The Holographic Dual of “Entanglement of Purification”

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Based on arXiv:1708.09393

Outline

- Introduction
- **Conjecture:** $E_W^{(AdS)} \cong E_P^{(CFT)}$
- Consistency check
- A heuristic proof

Introduction

Quantum states:

Vectors in Hilbert space $\mathcal{H}$

$$|\Psi\rangle$$

Pure states

↓

Density matrices acting on $\mathcal{H}$

$$\rho = \Sigma_n p_n |\Psi_n\rangle\langle\Psi_n|, \Sigma_n p_n = 1, p_n \geq 0$$

Mixed states

- Lack of information (e.g. thermal, noise).
- A state of subsystems:

$$|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \to \rho_A = \mathrm{Tr}_B[|\Psi\rangle\langle\Psi|] \text{ on } \mathcal{H}_A$$

reduced density matrices

Introduction

Correlation:

$(A \cup B \equiv AB)$

$$\rho_{AB} \text{ on } \mathcal{H}_A \otimes \mathcal{H}_B$$

How much do A and B correlate?

- **Information-theoretic** measures of correlation.

For pure states $|\Psi\rangle_{AB}$:

Entanglement entropy

$$\boldsymbol{S(\rho_A) := -\mathrm{Tr}\rho_A \log \rho_A \ \left(= S(\rho_B)\right).}$$

Introduction

Quantum correlation or **entanglement:**

$$|\mathrm{EPR}\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)$$
$$= \frac{1}{\sqrt{2}}(|\boldsymbol{\theta}\rangle_A \otimes |\boldsymbol{\theta}\rangle_B + |\boldsymbol{\theta}_\perp\rangle_A \otimes |\boldsymbol{\theta}_\perp\rangle_B)$$

$$|\boldsymbol{\theta}\rangle \quad \equiv \cos\boldsymbol{\theta}\,|0\rangle - \sin\boldsymbol{\theta}|1\rangle$$
$$|\boldsymbol{\theta}_\perp\rangle \quad \equiv \sin\boldsymbol{\theta}\,|0\rangle + \cos\boldsymbol{\theta}|1\rangle$$

Introduction

Quantum correlation or **entanglement:**

$$|\mathrm{EPR}\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\right)$$

$S_A (= S_B)$ is **the number of EPR pairs** which is **needed to produce/can be extracted from** ρ_{AB} using local operations and classical communications.

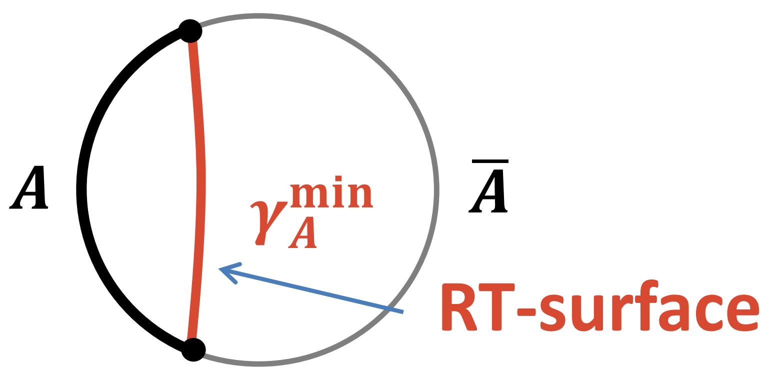
Introduction

Entanglement entropy in AdS/ CFT

Ryu-Takayanagi formula

[Ryu-Takayanagi '06]
[Hubeny-Rangamani-Takayanagi '07]

in CFT $$S_A = \min_{\gamma_A} \frac{\mathrm{Area}(\gamma_A)}{4G_N}$$ **in AdS**



γ_A: codimension-2 surfaces

1. $\partial\gamma_A = \partial A$
2. γ_A is homologous to A

Information-theoretic interpretation:

Number of EPR pairs ≈ Area of RT-surface

Introduction

Entanglement entropy

$$S(\rho_A) \equiv S_A = -\mathrm{Tr}\rho_A \log\rho_A \ \ (= S_B).$$

is a **unique** correlation measure **for pure states**.

[Donald-Horodecki-Rudolph '02]

(If $E(A:B)$ satisfies the axioms of measure of quantum correlations, $E(A:B) = S_A = S_B$ for pure states.)

Introduction

How about **mixed states**?

- **Entanglement entropy** is **no more entanglement!**
- $S_A \neq S_B$ in general.

In quantum information theory,
$\exists$ **Various** measures of correlation **for mixed states**.

- Mutual information $I(A:B)$,
- Squashed entanglement $E_{sq}(A:B)$,
- Entanglement of purification $E_P(A:B)$, etc.

Introduction

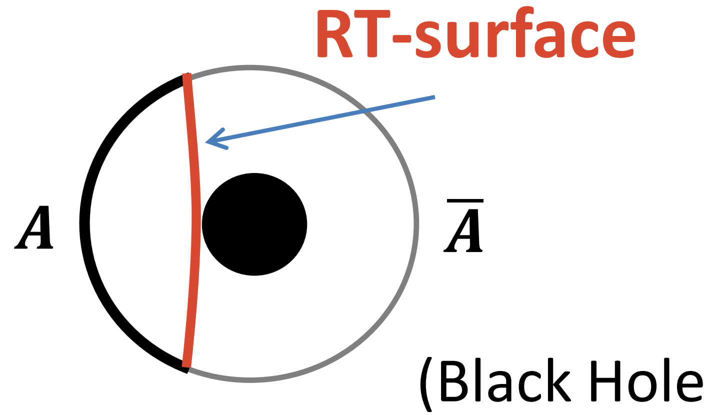
In AdS/ CFT ?

No duality of correlation measures was known.
(essentially)

Mixed states in AdS/CFT

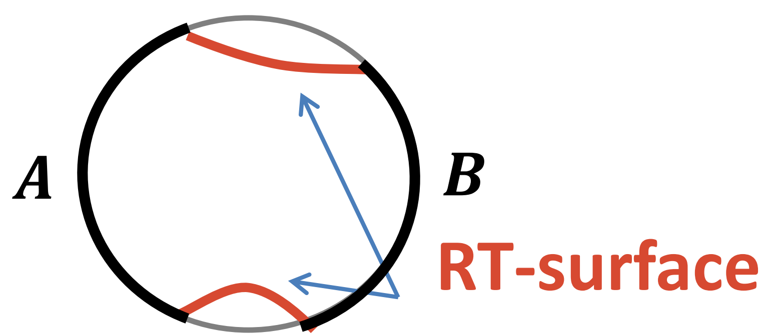
e.g.) Thermal state:

$$\rho_{Thermal} = \frac{e^{-\beta H}}{Z(\beta)}$$



A reduced density matrix:

$$\rho_{AB} = \mathrm{Tr}_{\overline{AB}}[|\Omega\rangle\langle\Omega|]$$



RT-formula still works, but S_A has no interpretation!

Introduction

We seek **a new duality**
between **correlation measures for mixed states**
and **geometry**,
which will bring us an information-theoretic
interpretation of AdS/CFT.

Our conjecture

Our conjecture

We suggest a new holographic duality:

$$E_W^{(AdS)} \cong E_P^{(CFT)}.$$

[Takayanagi-KU '17]

- E_W: **Entanglement wedge cross section** in AdS.

[Takayanagi-KU '17]

- E_P : **Entanglement of purification** in CFT.

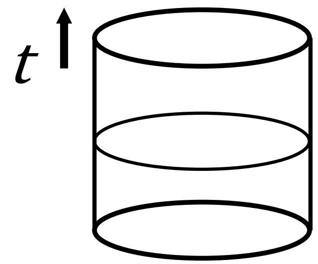
[Terhal-Horodecki-Leung-DiVincenzo '02]

[Note: *Nguyen-Devakul-Halbasch-Zaletel-Swingle [1709.07424]* also suggested the same duality.]

