu}|n, y\rangle\langle n, y|X_{\nu}|0, y\rangle}{(E_n(y) - E_0(y))^2}$$

This gives a metric on $\mathcal{M}$ in the large-N limit.

Symplectic Structure

For normalized state vectors $|\lambda\rangle$ labeled by parameters λ^a, let us consider

$$A_a = -i\langle\lambda|\frac{\partial}{\partial\lambda^a}|\lambda\rangle$$

Under the phase rotation $|\lambda\rangle \to e^{i\eta(\lambda)}|\lambda\rangle$, it transforms as $A_a \to A_a + \frac{\partial\eta}{\partial\lambda^a}$

A_a is called the Berry connection.

In our case ($|\lambda\rangle \to |0, y\rangle$), the Berry curvature gives a symplectic form on $\mathcal{M}$

$$\omega_{\mu\nu}(y) = \lim_{N\to\infty}\frac{2}{C_N}\sum_{n\neq 0}\mathrm{Im}\frac{\langle 0,y|X_\mu|n,y\rangle\langle n,y|X_\nu|0,y\rangle}{(E_n(y)-E_0(y))^2}$$

$$\begin{cases} \text{Non-degenerate on } \mathcal{M} \\ d\omega = 0 \end{cases}$$

Complex structure

Complex structure is given by a poler decomposition of ω

$$J = \frac{1}{\sqrt{\omega^T \omega}} \omega$$

$$J^\mu{}_\nu(y) = \lim_{N\to\infty} \sum_{n\neq 0} \mathrm{Im} \frac{\langle 0, y|X^\mu|n, y\rangle\langle n, y|X_\nu|0, y\rangle}{E_n(y) - E_0(y)}$$

$$\begin{cases} J^2 = -P & \text{(-1 on tangent space)} \\ \text{Integrable} & \end{cases}$$

(g, ω, J) satisfies the compatibility $\quad \omega(u, Jv) = g(u, v)$

$\Rightarrow$ Kahler structure

Poisson structure

$$W^{\mu\nu}(y) = -i \lim_{N\to\infty} C_N \langle 0, y | [X^\mu, X^\nu] | 0, y \rangle$$

This satisfies

$$\begin{cases} P \cdot W = W & (\Leftrightarrow W \text{ is a tangent bivector on } \mathcal{M}) \\ \text{Jacobi identity} \\ W \cdot \omega = P \end{cases}$$

$\Rightarrow$ It gives a Poisson tensor on $\mathcal{M}$.

Example 1 : Fuzzy sphere

$$X^\mu = \frac{2}{\sqrt{N^2-1}} L^\mu, \quad (\mu = 1, 2, 3) \qquad \sum_{\mu=1}^{3} (X^\mu)^2 = 1$$

$$\begin{cases} P^\mu{}_\nu = \dfrac{1}{|y|}\left(\delta^\mu_\nu - \dfrac{y^\mu y_\nu}{|y|^2}\right) \\[2ex] g_{\mu\nu} = \dfrac{1}{2|y|^2}\left(\delta_{\mu\nu} - \dfrac{y_\mu y_\nu}{|y|^2}\right) \\[2ex] \omega_{\mu\nu} = -\dfrac{1}{2}\epsilon_{\mu\nu\rho}\dfrac{y^\rho}{|y|^3} \\[2ex] J^\mu{}_\nu = \epsilon^\mu{}_{\nu\rho}\dfrac{y_\rho}{|y|^2} \\[2ex] W^{\mu\nu} = \epsilon^{\mu\nu\rho} y_\rho \end{cases}$$

For $y \in \mathcal{M} \ (|y| = 1)$

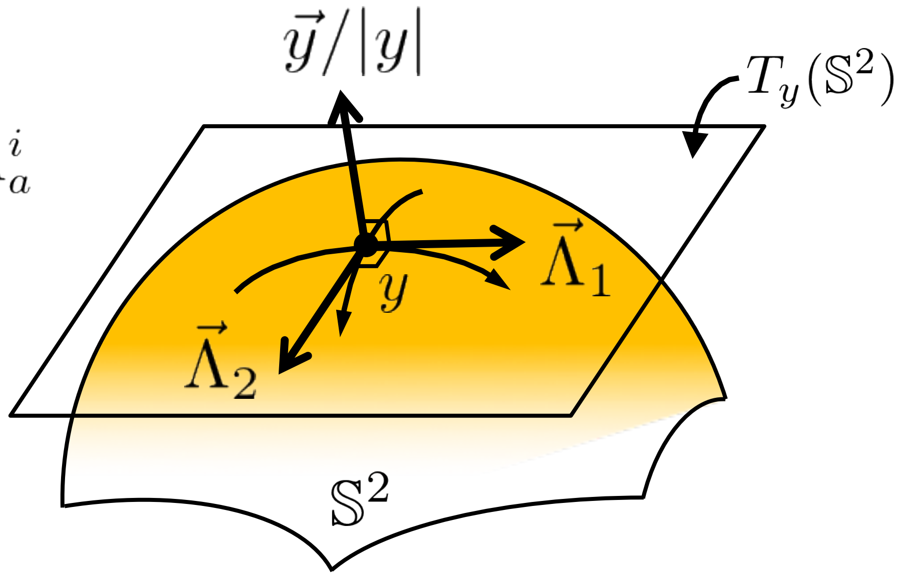
These satisfy mathematical definitions like

