Duality Constraints on String Theory

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based on collaborations with

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Main Reference

[CIY4], "Analytic Study for the String Theory Landscapes via Matrix Models,"
Phys.Rev. D86 (2012) 126001 [arXiv:1206.2351 [hep-th]]

[CIY5], "Duality Constraints on String Theory I: spectral networks and instantons,"
arXiv:1308.6603 [hep-th]

[CIY6], "Duality Constraints on String Theory II: spectral p-q duality and Riemann-Hilbert calculus," **in progress**

General Motivations

We wish to know

Definition of Non-perturbative String Theory

- **Perturbative string theory** is well-known

- Several **non-perturbative formulations** are proposed: (**Matrix models** (**Gauge theory**) and **String field theory**)

Non-perturbative Principle is still unclear

(Only rely on inclusion of perturbative string theory)

- Study of **Stokes phenomenon** is a way to tackle this issue **by directly capturing the vacuum structure of string theory** (interrelationships among string saddle points)

Physics beyond perturbative (Large N) expansions

Today, we check String duality beyond Large N !

Duality in question

Two-matrix model

$$\mathcal{Z} = \int dX dY \, e^{-N\mathrm{tr}[V_1(X)+V_2(Y)-XY]}$$

Integrate Y → X-system

Integrate X → Y-system

X-system ⟷ Y-system

spectral (p-q) dual

Today, we argue that **this duality is generally broken non-perturbatively**, although perturbatively it does work completely.

Plan of the talk

1. Perturbative strings and matrix models
2. Check of the Duality in large N
3. Stokes Phenomena in Matrix Models
4. Duality Constraints on String Theory

1. Perturbative Strings and Matrix Models

String Theory and Matrix Models

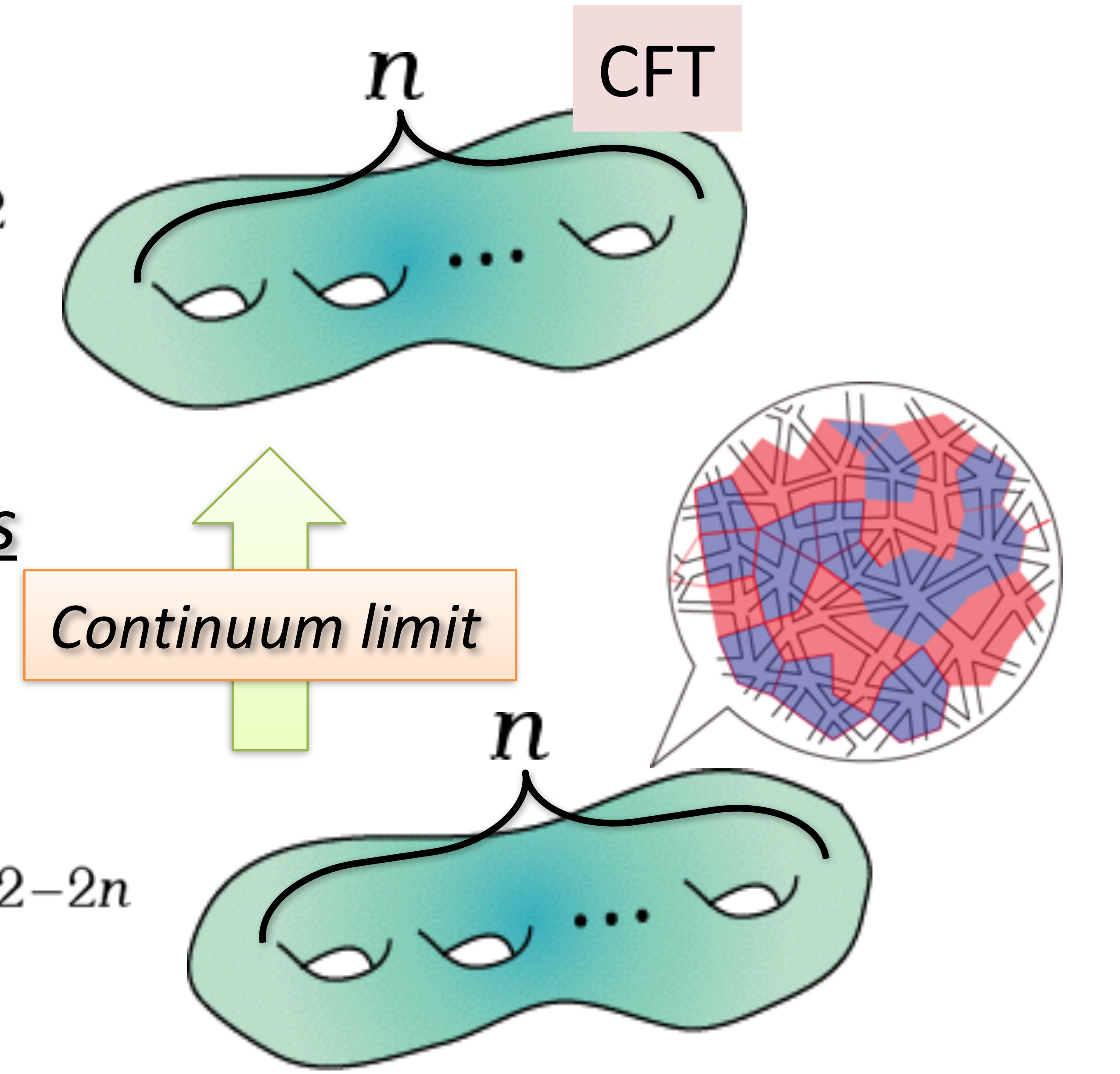
(Non-critical) String theory

$$\mathcal{F} = \ln Z = \sum_{\substack{\text{possible WS} \\ (\text{genus: } n=0,1,\cdots)}} g^{2n-2}$$

Large N expansion of matrix models

$$Z_{\text{MM}} = \int_{\text{N x N matrices}} dM e^{-N \text{tr} V(M)}$$

$$\mathcal{F} = \ln \mathcal{Z}_{MM} = \sum_{\substack{\text{Feynman Graph: } G \\ (\text{genus: } n=0,1,\cdots)}} N^{2-2n}$$



Triangulation (Lattice Gravity)

[Brezin-Kazakov '90][Gross-Migdal '90][Douglas-Shenker '90]

$g = N^{-1}$ *(Large N expansion ⇔ Perturbation theory of string coupling g)*

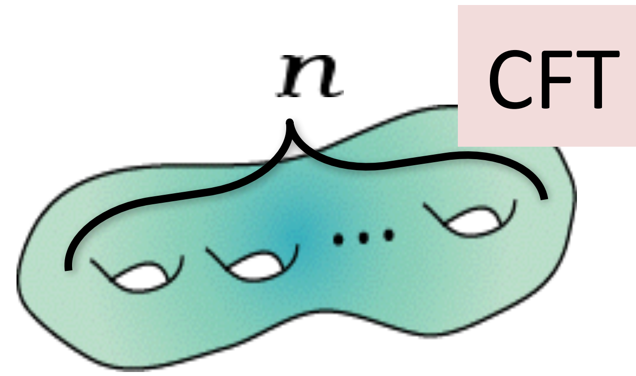
Non-perturbative Corrections

D-instanton Chemical Potential

$$\mathcal{F} = \ln \mathcal{Z} \simeq \underbrace{\sum_{n=0}^{\infty} g^{2n-2}\mathcal{F}_n}_{\text{perturbative corrections}} + \underbrace{\sum_{I} \theta_I \exp\Big[\sum_{n=0}^{\infty} g^{2n+b_I-2}\mathcal{F}_n^{(I)}\Big]}_{\text{non-perturbative (instanton) corrections}}$$

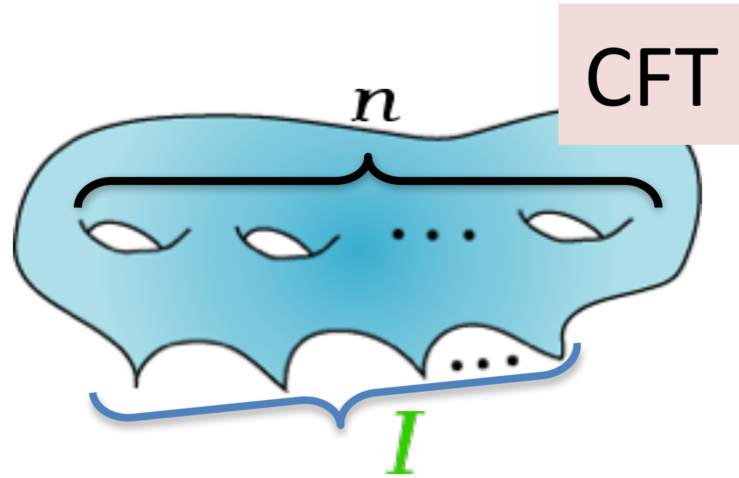
1. Perturbative amplitudes of WS_n:

$\mathcal{F}_n \leftrightarrow$



2. Non-perturbative amplitudes are D-instantons! **[Shenker '90, Polchinski '94]**

$, \mathcal{F}_n^{(I)} \leftrightarrow$



WS with Boundaries
= **open string theory**

3. The overall weight ϑ's (=Chemical Potentials) are out of the perturbation theory

This will be considered later

Spectral Curve

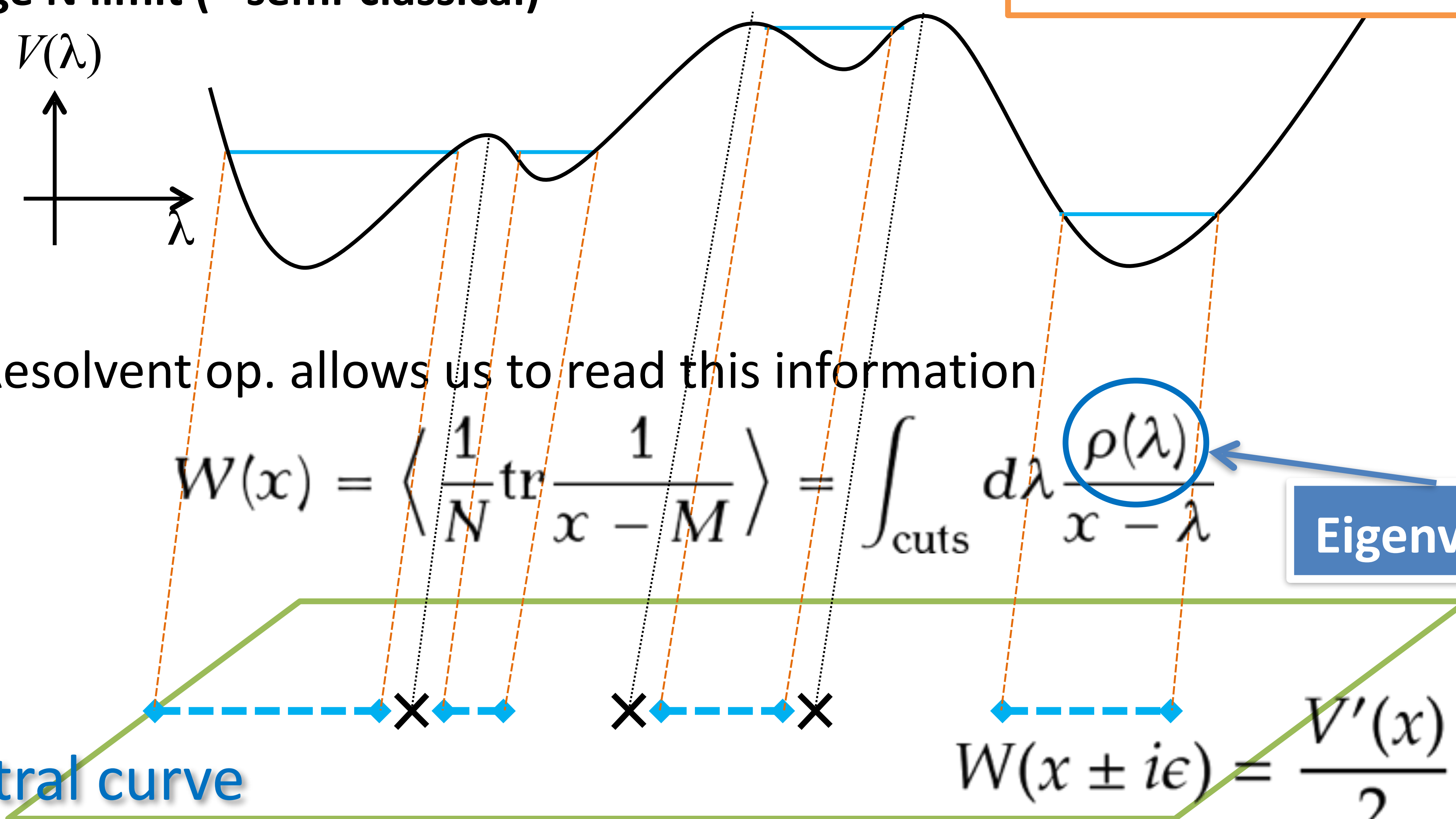
Resolvent: $$W(x) = \left\langle \frac{1}{N} \mathrm{tr} \frac{1}{x - M} \right\rangle$$

Diagonalization: $$U^{\dagger} M U = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$$

$$\mathcal{Z} = \int dM\, e^{-N\mathrm{tr}V(M)} \quad \Rightarrow \quad \mathcal{Z} = \int d^N\lambda \prod_{i>j} (\lambda_i - \lambda_j)^2 e^{-N \sum_i V(\lambda_i)}$$

N-body problem in the potential V

In Large N limit (= semi-classical)



The Resolvent op. allows us to read this information

$$W(x) = \left\langle \frac{1}{N} \mathrm{tr} \frac{1}{x - M} \right\rangle = \int_{\mathrm{cuts}} d\lambda \frac{\rho(\lambda)}{x - \lambda}$$

Eigenvalue density

$$W(x \pm i\epsilon) = \frac{V'(x)}{2} \mp \pi i \rho(x)$$

Why is it important?

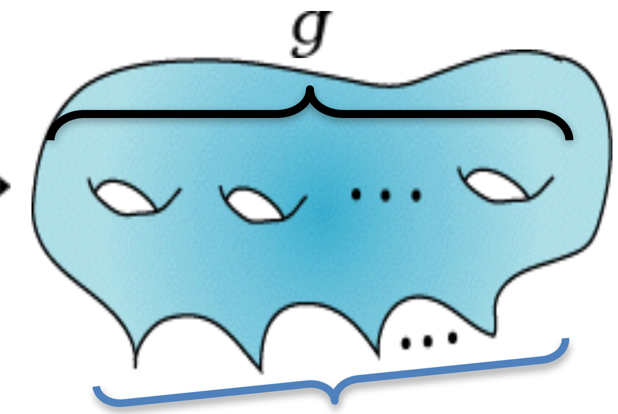
Spectral curve ⇔ Perturbative string theory

Perturbative correlators

$$W_n(z_1, z_2, \cdots, z_n) \equiv \left\langle \prod_{j=1}^{n} \frac{1}{N} \mathrm{tr} \frac{1}{z_j - X} \right\rangle_c$$

$$\simeq \sum_{g=0}^{\infty} N^{2-2g-n} W_n^{(g)}(z_1, \cdots, z_n)$$

$$W_n^{(g)}(J) \longleftrightarrow$$



$$J = (z_1, z_2, \cdots, z_n)$$

are all obtained recursively from the resolvent (e.g. by Loop eq.)

Topological Recursions [Eynard'04, Eynard-Orantin '07]

$$W_{n+1}^{(g)} = \sum_i \operatorname*{Res}_{z \to a_i} K(z_0, z) \left[W_{n+2}^{(g-1)}(z, \bar{z}, J) + \sum_{h=0}^{g} \sum_{I \subset J} W_{1+|I|}^{(h)}(z, I) W_{1+n-|I|}^{(g-h)}(\bar{z}, J \setminus I) \right]$$

Input: $W_1^{(0)}(z) = \lim_{N \to \infty} W_1(z)$, $K(z_0, z) \sim W_2^{(0)}(z_0, z)$:Bergman Kernel

Therefore, we symbolically write the free energy as

$$\mathcal{F}_{\mathrm{pert}}(\mathcal{C}) = \ln \mathcal{Z}_{\mathrm{pert}}(\mathcal{C}) = \sum_{n=0}^{\infty} g^{2n-2} \mathcal{F}_n(\mathcal{C}) \qquad (\mathcal{C} : \text{ spectral curve})$$

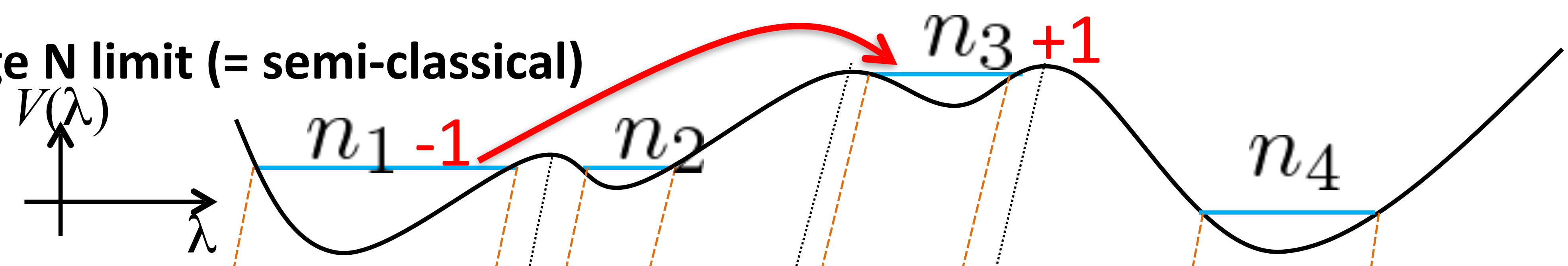
Why is it important?