Entanglement wedge cross section

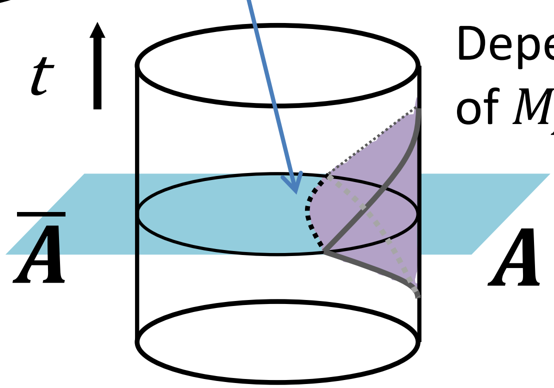
Entanglement wedge

Total states: $\rho_{tot}^{(CFT)} \cong$



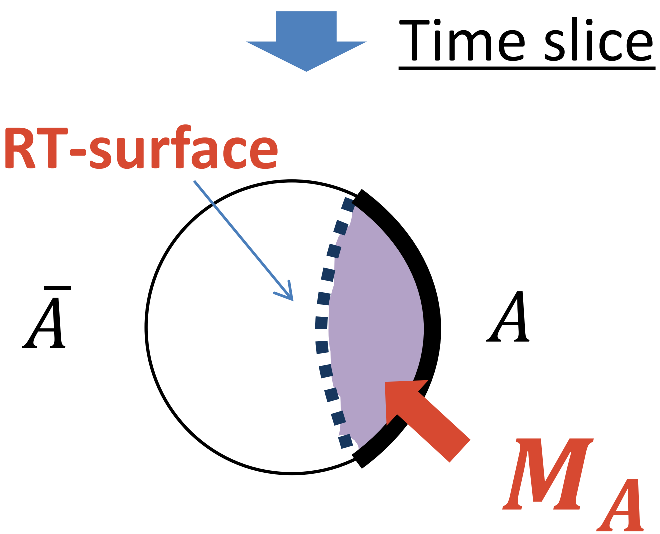
Partial states:

$$\rho_A^{(CFT)} = \mathrm{Tr}_{\bar{A}}\rho_{tot}^{(CFT)} \cong$$



Entanglement wedge in AdS

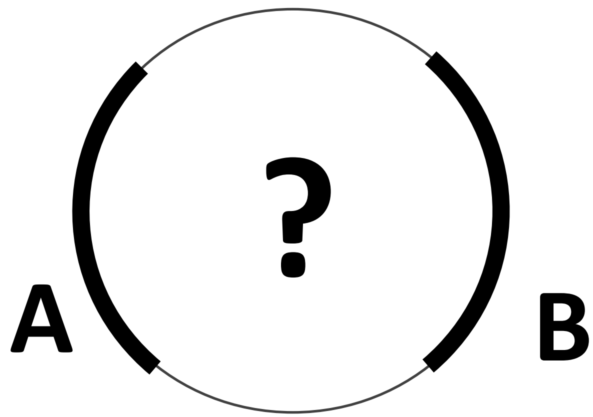
[Czech-Karczmarek-Nogueira-Raamsdonk '12]
[Wall '12][Headrick-Hubeny-Lawrence-Rangamani '14]



Cf. Bulk local operator reconstruction, kinematic space

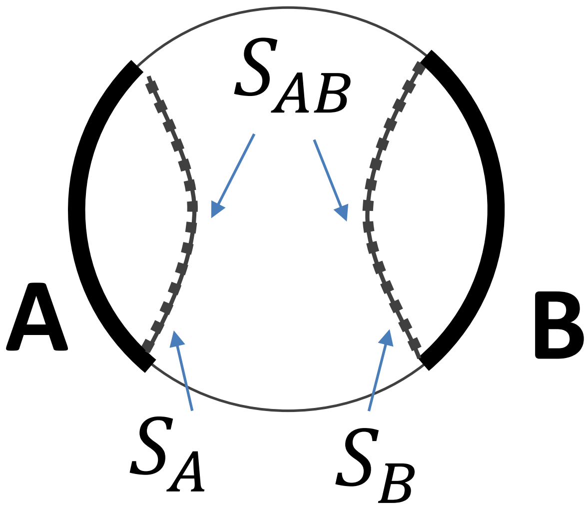
[Hamilton-Kabat-Lifschytz-Lowe '06]
[Czech-Lamprou-McCandlish-Sully '15]
[Kabat-Lifschytz '17] etc.

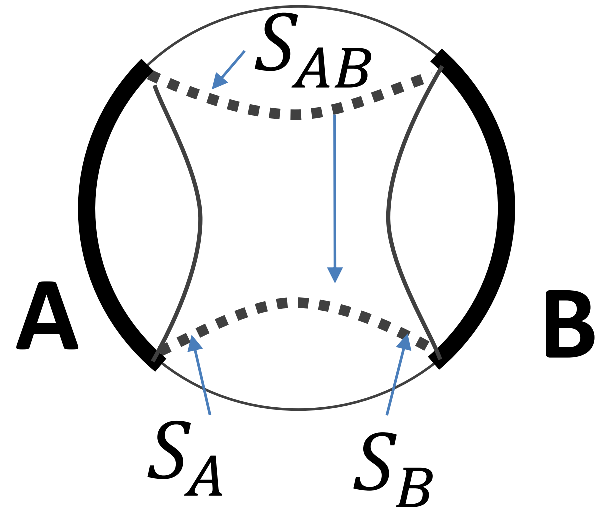
Entanglement wedge
of two disjoint subsystems ρ_{AB}



Entanglement wedge

RT-surface of S_{AB}





When:

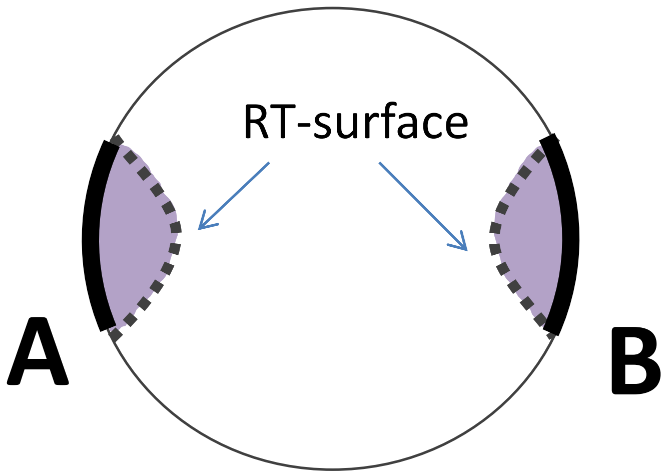
$\underline{S_A + S_B = S_{AB}}$

$\underline{S_A + S_B > S_{AB}}$

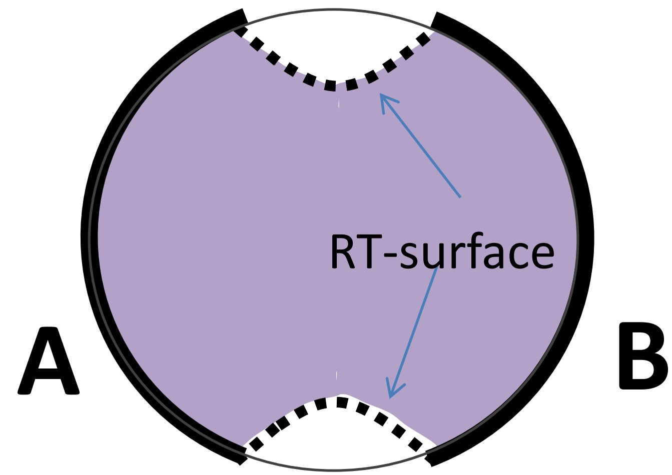
- **Mutual information:** $\boldsymbol{I(A:B) \equiv S_A + S_B - S_{AB}}$

