B-model topological strings and BV algebra

Costello-Gaiotto 1812.09257

B-model topological string on a CY 3-fold X has no world-sheet instanton corrections.

$\Longrightarrow$ The topological string theory can be described by some "topological" field theory on the space-time X

$\searrow$ Kodaira-Spencer gravity (BCOV theory)
(closed string sector)
Holomorphic Chern-Simons theory (Witten)
(open string sector)

cf A-model $\exists$ world-sheet instanton correction
Space-time theory is very difficult (except open string sector)

Basic fields of BCOV theory are poly-vector fields on X^{L2}

$$PV^{j,i}(X) = \Omega^{0,i}(X, \Lambda^j TX)$$

the space of $(0,i)$-forms with coefficients in the j-th exterior product of the holomorphic tangent bundle TX.

We denote $\quad PV^{*,*}(X) = \bigoplus_{i,j} PV^{j,i}(X)$

with the cohomological degreee $(i+j)$

By the wedge product $PV^{*,*}(X)$ is a graded commutative algebra.

We have the standard $\bar{\partial}$-operation with degree one

$$\bar{\partial} : \quad P^{j,i}(X) \longrightarrow P^{j,i+1}(X)$$

On a CY-3 fold X, we have a holomorphic volume form Ω_X, [3]
non-vanishing (3,0)-form
which leads to an isom. $PV^{j,i}(X) \to \Omega^{3-j,i}(X)$.
$\Omega^{p,q}(X)$: the space of (p,q)-forms

By the isom, the ∂-operator $\partial : \Omega^{p,q} \longrightarrow \Omega^{p+1,q}$
induces $\partial : PV^{j,i}(X) \longrightarrow PV^{j-1,i}(X)$
with degree (-1).

Let us take local coordinates $\{z^I, \bar{z}^{\bar{I}}\}$ on a patch $U \subset \mathbb{C}^3$

then $PV^{*,*}(U) \simeq C^\infty(U)[\eta_I, d\bar{z}^{\bar{I}}]$

the ring of C^∞-fns on U

(In the previous discussion on B-model, η_I was denoted θ_I)

$\eta_I \sim \dfrac{\partial}{\partial z^I}$ and $d\bar{z}^{\bar{I}}$ are odd variables with degree one
(fermionic) (ghost number)

On U, $\bar{\partial}$ and ∂ are expressed as

$$\bar{\partial} = \sum_{\bar{I}} d\bar{z}^{\bar{I}} \frac{\partial}{\partial \bar{z}^{\bar{I}}}, \qquad \partial = \sum_{I} \frac{\partial}{\partial \eta_I} \frac{\partial}{\partial z^I}$$

cf Δ-operator in Batalin-Vilkovisky.

$\eta_I \leftrightarrow dz^I$ dual $\quad dz^I\left(\frac{\partial}{\partial z^J}\right) = \delta_J^I$

The isom. by the hol. volume form exchanges dz^I and $\frac{\partial}{\partial \eta_I}$

η_I, dz^I : creation ops of "particle" and "anti-particle"

$\frac{\partial}{\partial \eta_I}$, $\frac{\partial}{\partial z^I}$: anihilation ops of 〃

anihilation of particle $\longleftrightarrow$ creation of anti-particle

(Take local coordinates s.t $\Omega_X \sim \varepsilon_{IJK}\, dz^I \wedge dz^J \wedge dz^K$)

Both $\bar\partial$ and ∂ are coboundary ops. $\bar\partial^2 = \partial^2 = 0$

$\bar\partial$ is a derivation (satisfies the Leibnitz rule)

but ∂ is NOT, since $PV^{*,*} \simeq \Omega^{3-*,*}$ is NOT alg. isom.

$$\partial(\alpha\wedge\beta) = (\partial\alpha)\wedge\beta + (-1)^{|\alpha|}\alpha\wedge\partial\beta + \underbrace{\{\alpha,\beta\}}_{\neq 0}$$

Define $\{\alpha,\beta\} := \partial(\alpha\wedge\beta) - (\partial\alpha)\wedge\beta - (-1)^{|\alpha|}\alpha\wedge(\partial\beta)$.