Spectral curve ⇔ Perturbative string theory

Non-perturbative corrections

$$n_1 + n_2 + n_3 + n_4 = N$$

In Large N limit (= semi-classical)



$$W(x) = \left\langle \frac{1}{N} \mathrm{tr} \frac{1}{x - M} \right\rangle = \int_{\mathrm{cuts}} d\lambda \frac{\rho(\lambda)}{x - \lambda}$$

$$\mathcal{F} = \ln \mathcal{Z} \simeq \sum_{n=0}^{\infty} g^{2n-2} \mathcal{F}_n + \sum_I \theta_I \exp\left[\sum_{n=0}^{\infty} g^{2n+b_I-2} \mathcal{F}_n^{(I)}\right]$$

Non-perturbative partition functions: [Eynard '08, Eynard-Marino '08]

$$\mathcal{Z}(\mathcal{C}) = \sum_{n_1+\cdots+n_K=N} \theta_1^{n_1} \cdots \theta_K^{n_K} e^{\mathcal{F}_{\mathrm{pert}}(\mathcal{C}_{n_1,\cdots,c_K})}$$

with some free parameters

Summation over all the possible configurations

difference $\delta\mathcal{F}_{\mathrm{pert}}$ wrt n_i

2. Perturbative Check of the Duality

Duality in question

Two-matrix model

$$\mathcal{Z} = \int dX dY \, e^{-N\mathrm{tr}[V_1(X)+V_2(Y)-XY]}$$

Integrate Y → X-system

Integrate X → Y-system

X-system ⟷ Y-system: spectral (p-q) dual

NOTE) X and Y describe dual spacetimes

≒ Double Field Theory

See it in Large N

Spectral Curves

X-system ⟷ Y-system

Resolvent of X	Resolvent of Y
$R(x) = \left\langle \frac{1}{N} \mathrm{tr} \frac{1}{x - X} \right\rangle$	$\widetilde{R}(y) = \left\langle \frac{1}{N} \mathrm{tr} \frac{1}{y - Y} \right\rangle$

probe eigenvalues of X

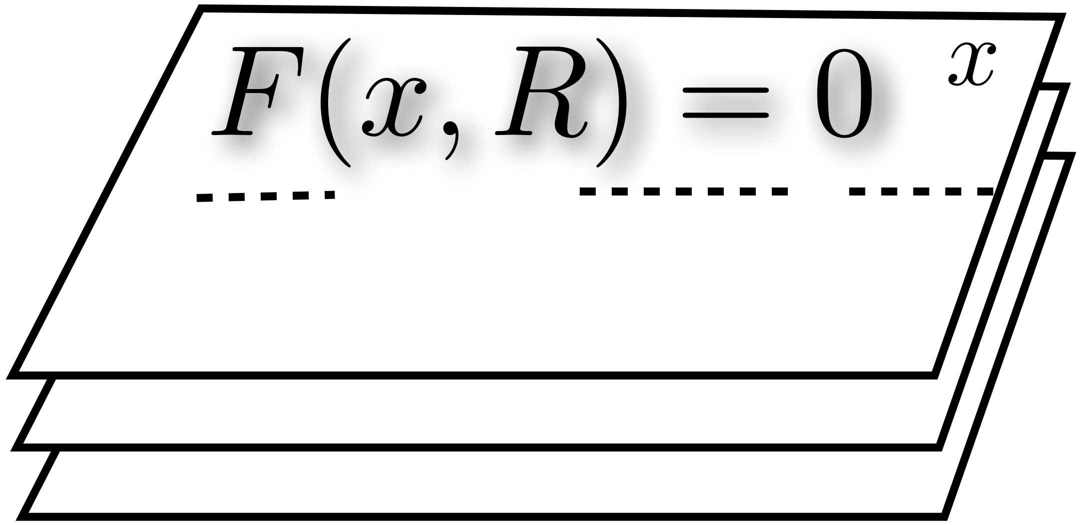
probe eigenvalues of Y

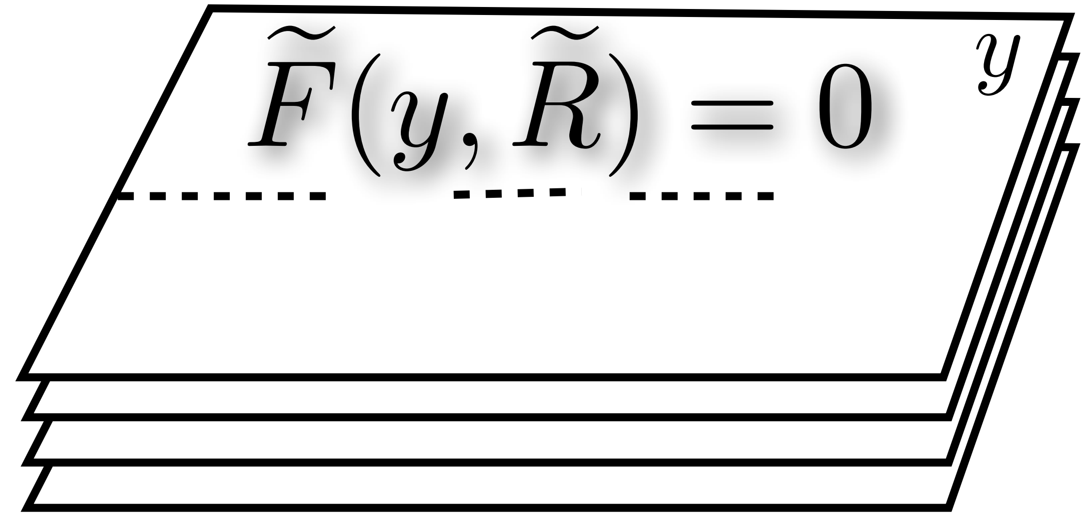
$$F(x, R) = 0 \quad x$$

$$\widetilde{F}(y, \widetilde{R}) = 0 \quad y$$

Actually, they are dual

Actually, they are dual





They are given by **the same algebraic equation**:

$$F(x,R)=f\big(x,R-V_1'(x)\big)=0$$

$$\widetilde{F}(y,\widetilde{R})=f\big(\widetilde{R}-V_2'(y),y\big)=0$$

We can identify

That is, $$f(x,y)=0$$

if x is spacetime → X-system

if y is spacetime → Y-system

X-system ⟷ p-q dual ⟷ Y-system

Duality in spectral curve

$$f(x,y)=0 \quad x \leftrightarrow y$$

Symplectic Invariance

Equivalent under All-order perturb. theory

(Including all-order instanton corrections)

X-system ⟷ Y-system

Perturbative coefficients are the same for all-order

$$\mathcal{F}_{\mathrm{pert}}(g)=\sum_{n=0}^{\infty} g^{2n-2}\mathcal{F}_n$$

Instantons (deform. of spectral curves) are also

$$\mathcal{F}^{(I)}_{\mathrm{Inst}}(g)=\sum_{n=0}^{\infty} g^{n-1}\mathcal{F}^{(I)}_n$$

$I \in \{\mathrm{instantons}\}$

World-sheet（Liouville Theory）

$$S = \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\Big[g^{ab}\partial_a\phi\partial_b\phi + QR\phi + 4\pi\mu e^{2b\phi}\Big] +$$

Liouville theory

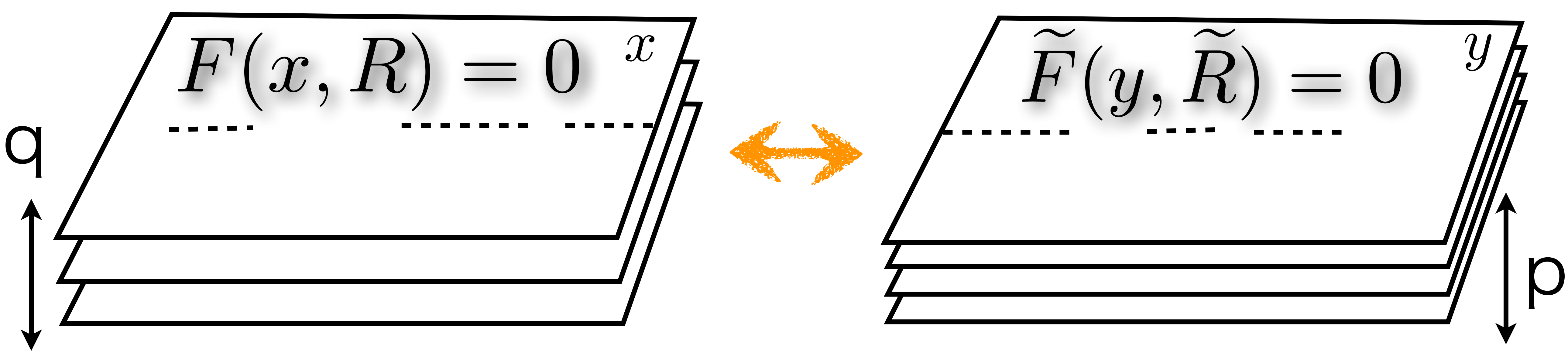
$$+\frac{1}{4\pi}\int d^2\sigma\sqrt{g}\Big[g^{ab}\partial_a X\partial_b X + i\widetilde{Q}RX\Big] + S_{\mathrm{ghost}}$$

(p,q) minimal CFT

$$Q = b + \frac{1}{b}, \quad \widetilde{Q} = b - \frac{1}{b} \qquad b = \sqrt{\frac{p}{q}}$$

p-q duality: p ↔ q $\Leftrightarrow$ $b \leftrightarrow \frac{1}{b}$ (≒T-duality) b ≒ radius of MCFT

A basic assumption for 3-pt function（DOZZ）!



p-q duality and T-duality: References

- [Fukuma-Kawai-Nakayama '92]

  [P,Q]=1 <=> [Q,-P]=1

- [Kharchev-Marshakov '92]

  Kontsevich MM: from (p,1) to (p,q)

- [Bertola-Eynard-Harnad '01-04]

  Two-matrix models, saddle point analysis

- [Asatani-Kuroki-Okawa-Sugino-Yoneya '96]

  Kramers-Wannier duality in Random Surfacs

- [Kuroki-Sugino '07]

  D-instanton fugacity

Non-perturbative Check of the Duality

Nonperturbative comparison

Two-matrix model

$$\mathcal{Z} = \int dX dY \, e^{-N\mathrm{tr}[V_1(X)+V_2(Y)-XY]}$$

[David'91,93];[Hanada-Hayakawa-Ishibashi-Kawai-Kuroki-Matuso-Tada '04];[Kawai-Kuroki-Matsuo '04];[Sato-Tsuchiya '04];[Ishibashi-Yamaguchi '05];[Ishibashi-Kuroki-Yamaguchi '05];[Matsuo '05];[Kuroki-Sugino '06]…

X-system ⟷ system

spectral (p-q)

D-instanton Chemical Potential

$$\mathcal{F} = \ln \mathcal{Z} \simeq \sum_{n=0}^{\infty} g^{2n-2}\mathcal{F}_n + \sum_I \theta_I \exp\Big[\sum_{n=0}^{\infty} g^{2n+b_I-2}\mathcal{F}_n^{(I)}\Big]$$

Nonperturbative comparison

Two-matrix model

$$\mathcal{Z} = \int dX dY \, e^{-N\mathrm{tr}[V_1(X)+V_2(Y)-XY]}$$

Integrate Y

Integrate X

X-system

Y-system

spectral (p-q) dual

For simplicity, we choose $V_2(Y)$ as gaussian

Nonperturbative comparison

Two-matrix model

gaussian

$$\mathcal{Z} = \int dX dY\, e^{-N\mathrm{tr}[V_1(X)+\frac{Y^2}{2}-XY]}$$

We can perform !

Integrate Y

Integrate X

X-system

Y-system

$$\mathcal{Z} = \int dX e^{-N\mathrm{tr}V(X)}$$

one-matrix model

(We know well)

Non-perturbative in One-matrix models

$$\mathcal{Z} = \int dX e^{-N\mathrm{tr}V(X)}$$

Several ways to converge integral

There are an number of **ambiguities** to define non-perturbative string theory

Non-Pert. in one-matrix

$$\mathcal{Z} = \int dX e^{-N \mathrm{tr} V(X)}$$

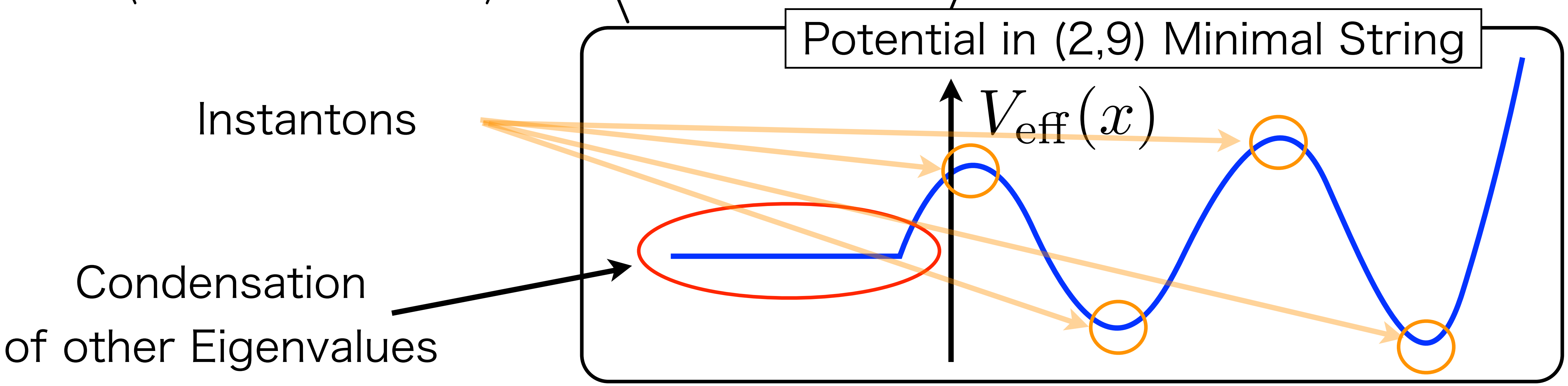
Mean-field Potential [David '91] Look at a single eigenvalue x

$$\mathcal{Z} \simeq \int dx e^{-N V_{\mathrm{eff}}(x)} = \int dx \left\langle \det(x-X)^2 \right\rangle e^{-NV(x)}$$

$\left\langle \det(x-X)^2 \right\rangle$ is resolvent in Large N :

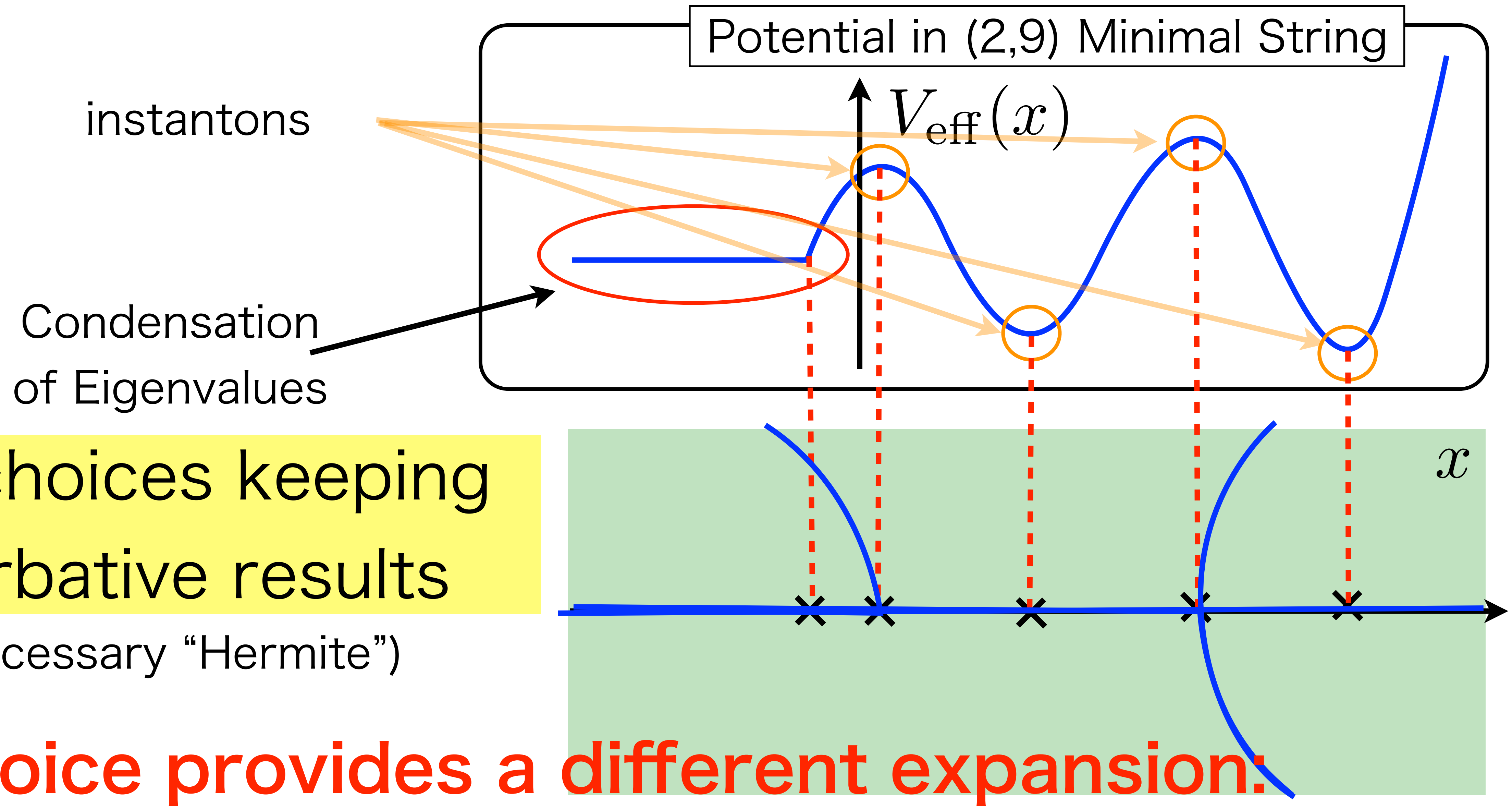
$$\left\langle \det(x-X)^2 \right\rangle = \left\langle e^{2 \mathrm{tr} \ln(x-X)} \right\rangle \simeq e^{2N \int^x dx' \left\langle \frac{1}{N} \frac{1}{x'-X} \right\rangle}$$