Entanglement wedge

Entanglement wedges



$\underline{I(A:B) = 0}$



$\underline{I(A:B) > 0}$

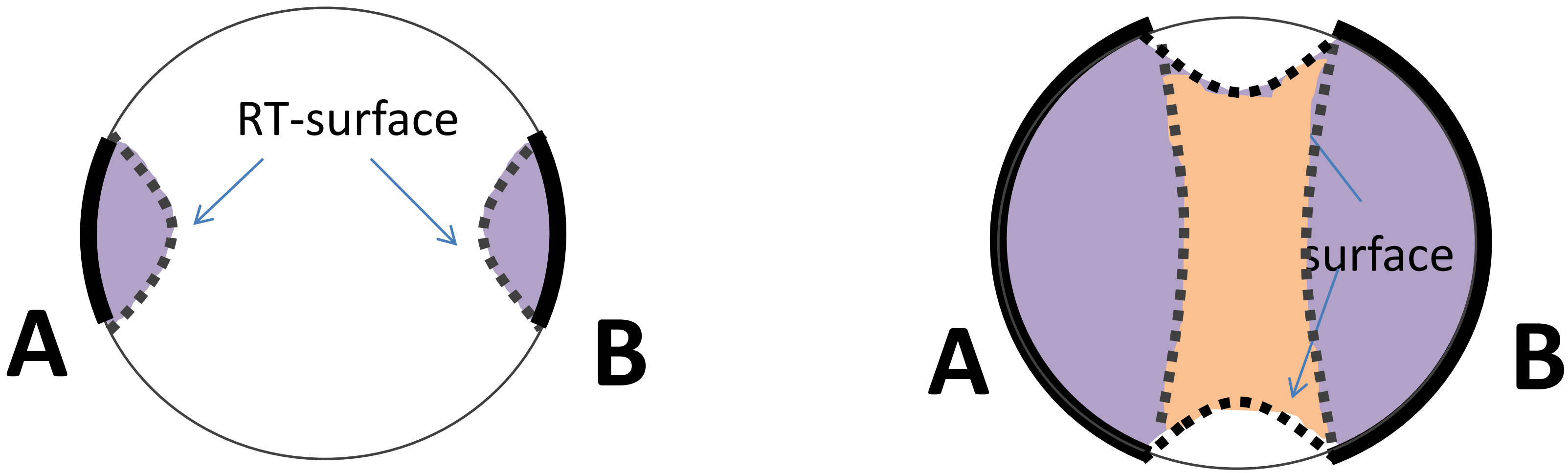
- **Mutual information**: $\boldsymbol{I(A:B) \equiv S_A + S_B - S_{AB}}$

$$I(A:B) = 0 \Leftrightarrow \rho_{AB} = \rho_A \otimes \rho_B.$$

Entanglement wedge

Entanglement wedges

Come from correlations



$\underline{I(A:B) = 0}$ $\underline{I(A:B) > 0}$

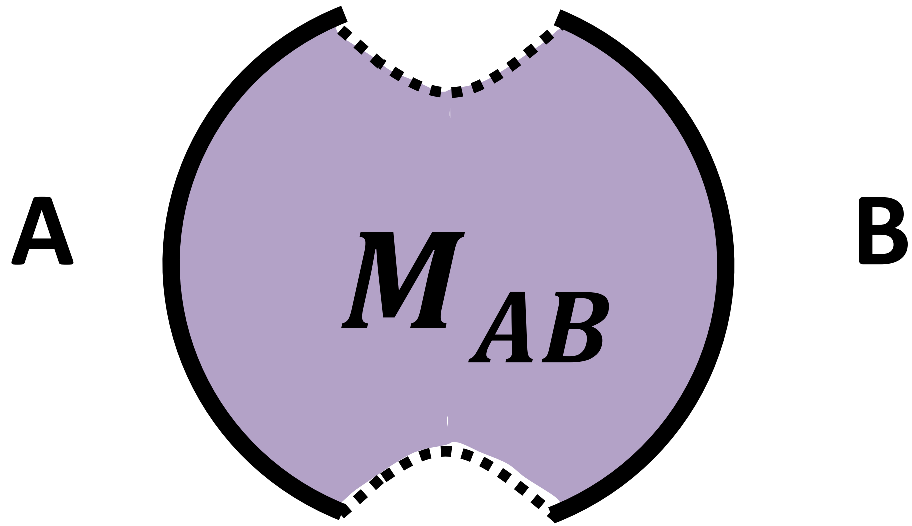
- **Mutual information**: $I(A:B) \equiv S_A + S_B - S_{AB}$

$$I(A:B) = 0 \Leftrightarrow \rho_{AB} = \rho_A \otimes \rho_B .$$

Entanglement wedge cross section

Definition of "entanglement wedge cross section"

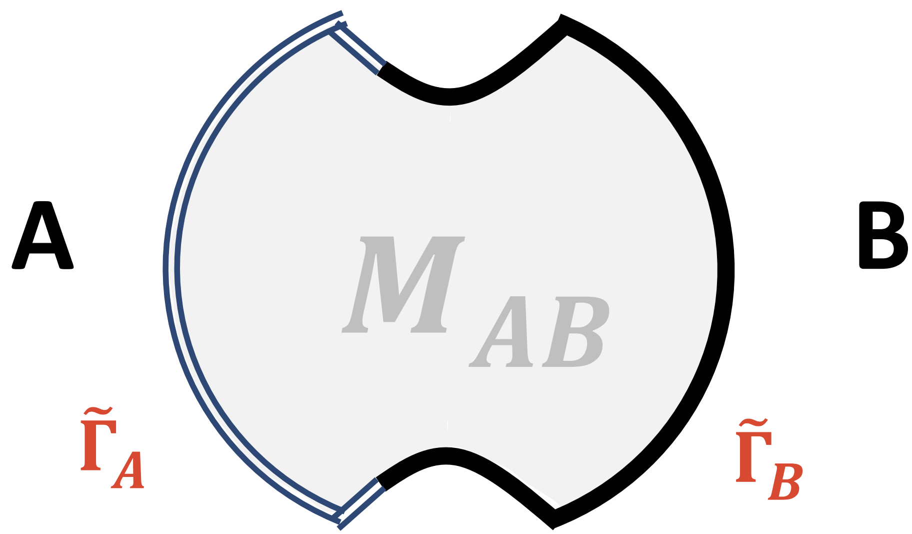
Step 1. Draw an **entanglement wedge M_{AB}**
(and forget all the other part of geometry):



Entanglement wedge cross section

Definition of "entanglement wedge cross section"

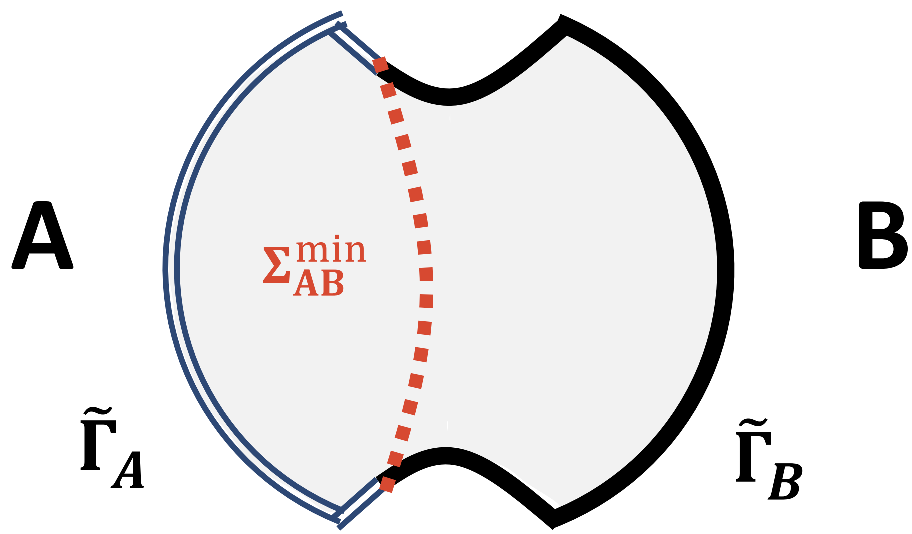
Step 2. **Divide the boundary ∂M_{AB}** into two subsets $\tilde{\Gamma}_A$ and $\tilde{\Gamma}_B$ such that $A, B \subset \tilde{\Gamma}_{A,B}$ respectively:



Entanglement wedge cross section

Definition of “entanglement wedge cross section”

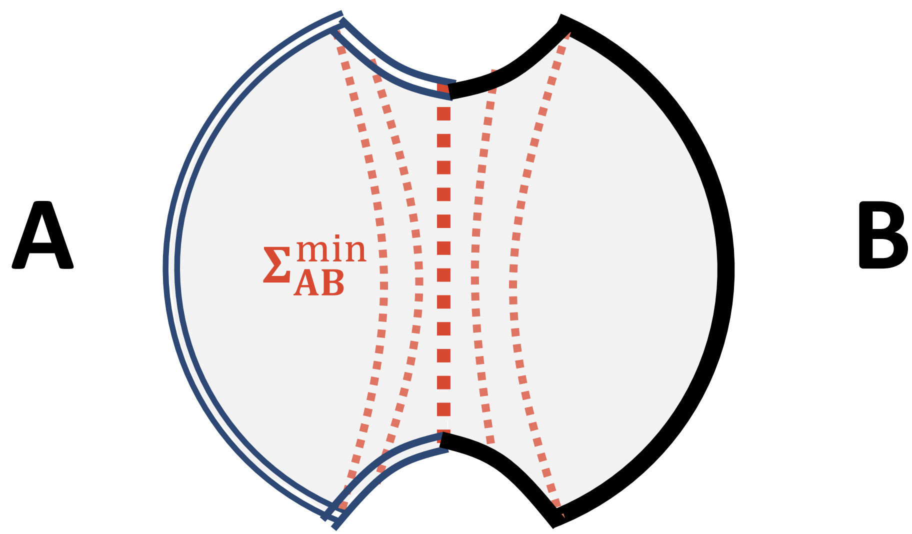
<u>Step 3</u>. Find the **RT-surface Σ_{AB}^{min}** of $\tilde{\Gamma}_A$ (or $\tilde{\Gamma}_B$, either gives the same RT-surface):



