On the infinite gradient-flow for the domain-wall formulation of chiral lattice gauge theories

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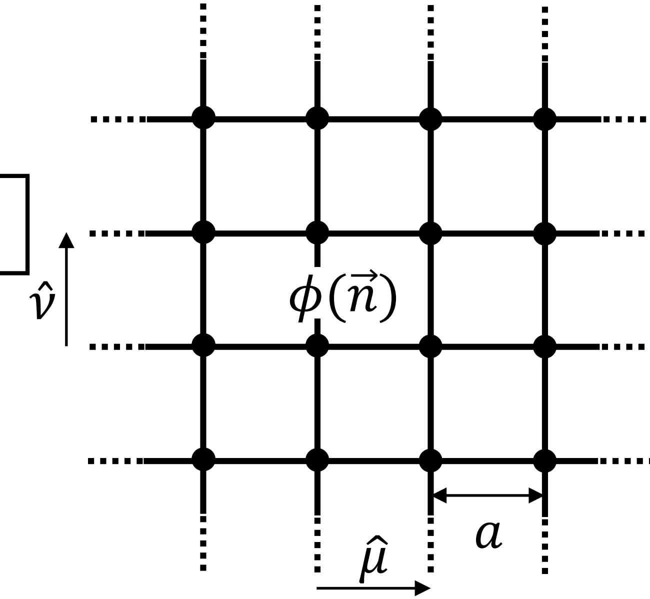
TA, Yoshio Kikukawa, 1911.10925 [hep-lat]

1. Introduction

Background



(scalar field)

Lattice theory: discretization of a spacetime

expectation value = integration w.r.t. $\phi(\vec{n})$

$$\langle \mathcal{O} \rangle = \int \left(\prod d\phi(\vec{n})\right)\mathcal{O}(\phi)\exp(-S(\phi))$$

approximated by the finite DOF

→ **numerical simulation**

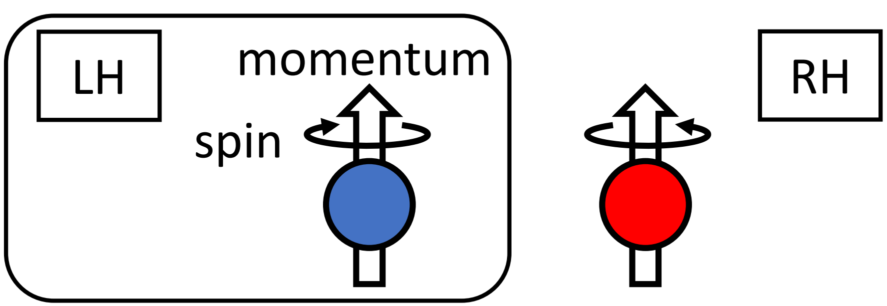
Big problem in lattice gauge theories

(Non-Abelian) chiral gauge theories are not formulated on the lattice

- Standard model is a chiral gauge theory

Chiral gauge theories

LH- and RH-fermions couple to gauge field differently.

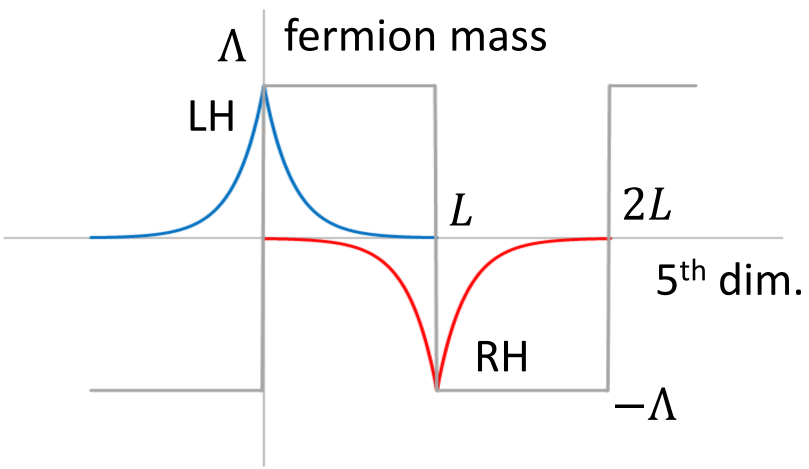


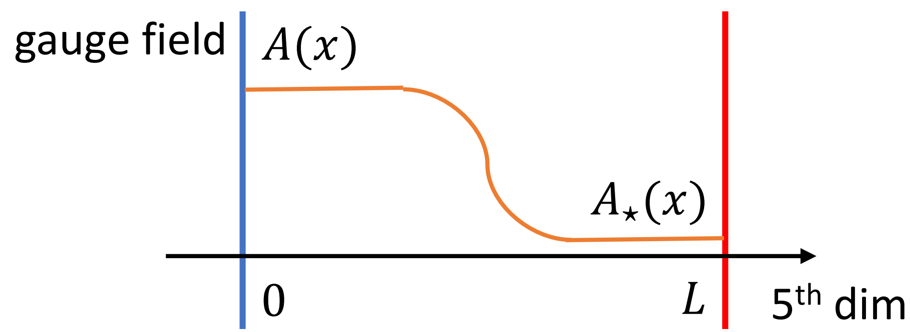
Grabowska-Kaplan's formulation

Proposal to formulate chiral gauge theories on the lattice

Grabowska-Kaplan (2015)

Domain-wall [Kaplan (1992)] + Gradient flow [Luscher (2010)]





- LH and RH are localized around defects
- gauge fields are damped along 5th dim.

→ LH fermions couple to the original gauge field $A_\mu(x)$

Plan of this talk

Problems of GK's proposal

- Formulation and properties of gradient flow on the lattice ?
- Infinite gradient-flow maps gauge fields non-locally

⇓ $\mathrm{U}(1)$ gauge theory

We examined

- The formulation of gradient flow which satisfies "admissibility"
- Relation of GK's formulation and Luscher's one
- Locality of GK's formulation

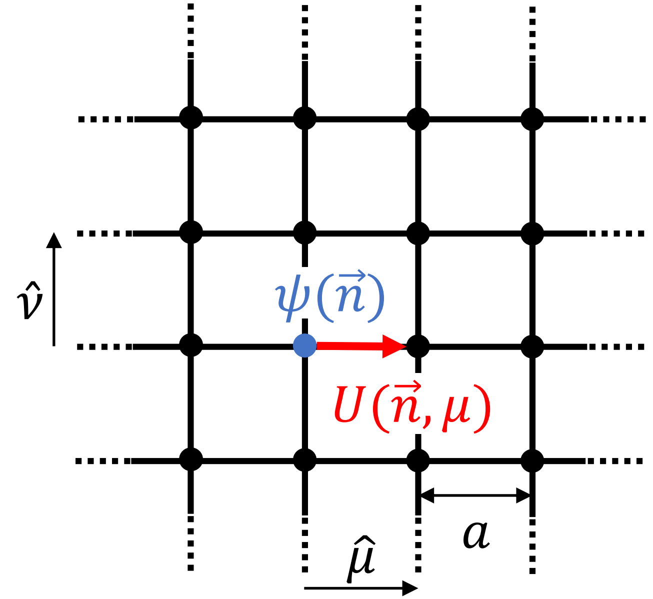
<u>Outline</u>

1. Introduction
2. Luscher's formulation
3. Grabowska-Kaplan's formulation
4. Relation of two formulations
5. Summary

2. Luscher’s formulation

Lattice fermions

Fermion fields $\psi(\vec{n})$ are defined on the cites
Gauge fields $U(\vec{n},\mu)$ are defined on the links



continuum ⟹ lattice

covariant derivative

$$(\partial_\mu + A_\mu)\psi(x)$$

"covariant difference"

$$\frac{1}{2a}\left[U(n,\mu)\psi(n+\hat{\mu}) - U^\dagger(n-\mu,\mu)\psi(n-\hat{\mu})\right]$$

- Gauge covariance

$$U(n,\mu) \to g_n U(n,\mu) g_{n+\mu}^\dagger$$

- Continuum limit

$$U(n,\mu) \sim \exp[iagA_\mu(na + 1/2\hat{\mu}a)]$$

Lattice fermions

continuum → ? → lattice

fermion action

$$S = \int dx^4\, \bar{\psi}(x)\left(\gamma_\mu D_\mu\right)\psi(x)$$

"naïve" lattice fermion action

$$S_{\mathrm{lat.}} = a^4 \sum_n \bar{\psi}(n) \sum_\mu \frac{U(n,\mu)\psi(n+\hat{\mu}) - U^\dagger(n-\mu,\mu)\psi(n-\hat{\mu})}{2a}$$