On a local coordinate patch

$$\partial(\alpha\wedge\beta) = \sum_I \frac{\partial}{\partial\eta_I}\frac{\partial}{\partial z^I}(\alpha\wedge\beta) = \sum_I \frac{\partial}{\partial\eta_I}\left(\frac{\partial\alpha}{\partial z^I}\wedge\beta + \alpha\wedge\frac{\partial\beta}{\partial z^I}\right)$$

$$= (\partial\alpha)\wedge\beta + (-1)^{|\alpha|}\alpha\wedge\partial\beta + \sum_I\left((-1)^{|\alpha|}\frac{\partial\alpha}{\partial z^I}\wedge\frac{\partial\beta}{\partial\eta_I} + \frac{\partial\alpha}{\partial\eta_I}\wedge\frac{\partial\beta}{\partial z^I}\right)$$

$$\{\alpha, \beta\} = \sum_I \frac{\partial\alpha}{\partial\eta_I}\frac{\partial\beta}{\partial z^I} + (-1)^{|\alpha|}\frac{\partial\alpha}{\partial z^I}\frac{\partial\beta}{\partial\eta_I}$$

Odd Poisson bracket (Schouten bracket, BV anti-bracket)

<u>Lemma</u> $\partial(\alpha\beta\gamma) = \partial(\alpha\beta)\gamma + (-1)^{|\beta|(|\alpha|+1)}\beta\,\partial(\alpha\gamma)$

$+ (-1)^{|\alpha|}\alpha\,\partial(\beta\gamma) - (\partial\alpha)\beta\gamma - (-1)^{|\alpha|}\alpha(\partial\beta)\gamma - (-1)^{|\alpha|+|\beta|}\alpha\beta(\partial\gamma)$

(We omit $\wedge$ for simplicity)

$(\because)$ $$\partial(\alpha\beta\gamma) = \sum_I \frac{\partial}{\partial\eta^I}\left(\frac{\partial\alpha}{\partial z^I}\beta\gamma + \alpha\frac{\partial\beta}{\partial z^I}\gamma + \alpha\beta\frac{\partial\gamma}{\partial\eta^I}\right)$$

$$= (\partial\alpha)\beta\gamma + (-1)^{|\alpha|}\alpha(\partial\beta)\gamma + (-1)^{|\alpha|+|\beta|}\alpha\beta(\partial\gamma)$$

$$+ (-1)^{|\alpha|}\alpha\{\beta,\gamma\} + \{\alpha,\beta\}\gamma$$

$$+ (-1)^{|\beta|(|\alpha|+1)}\beta\{\alpha,\gamma\}$$

$$
\begin{aligned}
\therefore\ \partial(\alpha\beta\gamma) &= \partial(\alpha\beta)\gamma + (-1)^{|\beta|(|\alpha|+1)}\beta\,\partial(\alpha\gamma) + (-1)^{|\alpha|}\alpha\,\partial(\beta\gamma) \\
&\quad + (\partial\alpha)\beta\gamma + \cancel{(-1)^{|\alpha|}\alpha(\partial\beta)\gamma} + (-1)^{|\alpha|+|\beta|}\alpha\beta(\partial\gamma) \\
&\quad - (\partial\alpha)\beta\gamma - \cancel{(-1)^{|\alpha|}\alpha(\partial\beta)\gamma} \\
&\quad - (-1)^{|\beta|(|\alpha|+1)}\left(\beta\,\partial\alpha\,\gamma + (-1)^{|\beta|}\beta\alpha(\partial\gamma)\right) \\
&\quad - (-1)^{|\alpha|}\left(\alpha(\partial\beta)\gamma + (-1)^{|\beta|}\alpha\beta(\partial\gamma)\right) \\
&= \partial(\alpha\beta)\gamma + (-1)^{|\beta|(|\alpha|+1)}\beta\,\partial(\alpha\gamma) + (-1)^{|\alpha|}\alpha\,\partial(\beta\gamma) \\
&\quad - (\partial\alpha)\beta\gamma - (-1)^{|\alpha|}\alpha(\partial\beta)\gamma - (-1)^{|\alpha|+|\beta|}\alpha\beta(\partial\gamma) \qquad //
\end{aligned}
$$

Prop With the differential ∂ the space of polyvector fields $PV^{*,*}(X)$ on X is a bi-graded Batalin-Vilkovisky alg.