Entanglement wedge cross section

Definition of "entanglement wedge cross section"

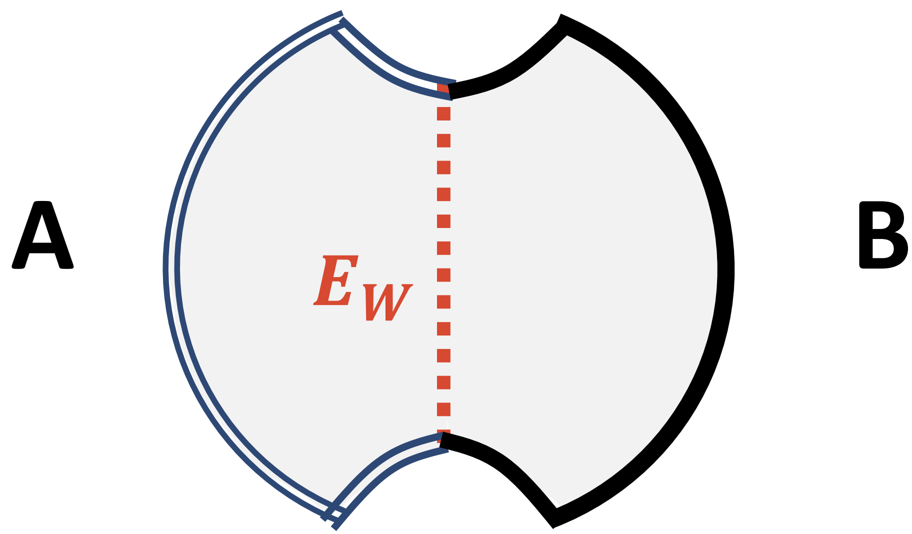
<u>Step 4</u>. **Minimize the area** of $\Sigma_{AB}^{\min}$
over **all possible divisions** of ∂M_{AB}:



Entanglement wedge cross section

Definition of "entanglement wedge cross section"

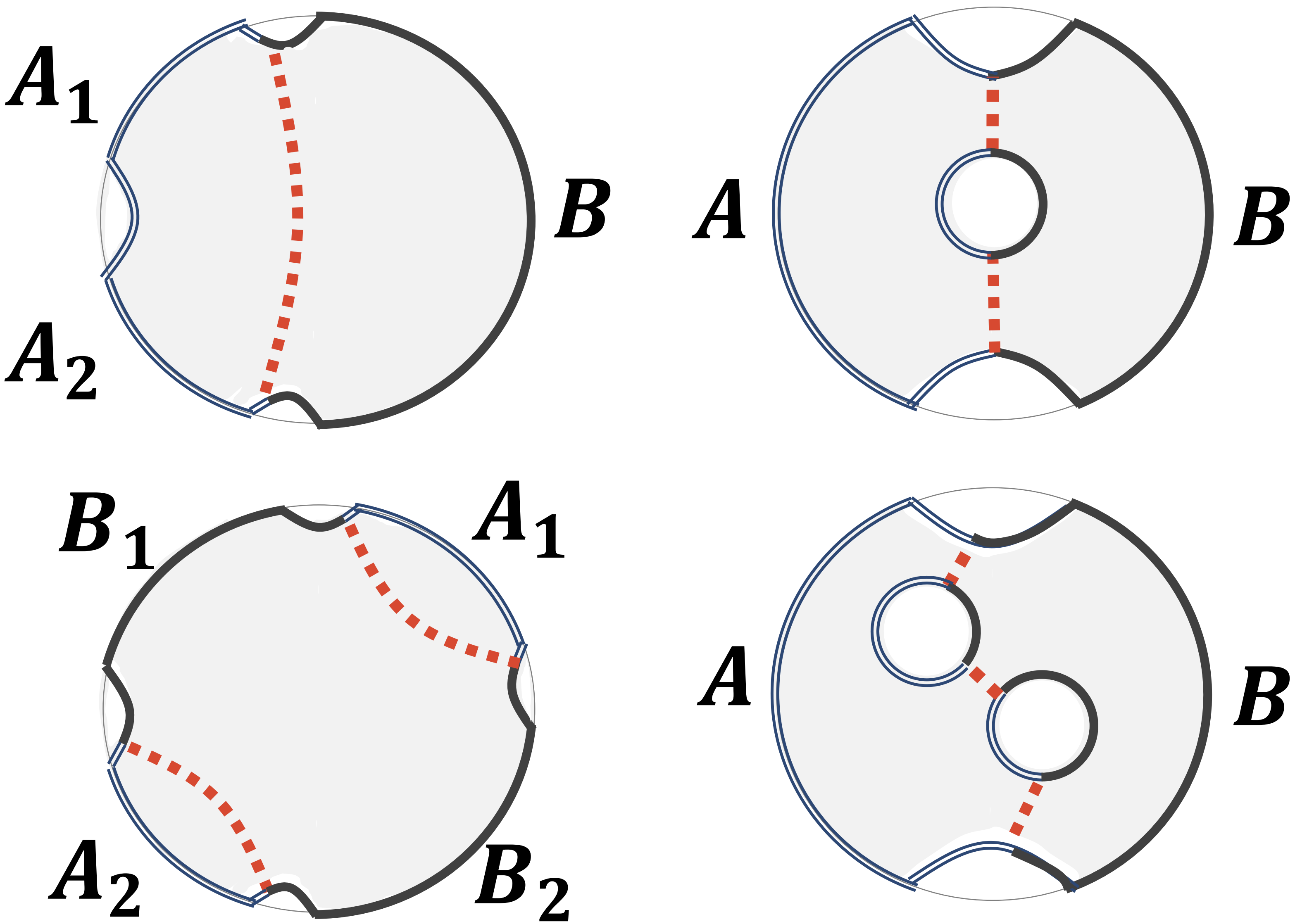
Step 5. **Its minimal area** (divided by $4G_N$) is defined as the **entanglement wedge cross section** of ρ_{AB}.



$$E_W(\rho_{AB}) := \min_{\tilde{\Gamma}_A} \frac{Area(\Sigma_{AB}^{min})}{4G_N}$$

Entanglement wedge cross section

Examples



Entanglement wedge cross section

Formula example in Poincaré pure AdS_3

Subsystems: $A = [a_1, a_2]$, $B = [b_1, b_2]$

$$E_W(\rho_{AB}) = \frac{\mathrm{c}}{6}\log[1 + 2z + 2\sqrt{z(z+1)}]$$

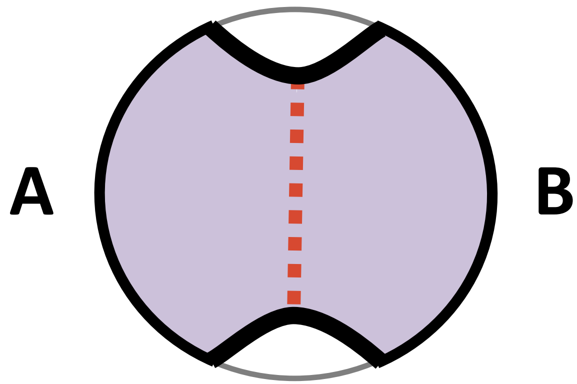
cross ratio: $z \equiv \frac{(a_2 - a_1)(b_2 - b_1)}{(b_2 - a_1)(b_1 - a_2)}$

Cf. mutual information : $I(A{:}B) = \log[z]$

∴ For $z > 1$ the E.W. is connecting A with B

Entanglement wedge cross section

The **entanglement wedge cross section**



- **is contained** in M_{AB} corresponding to ρ_{AB}.
- **vanishes** for $\rho_A \otimes \rho_B$.
- **returns to RT-surface** for pure states: $A \cup B = total$.