✓ chiral symmetry + gauge invariance

- But propagator

$$G_F = \left(a^{-1} i\gamma_\mu \sin p_\mu a + m\right)^{-1}$$

has 16 poles

⇒16 particles (=doublers)

- Fermion measure is invariant under chiral transformation

⇒No chiral anomaly (physically necessary to explain U(1) problem)

Nielsen-Ninomiya theorem

Nielsen-Ninomiya (1981)

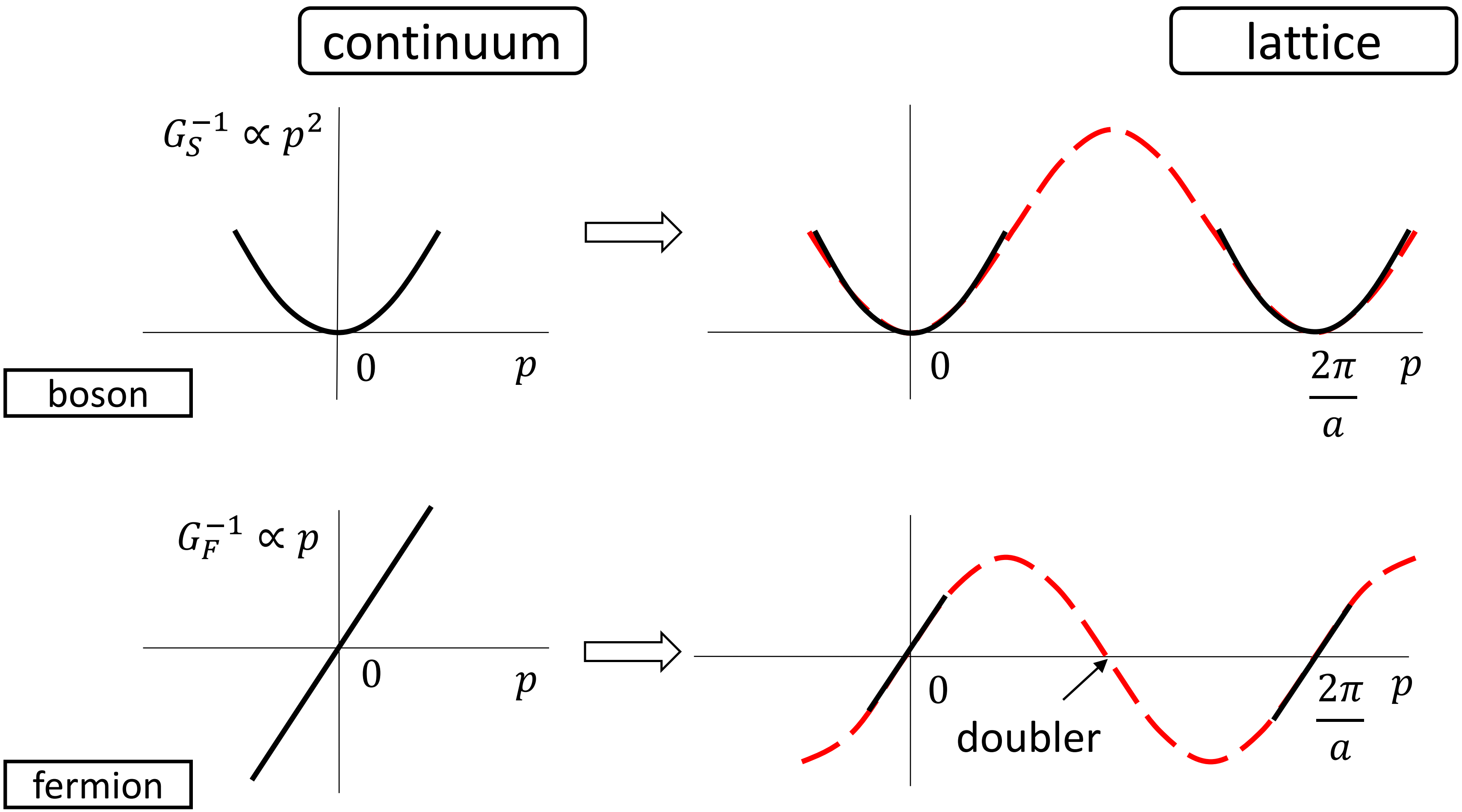
A lattice fermion action

$$S_{\text{lat.}} = \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4} \bar{\psi}(-p) D_{\text{lat.}}(p) \psi(p)$$

cannot have the operator $D_{\text{lat.}}(p)$ satisfying the following conditions simultaneously:

1. $D_{\text{lat.}}(p)$ is periodic, analytic function of p_μ (locality)
2. $D_{\text{lat.}}(p) \propto \gamma_\mu p_\mu$ for $a|p_\mu| \ll 1$ (continuum limit)
3. $D_{\text{lat.}}(p) = 0$ only if $p_\mu = 0$ (no doublers)
4. $\gamma_5 D_{\text{lat.}}(p) + D_{\text{lat.}}(p)\gamma_5 = 0$ (chiral symmetry)

Intuitive explanation



Ginsparg-Wilson relation

To avoid the doubling problem, one of the conditions need be violated. ⇒ chiral symmetry is not exact

Ginsparg-Wilson relation [Ginsparg-Wilson (1982)]

$$D\gamma_5 + \gamma_5 D = aD\gamma_5 D$$

- Exact “lattice chiral symmetry” [Luscher (1998)]

$$\delta\psi = \gamma_5(1 - aD)\psi, \qquad \delta\bar{\psi} = \bar{\psi}\gamma_5$$

$$\Rightarrow \delta[\Sigma_n \bar{\psi}(n) D\psi(n)] = 0$$

- Chiral anomaly [Luscher (1998)]

$$\delta \prod_x d\psi(x) d\bar{\psi}(x) = 2N_f \mathrm{index}(D) \prod_x d\psi(x) d\bar{\psi}(x)$$

- Overlap fermion [Neuberger (1998)] satisfies GW relation

Luscher's formulation

U(1) chiral lattice gauge theories with exact gauge invariance

Luscher (1999)

Starting point

Ginsparg-Wilson relation ("chiral symmetry" on the lattice)

$$\gamma_5 D + D\hat{\gamma}_5 = 0, \qquad \hat{\gamma}_5 = \gamma_5(1 - D)$$

Projection operators: $P_\pm = \dfrac{1 \pm \gamma_5}{2}, \qquad \hat{P}_\pm = \dfrac{1 \pm \hat{\gamma}_5}{2}$

LH-fermions: $\psi_-(x) = \hat{P}_-\psi_-(x), \qquad \bar{\psi}_-(x) = \bar{\psi}_-(x)P_+$

⇒ Action of LH-fermions: $S_F = \sum_x \bar{\psi}_-(x)D\psi_-(x)$

(Invariant under "lattice chiral transformation")

Luscher's formulation

Fermion Measure: $D[\psi_-]D[\bar{\psi}_-] = \prod_j dc_j \prod_k d\bar{c}_k$

$\psi_-(x) = \sum_j v_j(x)c_j\,, \qquad \bar{\psi}_-(x) = \sum_k \bar{c}_k \bar{v}_k(x)$

($v_j(x), \bar{v}_k(x)$: basis vectors)

$$\hat{P}_- v_j = v_j, \qquad (v_k, v_j) = \delta_{kj}$$

$$\bar{v}_k P_+ = \bar{v}_k, \qquad (\bar{v}_k, \bar{v}_j) = \delta_{kj}$$

⇒ Effective action: $\Gamma[U] = \log \det M, \qquad M_{kj} = \bar{v}_k D v_j$

<u>Effective action is gauge invariant ?</u>

Luscher's formulation

- $\hat{P}_-$ depends on the gauge fields $U(x,\mu)$ (basis vectors v_j also do)

Variation of a link field: $\delta_\eta U(x,\mu) = i\eta_\mu(x)U(x,\mu)$

$$\delta_\eta \Gamma[U] = \mathrm{Tr}\, P_+ \delta_\eta D D^{-1} + \underline{\sum_j (v_j, \delta_\eta v_j)}$$