Def [BV algebra] (Getzler CMP 159 (1994) 265.)

A differential-graded (d-g) commutative algebra A is called BV algebra, if the differential $\Delta : A^{*} \longrightarrow A^{*+1}$ with $\Delta^2 = 0$ satisfies

$$\Delta(abc) = \Delta(ab)c + (-1)^{|a|} a\Delta(bc) + (-1)^{(|a|-1)|b|} b\Delta(ac)$$
$$- (\Delta a)bc - (-1)^{|a|} a(\Delta b)c - (-1)^{|a|+|b|} ab(\Delta c)$$

" Δ is an order two differential "

V : graded vector space $\quad v \in V$, $\quad |v|$: the grading of v

$[\ ,\ ] : V \otimes V \longrightarrow V$ (bilinear) is called a Lie bracket of degree m

$$\Longleftrightarrow \quad [v, w] = -(-1)^{(|v|-m)(|w|-m)} [w, v]$$
$$[u, [v, w]] = [[u, v], w] + (-1)^{(|u|-m)(|v|-m)} [v, [u, w]]$$

$\mathbb{A}$: graded commutative algebra

$[\ ,\ ] : \mathbb{A} \otimes \mathbb{A} \longrightarrow \mathbb{A}$ a Lie bracket of degree m

is called Poisson bracket of degree m

$$\Longleftrightarrow \quad [u, v \cdot w] = [u, v]w + (-1)^{|u|(|v|-1)} v[u, w]$$

"Poisson rule"

<u>Def</u> [Gerstenhaber alg.] (Lian-Zuckermann hep-th/9211072)

A graded commutative alg. $\mathbb{A}$ is called Gerstenhaber alg.

if $\mathbb{A}$ has a Poisson bracket of degree 1 //

Prop BV algebra is a Gerstenhaber algebra.

Namely, if we define

$$[a, b] = (-1)^{|a|}\Delta(ab) - (-1)^{|a|}(\Delta a)b - a(\Delta b)$$

$[\,,\,]$ is a Poisson bracket of degree one and we have

$$\Delta [a, b] = [\Delta a, b] + (-1)^{|a|-1}[a, \Delta b] \quad //$$

"Leibniz rule"

$(\because)$

$$\begin{aligned}
[a, b] &= (-1)^{|a|}\Delta(ab) - (-1)^{|a|}\Delta(a)b - a(\Delta b) \\
&= (-1)^{|a|+|a||b|}\left(\Delta(ba) - (-1)^{|b|}b\Delta(a) - (\Delta b)a\right) \\
&= -(-1)^{(|a|-1)(|b|-1)}\left((-1)^{|b|}\Delta(ba) - (-1)^{|b|}(\Delta b)a - b(\Delta a)\right) \\
&= -(-1)^{(|a|-1)(|b|-1)}[b, a] \quad //
\end{aligned}$$

Leibniz rule check //

$$\Delta[a,b] = (-1)^{|a|}\Delta(\Delta(ab)) - (-1)^{|a|}\Delta(\Delta(a)b) - \Delta(a\Delta b)$$

$$= -(-1)^{|a|}\Delta(\Delta(a)b) - \Delta(a\Delta b)$$

$$[\Delta(a), b] + (-1)^{|a|-1}[a, \Delta b]$$

$$= (-1)^{|a|+1}\Delta(\Delta(a)b) - (-1)^{|a|+1}(\Delta^2(a)b) - (\Delta(a)\Delta b) - \Delta(a\Delta(b)) + \Delta(a)\Delta(b) + (-1)^{|a|-1}(a\Delta^2 b)$$

$$= -(-1)^{|a|}\Delta(\Delta(a)b) - \Delta(a\Delta(b))$$

//

Jacobi id. $\longleftarrow$ $\begin{cases} \text{Poisson rule for } [u, v\cdot w],\ [u, (\Delta v)w] \\ \qquad [u, v(\Delta w)] \\ \text{Leibniz rule for } \Delta[u,v],\ \Delta[u,w] \end{cases}$