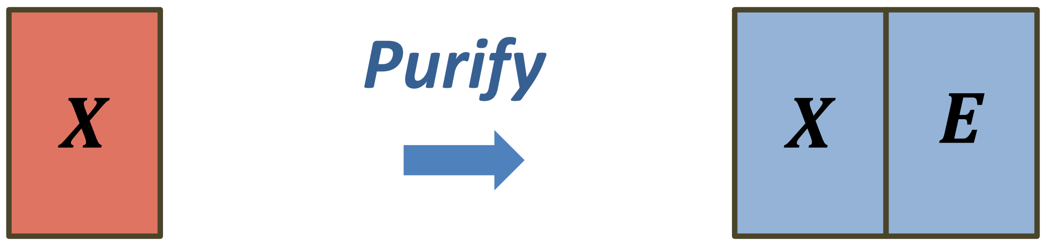
It may be a **geometrical counterpart** of
a measure of correlation!

Entanglement of Purification

Purification for mixed states

Purification

$\mathcal{H}_X$ $\qquad$ $\mathcal{H}_X \otimes \mathcal{H}_E$



$\boldsymbol{\rho_X}$ $\qquad$ $|\boldsymbol{\Psi}\rangle_{\boldsymbol{XE}}$

s.t. $\mathrm{Tr}_{\boldsymbol{E}}[|\boldsymbol{\Psi}\rangle\langle\boldsymbol{\Psi}|_{\boldsymbol{XE}}] = \boldsymbol{\rho_X}$.

Purification for mixed states

Example: Thermal state

Mixed:

$$\rho_X = \frac{1}{Z(\beta)} \sum_n e^{-\beta E_n} |E_n\rangle\langle E_n|_X$$

Purify

Pure:

$$|\mathrm{TFD}\rangle_{XE} = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\frac{\beta E_n}{2}} |E_n\rangle_X \otimes |E_n\rangle_E$$

ThermoField Double State (TFD)

$$\mathrm{Tr}_E[|\mathrm{TFD}\rangle\langle\mathrm{TFD}|_{XE}] = \rho_X .$$

Purification for mixed states

- **Any mixed state** ρ_X can be **purified**.

$$\forall \rho_X,\ \exists |\boldsymbol{\Psi}\rangle_{\boldsymbol{XE}}\ \ \mathrm{s.t.}\ \ \mathrm{Tr}_E[|\Psi\rangle\langle\Psi|_{XE}] = \rho_X.$$

$\uparrow$

“a purification of $\boldsymbol{\rho_X}$”

- Purification $|\boldsymbol{\Psi}\rangle_{\boldsymbol{XE}}$ is **NOT unique**.

Entanglement of purification

Definition of "entanglement of purification"

$$E_P\,(\rho_{AB}) \coloneqq \min_{\substack{|\Psi\rangle_{AA'BB'} \in \mathcal{H}_{AB} \otimes \mathcal{H}_{A'B'} \text{ s.t.} \\ \mathrm{Tr}_{A'B'}[|\Psi\rangle\langle\Psi|_{AA'BB'}] = \rho_{AB}}} S(\mathrm{Tr}_{AA'}[|\Psi\rangle\langle\Psi|_{AA'BB'}])$$

[Terhal-Horodecki-Leung-DiVincenzo '02]

Step1. Given a state ρ_{AB}, consider **a purification** :$|\boldsymbol{\Psi}\rangle_{\boldsymbol{ABE}}$

Step2. Divide the environmental system
into **two subsystems** : $\boldsymbol{E \equiv A' \cup B'}$

Step3. Calculate the **entanglement entropy** between AA'
and BB': $\boldsymbol{S_{AA'}} = S(\mathrm{Tr}_{AA'}[|\Psi\rangle\langle\Psi|_{AA'BB'}])$

Step4. E_P is **the minimal $\boldsymbol{S_{AA'}}$** over all purifications
and all divisions of E.

Entanglement of purification

Meanings

- A **total correlation** measure between A and B.
  (**NOT a measure of entanglement**.)
- It allows an interpretation based on EPR pairs:

  $\boldsymbol{E_P(\rho_{AB})}$ **is the minimal number of EPR pairs** which is **needed to produce** $\boldsymbol{\rho_{AB}}$ using only local operations and almost zero communications.

Our conjecture

We suggest a new holographic duality:

$$E_W^{(AdS)} \cong E_P^{(CFT)}.$$

[Takayanagi-KU '17]

- E_W: **Entanglement wedge cross section** in AdS.

[Takayanagi-KU '17]

- E_P : **Entanglement of purification** in CFT.

[Terhal-Horodecki-Leung-DiVincenzo '02]

[Note: *Nguyen-Devakul-Halbasch-Zaletel-Swingle [1709.07424]* also suggested the same duality.]

Consistency check

Consistency check

$$E_W \cong E_P$$

- The calculation of $\boldsymbol{E_P}$ is hard (because of the optimization).
- We will check that the **properties of them** are consistent.

Consistency check

Properties of E_P

[Terhal-Horodecki-Leung-DiVincenzo '02]

[Bagchi-Pati '15]

- For pure states: $E_P(|\Psi\rangle_{AB}) = S(\rho_A) = S(\rho_B)$.
- For product states: $E_P(\rho_A \otimes \sigma_B) = 0.$
- $E_W(\rho_{AB}) \leq \min\{S(\rho_A), S(\rho_B)\}.$
- $E_W\left(\rho_{A(BC)}\right) \geq E_W(\rho_{AB}).$
- …

⬇

We check these properties of E_W.

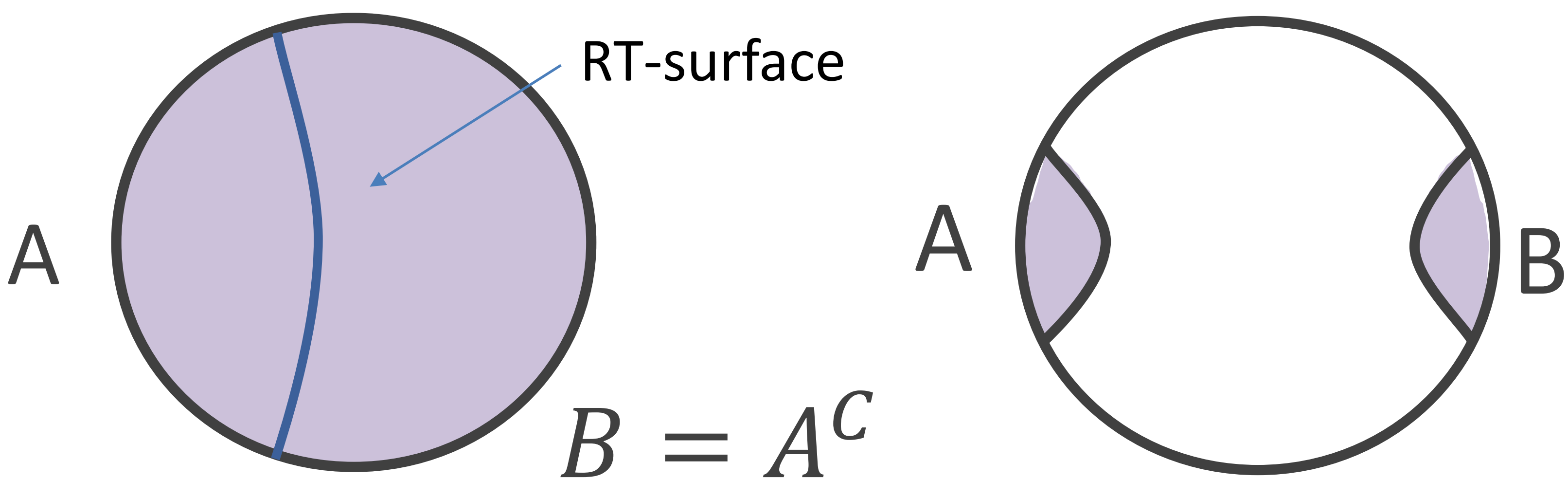
Consistency check

☑ Returns to entanglement entropy **for pure states**.

$$\boldsymbol{E_W(\rho_{AB}) = S_A = S_B} \text{ when } \rho_{AB} \text{ is pure.}$$

☑ Non-negative & vanishes only **for product states**.

$$\boldsymbol{E_W(\rho_{AB}) = 0} \text{ if and only if } \boldsymbol{\rho_{AB} = \rho_A \otimes \rho_B}.$$



Consistency check

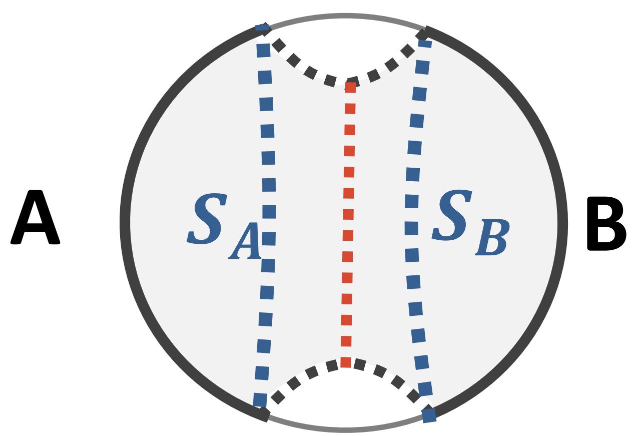
☑ **Less than** the entanglement entropies.