$$2N \int dx R(x)$$



Non-Pert. in one-matrix

Model is defined by a choice of the contour



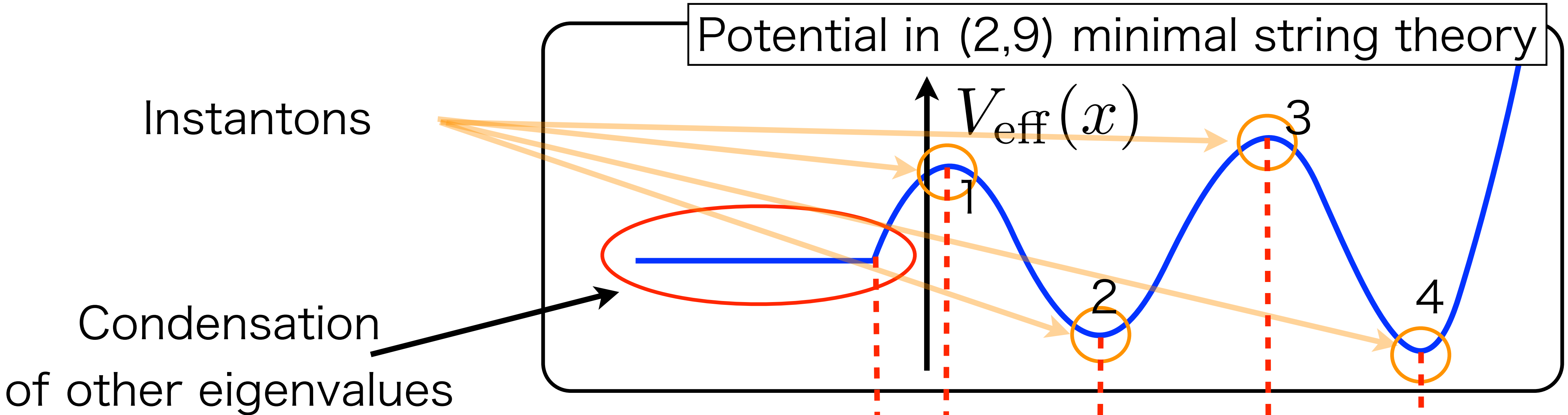
Many choices keeping perturbative results

(Not necessary “Hermite”)

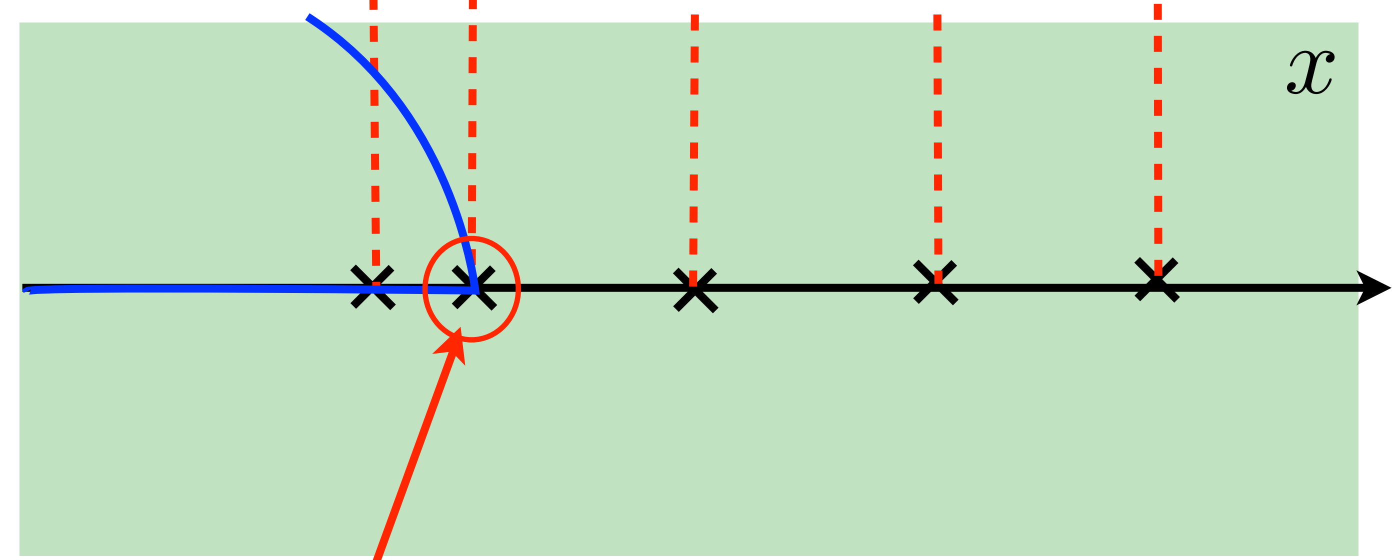
Each choice provides a different expansion:

$$\mathcal{F} \simeq \sum_{n=0}^{\infty} g^{2n-2}\mathcal{F}_n + \sum_{I\in\mathfrak{J}_{\rm relev}} \theta_I \times g^{1/2}\exp\Big[\frac{1}{g}\sum_{n=0}^{\infty}\mathcal{F}_n^{(I)}\Big]$$

Examples [CIY4 '12]



Model 1:



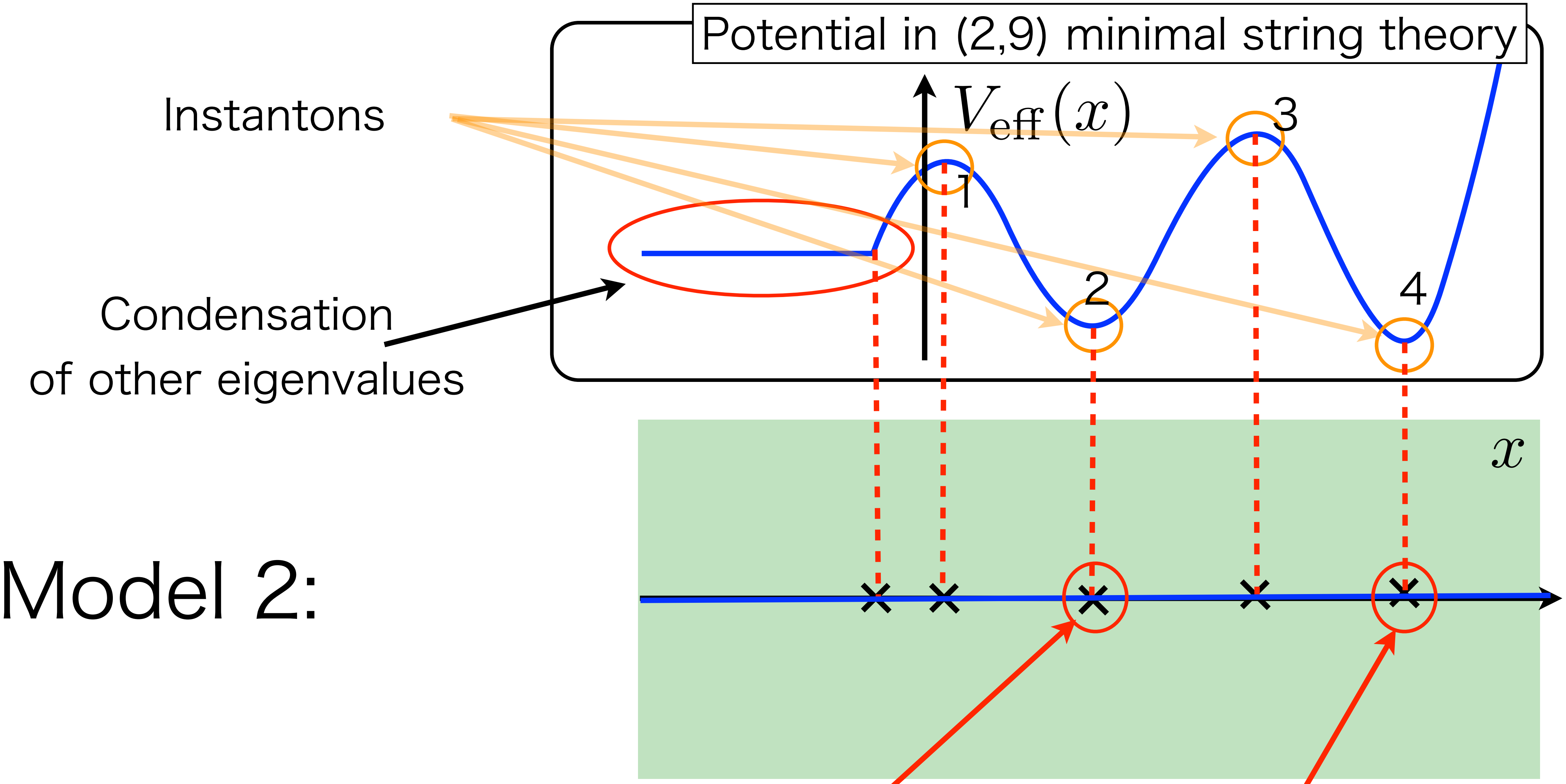
$$\Leftrightarrow \quad \mathcal{F}(g) \simeq \sum_{n=0}^{\infty} g^{2n-2}\mathcal{F}_n + \theta_1 e^{-\frac{1}{g}\mathcal{S}_1^{(\mathrm{inst.})}} + \cdots$$

Small instanton corrections

In this model, minimal string theory

is true vacuum of this system

Examples [CIY4 '12]



Model 2:

$$\Leftrightarrow \quad \mathcal{F}(g) \simeq \sum_{n=0}^{\infty} g^{2n-2}\mathcal{F}_n + \theta_2 e^{+\frac{1}{g}\mathcal{S}_2^{(\rm inst.)}} + \theta_4 e^{+\frac{1}{g}\mathcal{S}_4^{(\rm inst.)}} \cdots$$

Large instanton corrections

In this model, the string theory is meta-stable

With the ambiguity, even stability can change

Nonperturbative completion

Two-matrix model

gaussian

$$\mathcal{Z} = \int dX dY\, e^{-N\mathrm{tr}[V_1(X) + \frac{Y^2}{2} - XY]}$$

We can perform !

Integrate Y

Integrate X

X-system

Y-system

$$\mathcal{Z} = \int dX e^{-N\mathrm{tr}V(X)}$$

one-matrix model

Next !

Non-Pert. in Y-system

Mean-field Potential ([Kazakov-Kostov'04])

$$\mathcal{Z} = \int dX dY \, e^{-N\mathrm{tr}[V_1(X)+\frac{Y^2}{2}-XY]}$$

$$\simeq \int dx dy \langle \det(x-X)\det(y-Y)\rangle \, e^{-N[V_1(x)+V_2(y)-xy]}$$

is evaluated in Large N as

integral of resolvents

$$\simeq e^{-N[\Phi_1(x)+\Phi_2(y)-xy]}$$

Integrate X: $\int dx e^{-N[\Phi_1(x)-xy]}$: like Airy func.

Non-Pert. requires Stokes phenomena [CIY5]

Saddle point equation gives:

$$e^{-NV_{\mathrm{eff}}(y)} \sim \sum_{j,l} (*) \times e^{\frac{1}{g}\varphi^{(j,l)}(y)}$$

$$\varphi^{(j,l)}(y) = \int dy \left[x^{(j)}(y) - x^{(l)}(y)\right]$$

$$f(x^{(j)}(y), y) = 0$$

4. Stokes Phenomena in Matrix Models

Stokes Phenomenon in Airy function

Airy function: $\left(\frac{d^2}{d\zeta^2} - \zeta\right)\psi(\zeta) = 0 \qquad \psi(\zeta) = Ai(\zeta),\, Bi(\zeta)$

$$\left[Bi(\zeta) = e^{\frac{\pi}{6}i} Ai(e^{\frac{2}{3}\pi i}\zeta) + e^{-\frac{\pi}{6}i} Ai(e^{-\frac{2}{3}\pi i}\zeta) \right]$$

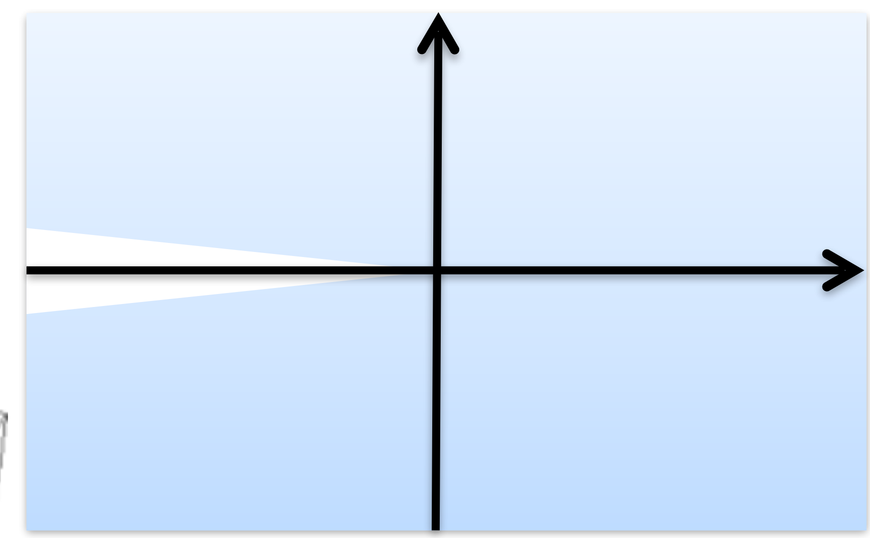
$\zeta \to +\infty$

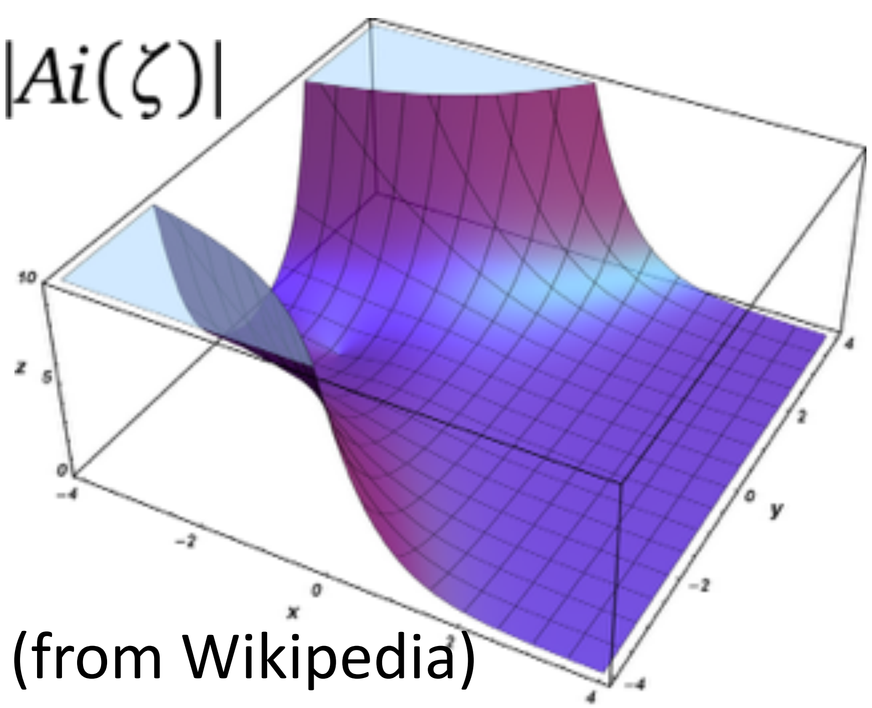
$$Ai(\zeta) \simeq \frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}} \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{2\pi} \left(\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{1}{6})\Gamma(n+\frac{5}{6})}{n!} \zeta^{-\frac{3}{2}n} + \mathcal{O}(e^{-*\zeta^*}) \right]$$

$\sim n!$

Asymptotic expansion!