Poisson rule check

$$[a, bc] - [a, b]c - (-1)^{(|a|-1)|b|} b[a, c]$$

$$= (-1)^{|a|}\Delta(abc) - \cancel{(-1)^{|a|}\Delta(a)\cdot bc} - a\Delta(bc)$$

$$- (-1)^{|a|}\Delta(ab)c + \cancel{(-1)^{|a|}\Delta(a)\cdot bc} + a(\Delta b)c$$

$$- (-1)^{(|a|-1)|b|}\left[(-1)^{|a|} b\Delta(ac) - (-1)^{|a|} b(\Delta a)c - ba(\Delta c)\right]$$

$$= (-1)^{|a|}\Big[\Delta(abc) - (-1)^{|a|} a\Delta(bc) - \Delta(ab)c + (-1)^{|a|} a(\Delta b)c$$

$$- (-1)^{(|a|-1)|b|} b\Delta(ac) + \cancel{(-1)^{(|a|-1)|b|}}\, b(\Delta a)c + (-1)^{|a|+|b|} ab(\Delta c)\Big]$$

$(\Delta a)b$

$$= 0 \quad (\text{by the property of } \Delta)$$

More on BV algebra

(S_I L_I 1612.01292)

Def Differential BV algebra is a triple $(\mathcal{A}, Q, \Delta)$

- $\mathcal{A}$ is $\mathbb{Z}$-graded commutative assciative unital algebra
- Q is a derivation of degree one s.t $Q^2 = 0$
- Δ is a linear op. (not derivation) of degree one s.t. $\Delta^2 = 0$
- Δ is a differential of order two, namely if we define

$$\{a, b\} := \Delta(ab) - (\Delta a) b - (-1)^{|a|} a \Delta b$$

then $$\{a, bc\} = \{a, b\} c + (-1)^{(|a|+1)|b|} b \{a, c\}$$

- $$[Q, \Delta] = Q\Delta + \Delta Q = 0$$

$$\Longleftrightarrow \quad Q\{a, b\} = -\{Qa, b\} - (-1)^{|a|}\{a, Qb\}$$

We call Δ the BV operator, $\{\ ,\ \}$ the BV bracket

Prop $\{\ ,\ \}$ defines a Poisson bracket of degree one

- $\{a, b\} = (-1)^{|a||b|}\{b, a\}$
- $\{a, bc\} = \{a, b\}c + (-1)^{(|a|+1)|b|} b\{a, c\}$
- $\Delta\{a, b\} = -\{\Delta a, b\} - (-1)^{|a|}\{a, \Delta b\}$
- Jacobi identity

Prop X : Calabi-Yau $(PV(X), \bar\partial, \partial)$ is
a differential BV algebra with $Q = \bar\partial$, $\Delta = \partial$ //

[Notes] In Witten Mod. Phys. Lett. A5 (1990) 487-494
only Δ was employed, Q did not appear.

Def $(\mathcal{A}, Q, \Delta)$: differential BV algebra

for $I \in \mathcal{A}_0$ (degree 0) the classical master equation is

$$QI + \frac{1}{2}\{I, I\} = 0$$

Remark I : a solution to the classical master equation

$\Rightarrow$ $Q + \{I, *\}$ defines a differential on $\mathcal{A}$

However this may not be compatible (anti-commute) with Δ.

A sufficient condition for it is to impose the "divergence free" condition $\Delta I = 0$.

Remark When we regard $(PV(X), \bar{\partial}, \partial)$ as a diff. BV algebra, the classical master equation is nothing but the KS equation!

Def $(\mathcal{A}, Q, \Delta)$ a differential BV algebra

$\hbar$: a formal variable

For $I \in \mathcal{A}[[\hbar]]$ ~ formal power series in $\hbar$

the quantum master equation is

$$QI + \hbar \Delta I + \frac{1}{2}\{I, I\} = 0$$

Remark The "order two" property of Δ implies that formally QME is equivalent to

$$(Q + \hbar\Delta) e^{I/\hbar} = 0 .$$

If I is a solution to QME

$Q + \hbar\Delta + \{I, *\}$ defines a coboundary operator.

"quantum BRST" (squares to zero)