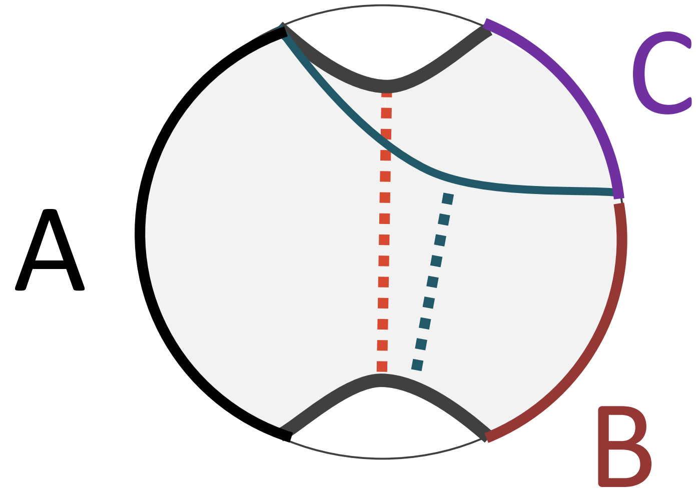
$$E_W(\rho_{AB}) \leq min\{S_A, S_B\}.$$

☑ Never increasing upon **discarding ancilla**.

$$E_W(\rho_{A(BC)}) \geq E_W(\rho_{AB}).$$

From *[Wall '12]*
"Entanglement wedge nesting"

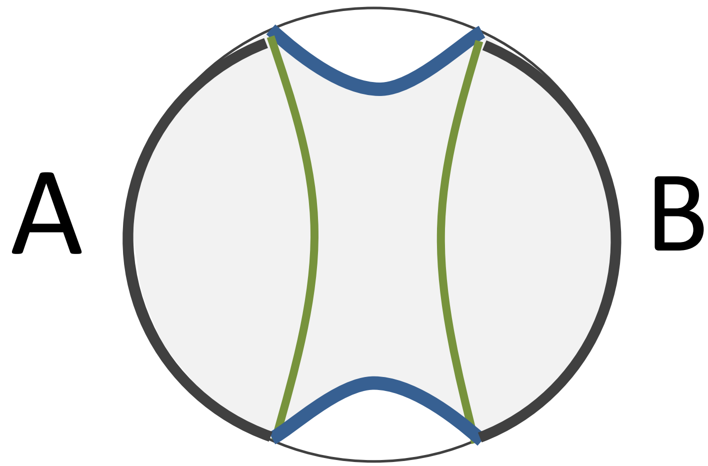




Consistency check

Quantum mutual information

$$I(A:B) \coloneqq S_A + S_B - S_{AB}.$$



☑ Larger than half of mutual information.

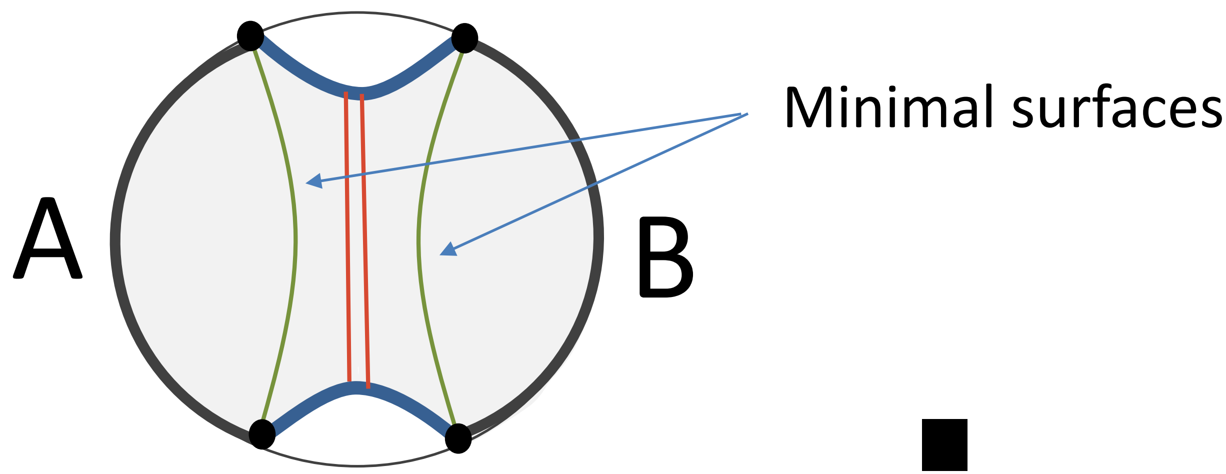
$$\boldsymbol{E_W(\rho_{AB}) \geq \frac{I(A:B)}{2}}.$$

Consistency check

Proof

$$E_W(\rho_{AB}) \geq \frac{I(A:B)}{2}.$$

$$\Leftrightarrow \quad 2E_W(\rho_{AB}) + S_{AB} \geq S_A + S_B.$$



∎

M. Freedman and M. Headrick has proved this inequality using bit threads formalism in [Commun. Math. Phys. 352 (2017)].

Consistency check

Mutual information satisfies **monogamy** inequality **in holographic theories**. *[Hayden-Headrick-Maloney '11]*

$$\boldsymbol{I(A:BC) \geq I(A:B) + I(A:C)}.$$

$$(\therefore\ E_W(\rho_{A(BC)}) \geq \frac{I(A:BC)}{2} \geq \frac{I(A:B)+I(A:C)}{2})$$

☑ $$\boldsymbol{E_W(\rho_{A(BC)}) \geq \frac{I(A:B)}{2} + \frac{I(A:C)}{2}}.$$

Remark: $\boldsymbol{E_P}$ **always satisfies** this inequality regardless of monogamy of M.I. *[Bagchi-Pati '15]*

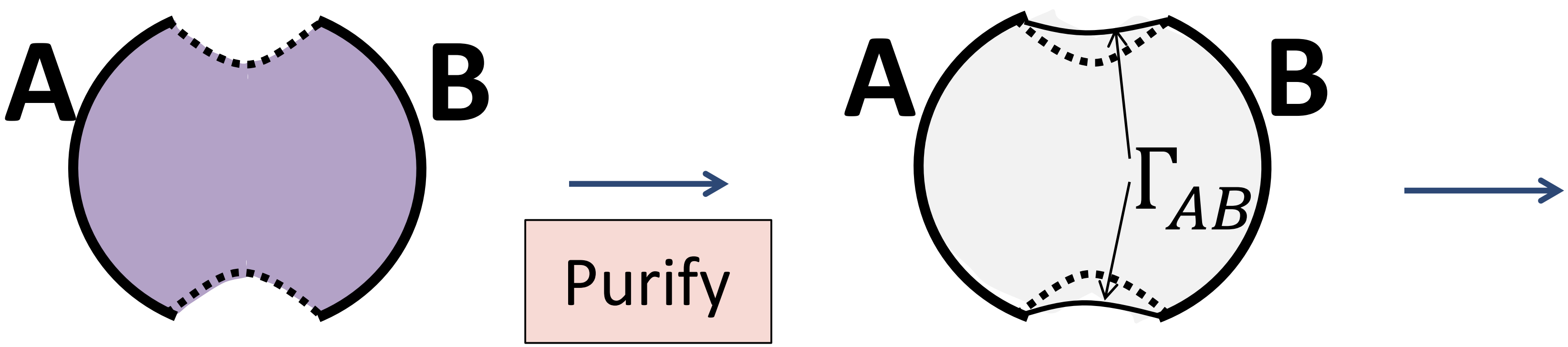
A heuristic proof

A heuristic proof

Holographic picture of "purification"

