Deformation of complex structure

[Ref] 小平「複素多様体論」 §5-3 (岩波)

$t \in \mathbb{C}$; deformation parameter $\Delta := \Delta_r = \{ t \in \mathbb{C} \mid |t| < r \}$

In general we have multiparameters $t = (t_1, \cdots, t_m) \in \mathbb{C}^m$,
but for simplicity we take $m = 1$ in the following,

We define complex structure by local complex coordinates on $M \times \Delta$

$\{ (\zeta_j, t) \}_{j \in J}$ $M = \bigcup_{j \in J} U_j$ $U_j \subset M$ open

$(\zeta_j, t) = (\zeta_j^1(z,t), \cdots, \zeta_j^n(z,t), t)$ $\dim_{\mathbb{C}} M = n$

$\zeta_j^\alpha(z,t) = \zeta_j^\alpha(z^1, \cdots, z^n, t)$ is a C^∞-fn

$(z^1, \cdots, z^n)$ local complex coordinates of $M = M_0$
any coordinates but fixed as a reference

Since $(\zeta_j^1(z,0), \cdots, \zeta_j^n(z,0))$ and $(z^1, \dots, z^n)$ are local complex coordinate system $\zeta_j^\alpha(z,0)$ is a holomorphic function of $z^1, \cdots, z^n$ and we have

$$\det\left(\frac{\partial \zeta_j^\alpha(z,0)}{\partial z^\beta}\right)_{1\le\alpha,\beta\le n} \neq 0$$

(complex structure defined by $\{\zeta_j^\alpha(z,0)\}_{j\in J}$ is the same as the original.)

Taking r of Δr sufficiently small, we can assume

$$\forall t\in\Delta \qquad \det\left(\frac{\partial \zeta_j^\alpha(z,t)}{\partial z^\beta}\right)_{1\le\alpha,\beta\le n} \neq 0 \quad \cdots (*)$$

By (*) there a unique $(0,1)$-form $\varphi_j^{\alpha}(z,t) = \varphi_{j\,\bar\beta}^{\alpha}\, d\bar z^{\bar\beta}$ L3

s.t $\bar\partial \zeta_j^{\alpha} = \varphi_j^{\beta}\, \underbrace{\partial_\beta \zeta_j^{\alpha}}_{\det \neq 0} = \varphi_{j\,\bar\gamma}^{\beta}\, d\bar z^{\bar\gamma}\, \partial_\beta \zeta_j^{\alpha}$

" $\varphi_j^{\beta} = (\partial_\beta \zeta^\alpha)^{-1}\, \bar\partial \zeta_j^{\alpha}$ "

<u>Lemma</u> $U_j \cap U_k \neq 0 \Longrightarrow \varphi_j^{\alpha}(z,t)\,\partial_\alpha = \varphi_k^{\beta}(z,t)\,\partial_\beta$

[pf] ~ exercise

Hence we can define a global C^∞- vector valued $(0,1)$ form on M

by $\varphi(z,t) = \varphi_j^{\alpha}\,\partial_\alpha$ (indep. of j)

$$\varphi^{\alpha}(z,t) = \underbrace{\varphi^{\alpha}_{\bar\beta}(z,t)}_{C^\infty\text{-fn}}\, d\bar z^{\bar\beta}$$

We can regard $\varphi(z,t)$ as a differential operator

$$\varphi(z,t) : \quad f(z) \longmapsto \varphi^{\alpha}(z,t)\,\partial_\alpha f(z).$$

local C^∞-fn

In summary we obtain

indep. of $\dot{\jmath}$

$$\left(\bar\partial - \varphi(z,t)\right) \zeta^{\alpha}_{\dot{\jmath}}(z,t) = 0 \qquad (\forall \dot{\jmath})$$

M_t : $M_t = M_0$ as a C^∞-manifold, but the complex st. is defined by $\{\zeta_{\dot{\jmath}}(z,t)\}_{\dot{\jmath}\in J}$