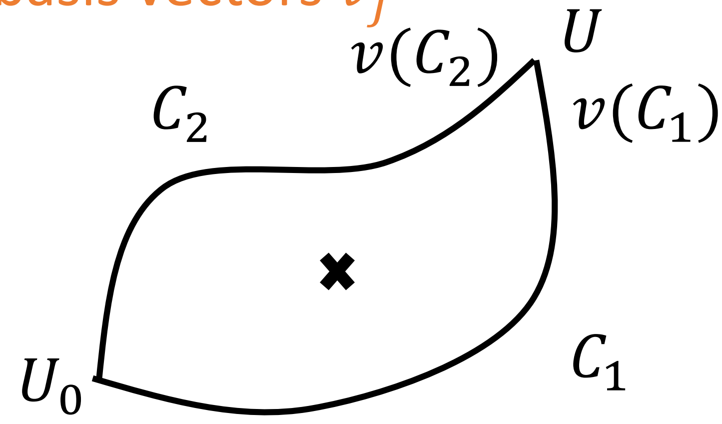
$\equiv -i\mathcal{L}_\eta$ (measure term)

$\Gamma[U]$ depends on the choice of basis vectors v_j

<u>How to fix basis vectors v_j?</u>

$$v_j(C_2) = v_i(C_1) Q_{ij}^{-1}$$

$$\Rightarrow \mathcal{L}_\eta(C_2) = \mathcal{L}_\eta(C_1) - i\delta_\eta \ln\det Q$$



Luscher's formulation

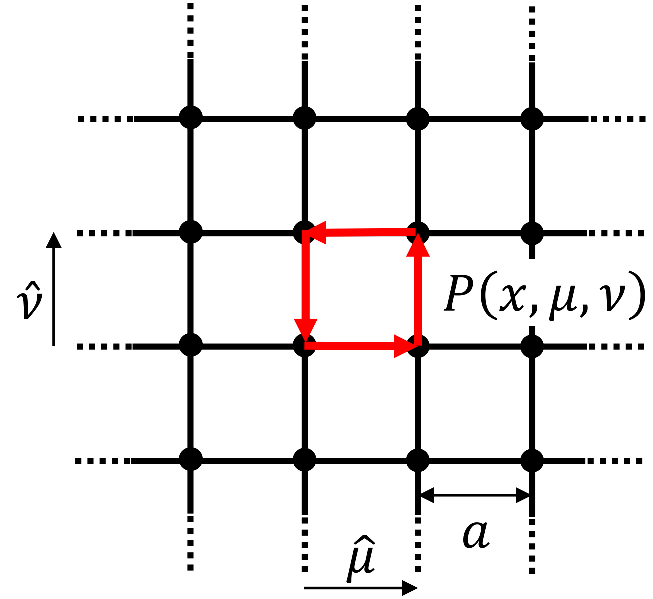
Suppose $\mathcal{L}_\eta = \sum_x \eta_\mu(x) j_\mu(x)$ satisfies the following condition
Then there exist a smooth fermion measure

1. $j_\mu(x)$ depends smoothly on the "admissible" link fields $U(x,\mu)$
2. $j_\mu(x)$ is gauge-invariant and transforms as an axial vector
3. Integrability condition: $\delta_\eta \mathcal{L}_\zeta - \delta_\zeta \mathcal{L}_\eta = i\mathrm{Tr}\hat{P}_-[\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_-]$
4. anomalous conservation law: $\partial_\mu^* j_\mu(x) = \mathrm{tr}\, Q\gamma_5(1-D)(x,x)$

Such a $j_\mu(x)$ can be explicitly constructed in U(1) gauge theories
if the anomaly cancellation condition is satisfied

Problem: non-Abelian case ?

Admissibility



- Admissibility condition

$$\left|F_{\mu\nu}(x)\right| < \epsilon \qquad \text{for all } x,\mu,\nu$$

$$F_{\mu\nu}(x) = i^{-1}\log P(x,\mu,\nu), \qquad -\pi < F_{\mu\nu} < \pi,$$
$$P(x,\mu,\nu) = U(x,\mu)U(x+\mu,\nu)U(x+\nu,\mu)^{-1}U(x,\nu)^{-1}$$

Admissibility is the constraint on the gauge fields

- Overlap operator is local $\|D(x,y)\| \leq \kappa e^{-\|x-y\|_1/\rho}$
- Topological structure $\sum_x[-\mathrm{tr}\,\gamma_5 D(x,x)] = (\text{integer})$

Gradient flow should satisfy the admissibility condition

3. Grabowska-Kaplan’s formulation

Domain-wall fermion

Kaplan (1992)

- 5D Dirac equation with s-dependent mass in $\mathbb{R}^4 \times \mathbb{R}$

$$\left(\gamma_\mu D_\mu + \gamma_5 \partial_s + m(s)\right)\Psi(x,s) = 0$$

$$m(s) = \begin{cases} +\Lambda\ (s > 0) \\ -\Lambda\ (s < 0) \end{cases}$$

Massless solution: $\Psi(x,s) = e^{-\Lambda|s|}P_+\psi(x),\ \gamma_\mu D_\mu \psi(x) = 0$

Massive solution: $(\mathrm{mass}) > \mathcal{O}(\Lambda)$

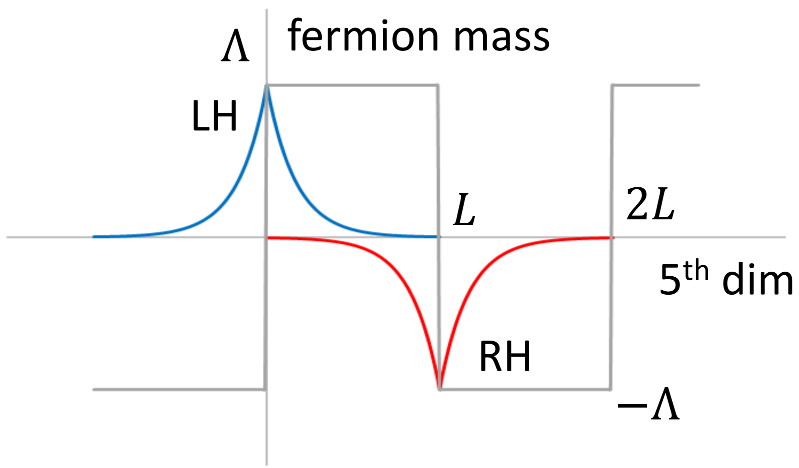
- Periodic boundary condition

$$\Psi(x, s+2L) = \Psi(x,s)$$

$\Rightarrow$LH is localized around $s = 0$

RH is localized around $s = L$

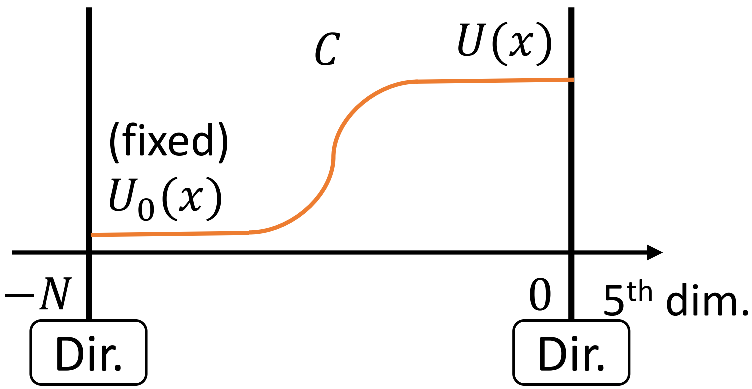
- Heavy modes do not contribute to the effective low energy theory



CGT with DW fermions Kikukawa (2002)

Domain wall fermion + interpolation of the gauge field

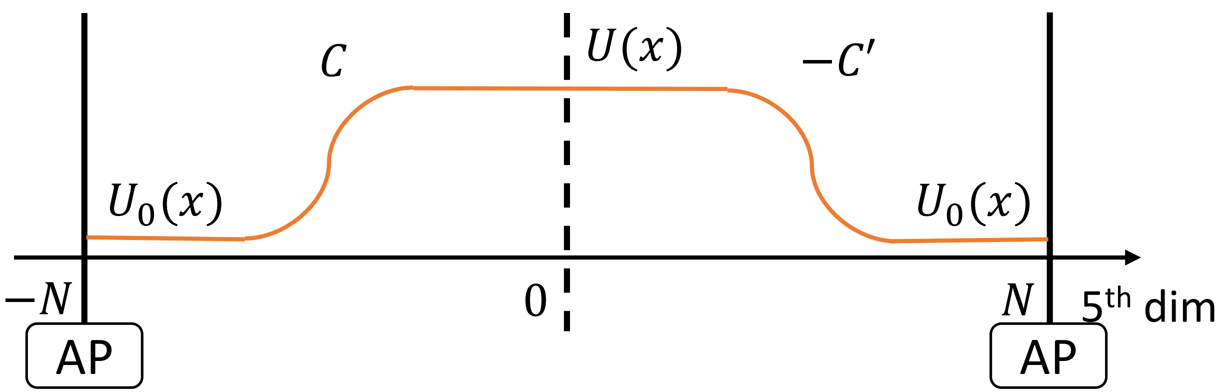
Dirichlet boundary condition



$\det(D_{5\mathrm{w}} - m_0)\,|_{\mathrm{Dir.}}$ depends on the path C (not gauge invariant)