$E_P(\rho_{AB})$:
- $\mathrm{Tr}_E[|\Psi\rangle\langle\Psi|_{ABE}] = \rho_{AB} \Rightarrow M_{ABE} \supset M_{AB}$.
- pure $\Rightarrow S_A = S_{\bar{A}}$

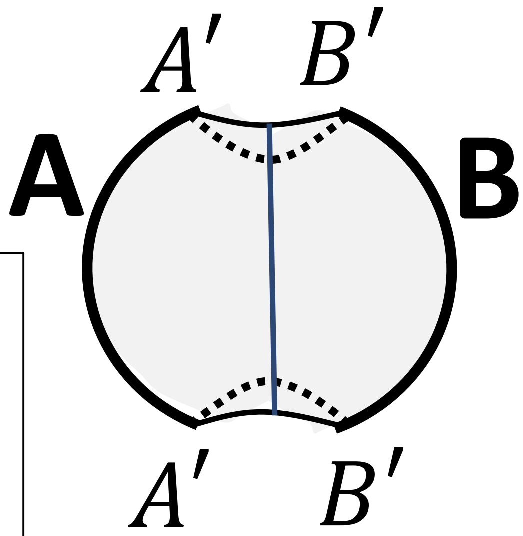
  $\Rightarrow \partial M_{ABE}$ is closed & no holes in M_{ABE}.
- RT − formula $\Rightarrow \partial M_{ABE}$ is convex.



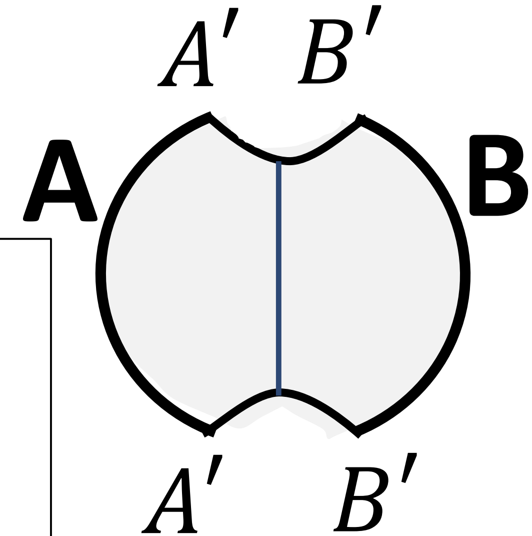
Dual to ρ_{AB} Dual to $|\Psi\rangle_{A\cup B\cup\Gamma_{AB}}$

A heuristic proof

Devide Γ_{AB}
&
RT-formula



Minimize
over
$|\Psi\rangle_{\Gamma'}$ & A'



$$= E_W(\rho_{AB}).$$

∴The holographic definition of E_P

$\cong$ **the definition of E_W**

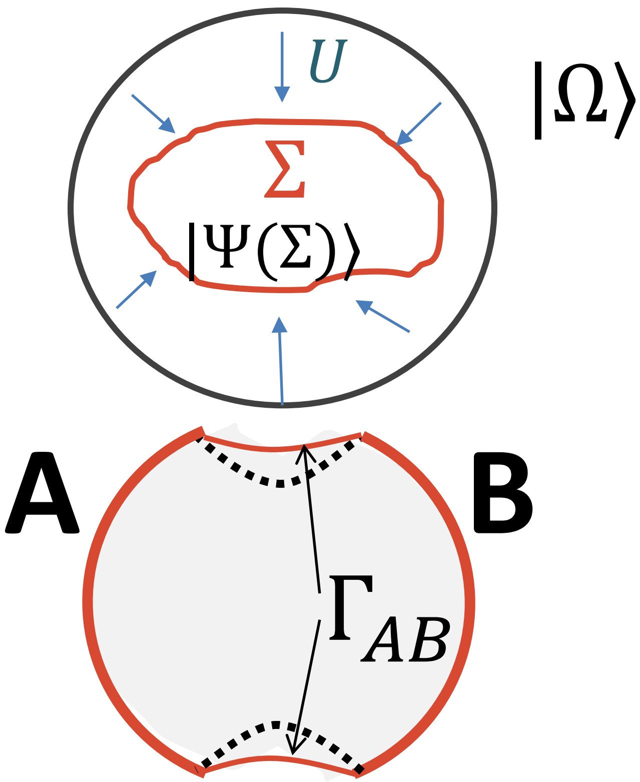
A heuristic proof

The surface/state correspondence

of tensor network description of AdS/CFT justifies **the purification step.**

[Swingle '09]
[Miyaji-Takayanagi '15]
[Caputa-Kundu-Miyaji-Takayanagi-Watanabe '17]

Σ: any closed convex surfaces



$$|\Psi(\Sigma)\rangle_\Sigma \equiv \boldsymbol{U}(\boldsymbol{\Sigma})|\Omega\rangle_{total},$$
$$U^\dagger U = I$$

$$\Sigma = A \cup B \cup \Gamma_{AB},$$
$$\mathrm{Tr}_{\Gamma_{AB}}[|\Psi(\Sigma)\rangle\langle\Psi(\Sigma)|]$$
$$= \mathrm{Tr}_{\overline{AB}}[|\Omega\rangle\langle\Omega)|] = \rho_{AB}.$$

Conclusion

We conjectured **a new duality** between information and geometry:

$$\boldsymbol{E_W \cong E_P}.$$

Future works

[Work in progress with Bhattacharyya-Takayanagi]

- Calculation of E_P in holographic CFTs
- Proof of the conjecture
- Holographic counterpart of LOCC/LOq in AdS/CFT
- Dual of multipartite correlation measures

Thank you for your attention.

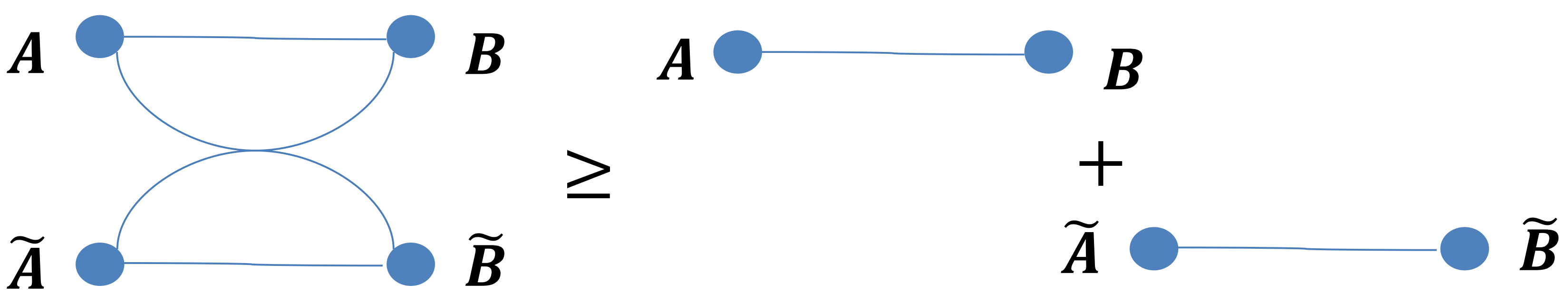
Appendix

Strong superadditivity

Definition:

$$E\left(\rho_{(A\tilde{A})(B\tilde{B})}\right) \geq E(\rho_{AB}) + E(\rho_{\tilde{A}\tilde{B}}).$$