$$P \cdot P = P$$
$$J \cdot J = -P$$
$$\vdots$$

Example 2 : fuzzy sphere + fluctuation

Tangential/Normal decomposition

$$Y^i(y) = \Phi(y)\frac{y^i}{|y|} + \sum_{a=1,2} C_a(y)\Lambda_a^i$$



Consider matrices

$$X_\mu = X_\mu^{(0)} + Y_\mu(X^{(0)})$$

Fuzzy sphere   Fluctuation (Degree << N)

Hamiltonian

perturbation

$$H'(y) = \frac{1}{2}\sum_{i=1}^{3}(X^i + Y^i(X) - y^i)^2 = H^{(0)}(y) + \underline{\sum_{i=1}^{3}\frac{1}{2}\{Y^i(X), X^i - y^i\} + \mathcal{O}(Y^2)}$$

$$\mathcal{M} = \left\{ y \in \mathbb{R}^3 \middle|\, |y| = 1 + \Phi(y) + \mathcal{O}(Y^2) \right\}$$

Geometric objects: e.g. symplectic form

$$\omega_{ij}(y) = \frac{1}{2}\epsilon_{ijk}\left[-\frac{y_k}{|y|^3} + \Lambda_{ak}\nabla_a\Phi(y) + \frac{y_k}{|y|}\nabla_a C_a(y) \right.$$
$$\left. + \frac{y_k}{|y|}\Lambda_{c\ell}(\nabla_c\Lambda_{b\ell})C_b(y) + \mathcal{O}(Y^2) \right]$$

Fluctuation of matrices $\Leftrightarrow$ Fluctuations of $\mathcal{M}$ & geometric structures

[cf. Badyn-Karczmarek-Garnier-Yeh]

Physical interpretations

◇ If we consider X^μ as bosons of the low energy theory of D 0 -branes (eg. BMN mm),

Tangential fluctuations ⇔ gauge fields on D2-brane

[cf. Maldacena-Sheiki-Jabbari-Raamsdonk]

On the other hand, in our computations

Tangential fluctuations ⇔ fluctuations of the Berry connection

From this, it is suggested that the Berry connection corresponds to gauge field on the emergent D-brane

◇ g corresponds to the open string metric [Seiberg-Witten]

It contains some intrinsic information like the density of D 0 -branes

4. EXTENSION TO FINITE N

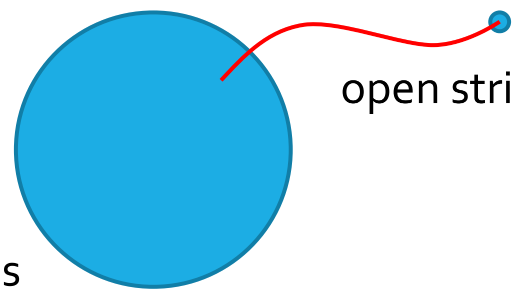
[Berenstein-Dzienkowski]

Geometry at finite N

Consider D 0 -branes forming a fuzzy sphere + another probe D 0 -brane

$$X^\mu \propto L^\mu$$

Fuzzy sphere
= N D 0 -branes
(a kind of bound state)



Let consider the lowest energy (non-oscillating) modes of the open string

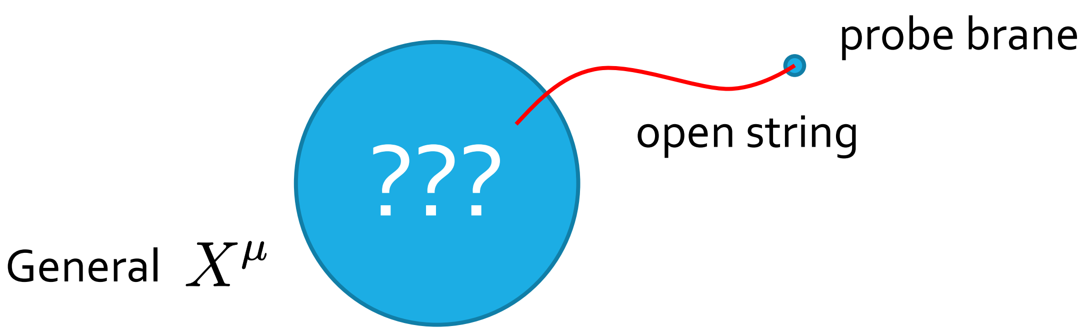
Energy $\propto$ length of the open string

The open string has zero energy state only if the probe brane is touching the fuzzy sphere

A set of points s.t. the open string has zero energy states $\Rightarrow$ Shape of fuzzy sphere.