$\Rightarrow$subtract by $\det(D_{5\mathrm{w}} - m_0)\,|_{\mathrm{AP}}$

Antiperiodic boundary condition



CGT with DW fermions   Kikukawa (2002)

- Path dependence of the real part is subtracted by $\det(D_{5\mathrm{w}} - m_0)\,|_{\mathrm{AP}}$

$$\left(\ln\det(D_{5\mathrm{w}} - m_0)\Big|_{\mathrm{Dir.}}^{C_2} - 1/2\ln\det(D_{5\mathrm{w}} - m_0)\Big|_{\mathrm{AP}}^{C_2+(-C_2)}\right) - (C_2 \leftrightarrow C_1)$$

$$= i\mathrm{Im}\ln\det(D_{5\mathrm{w}} - m_0)\Big|_{\mathrm{AP}}^{C_2+(-C_1)} \quad (N \to \infty)$$

- Suppose the lattice Chern-Simons term is expressed by a local gauge invariant function $q_{5\mathrm{w}}(x,s)$

$$\mathrm{Im}\ln\det(D_{5\mathrm{w}} - m_0)\Big|_{\mathrm{AP}}^{C_2+(-C_1)} \to \sum_s \sum_x q_{5\mathrm{w}}(x,s) = C_{5\mathrm{w}}^{C_2} - C_{5\mathrm{w}}^{C_1}$$

$\Rightarrow \dfrac{\det(D_{5\mathrm{w}}-m_0)|_{\mathrm{Dir.}}^{C}}{\left|\det(D_{5\mathrm{w}}-m_0)|_{\mathrm{AP}}^{C+(-C)}\right|^{1/2}} e^{-iC_{5\mathrm{w}}^{C}}$ is path independent and gauge-invariant

- $q_{5\mathrm{w}}(x,s)$ can be explicitly constructed in U(1) gauge theories

Problem: non-Abelian case ? ⇒ proposal: gradient flow

Gradient flow   Lüscher (2010)

$$\partial_t \bar{A}_\mu(x,t) = -g_0^2 \frac{\delta S_{\mathcal{G}}}{\delta \bar{A}_\mu} = -D_\nu \bar{F}_{\mu\nu}(x,t), \qquad \bar{A}_\mu(x,0) = A_\mu(x)$$

variables of path integral

ex: 2D U(1) gauge field

$\bar{A}_\mu$ is decomposed into gauge and physical DOF

$$\bar{A}_\mu = \partial_\mu \omega + \epsilon_{\mu\nu}\partial_\nu \lambda$$

Each DOF is evolved by the following equations:

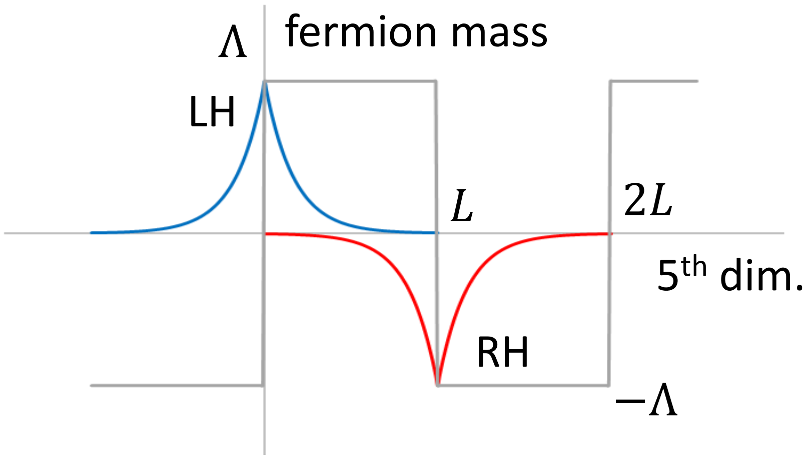
$$\partial_t \omega = 0 \quad \text{(constant)}$$
$$\partial_t \lambda = \partial_\mu \partial_\mu \lambda \quad \text{(damping)}$$

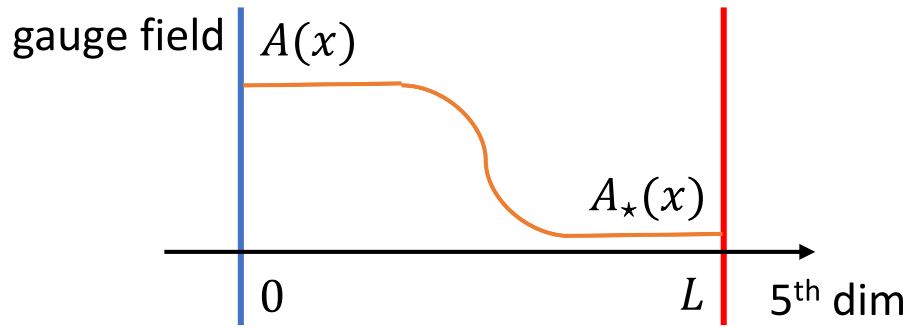
- Gradient flow is a gauge covariant equation
- $\bar{A}_\mu(x,\infty)$ is a pure gauge in the U(1) gauge theories
- In non-Abelian cases, $\bar{A}_\mu(x,\infty)$ is a local minimum of the action $S_{\mathcal{G}}$

DW fermion + gradient flow

Grabowska-Kaplan (2015)

Domain-wall + Gradient flow





- LH and RH are localized around defects
- gauge fields are damped along 5^{th} dim.

$$\Lambda \to \infty, L \to \infty$$

⇒only LH-fermions couple to the physical modes of A_μ

(In U(1) theories, RH-fermions couple to the gauge modes of A_μ)