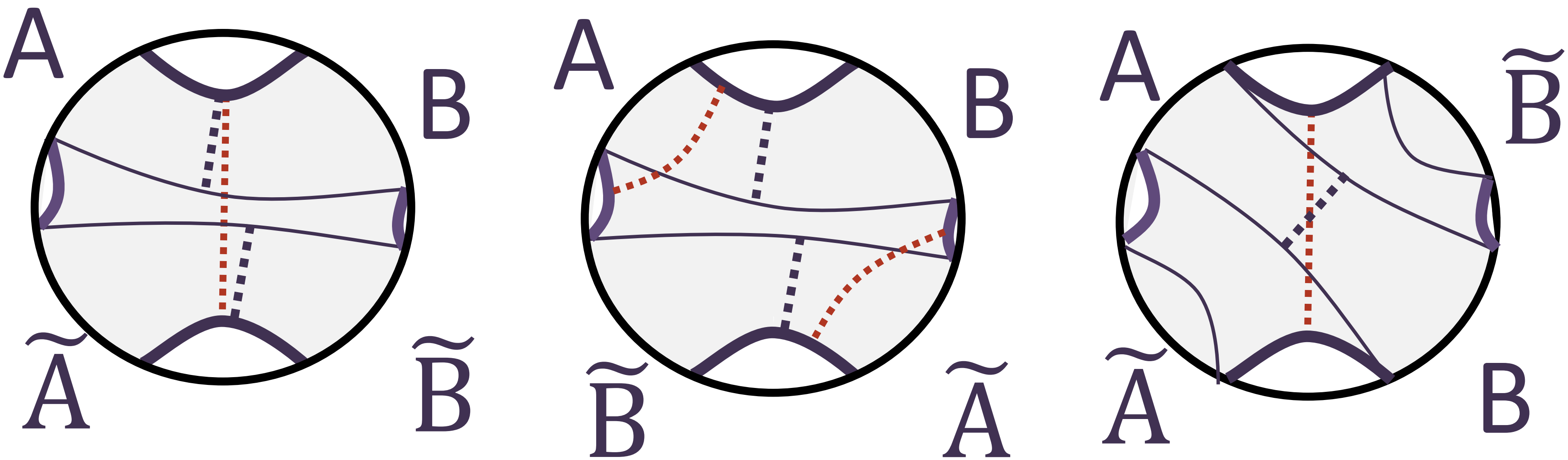
- It is thought to be a nature of **quantum** correlation.
- Monogamy $E(A{:}BC) \geq E(A{:}B) + E(A{:}C)$ immediately implies SSA.



Strong superadditivity

Strong superadditivity of E_W

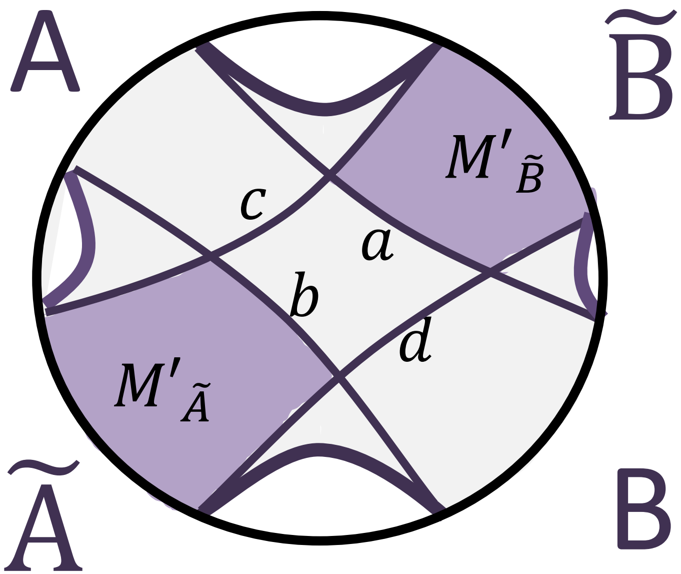
$$E_W\left(\rho_{(A\tilde{A})(B\tilde{B})}\right) \geq E_W(\rho_{AB}) + E_W(\rho_{\tilde{A}\tilde{B}}).$$



etc.

Strong superadditivity

For the third case: <u>*No crossing bridge*</u>



If M_{AB} is connected, then $M_{\tilde{A}\tilde{B}}$ is disconnected (and vice versa).

Proof:
If M_{AB} is connected, $a + b < c + d$ should hold.
Then, for $M_{\tilde{A}\tilde{B}}$, at least the disconnected wedge $M'_{\tilde{A}} \cup M'_{\tilde{B}}$ is preferred.

Implications

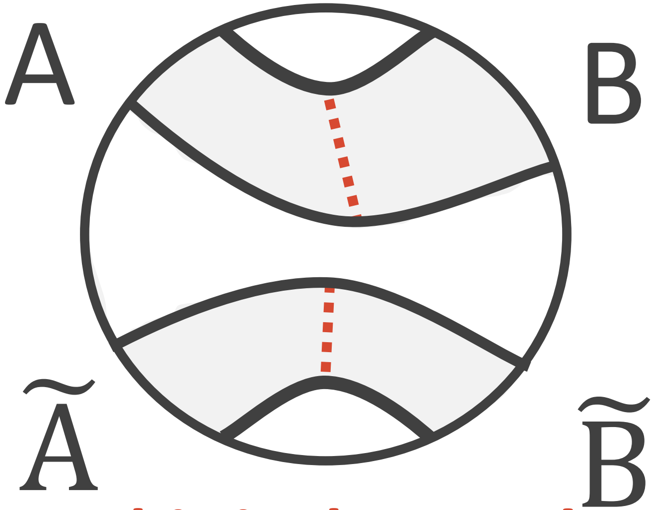
Strong superadditivity

- We expect $\boldsymbol{E_P}$ **to be strong superadditive in holographic CFTs**.
- It tell us some "quantum" aspect of holographic correlations.

Cf. Monogamy of mutual information.

Additivity

- $E_P(\rho_{AB} \otimes \sigma_{\tilde{A}\tilde{B}})$ is known to be additive **if and only if** an optimal purification of $\rho_{AB} \otimes \sigma_{\tilde{A}\tilde{B}}$ is just a tensor product of optimal purifications of ρ_{AB} and $\sigma_{\tilde{A}\tilde{B}}$ (up to unitary equivalence).



This is the case!

$$E_W(\rho_{AB} \otimes \sigma_{\tilde{A}\tilde{B}}) = E_W(\rho_{AB}) + E_W(\sigma_{\tilde{A}\tilde{B}}).$$

"Regularized" E_P

"The minimal number of EPR pairs
which is **needed to produce** $\boldsymbol{\rho_{AB}}$ using only
local operations and **vanishing communications**."

$$E_{LOq}(\rho_{AB}) \coloneqq$$
$$\inf_r \left\{ r \middle| \lim_{n\to\infty} \left[\inf_{\Lambda\in\mathrm{LOq}} D_{tr}\left(\rho_{AB}^{\otimes n}, \Lambda\left(\Phi_{2^{rn}}^{+}\right)\right)\right] = 0 \right\}.$$

Thm. $E_{LOq}(\rho_{AB}) = \lim_{n\to\infty} \frac{E_P\left(\rho_{AB}^{\otimes n}\right)}{n}$.

[Terhal-Horodecki-Leung-DiVincenzo '02]

$\therefore$ When it's additive, $E_{LOq} = E_P$.

Time-dependent case

- Replacing the "minimal surface $\Sigma_{AB}^{\min}$"
  → "**extremal surface $\boldsymbol{\Sigma_{AB}^{ext}}$** "
  following HRT formula.

[Hubeny-Rangamani-Takayanagi '07]

- All properties are proven by using of
  the **"maximin surfaces" prescription**
  discussed by A.Wall in
  [Class. Quant. Grav. 31 (2014) no.22, 225007]

Appendix

Relative entropy of entanglement

$$E_R(\rho_{AB}) := \min_{\sigma_{AB} \in \text{Seprable states}} R(\rho_{AB}||\sigma_{AB}).$$

where $R(\rho_{AB}||\sigma_{AB})$ is relative entropy.

- However... It must be less than $I(A:B)$:

$$E_R(\rho_{AB}) \leq I(A:B).$$

Origin of the monogamy of M.I.

- "Squashed entanglement":

$$E_{sq}(\rho_{AB}) := \frac{1}{2} \min_{\mathrm{Tr}_C \rho_{ABC} = \rho_{AB}} I(A:B|C)$$
$$= \frac{1}{2} \min_{\mathrm{Tr}_C \rho_{ABC} = \rho_{AB}} [S_{AC} + S_{BC} - S_{ABC} - S_C].$$

- E_{sq} is the most promising measure of entanglement for mixed states, and known to be **always monogamous**.
- In our picture $\boldsymbol{E_{sq} = \frac{I}{2}}$ in holography.
- This is discussed in [Hayden-Headrick-Maloney '11].