This expansion is valid in $\zeta \to \infty,\ |\arg(\zeta)| < \pi$





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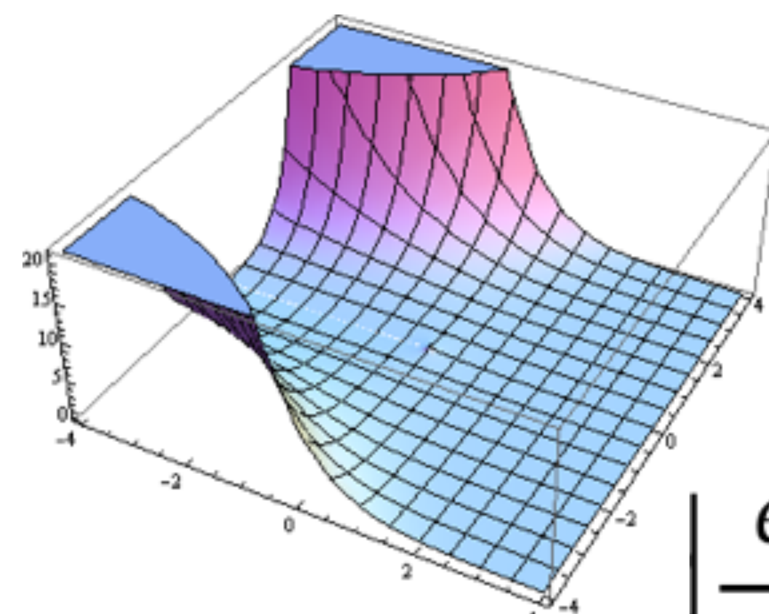


$$\left| \frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}} \right|$$

Stokes Phenomenon in Airy function

Airy function:

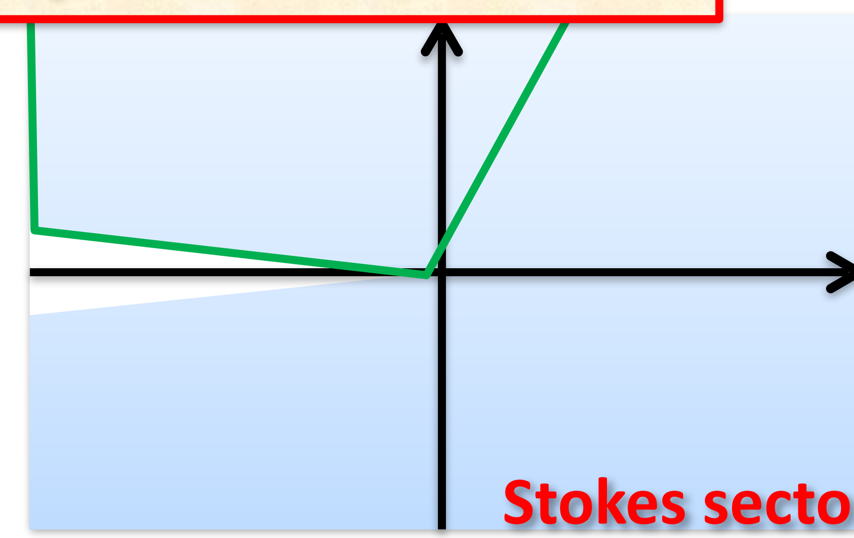
1. Asymptotic expansions are only applied in specific angular domains (**Stokes sectors**)
2. Differences of the expansions in the intersections are only by **relatively and exponentially small terms**

$\frac{2}{3}\pi i \zeta)$

$\zeta \to +\infty$

$$Ai(\zeta) \simeq \frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\left[1+\cdots\right]$$

(valid in $\zeta \to \infty,\ |\arg(\zeta)| < \pi$)



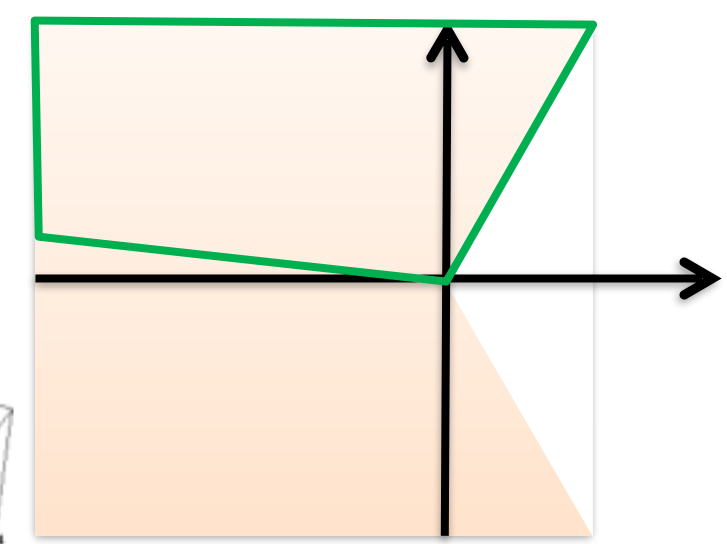
Stokes multiplier

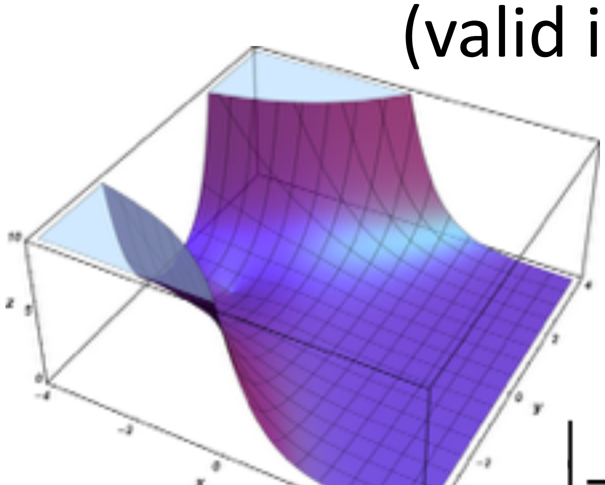
$\zeta \to \infty \times e^{\pi i}$

(relatively) Exponentially small !

$$Ai(\zeta) \simeq \frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\left[1+\cdots\right] + i\frac{e^{+\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\left[1+\cdots\right]$$

(valid in $\zeta \to \infty,\ |\pi - \arg(\zeta)| < \frac{2\pi}{3}$)





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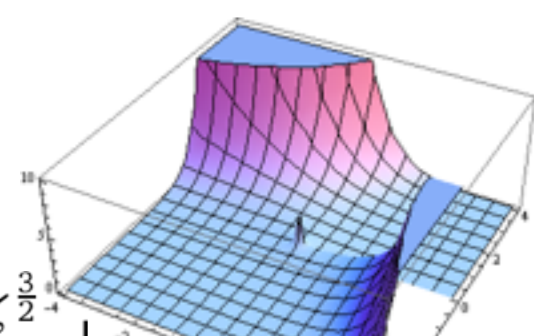
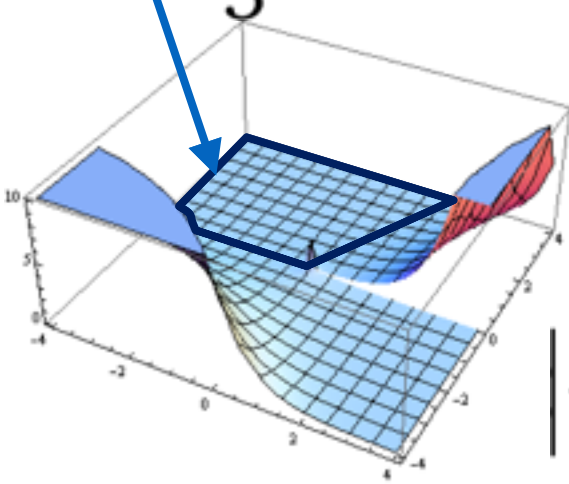


$\left|\frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\right|$

+



$\left|\frac{e^{+\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\right|$

Stokes Phenomenon in Airy function

Airy function: $\left(\frac{d^2}{d\zeta^2} - \zeta\right)\psi(\zeta) = 0 \qquad \psi(\zeta) = Ai(\zeta),\, Bi(\zeta)$

$$Bi(\zeta) = e^{\frac{\pi}{6}i} Ai(e^{\frac{2}{3}\pi i}\zeta) + e^{-\frac{\pi}{6}i} Ai(e^{-\frac{2}{3}\pi i}\zeta)$$

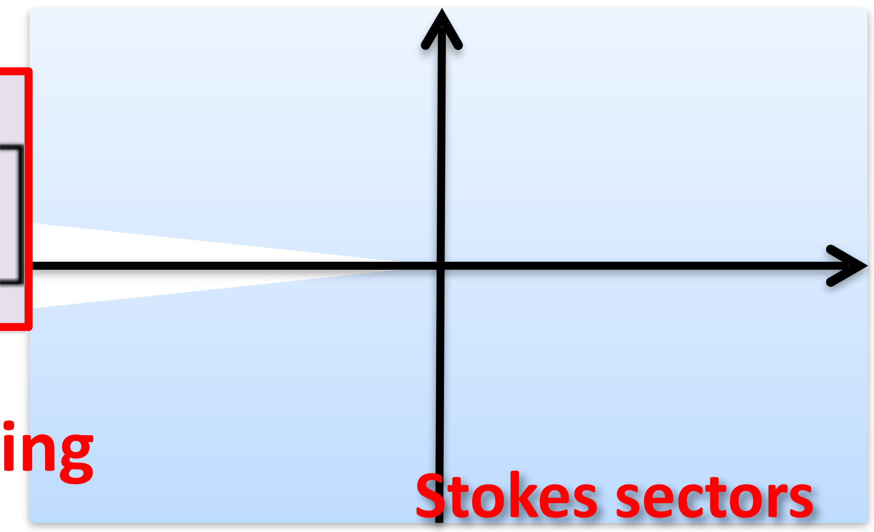
$\zeta \to +\infty$

$$Ai(\zeta) \qquad \simeq \frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\left[1 + \cdots\right]$$

(valid in $\zeta \to \infty,\ |\arg(\zeta)| < \pi$)

different

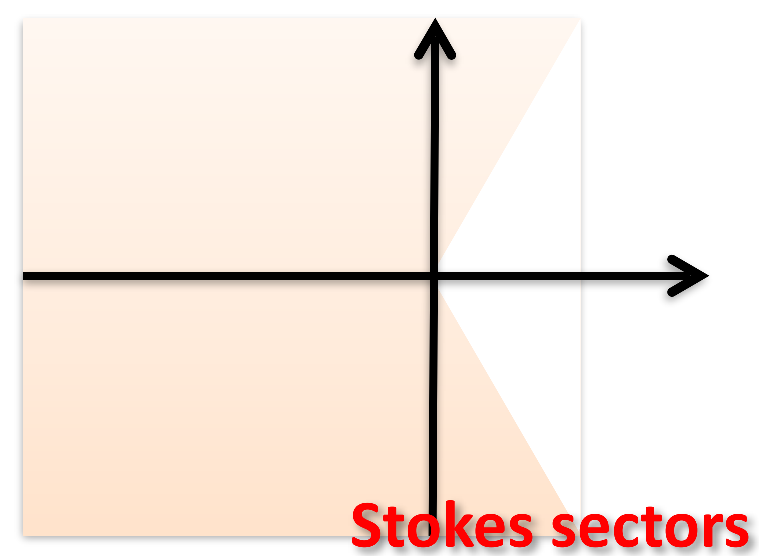
Keep using



$\zeta \to \infty \times e^{\pi i}$

$$Ai(\zeta) - i\left(e^{-\frac{\pi}{6}i} Ai(e^{-\frac{2}{3}\pi i}\zeta)\right) \simeq \frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\left[1 + \cdots\right]$$

(valid in $\zeta \to \infty,\ |\pi - \arg(\zeta)| < \frac{2\pi}{3}$)



Two results on Stokes phenomena

1) Position of Cuts = Stokes lines

Airy system ⇔ (2,1) minimal string theory

$$Ai(x) \simeq \langle \det(x - X) \rangle$$

$$\langle \det(x - X) \rangle = \left\langle e^{\mathrm{tr}\ln(x-X)} \right\rangle \simeq e^{N\int dx \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{x-X} \right\rangle}$$

Resolvent

entire function

it has jump (i.e. cuts)

(Physical/eigenvalue) cuts are the lines of changing the behavior of $\langle \det(x - X) \rangle$ [Maldacena-Moore-Seiberg-Shih '05]

Stokes Phenomenon in Airy function

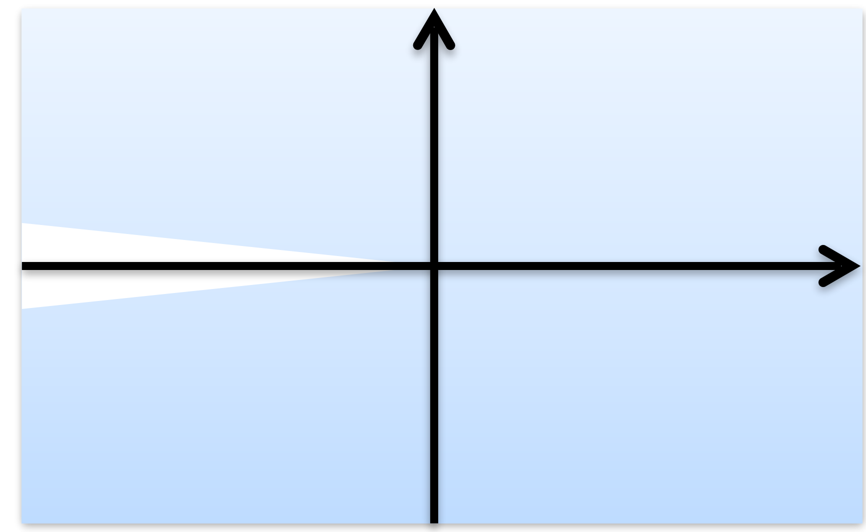
Airy function: $\left(\frac{d^2}{d\zeta^2} - \zeta\right)\psi(\zeta) = 0 \qquad \psi(\zeta) = Ai(\zeta), Bi(\zeta)$

$$Bi(\zeta) = e^{\frac{\pi}{6}i} Ai(e^{\frac{2}{3}\pi i}\zeta) + e^{-\frac{\pi}{6}i} Ai(e^{-\frac{2}{3}\pi i}\zeta)$$

$\zeta \to +\infty$

$$Ai(\zeta) \simeq \frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\left[1+\cdots\right]$$

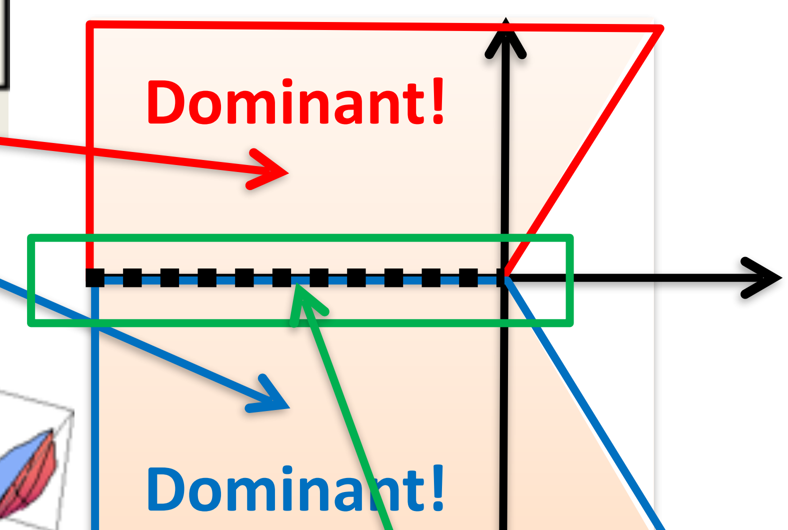
(valid in $\zeta \to \infty, \ |\arg(\zeta)| < \pi$)



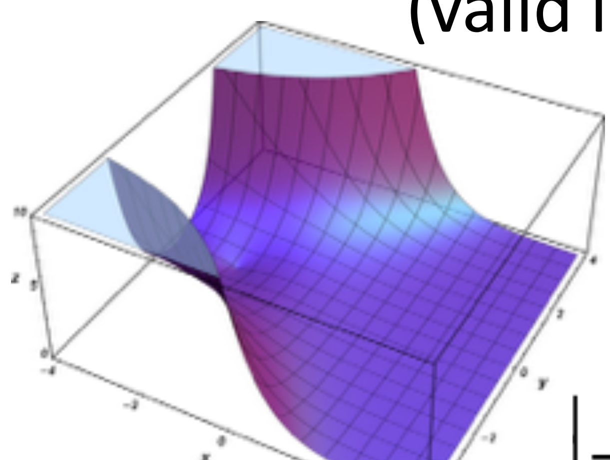
$\zeta \to \infty \times e^{\pi i}$

$$Ai(\zeta) \simeq \frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\left[1+\cdots\right] + i\frac{e^{+\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\left[1+\cdots\right]$$

(valid in $\zeta \to \infty, \ |\pi - \arg(\zeta)| < \frac{2\pi}{3}$)

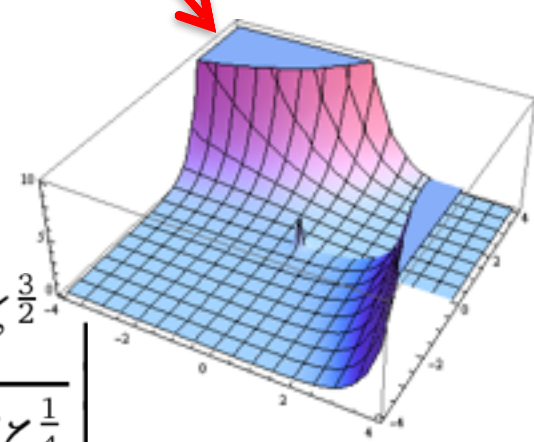


Change of dominance
(Stokes line)

