- Kodaira-Spencer equation in Tian's form

$A$ : holom. vector valued $(0,1)$-form

$$\bar{\partial} A + \frac{1}{2}[A, A] = 0 \qquad \text{(Kodaira-Spencer eq.)}$$

$$A = A^{I}{}_{\bar{J}} (\partial_I \otimes d\bar{z}^{\bar{J}})$$

$$\bar{\partial} A = \bar{\partial}_{\bar{k}} A^{I}{}_{\bar{J}} (\partial_I \otimes d\bar{z}^{\bar{K}} \wedge d\bar{z}^{\bar{J}})$$

$$\begin{aligned} [A, A] &= [A^{I}{}_{\bar{J}} (\partial_I \otimes d\bar{z}^{\bar{J}}), A^{L}{}_{\bar{K}} (\partial_L \otimes d\bar{z}^{\bar{K}})] \\ &= [A^{I}{}_{\bar{J}} \partial_I, A^{L}{}_{\bar{K}} \partial_L] \otimes d\bar{z}^{\bar{J}} \wedge d\bar{z}^{\bar{K}} \\ &= 2 (A^{I}{}_{\bar{J}} \partial_I A^{L}{}_{\bar{K}}) \partial_L \otimes d\bar{z}^{\bar{J}} \wedge d\bar{z}^{\bar{K}} \end{aligned}$$

$f$ is holomorphic $\Longleftrightarrow$ $(\bar{\partial} + A^I \partial_I) f = 0$

the integrablity $(\bar{\partial} + A^I \partial_I)^2 = 0$

$$(\bar{\partial} + A^I \partial_I)(\bar{\partial} + A^J \partial_J) = \bar{\partial}(A^J \partial_J) + (A^I \partial_I)\bar{\partial} + (A^I \partial_I)(A^J \partial_J)$$

$$= (\bar{\partial} A^J)\partial_J + (A^I \partial_I A^J)\partial_J \quad (A^I A^J \partial_I \partial_J = 0)$$

$$\therefore \quad \bar{\partial} A^J + (A^I \partial_I A^J) = 0 \qquad //$$

The linearized KS equation $\bar{\partial} A = 0$

The (infinitesimal) deformation is trivial, if it comes from a diffeomorphism $A \longrightarrow A + \bar{\partial}\varepsilon \leftarrow$ vector field

Hence infinitesimal deformations are parametrized by $\bar{\partial}$-cohomology.

Recall that we have an isom. $\vee : \Omega^{(0,p)}(\Lambda^q TM) \longrightarrow \Omega^{(3-q,p)}(M)$ $\lrcorner 3$

by $A^\vee = (A \cdot \Omega)$ $\qquad$ $\cdot$ contraction of vector fields with differential form

$\uparrow$ (3,0)-form

Let us rephrase the KS equation in terms of $A^\vee \in \Omega^{(2,1)}(M)$

It is convenient to impose a constraint: $\underline{\partial A^\vee = 0}$

cf $\bar{b}_0 \Psi = 0$ string field theory

We also impose a gauge fixing condition $\bar{\partial}^\dagger A^\vee = 0$ for the symmetry $A \to A + \bar{\partial}\varepsilon$

<u>Lemma</u> (Tian) $\partial A^\vee = \partial B^\vee = 0 \implies [A, B]^\vee = \partial (A \wedge B)^\vee$

cf BV $\{\alpha, \beta\} \propto \partial(\alpha \wedge \beta) - (\partial\alpha) \wedge \beta - \alpha \wedge (\partial\beta)$ //

In terms local coordinate B.V bracket is

$$\{A, B\} = \sum_I \frac{\partial A}{\partial \eta_I}\frac{\partial B}{\partial z^I} + (-1)^{|A|}\frac{\partial A}{\partial z^I}\frac{\partial B}{\partial \eta_I}$$