Probing the geometry

- Suppose that we do not know the shape of the D 0 bound state



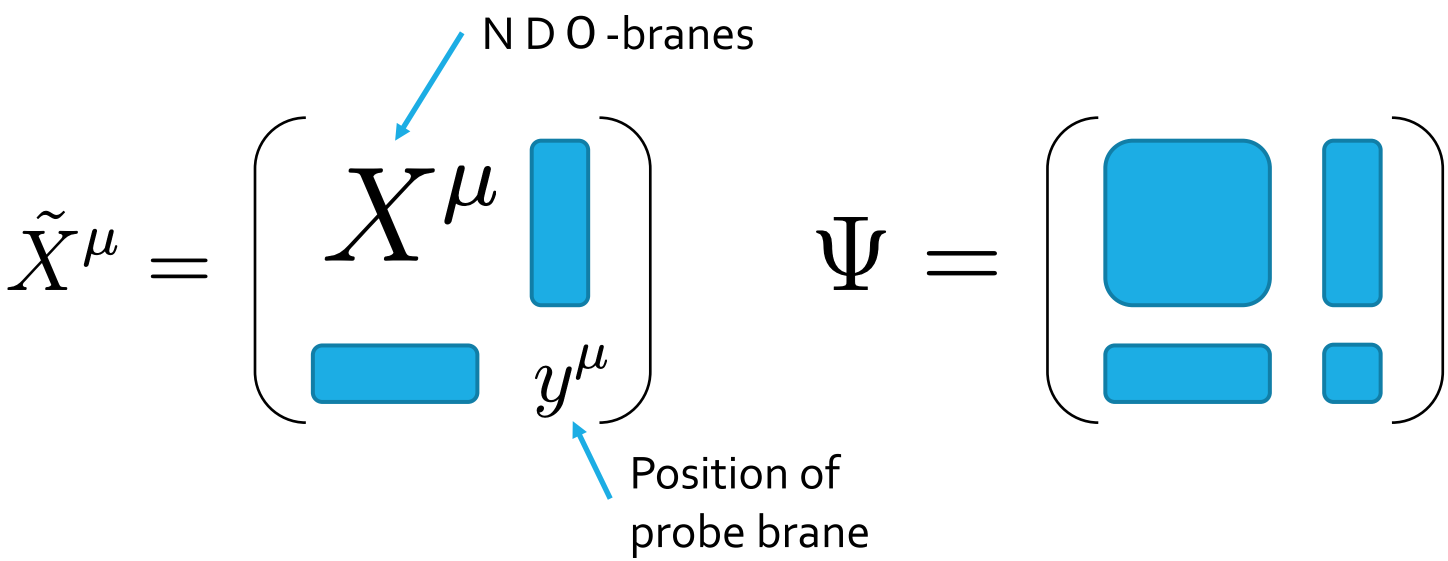
We can find the shape of the bound state by looking at the massless modes on the open string.

Geometry of the bound state = loci of the massless modes exist

We can find the geometry of X^μ in this way.

Action

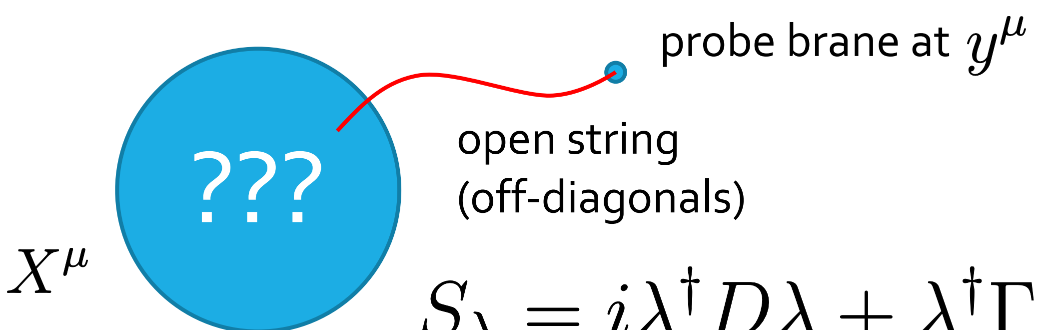
$$S = \int dt \mathrm{Tr}\left[\frac{1}{2}(D\tilde{X}^\mu)^2 + \frac{1}{4}[\tilde{X}^\mu, \tilde{X}^\nu]^2 + i\Psi^T D\Psi - \Psi^T\Gamma_\mu[\tilde{X}^\mu, \Psi]\right]$$



Off diagonal blocks correspond to the open string connecting N D 0 and the probe.