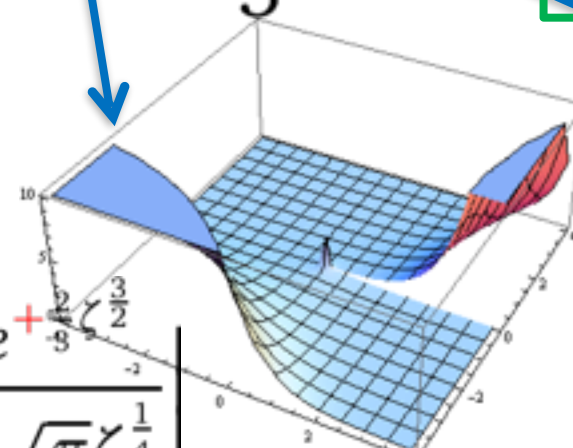


(from Wikipedia)

$\approx \left|\frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\right|$



$+ \left|\frac{e^{+\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\right|$



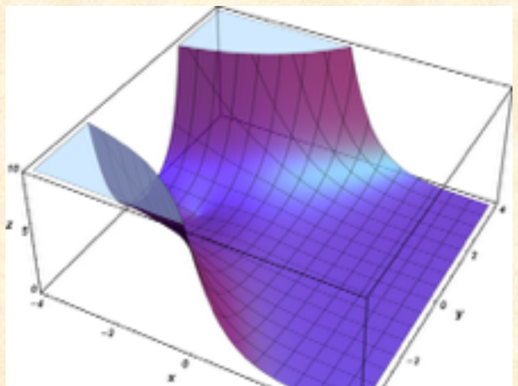
Stokes Phenomenon in Airy function

Airy system ⇔ (2,1) minimal string theory

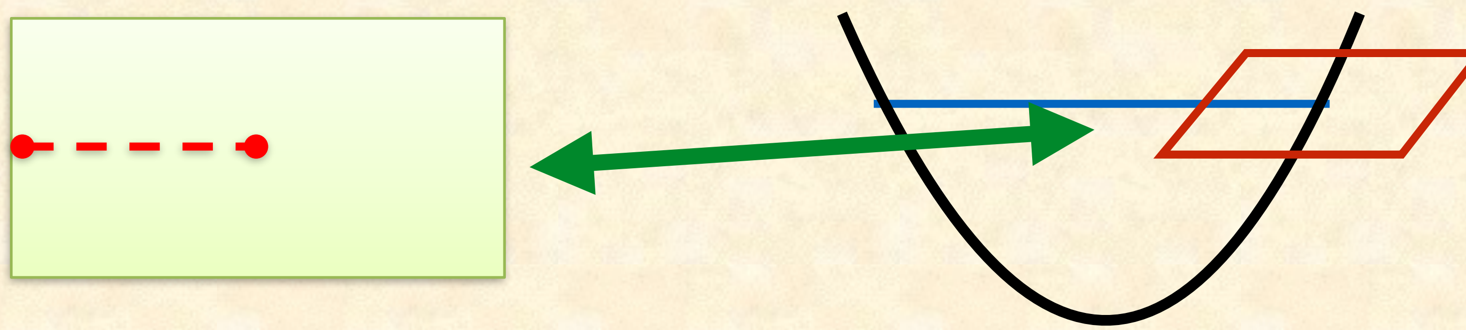
$$Ai(x) = \langle \det(x - X) \rangle \sim e^{\frac{1}{g}\varphi(x)}$$

$$W(x) = \partial_x \varphi(x)$$

gaussian matrix models



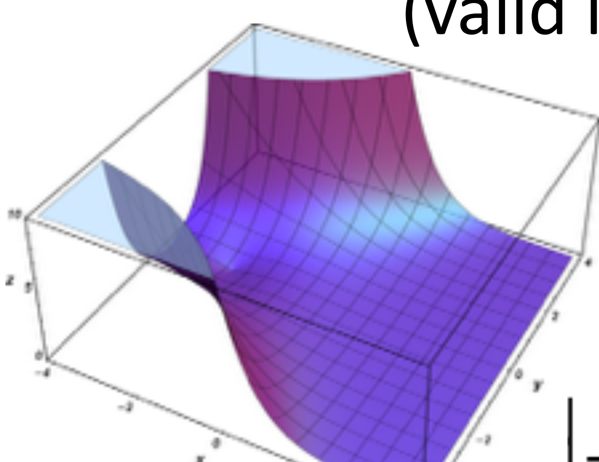
(from Wikipedia)



$\zeta \to \infty \times e^{\pi i}$

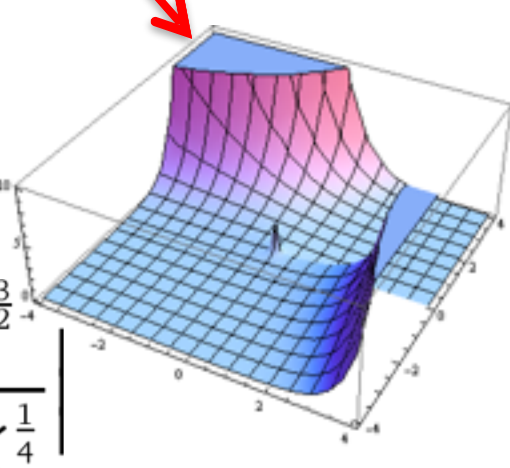
$$Ai(\zeta) \simeq \frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\Big[1+\cdots\Big] + i\,\frac{e^{+\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\Big[1+\cdots\Big]$$

(valid in $\zeta \to \infty,\ |\pi - \arg(\zeta)| < \frac{2\pi}{3}$)

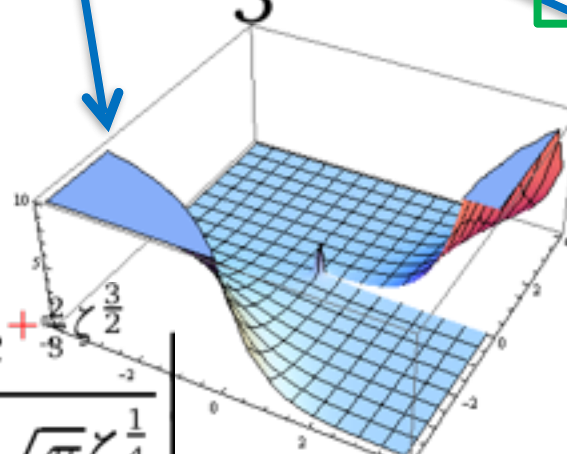


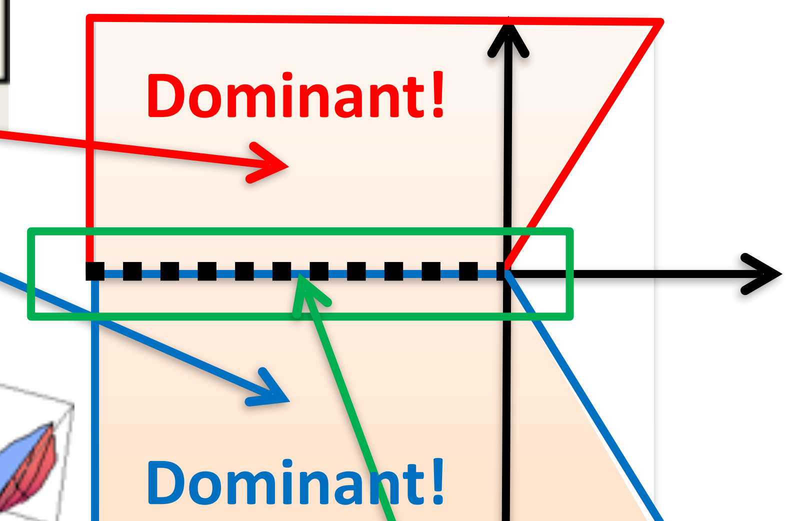
(from Wikipedia)

$\approx \left|\frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\right|$



$+ \left|\frac{e^{+\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\right|$



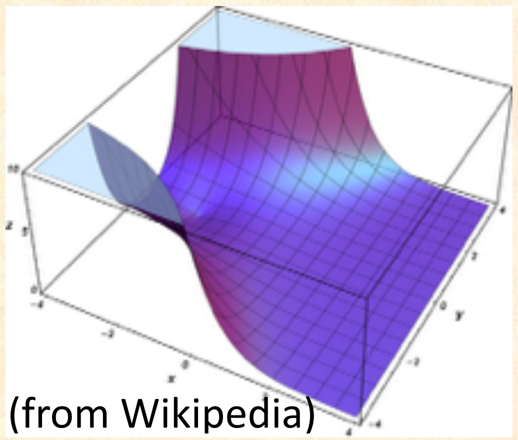


Change of dominance **(Stokes line)**

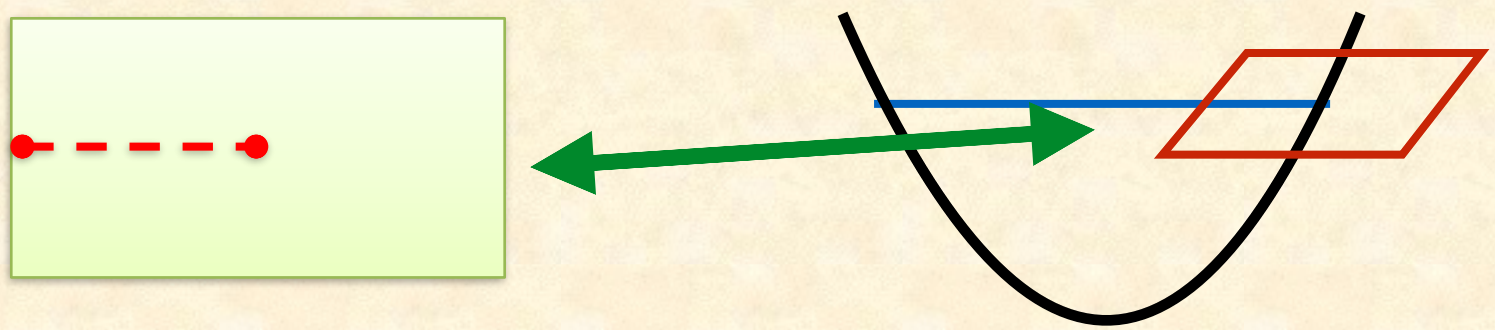
Stokes Phenomenon in Airy function

Airy system ⇔ (2,1) minimal string theory

$$Ai(x) = \langle \det(x - X) \rangle \sim e^{\frac{1}{g}\varphi(x)} \qquad W(x) = \partial_x \varphi(x)$$


(from Wikipedia)

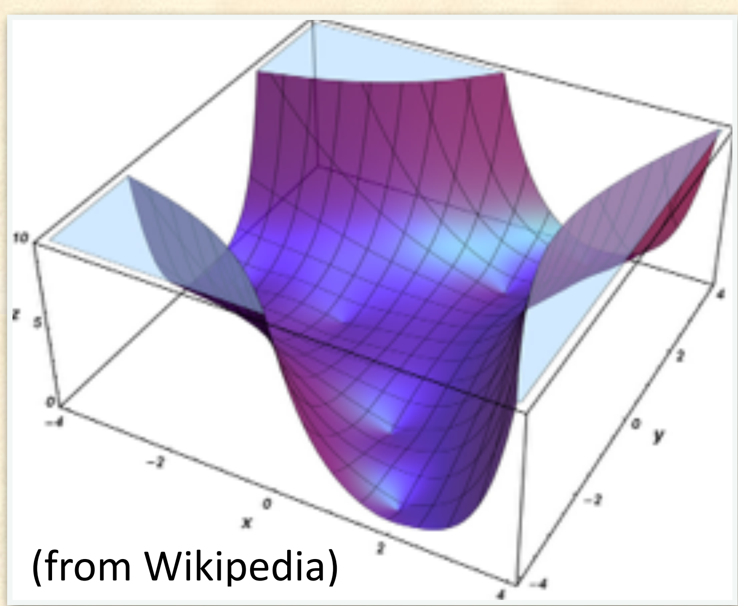
gaussian matrix models

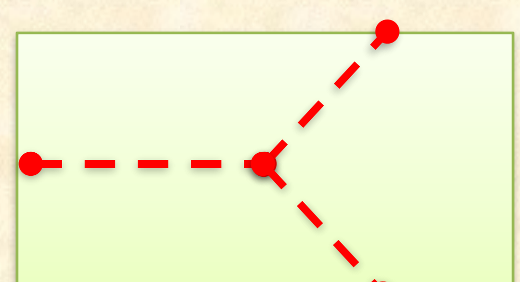


NOTE: If one considers Biry Function $\mathrm{Bi}(x) \simeq e^{\frac{1}{g}\varphi_{\mathrm{Bi}}(x)}$

of $$\Big(\frac{d^2}{dx^2} - x\Big)\mathrm{Bi}(x) = 0$$

the cut of the exponent is given as


(from Wikipedia)

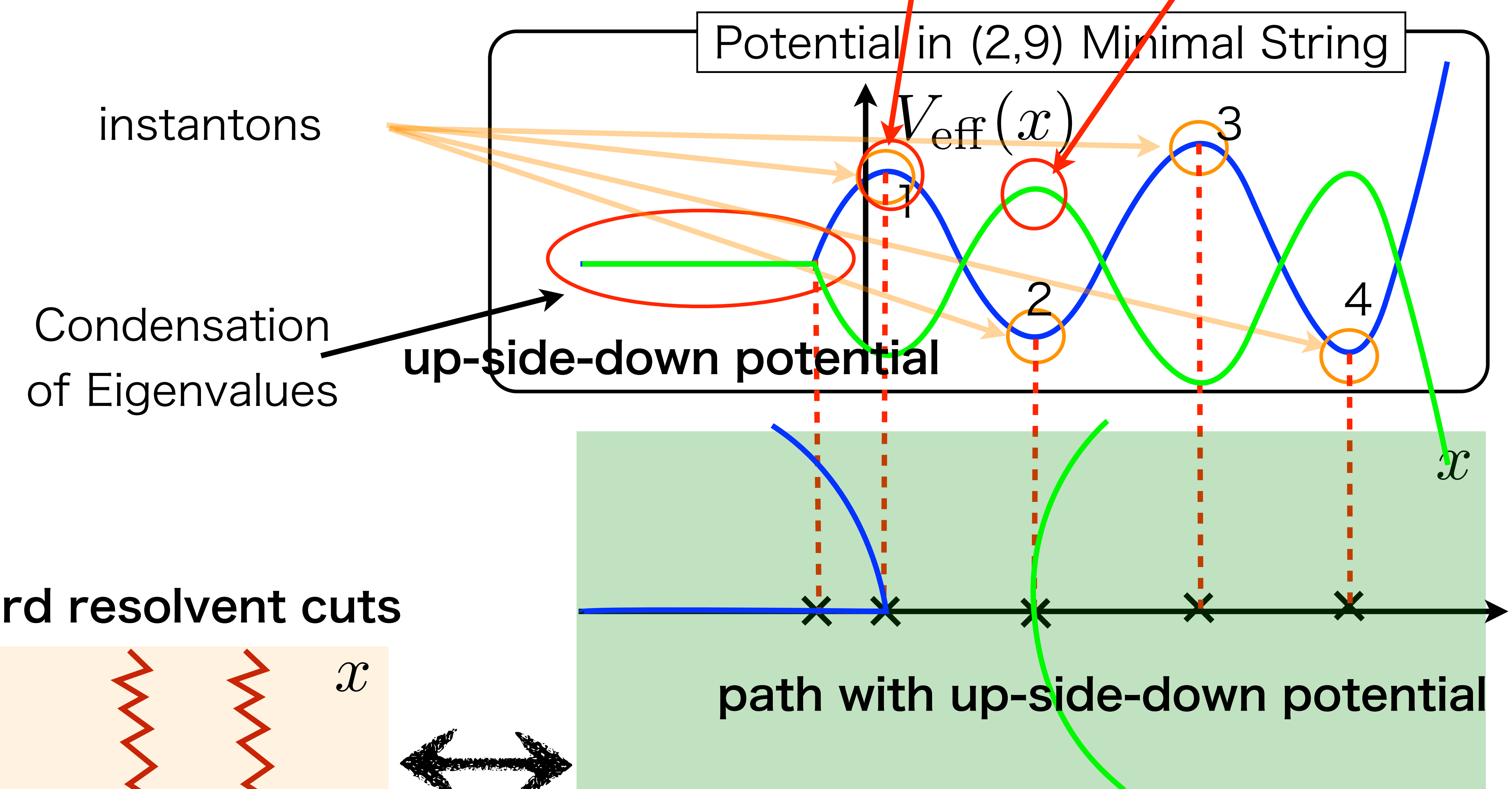


$$\partial_x \varphi_{\mathrm{Bi}}(x)$$

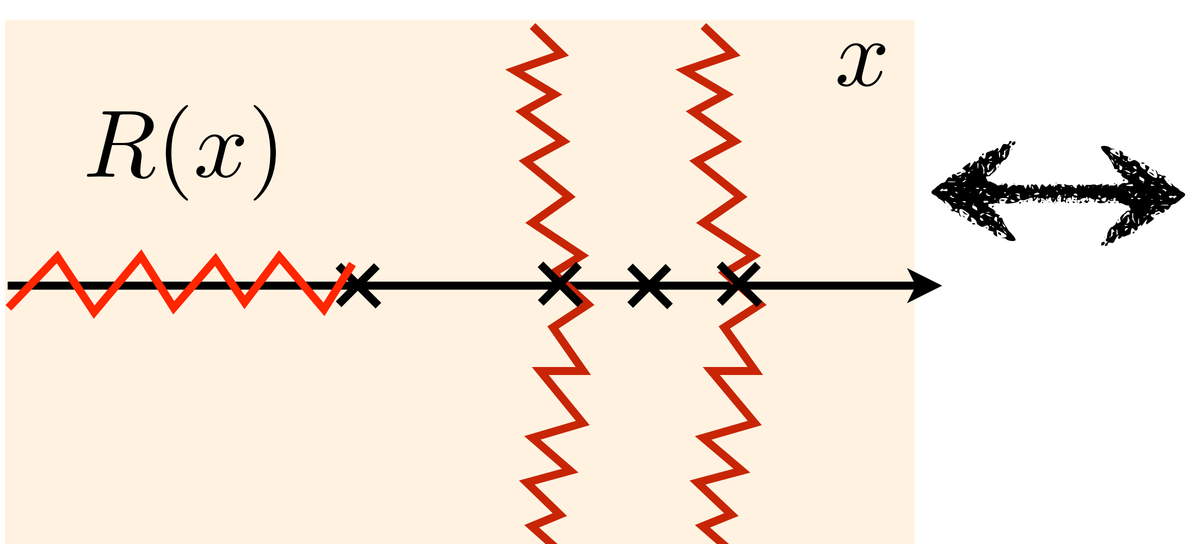
Multi-cut Boundary Condition [CIY4 '12]

small instantons

Model 3: $$\mathcal{F}(g) \simeq \sum_{n=0}^{\infty} g^{2n-2}\mathcal{F}_n + \theta_1 e^{-\frac{1}{g}\mathcal{S}_1} + \theta_2 e^{-\frac{1}{g}\mathcal{S}_2} + \cdots$$