Assume $A = A^I{}_{\bar J}\, \eta_I\, d\bar z^{\bar J}$, $B = B^K{}_{\bar L}\, \eta_K\, d\bar z^{\bar L} \in PV^{1,1}(X)$

$$\{A, B\} = A^I{}_{\bar J}\, d\bar z^{\bar J}\, \frac{\partial B^K{}_{\bar L}}{\partial z^I}\, \eta_K\, d\bar z^{\bar L} + \frac{\partial A^K{}_{\bar J}}{\partial z^I}\, \eta_K\, d\bar z^{\bar J}\, B^I{}_{\bar L}\, d\bar z^{\bar L}$$

$$= \left(-A^I{}_{\bar J}\frac{\partial B^K{}_{\bar L}}{\partial z^I} + B^I{}_{\bar L}\frac{\partial A^K{}_{\bar J}}{\partial z^I}\right)\eta_K\, d\bar z^{\bar J}\, d\bar z^{\bar L}$$

$$= -\,[A_{\bar J}, B_{\bar L}]^K\, \eta_K\, d\bar z^{\bar J}\, d\bar z^{\bar L} \qquad //$$

符号 ?

fundamental dynamical varivable sign flip $(A \to -A)$

$$\bar\partial A + \frac{1}{2}[A, A] = 0 \quad \to \quad \bar\partial A - \frac{1}{2}[A, A] = 0$$

By the lemma we can rewrite the KS equation as follows;

$$\bar{\partial} A^{\vee} + \frac{1}{2} \partial (A \wedge A)^{\vee} = 0 \quad \in \Omega^{(2,2)}(M)$$

$$A \wedge A \in \Omega^{(0,2)}(\wedge^2 TM), \quad (A \wedge A)^{\vee} \in \Omega^{(1,2)}(M)$$

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<u>Prop</u> [Tian - Tororov] Todorov CMP <u>126</u> (1989) 325.

Given $A_1 \in H^{(0,1)}(TM)$ with $\partial A_1^{\vee} = \bar{\partial}^{\dagger} A_1^{\vee} = 0$

there is a unique solution to the K-S equation in the form

$$A = \sum_{n=1}^{\infty} \varepsilon^n A_n \quad \text{with} \quad \partial A_n^{\vee} = \bar{\partial}^{\dagger} A_n^{\vee} = 0$$

which is obtained by solving the K-S equation perturbatively in $\varepsilon$.

(The series has a finite radius of convergence.)

"Any infinitesimal deformation of complex str. is unobstructed"

Physically $A_1^\vee$ is massless modes and $A_n^\vee$ $(n>1)$ are massive modes

massive modes $\sim$ linear combi. of eigenstates of Laplacian with positive eigenvalues

Let us look at how we can solve the KS eq. perturbatively.

$O(\varepsilon^2)$ KS eq. in Tian's form is $\bar\partial A_2^\vee + \frac{1}{2}\partial(A_1 \wedge A_1)^\vee = 0$

we have a symmetry $A_2^\vee \longrightarrow A_2^\vee + \bar\partial v$

we fix the gauge sym. by imposing $\bar\partial^\dagger A_2^\vee = 0$ cf Lorentz gauge.

then a unique solution is given by

$$A_2^\vee = -\bar\partial^\dagger \frac{1}{\Delta} \partial (A_1 \wedge A_1)^\vee$$

where $\Delta = 2(\bar\partial \bar\partial^\dagger + \bar\partial^\dagger \bar\partial)$

<u>Remark</u> $\frac{1}{\Delta}$ is well-defined due to $\operatorname{Ker}\partial \subset \operatorname{Ker}\Delta$ (?)

Let us check.

$$\bar{\partial} A_2^\vee = -(\bar{\partial}\bar{\partial}^\dagger)\frac{1}{\Delta}\partial(A_1\wedge A_1)^\vee$$

$$= -\frac{1}{2}\partial(A_1\wedge A_1)^\vee + \frac{1}{2}\frac{\bar{\partial}^\dagger\bar{\partial}}{\Delta}\partial(A_1\wedge A_1)^\vee$$

$$\bar{\partial}\partial(A_1\wedge A_1)^\vee = -\partial\bar{\partial}(A_1\wedge A_1)^\vee = 0 \quad (\because \bar{\partial}\Omega = \bar{\partial}A_1 = 0)$$

$$\therefore \quad \bar{\partial} A_2^\vee + \frac{1}{2}\partial(A_1\wedge A_1)^\vee = 0$$

$$\partial A_2^\vee = \bar{\partial}^\dagger\frac{1}{\Delta}\partial^2(A_1\wedge A_1)^\vee = 0 \qquad //$$