3.5. Short summary

DW fermion and GW relation

Doubling problem

DW fermion and GW relation

Doubling problem

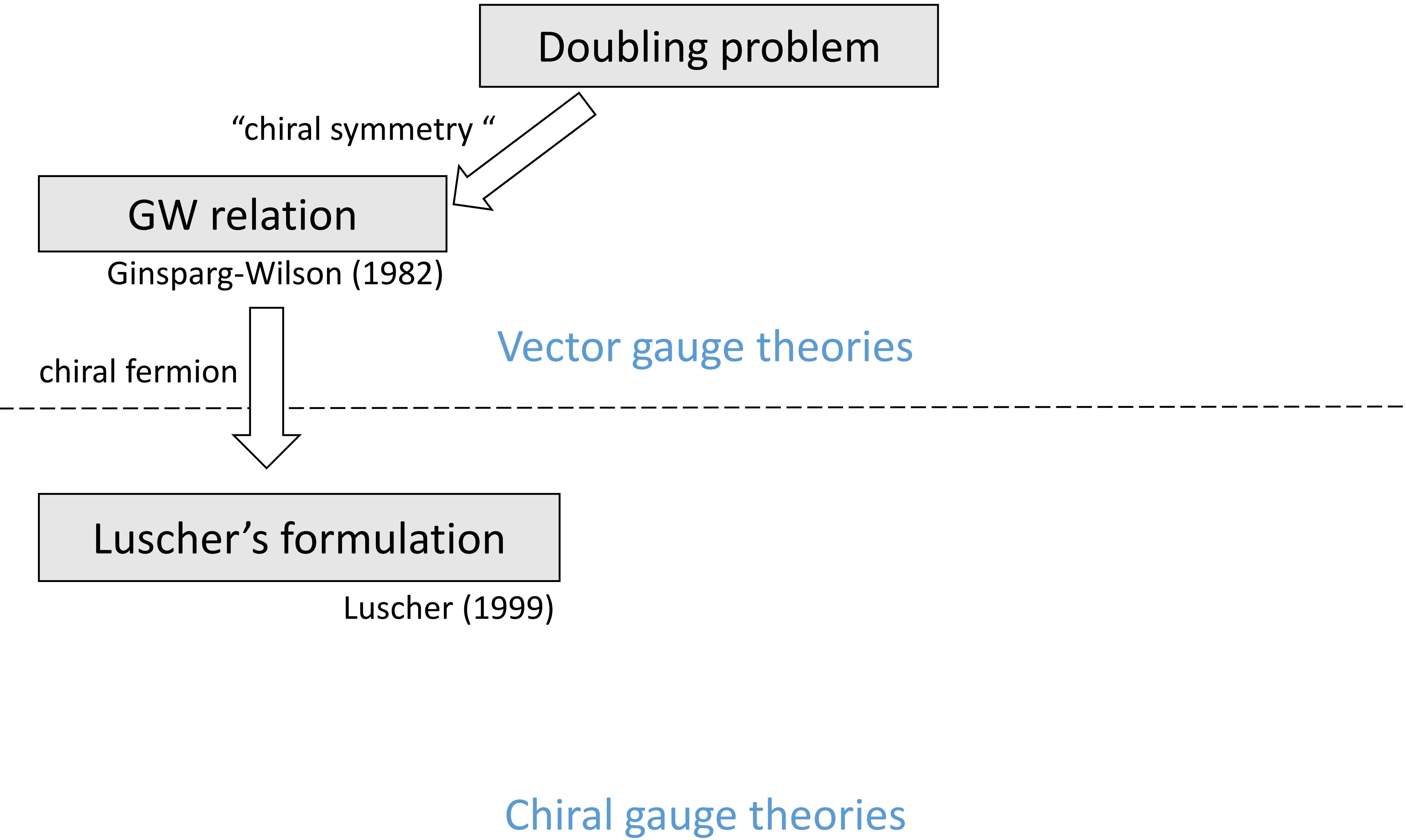
“chiral symmetry “

GW relation

Ginsparg-Wilson (1982)

Vector gauge theories

DW fermion and GW relation



DW fermion and GW relation

Doubling problem

"chiral symmetry "

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Ginsparg-Wilson (1982)

chiral fermion

Vector gauge theories

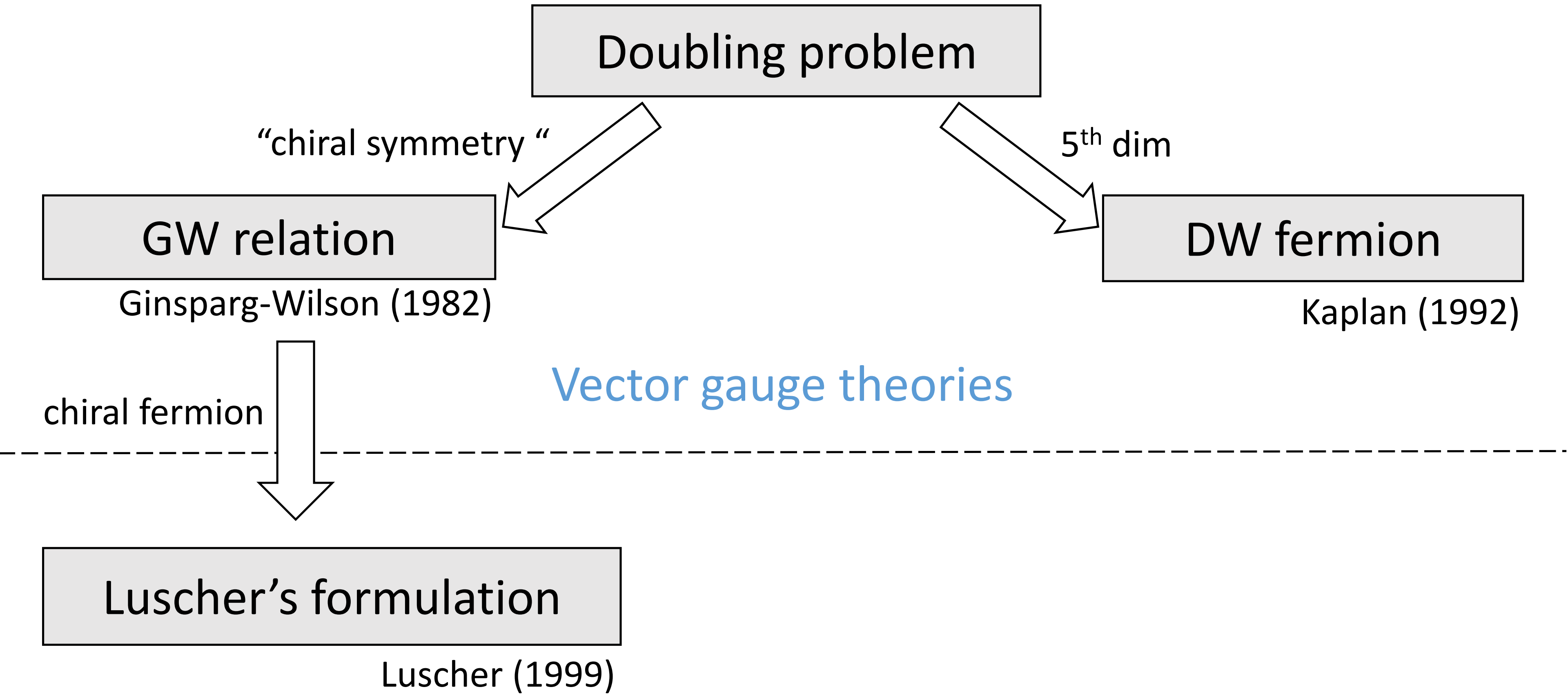
Luscher's formulation

Luscher (1999)

explicit solution in non-abelian gauge theories?

Chiral gauge theories

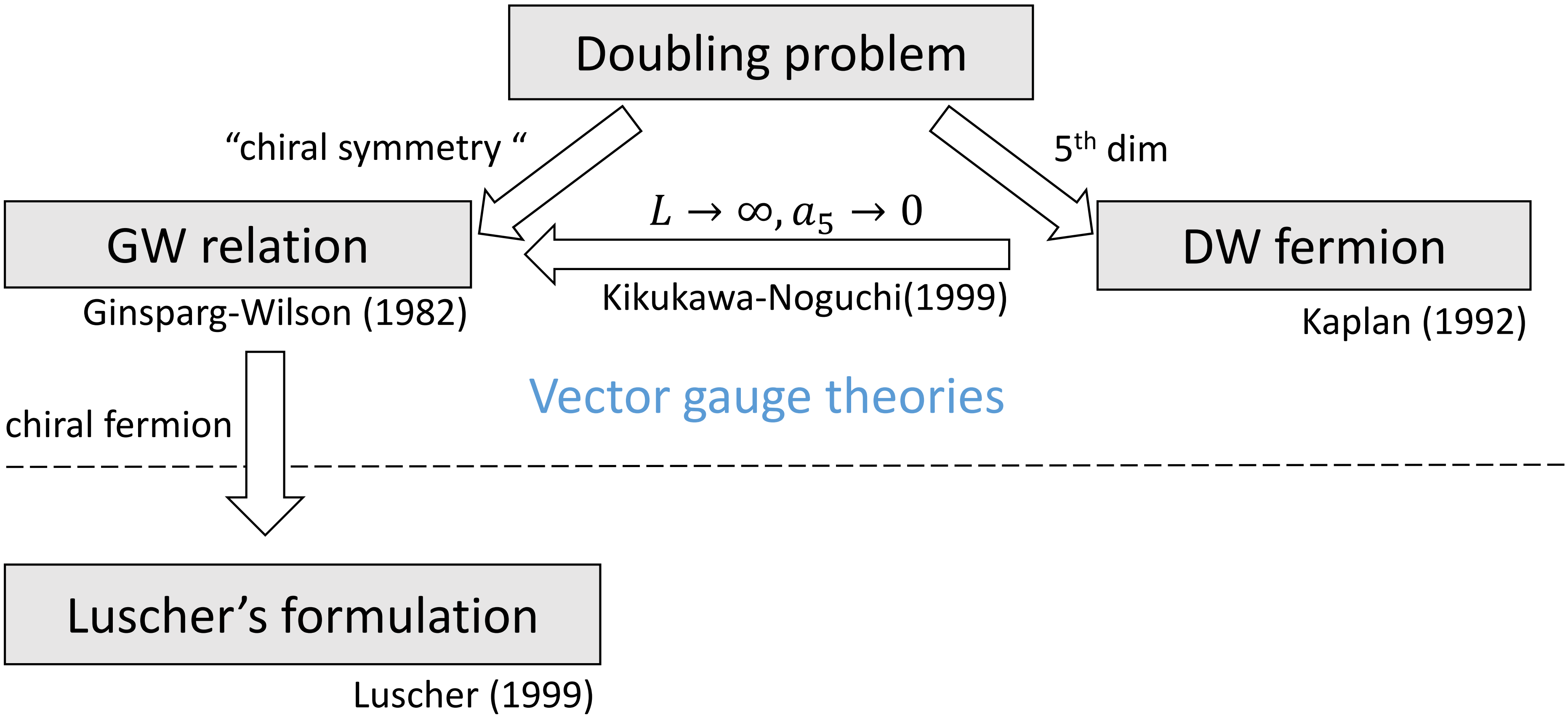
DW fermion and GW relation



explicit solution in non-abelian gauge theories?