No standard resolvent cuts



No matrix-model realization

Two results on Stokes phenomena

2) "Spectral Networks" = Matrix-Model Contours

Spectral Networks = Graphical way to express Stokes phenomena

[Deift-Zhou]

Airy Eqn $\left(\frac{d^2}{dx^2} - x\right)\psi(x) = 0$ → asymp. solutions $\psi_\pm(x) \simeq \frac{e^{\pm\frac{2}{3}x^{\frac{2}{3}}}}{2\sqrt{\pi}x^{\frac{1}{4}}}\left[1 + \cdots\right]$

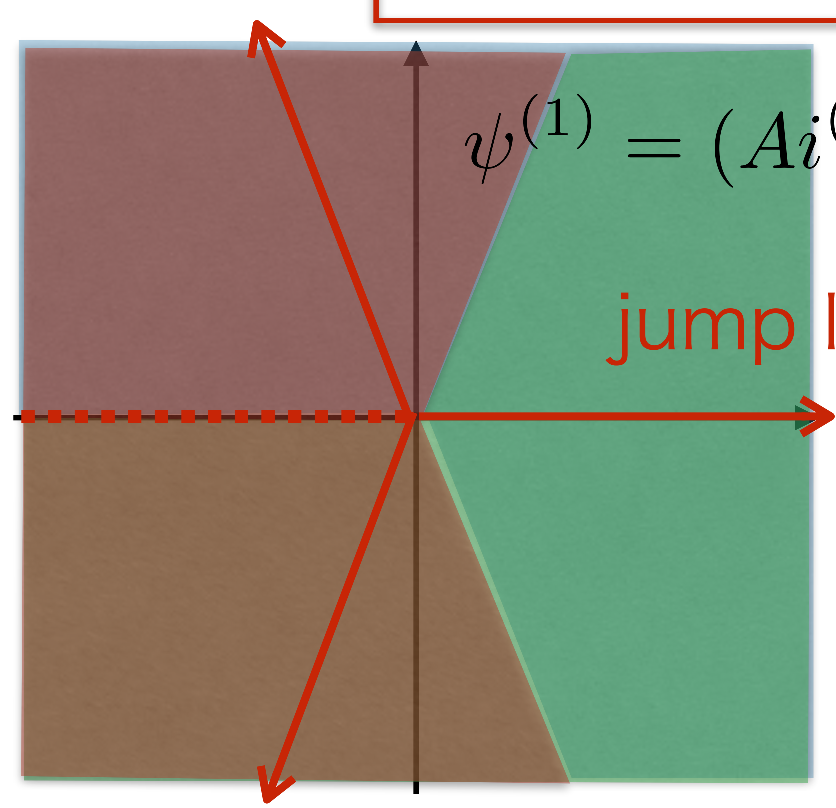
Keep using the same asymptotic solutions

$\psi^{(n)}(x) \simeq \left(\psi_-(x), \psi_+(x)\right)$ in each Stokes sector

$$\psi_{\rm RH}(x) =$$

$\psi^{(2)} = (Ai^{(2)}, Ai^{(1)})$

$\psi^{(1)} = (Ai^{(0)}, Ai^{(1)})$



$\psi^{(-1)} = (Ai^{(-2)}, Ai^{(-1)})$

$\psi^{(0)} = (Ai^{(0)}, Ai^{(-1)})$

$$Ai^{(n)}(x) = e^{-\frac{\pi i n}{6}} Ai(e^{-\frac{2\pi i n}{3}} x) \qquad Ai^{(n)} - i Ai^{(n+1)} = Ai^{(n+2)}$$

Two important considerations

2) "Spectral Networks" = Matrix-Model Contours [CIY4]

Spectral Networks = Graphical way to express Stokes phenomena

[Deift-Zhou]

Since each solution is related by matrices $\{S_n\}_n$

$$\psi^{(n+1)}(x) = \psi^{(n)}(x) S_n$$

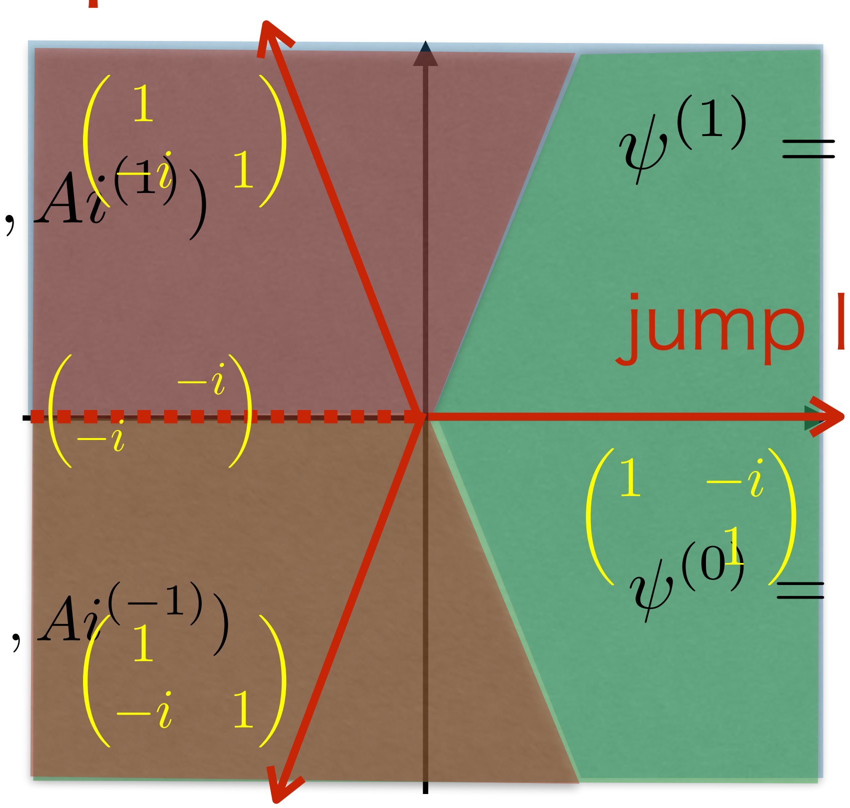
we assign the matrices to the jump lines

Jump lines + Stokes matrices = Spectral Network!

$$\psi^{(2)} = (Ai^{(2)}, Ai^{(1)})$$

$$\psi^{(1)} = (Ai^{(0)}, Ai^{(1)})$$

$$\psi_{\mathrm{RH}}(x) =$$



$$\psi^{(0)} = (Ai^{(0)}, Ai^{(-1)})$$

$$\psi^{(-1)} = (Ai^{(-2)}, Ai^{(-1)})$$

$$Ai^{(n)}(x) = e^{-\frac{\pi i n}{6}} Ai(e^{-\frac{2\pi i n}{3}} x) \qquad Ai^{(n)} - i Ai^{(n+1)} = Ai^{(n+2)}$$

Two important considerations

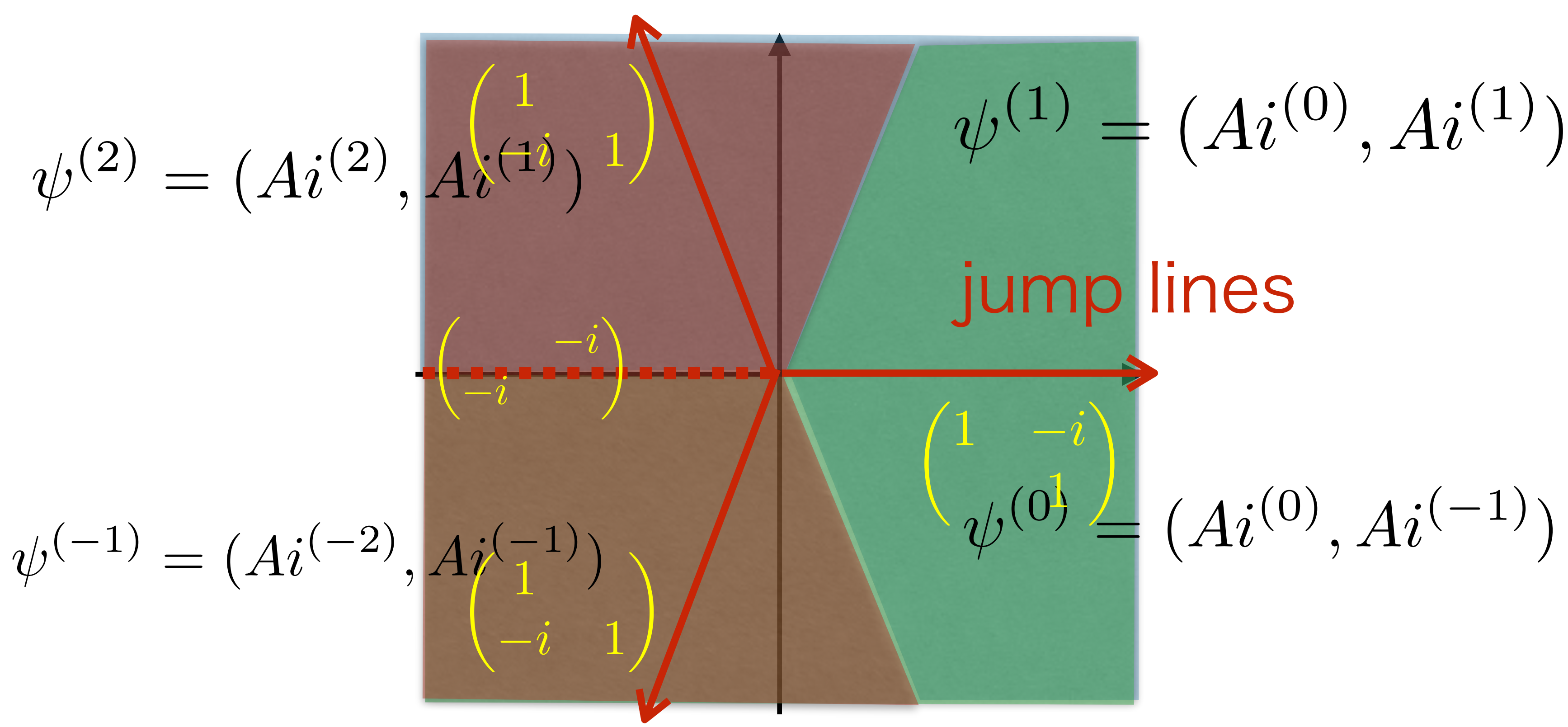
2) "Spectral Networks" = Matrix-Model Contours [CIY4]

Riemann-Hilbert problem [Miwa-Jimbo⋯]

Actually, If we know spectral network of $\psi^{(n+1)}(x) = \psi^{(n)}(x) S_n$

and with the leading asymptotics: $\psi_{\pm}(x) \simeq \frac{e^{\pm\frac{2}{3}x^{\frac{2}{3}}}}{2\sqrt{\pi}x^{\frac{1}{4}}}\left[1+\cdots\right]$

Then, we can reconstruct every other information



$$Ai^{(n)}(x) = e^{-\frac{\pi i n}{6}} Ai(e^{-\frac{2\pi i n}{3}} x) \qquad Ai^{(n)} - i Ai^{(n+1)} = Ai^{(n+2)}$$

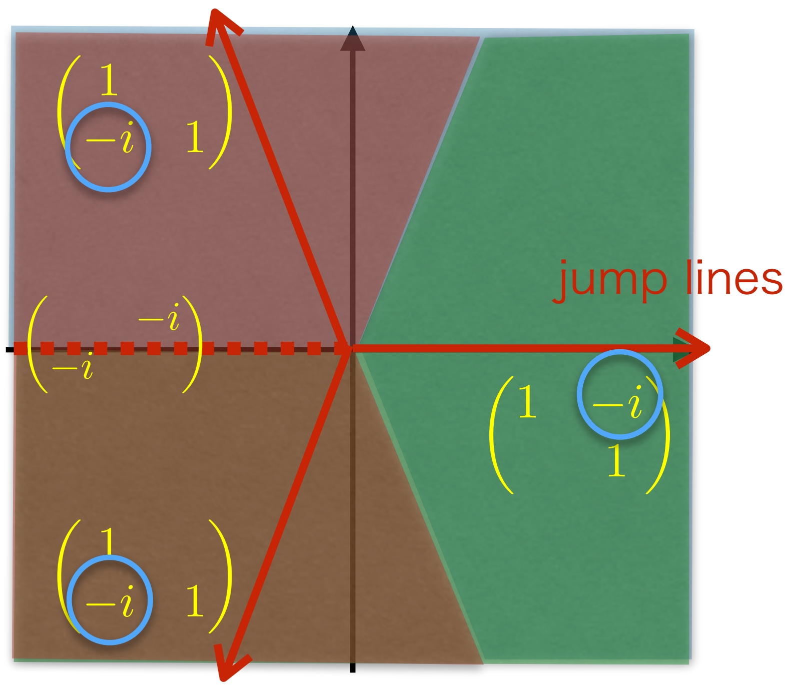
Two important considerations

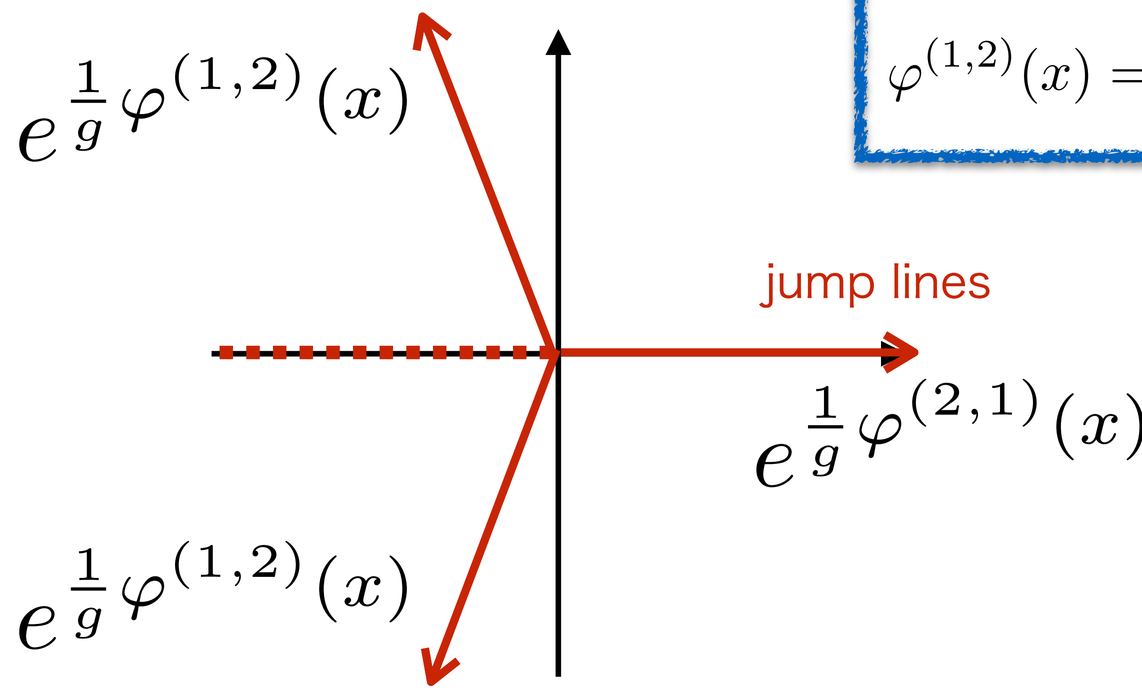
2) "Spectral Networks" = Matrix-Model Contours [CIY4]