**Remark** On $\operatorname{Ker}\bar{\partial} \cap \operatorname{Im}\partial$

$$-\bar{\partial}^{\dagger}\frac{1}{\Delta}\partial = -\frac{1}{2}\frac{\bar{\partial}^{\dagger}}{\bar{\partial}\bar{\partial}^{\dagger} + \cancel{\bar{\partial}^{\dagger}\bar{\partial}}}\partial = -\frac{1}{2}\frac{1}{\bar{\partial}}\partial$$

LHS is regarded as the propergator of the massive modes

KS eq. in Tian's form $=$ $A^{\vee} = \underbrace{-\frac{1}{2}\frac{1}{\bar{\partial}}}_{\text{Green's operator for KS-eq.}}\partial(A \wedge A)^{\vee}$

The next term of the KS eq. is $\bar{\partial}A_3^{\vee} + \partial(A_2 \wedge A_1)^{\vee} = 0$.

If the second term is in $\operatorname{Ker}\bar{\partial}$, we can use the same propergator (Green's function) to solve $A_3^{\vee}$.

Let us check $\bar{\partial}\left(\partial(A_2 \wedge A_1)^{\vee}\right) = 0$

$$\bar{\partial}\left(\partial(A_2 \wedge A_1)^{\vee}\right) = -\partial(\bar{\partial}A_2 \wedge A_1)^{\vee} \qquad (\because\ \bar{\partial}A_1 = 0)$$

$$= \partial\left(\frac{1}{2}[A_1, A_1] \wedge A_1\right)^{\vee} \qquad (\text{by KS-eq.})$$

$$= \frac{1}{2}\left([[A_1, A_1], A_1]\right)^{\vee} \qquad (\text{by Tian's lemma})$$

$$= 0 \qquad \text{by Jacobi identity} \qquad //$$

The n-th order equation is

$$\bar{\partial} A_n^{\vee} + \frac{1}{2}\sum_{i=1}^{n} \partial(A_{n-i} \wedge A_i)^{\vee} = 0$$

We can check $\bar{\partial}\left(\sum_{i=1}^{n} \partial(A_{n-i} \wedge A_i)\right)^{\vee} = 0$

We use $\partial([A,B]\wedge C)^{\vee} + (ABC, \text{cyclic}) = 0.$ 

which follows from the Jacobi identy

$$[[A,B],C] + (ABC, \text{cyclic}) = 0 \qquad A, B, C \in PV^{1,1}(X)$$

$$\partial A = \partial B = \partial C = 0$$

and $[[A,B]\, C]^{\vee} = \partial([A,B]\wedge C)^{\vee}$

$$\bar{\partial}\left(\partial(A_2\wedge A_2)^{\vee} + 2\,\partial(A_1\wedge A_3)^{\vee}\right)$$

$$= -2\,\partial\left[(\bar{\partial}A_2\wedge A_2)^{\vee} + (\bar{\partial}A_1\wedge A_3)^{\vee} - (A_1\wedge\bar{\partial}A_3)^{\vee}\right]$$

$$= \partial\left[([A_1,A_1]\wedge A_2)^{\vee} - (A_1\wedge[A_1,A_2])^{\vee} - (A_1\wedge[A_2,A_1])^{\vee}\right]$$

$$= 0 \qquad //$$

$$\bar{\partial}\left[\partial(A_1\wedge A_4)^\vee + \partial(A_2\wedge A_3)^\vee\right]$$

$$= -\partial\Big[-\big(A_1\wedge \tfrac{1}{2}[A_2,A_2]\big)^\vee - \big(A_1\wedge[A_1,A_3]\big)^\vee$$

$$+\tfrac{1}{2}\big([A_1,A_1]\wedge A_3\big)^\vee + \big(A_2\wedge[A_1,A_2]\big)^\vee\Big]$$

$$= 0$$

$$\bar{\partial}\Big(\sum_{i=1}^{n}\partial(A_{n-i}\wedge A_i)\Big)^\vee \propto \partial\Big(\sum_{i+j+k=n}\big([A_i,A_j]\wedge A_k\big)^\vee\Big)$$

$$= 0$$

$n=5$ $\quad (i,j,k) = (1,2,2)$ cyclic perm

$\qquad\qquad\qquad (1,1,3)$ cyclic perm