Off-diagonal fermionic modes

$$\tilde{X}^\mu = \begin{pmatrix} X^\mu & \\ & y^\mu \end{pmatrix} \qquad \Psi = \begin{pmatrix} & \lambda \\ \lambda^\dagger & \end{pmatrix}$$



$$S_\lambda = i\lambda^\dagger D\lambda + \lambda^\dagger\Gamma_\mu(X^\mu - y^\mu)\lambda$$

Geometry of X^μ is given by a set of points where λ becomes massless

$$\mathcal{M} = \{y \in R^9 | \Gamma_\mu(X^\mu - y^\mu) \text{ has zero eigenvalue}\}$$

Relation to my work

$$\mathcal{M} = \{y \in R^9 | \Gamma_\mu (X^\mu - y^\mu) \text{ has zero eigenvalue}\}$$

$$[\Gamma_\mu \otimes (X^\mu - y^\mu)]^2 = 1 \otimes (X^\mu - y^\mu)^2 + \frac{1}{2}\Gamma_{\mu\nu}[X^\mu, X^\nu]$$

Thus, if we assume $[X^\mu, X^\nu] = \mathcal{O}(1/N)$ and consider the large-N limit,

$$\mathcal{M} \sim \{y \in R^9 | (X^\mu - y^\mu)^2 \text{ has zero eigenvalue}\}$$

This is same as my definition.

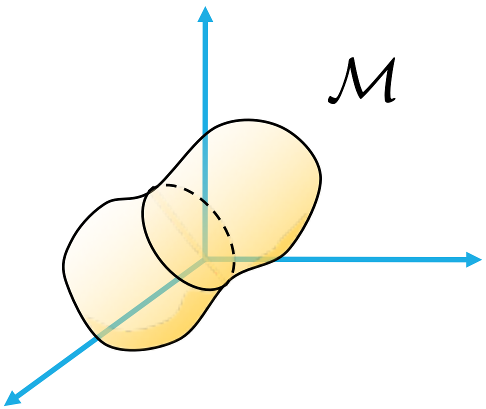
However, Berenstein's method has a great advantage that the space is defined at finite fixed N

For surfaces embedded in 3D

$$\not{\nabla}(y) = \sigma^i \otimes (X^i - y^i) \qquad (i = 1, 2, 3)$$

$$\not{\nabla}(y)|n, y\rangle = E_n(y)|n, y\rangle \qquad |E_0| \leq |E_1| \leq \cdots$$

$$\mathcal{M} = \{y \in R^3 | E_0(y) = 0\}$$

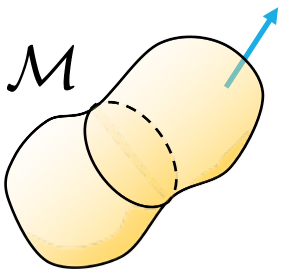


The surface is defined at finite fixed N

[cf. Badyn-Karczmarek-Garnier-Yeh]

Geometric structures

We consider the case where the zero eigenstate has a definite "chirality"



$\vec{v}(y)$: normal vector at each point $y \in \mathcal{M}$

$$\sigma_v = \vec{v} \cdot \vec{\sigma}$$

$$\boxed{\sigma_v |0, y\rangle = +|0, y\rangle \quad y \in \mathcal{M}}$$

In this case, we found that the geometric structures we found in the previous setup also work in this setup without taking the large-N limit. For example,

$$P^\mu{}_\nu(y) = 2 \sum_n \mathrm{Re} \frac{\langle 0, y|X^\mu|n, y\rangle\langle n, y|X_\nu|0, y\rangle}{E_n^2(y) - E_0^2(y)}$$

is a projection onto $T\mathcal{M}$

Examples

The chirality condition is satisfied in the following examples.

- Fuzzy plane
- Fuzzy sphere
- The large-N limit of matrices satisfying

$$[X^\mu, X^\nu] = \frac{1}{C_N} F^{\mu\nu}(X) + \mathcal{O}(1/C_N^2)$$

If we keep only the leading order terms in the large-N limit, the chirality condition is satisfied.

Summary and Outlook

- Coherent states $\Rightarrow$ Classical geometry
- Kahler structure can be expressed in terms of the matrix elements
- This holds even at finite N, if the zeromode of Dirac op. has a definite chirality. (3d)
- The Hamiltonian can also be understood as a tachyon configuration on non-BPS branes

  Zero eigenspace $\Leftrightarrow$ Stable D-branes after tachyon condensation

  [Asakasa-Sugimoto-Terashima
  + Work in progress]

- Dirac operators in higher dimensional cases?
- Geometric structures are gauge invariant $\Leftrightarrow$ observables in matrix models
- Geometric interpretation of matrix models? Emergent space-time?