matrix-model contours by comparison

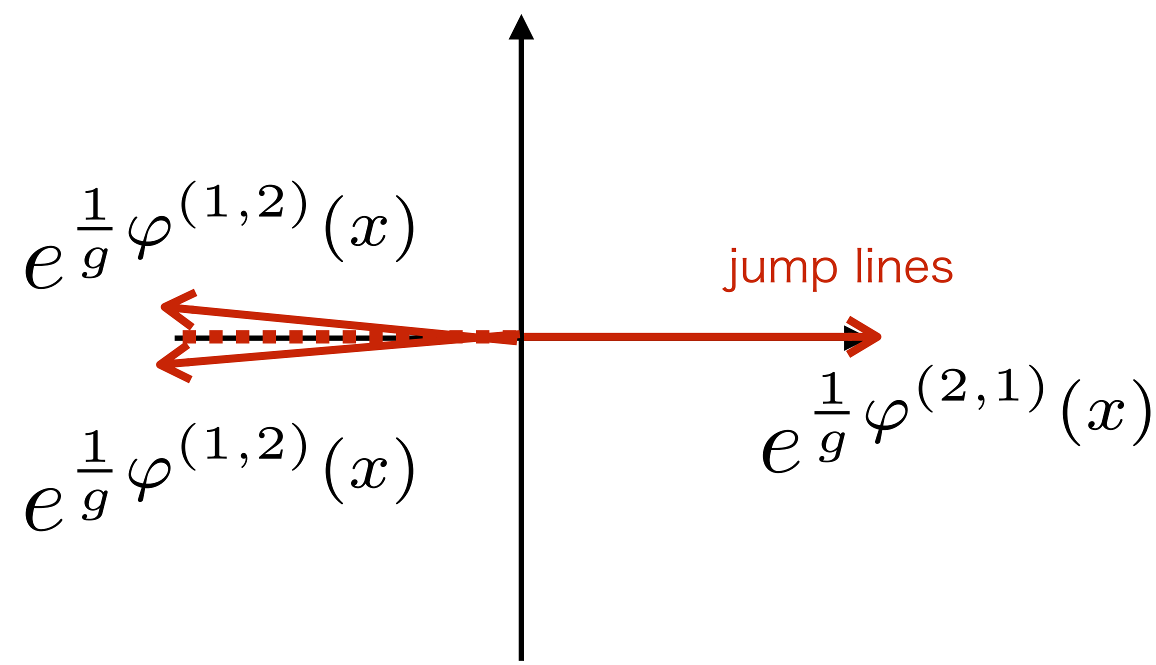
$$\varphi^{(2,1)}(x) = -\frac{4}{3}x^{\frac{3}{2}}$$

$$\varphi^{(1,2)}(x) = +\frac{4}{3}x^{\frac{3}{2}}$$





$$e^{\frac{1}{g}V_{\mathrm{eff}}(x)} \simeq$$



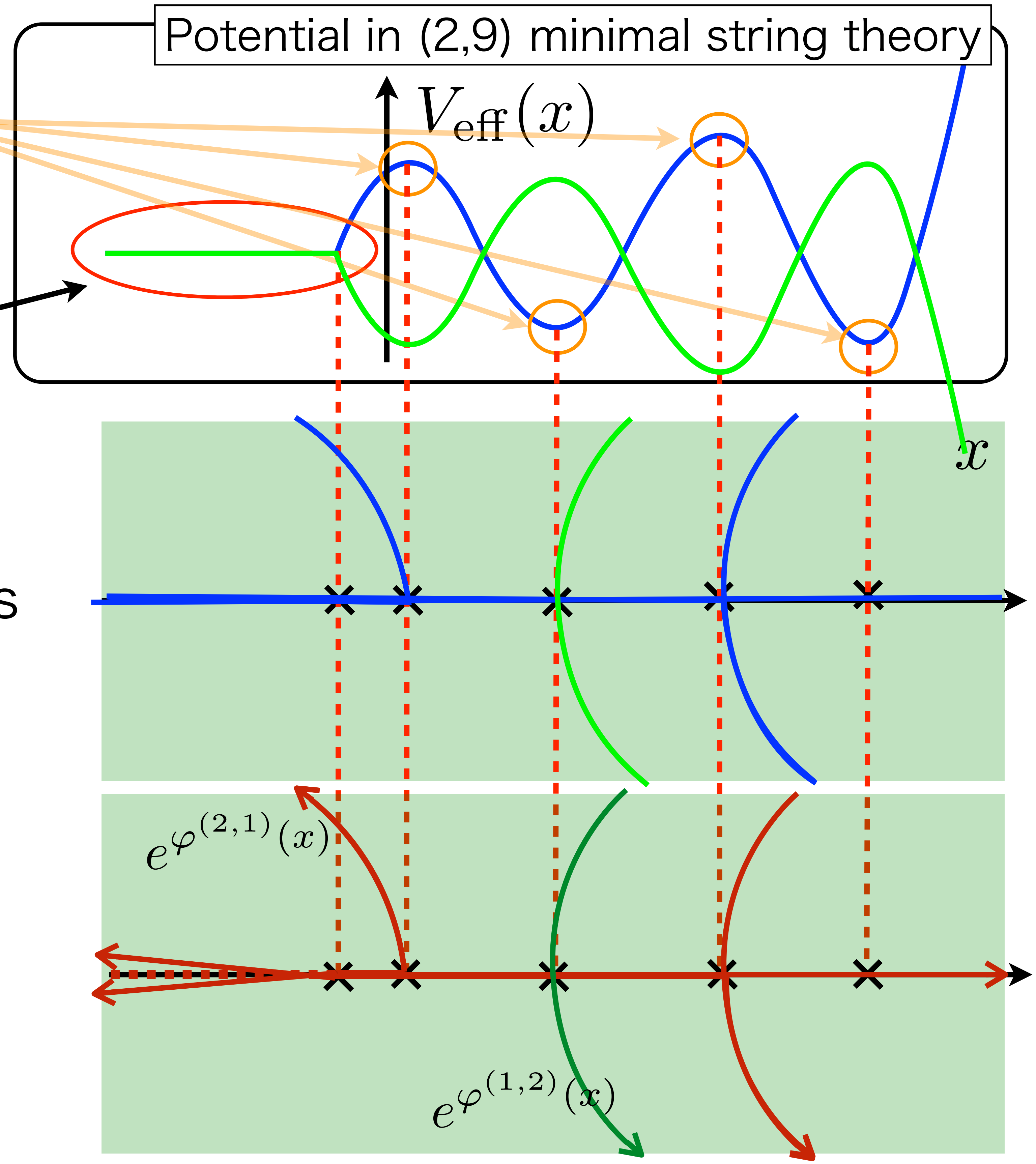
2) "Spectral Networks" = Matrix-Model Contours [CIY4]

Instantons

Condensation of other eigenvalues

Matrix-model contours

Spectral Networks (Stokes Phenom)



Stokes Phenomena (Result) [CIY5]

Generally:

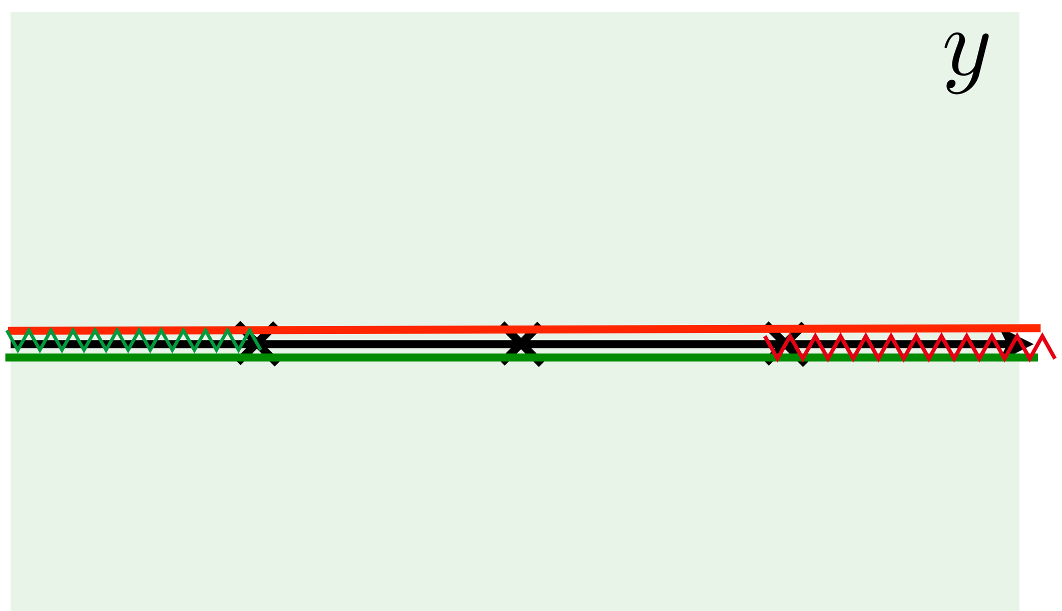
$$e^{-N\widetilde{V}_{\rm eff}(y)} = \sum_{j,l} (*)_{j,l} \times e^{\varphi^{(j,l)}(y)}$$

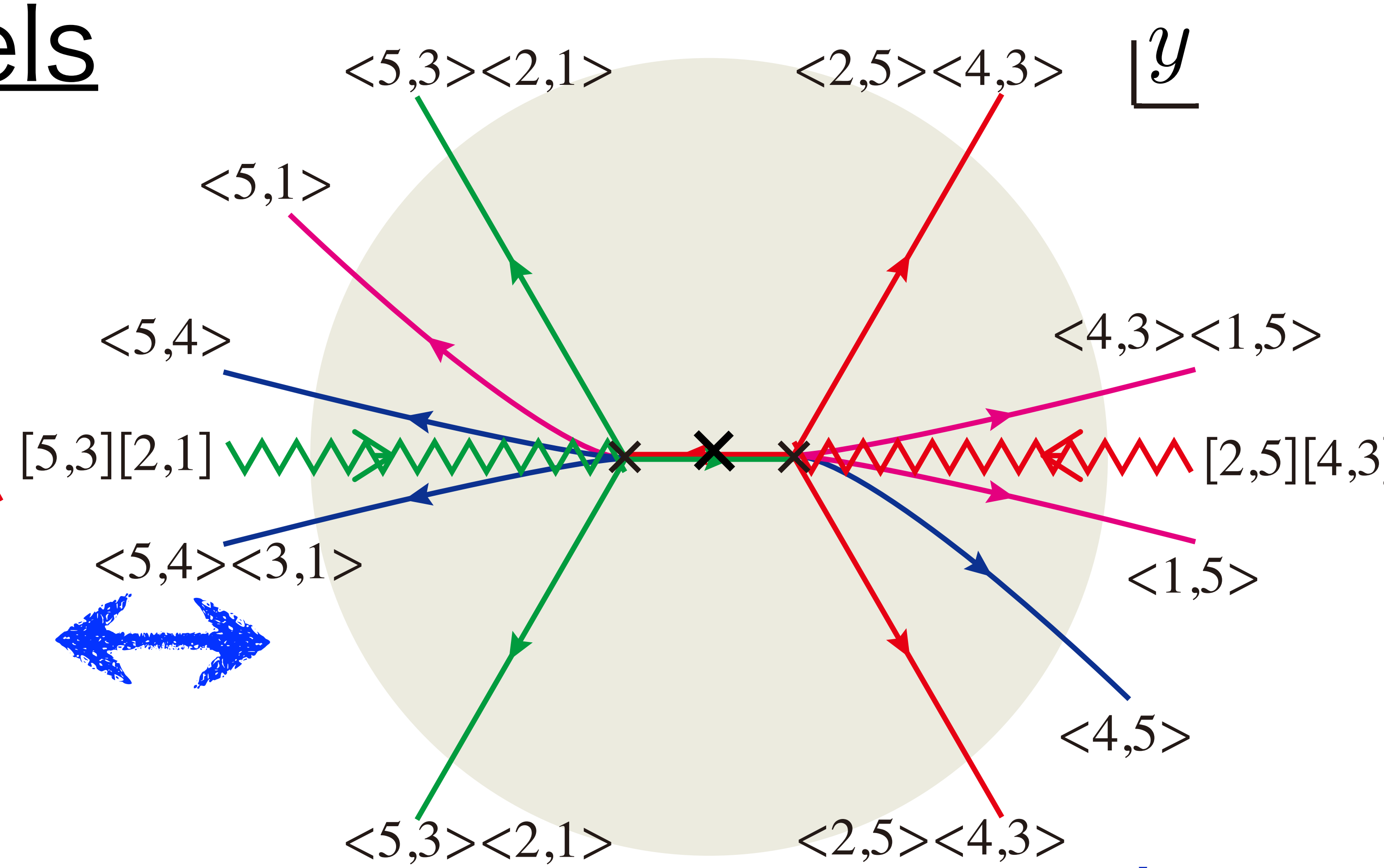
Sol to saddle Eqn

Stokes Coefficients

$$\left(\begin{array}{c} \varphi^{(j,l)}(y) = \int dy\,\left[x^{(j)}(y) - x^{(l)}(y)\right] \\ f(x^{(j)}(y), y) = 0 \end{array} \right)$$