Chiral gauge theories

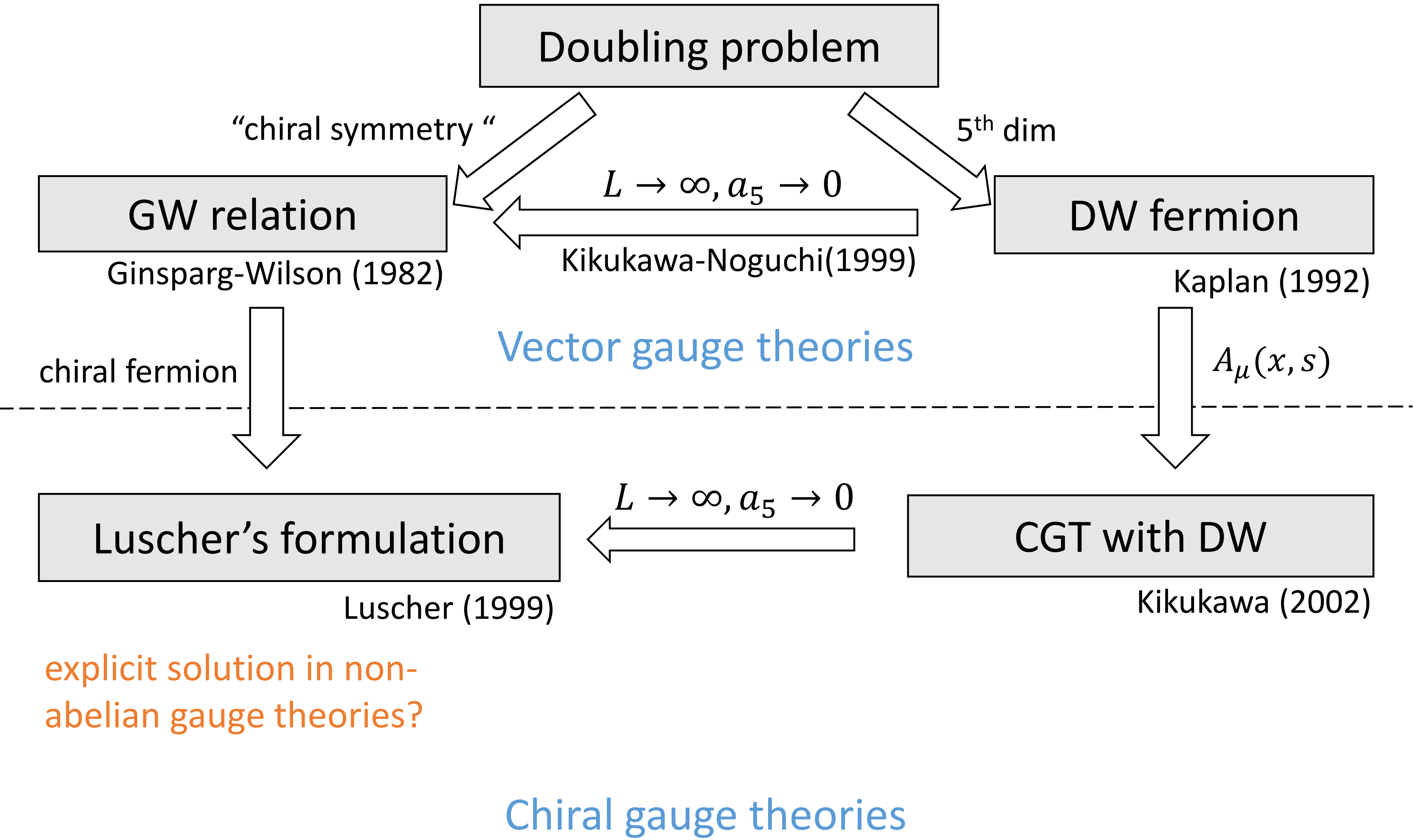
DW fermion and GW relation



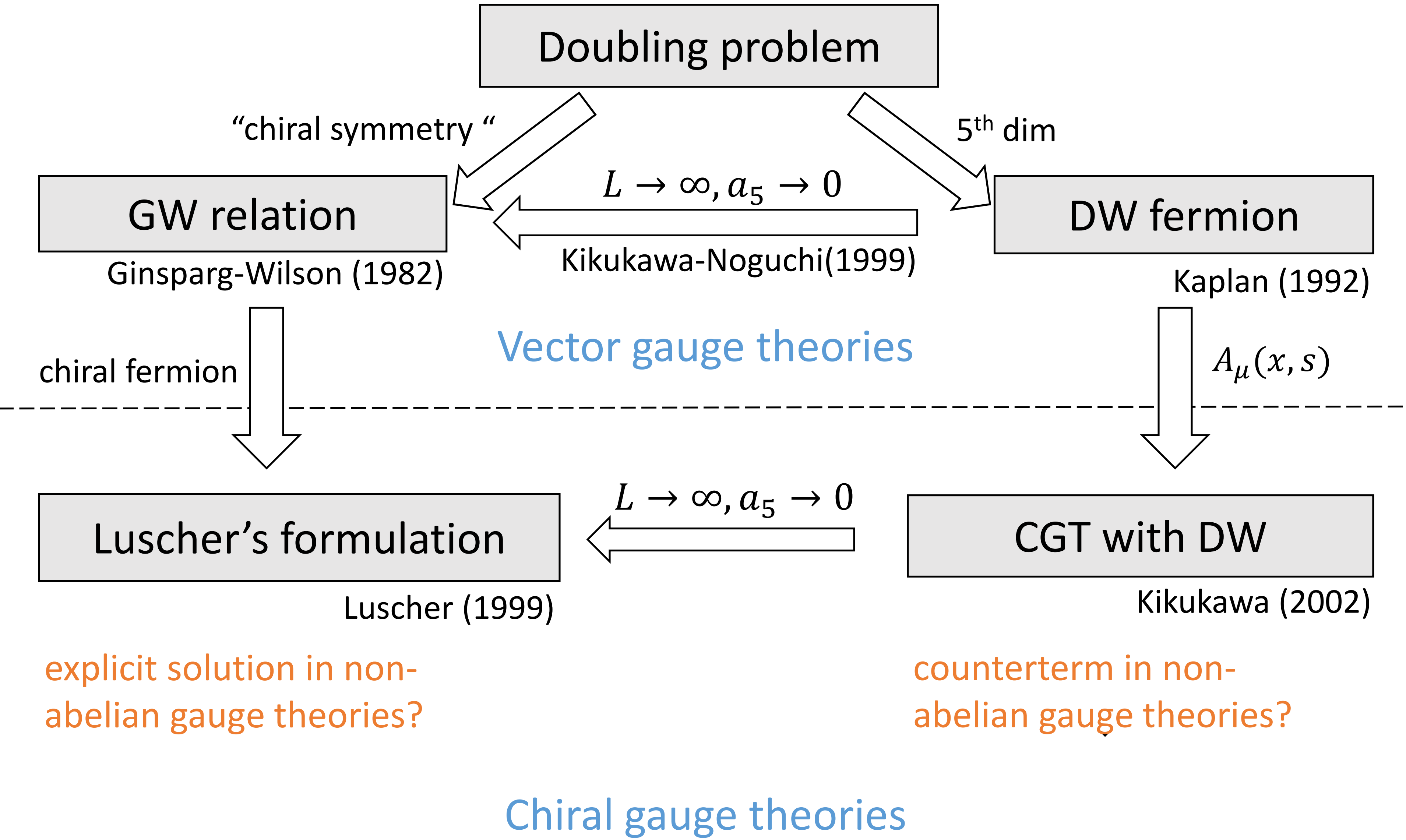
explicit solution in non-abelian gauge theories?

Chiral gauge theories

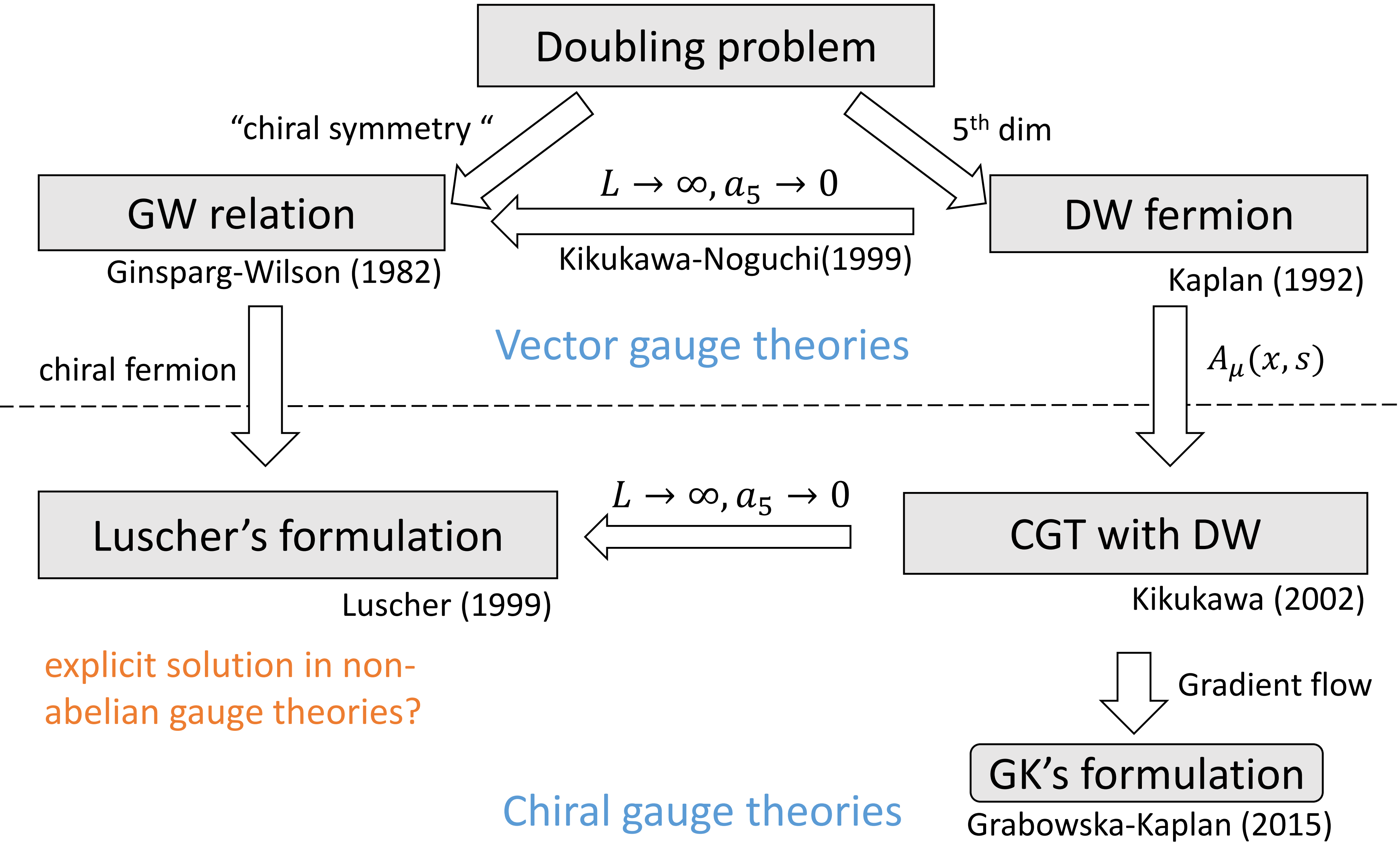
DW fermion and GW relation



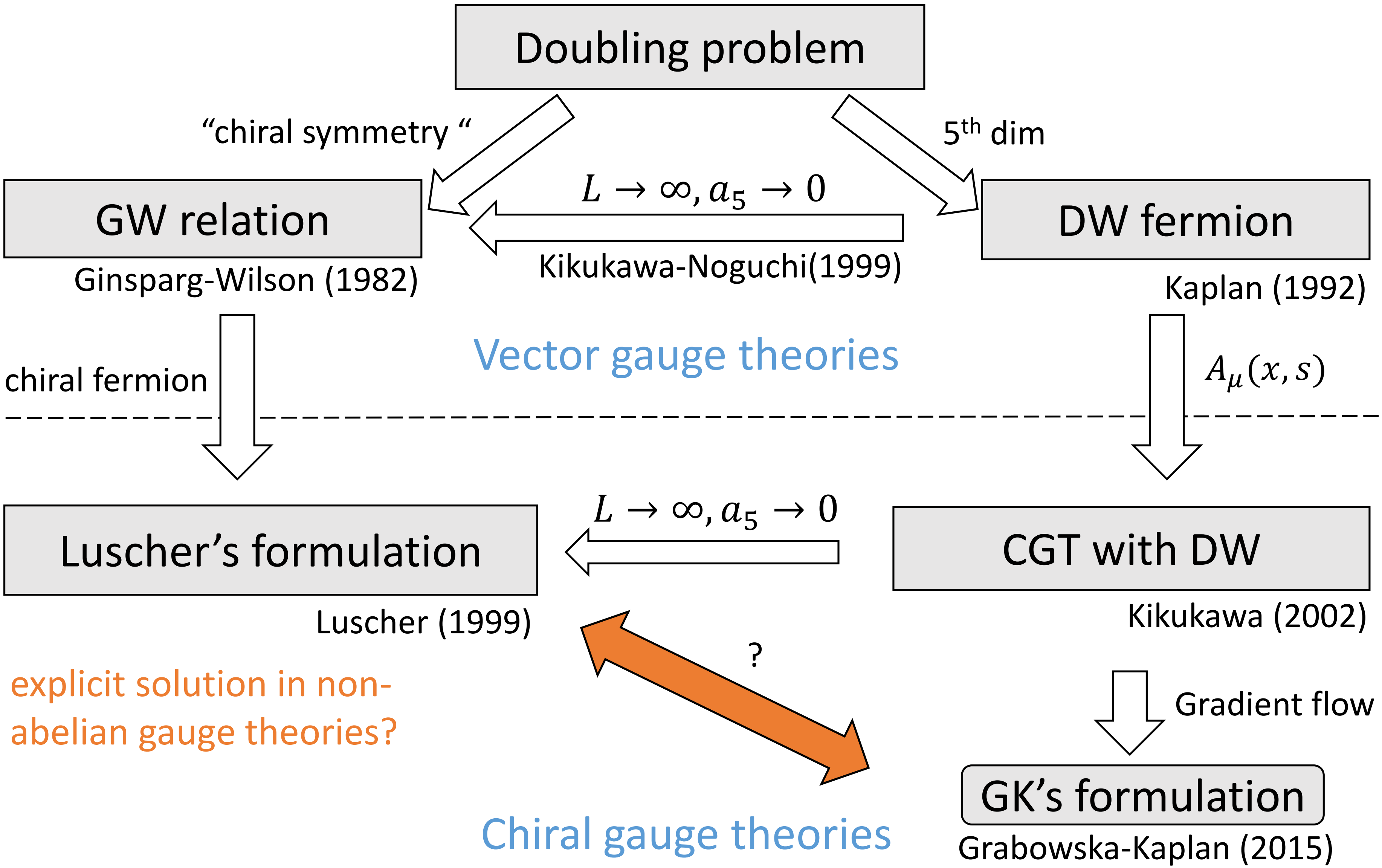
DW fermion and GW relation



DW fermion and GW relation



DW fermion and GW relation



4. Relation of two formulations

Gradient flow

$$\partial_t \bar{A}_\mu(x,t) = -g_0^2 \frac{\delta S_{\mathcal{G}}}{\delta \bar{A}_\mu} = -D_\nu \bar{F}_{\mu\nu}(x,t)$$

lattice ⇨

$$\frac{d}{dt} U_t(x,\mu) = -g_0^2 [\partial_{x,\mu} S(U_t)] U_t(x,\mu)$$

Choice of $S(U)$: $S(U) = 1/(4g_0^2) \sum_x F_{\mu\nu}(x)^2$

⇩ Assume admissibility

⇒Bianchi identity: $\epsilon_{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$