(5,2) ↔ (2,5) Models



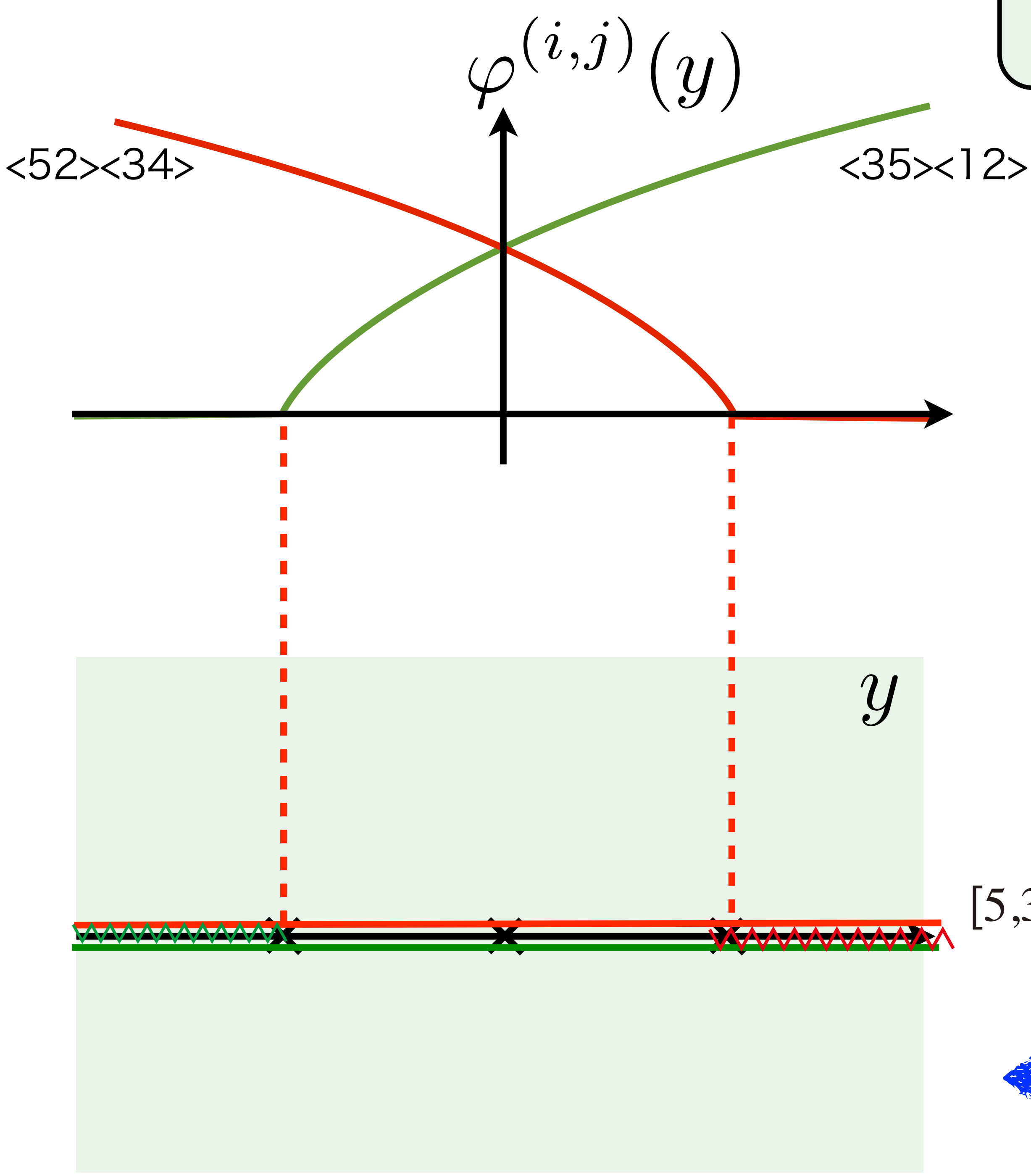


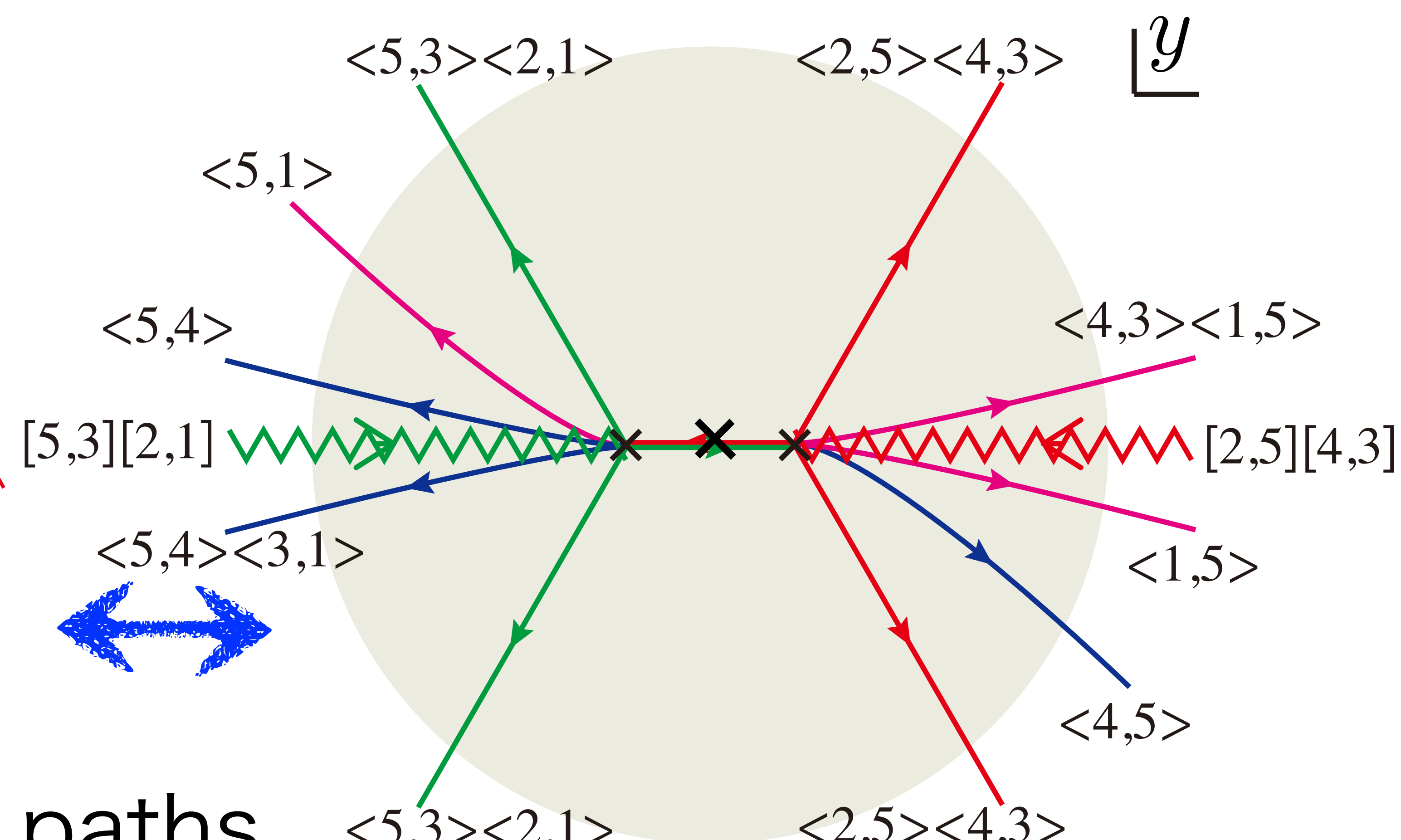
steepest descent

(5,2) ↔ (2,5) Models

$$e^{-N\widetilde{V}_{\rm eff}(y)} = \sum_{j,l} (*)_{j,l} \times e^{\varphi^{(j,l)}(y)}$$

Roughly speaking, it is a superposition of two different effective potentials



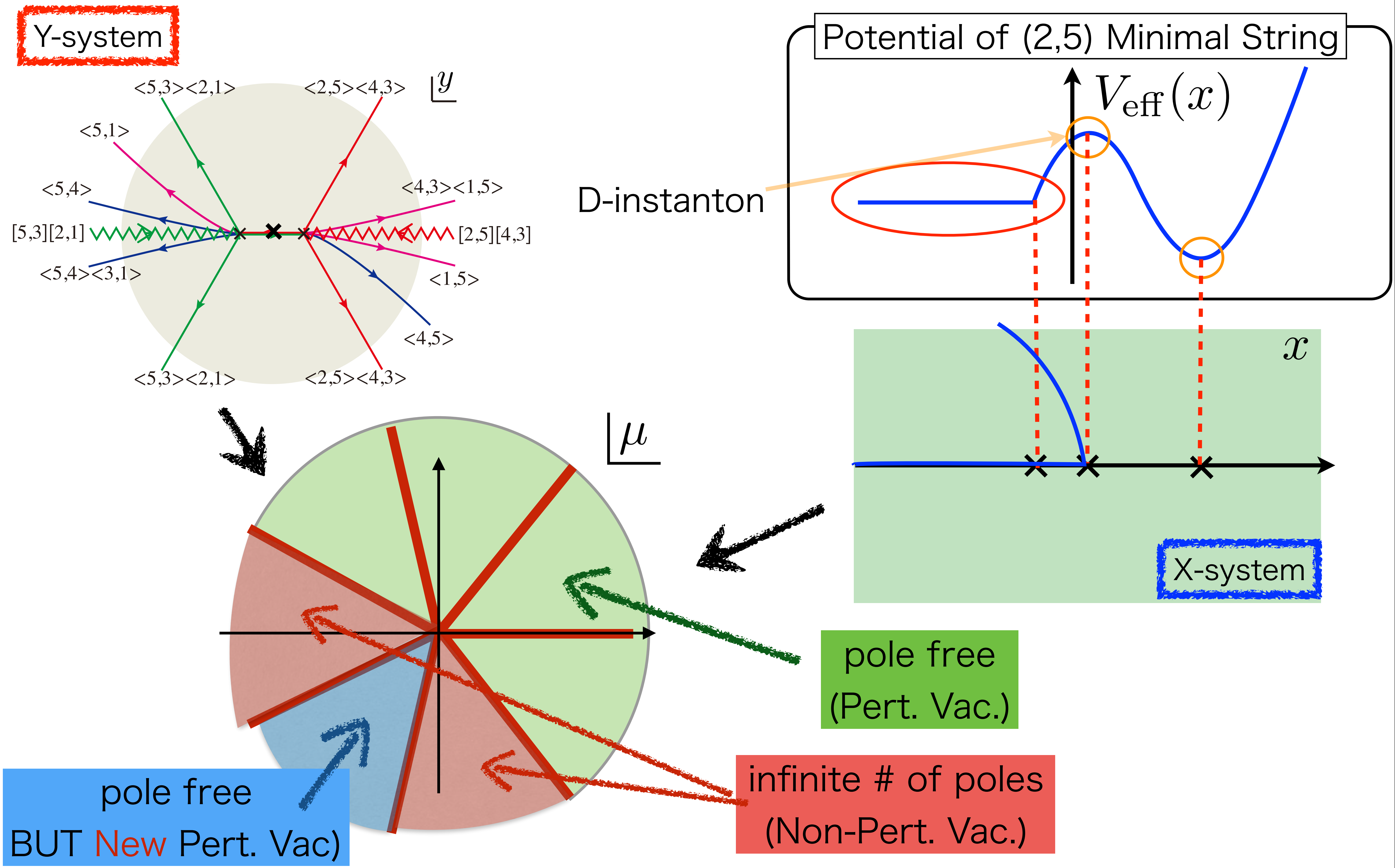


steepest descent

And we can add other paths

Duality Check of (5,2) ↔ (2,5) Models [CIY6]

We are now checking the phase structure in g -> 0 (or μ -> ∞)



4. Duality Constraints on String Theory

Are they Equivalent?

Two-matrix model

gaussian

$$\mathcal{Z} = \int dX dY \, e^{-N\mathrm{tr}[V_1(X) + \frac{Y^2}{2} - XY]}$$

We can perform !

Integrate Y — Integrate X

X-system ⟷ Y-system

$$\mathcal{Z} = \int dX e^{-N\mathrm{tr}V(X)}$$

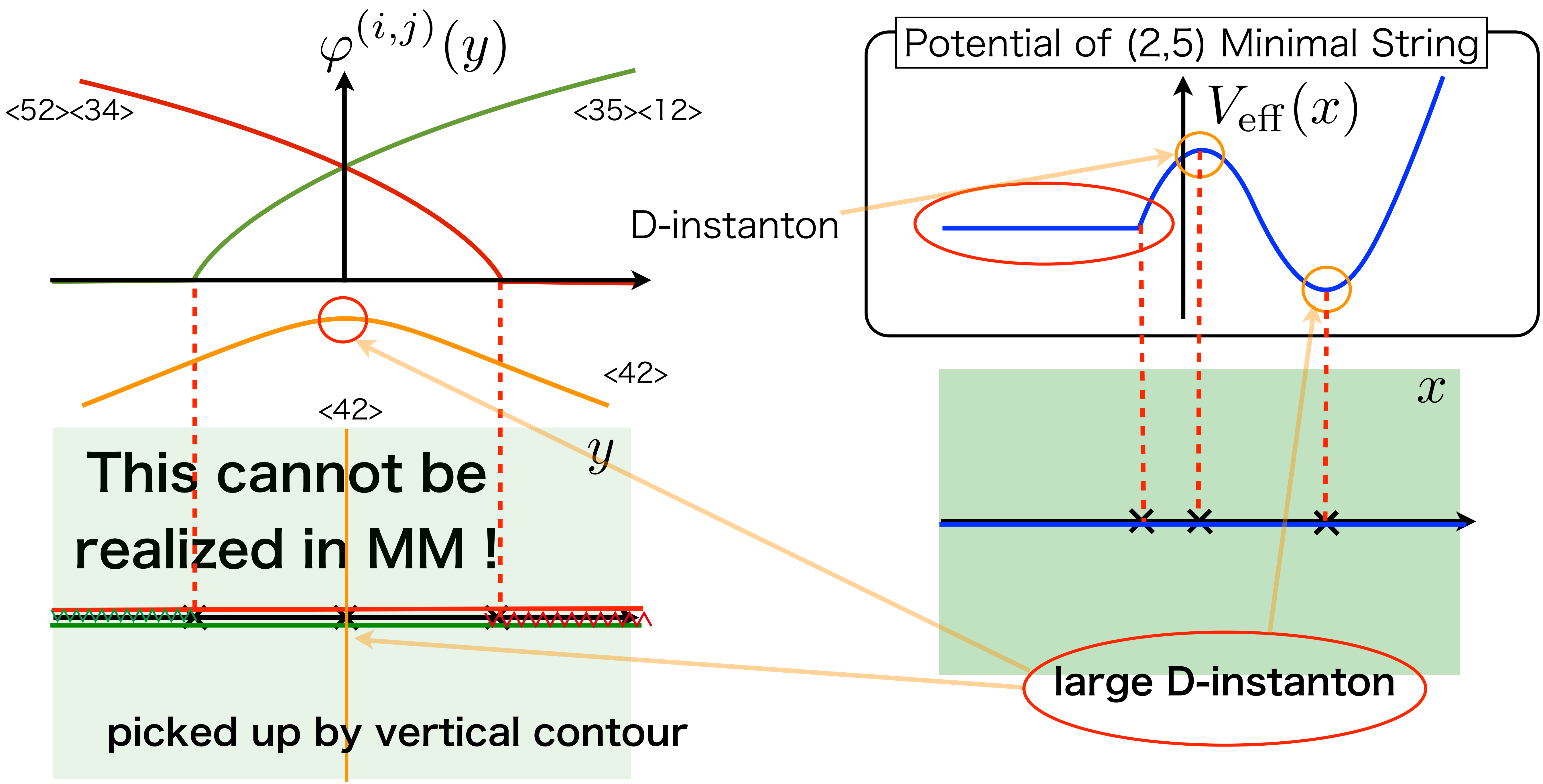
one-matrix model

See large instanton modes

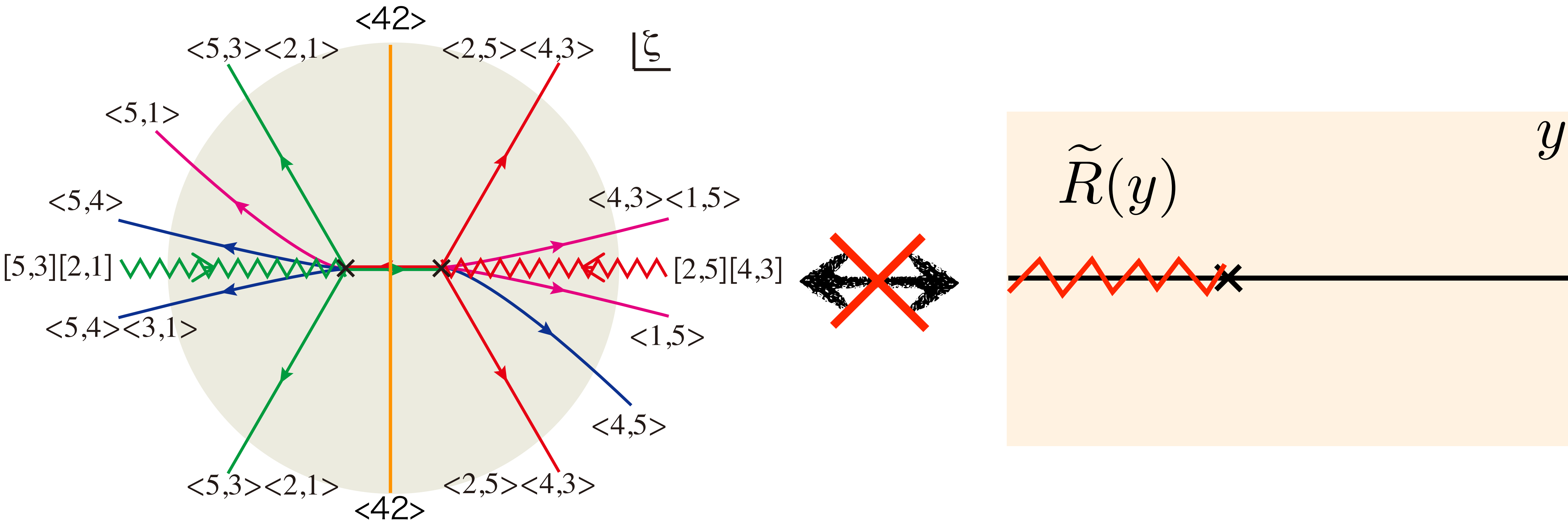
[CIY5]

$$\mathcal{F} \simeq \mathcal{F}_{\text{pert}} + e^{+\frac{1}{g}S_I} + \cdots$$

Large instantons → observed on both sides

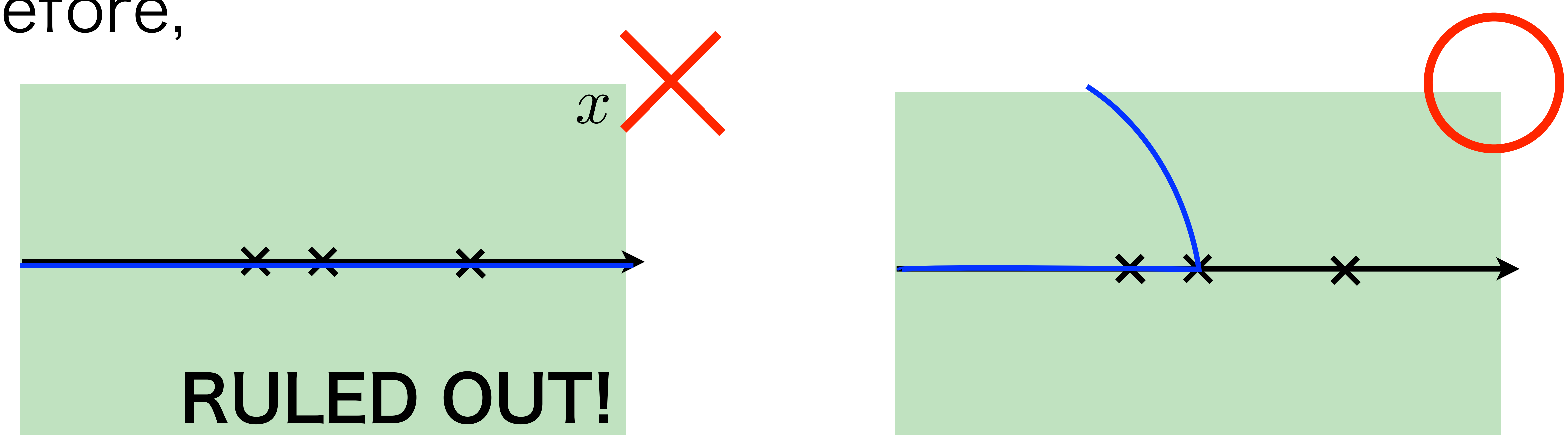


One-cut Boundary condition



This model cannot satisfy One-cut BC

Therefore,



Summary

- Duality in perturbation theory (Large N)
  v.s. Duality in Non-perturb. Completion
- Matrix models are known to possess non-perturbative [contour] ambiguity (because it only relies on *inclusion of perturbative string theory*).
- Duality may be broken non-perturbatively
- Therefore, if one requires "*string duality acts non-perturbatively,*" as a principle, then it provides a constraint on non-perturbative ambiguity of string theory

This is the first quantitative observation on non-perturbative principle of string theory

Thank you for your attention!