“diffusion eq.”: $\frac{d}{dt} F_{\mu\nu}(x,t) = \partial_\rho^* \partial_\rho F_{\mu\nu}(x,t)$

$$\partial_\mu f(x) = f(x+\hat{\mu}) - f(x)$$
$$\partial_\mu^* f(x) = f(x) - f(x-\hat{\mu})$$

Gradient flow

diffusion equation

$$\frac{d}{dt}F_{\mu\nu}(x,t) = \partial_\rho^* \partial_\rho F_{\mu\nu}(x,t)$$

Maximum principle

Suppose $f(x,t)$ is periodic w.r.t. x, and satisfies

$$\frac{d}{dt}f(x,t) = \partial_\mu^* \partial_\mu f(x,t)$$

Then $f(x,t)$ has a maximum at $t = 0$,

$$\max_{x\in\Gamma, t\in[0,T]} f(x,t) = \max_{x\in\Gamma} f(x,0)$$

if $F_{\mu\nu}(x,0)$ satisfies admissibility, $F_{\mu\nu}(x,t)$ respects this condition

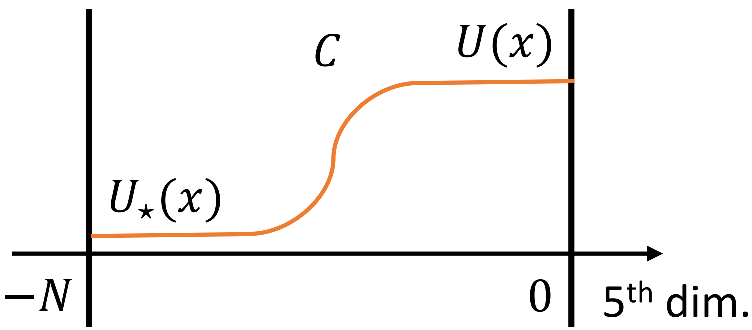
Relation of the two formulations

$$\exp(\Gamma_{\mathrm{GK}}) \equiv \frac{\det(D_{5\mathrm{w}} - m_0)|^{C}_{\mathrm{Dir.}}}{\left|\det(D_{5\mathrm{w}} - m_0)|^{C+(-C)}_{\mathrm{AP}}\right|^{1/2}}$$

cf. Kikukawa (2002)

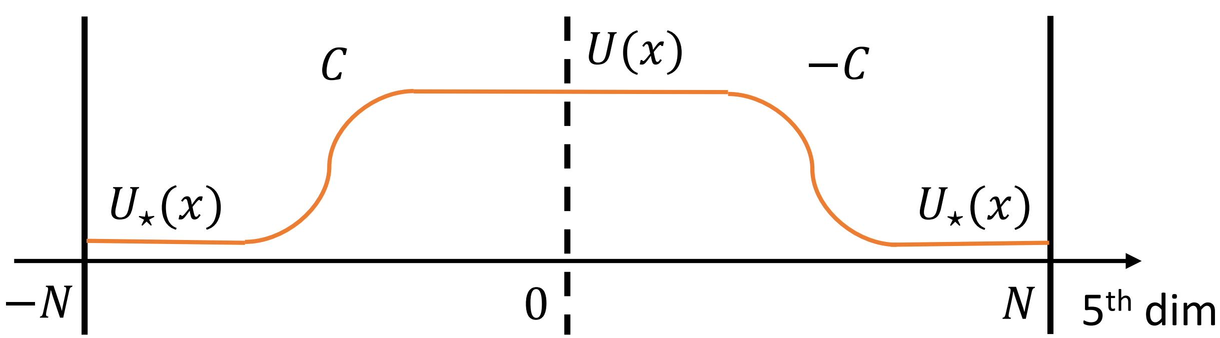
(No counterterm)

Domain wall fermion



LH + RH + massive modes

Pauli-Villars fermion



massive modes

Relation of the two formulations

$$\exp(\Gamma_{\rm GK}) \equiv \frac{\det(D_{5\rm w} - m_0)|^{C}_{\rm Dir.}}{\left|\det(D_{5\rm w} - m_0)|^{C+(-C)}_{\rm AP}\right|^{1/2}}$$

cf. Kikukawa (2002)

(No counterterm)

$(L \to \infty, a_5 \to 0)$

$$\partial_s \Gamma_{\rm GK}[U] \to {\rm Tr}\, P_+ \partial_s D D^{-1}|_{t=0} + ({\rm Tr}\, P_+ \partial_s D D^{-1})^*|_{t=-\infty} + \int_{-\infty}^{0} dt\, {\rm Tr} \hat{P}_- [\partial_t \hat{P}_-, \partial_s \hat{P}_-]$$

LH RH Bulk

Luscher's formulation: $\partial_t \Gamma_{\rm L}[U] = {\rm Tr}\, P_+ \delta_\eta D D^{-1} - i \mathcal{L}_\eta$

$$\Gamma_{\rm GK}[U] = \Gamma_{\rm L}[U] + \Gamma_{\rm L}[U_\star]^* + i \int_{-\infty}^{0} dt \mathcal{L}_\eta$$

$$\eta_\mu = i^{-1} \partial_t U_t(x,\mu) U_t(x,\mu)^{-1}$$

Locality

$$\mathcal{L}_\eta = \sum_x \left(e^{-\Box t} \partial_\nu^* F_{\nu\mu}(x) \right) j_\mu(x) \Big|_{U=U_t}$$

$j_\mu(x)$: gauge-invariant local field

- Perturbative analysis in the infinite volume limit

$$j_\mu(p) = \sum_{k\geq 4} \int V^{(k)}_{\mu\mu_1\nu_1\cdots\mu_k\nu_k}(p; p_1, \ldots, p_{k-1}) (2\pi)^4 \delta^{(4)}(\textstyle\sum p_i) \displaystyle\prod_{i=1}^{k} e^{\hat{p}_i^2 t} \tilde{F}_{\mu_i\nu_i}(p_i) \frac{dp_i^4}{(2\pi)^4}$$

$V^{(k)}_{\mu\mu_1\nu_1\cdots\mu_k\nu_k}(p; p_1, \ldots, p_{k-1})$: analytic function

$$\Rightarrow \int_{-\infty}^{0} dt \mathcal{L}_\eta = \sum_{k\geq 5} \int \frac{\tilde{V}^{(k)}_{\mu_1\cdots\mu_k}(p_1, \ldots, p_{k-1})}{\sum \hat{p}_i^2} (2\pi)^4 \delta^{(4)}(\textstyle\sum p_i) \displaystyle\prod_{i=1}^{k} \tilde{A}_{\mu_i}(p_i) \frac{dp_i^4}{(2\pi)^4}$$

$$\equiv \sum_{k\geq 5} \int \tilde{\Gamma}^{(k)}_{\mu_1\cdots\mu_k}(p_1, \ldots, p_{k-1}) (2\pi)^4 \delta^{(4)}(\textstyle\sum p_i) \displaystyle\prod_{i=1}^{k} \tilde{A}_{\mu_i}(p_i) \frac{dp_i^4}{(2\pi)^4}$$

$$\hat{p}^2 = \sum_\mu [2\sin(p_\mu/2)]^2$$

Locality

- $\tilde{\Gamma}^{(k)}_{\mu_1\cdots\mu_k}(p_1, \ldots, p_{k-1})$ are not analytic function
- For every positive integer N, there exist $n > N$ and $x_1, \cdots, x_{k-1}$ such that

$$\left|\Gamma^{(k)}_{\mu_1\cdots\mu_k}(x_1, \ldots, x_{k-1}, 0)\right| > \frac{c}{n^{5k-4}}, \qquad n = \|x_1\|_1 + \cdots + \|x_{k-1}\|_1$$

- There may be some effect on the critical behavior and the continuum limit of the lattice model
- (Numerical) approach to the dynamics is necessary to solve this question

5. Summary

Summary

- We considered U(1) chiral gauge theories
- the gradient flow is formulated for the admissible U(1) link fields
- GK’s effective action is the sum of Luscher’s effective actions and the measure term
- The measure term is non-local (not suppressed exponentially)
- There may be some effect on the critical behavior and the continuum limit of the lattice model