Thm Local C^∞-fn. f on M is holomorphic on M_t

$$\Longleftrightarrow \quad \left(\bar\partial - \varphi(t)\right) f = 0$$

Note BCOV convention $-\varphi \rightsquigarrow +A$

[pf] $(\bar\partial - \varphi(t)) f = \underbrace{(\bar\partial - \varphi(t))\, \zeta^{\alpha}_{j}(z,t)}_{= 0} \cdot \dfrac{\partial f}{\partial \zeta^{\alpha}_{j}} + (\bar\partial - \varphi(t))\, \bar\zeta^{\bar\alpha}_{j} \cdot \dfrac{\partial f}{\partial \bar\zeta^{\bar\alpha}_{j}}$

$$(\bar\partial - \varphi(t)) f = \left(\bar\partial_{\bar\nu} \bar\zeta^{\bar\alpha}_{j} - \varphi^{\mu}{}_{\bar\nu}\, \partial_\mu \bar\zeta^{\bar\alpha}_{j} \right) d\bar z^{\nu} \frac{\partial f}{\partial \bar\zeta^{\bar\alpha}_{j}}$$

$$\bar\partial_{\bar\mu} \zeta^{\alpha}_{j} = \varphi^{\lambda}{}_{\bar\mu}\, \partial_\lambda \zeta^{\alpha}_{j} \Longrightarrow \partial_\mu \bar\zeta^{\bar\alpha}_{j} = \bar\varphi^{\bar\lambda}{}_{\mu}\, \bar\partial_{\bar\lambda} \bar\zeta^{\bar\alpha}_{j}$$

$$(\bar\partial - \varphi(t)) f = \left(\delta^{\bar\lambda}{}_{\bar\nu} - \varphi^{\mu}{}_{\bar\nu}\, \bar\varphi^{\bar\lambda}{}_{\mu} \right) \boxed{\bar\partial_{\bar\lambda} \bar\zeta^{\bar\alpha}_{j}}\, d\bar z^{\bar\nu} \frac{\partial f}{\partial \bar\zeta^{\alpha}_{j}}$$

$\det \left(\delta^{\bar\lambda}{}_{\bar\nu} - \varphi^{\mu}{}_{\bar\nu}\, \bar\varphi^{\bar\lambda}{}_{\mu} \right) \neq 0$ for sufficiently small t

$(\varphi(0) = 0)$

$\det (\partial_\beta \zeta^{\alpha}_{j}) \neq 0 \Longrightarrow \det (\bar\partial_{\bar\beta} \bar\zeta^{\bar\alpha}_{j}) \neq 0 \quad t \in \Delta$

$\therefore \quad (\bar\partial - \varphi(t)) f = 0 \Longleftrightarrow \dfrac{\partial f}{\partial \bar\zeta^{\alpha}_{j}} = 0$ for small t //

Hence, (infinitesimal) deformation of complex structure is described by C^∞ vector valued $(0,1)$-form

$$\varphi(t) = \varphi^\lambda(z,t)\,\partial_\lambda = \varphi^\lambda{}_{\bar\mu}(z,t)\,d\bar z^{\bar\mu}\,\partial_\lambda$$

<u>Prop</u> $\varphi(t)$ satisfies $\bar\partial\varphi^\lambda(t) = \varphi^\mu(t)\wedge\partial_\mu\varphi^\lambda(t)$

$\uparrow$ $(0,2)$-form

$(\because)$ Recall $\bar\partial \zeta_j^\alpha = \varphi^\lambda\,\partial_\lambda \zeta_j^\alpha$.

Since $\bar\partial^2 = 0$, $\underbrace{\bar\partial(\varphi^\lambda\,\partial_\lambda\zeta_j^\alpha)}_{=0} = \underset{(0,2)}{(\bar\partial\varphi^\lambda)}\,\partial_\lambda\zeta_j^\alpha - \varphi^\mu\wedge\underset{(0,1)}{\bar\partial(\partial_\mu\zeta_j^\alpha)}$

$$\therefore\quad (\bar\partial\varphi^\lambda)\,\partial_\lambda\zeta_j^\alpha = \varphi^\lambda\,\bar\partial(\partial_\lambda\zeta_j^\alpha)$$

$$\bar\partial(\partial_\mu\zeta_j^\alpha) = (\partial_{\bar\nu}\,\partial_\mu\zeta_j^\alpha)\,d\bar z^{\bar\nu} = \partial_\mu(\varphi^\lambda{}_{\bar\nu}\,\partial_\lambda\zeta_j^\alpha)\,d\bar z^{\bar\nu}$$

$$= (\partial_\mu\varphi^\lambda{}_{\bar\nu}\,\partial_\lambda\zeta_j^\alpha + \varphi^\lambda{}_{\bar\nu}\,\partial_\mu\partial_\lambda\zeta_j^\alpha)\,d\bar z^{\bar\nu}$$

$$\therefore \quad \varphi^\mu \wedge \bar\partial \partial_\mu \zeta^\alpha_j = (\varphi^\mu \wedge \partial_\mu \varphi^\lambda)\, \partial_\lambda \zeta^\alpha_j + \underbrace{\varphi^\mu \wedge \varphi^\lambda \; \partial_\mu \partial_\lambda \zeta^\alpha_j}_{=0} \quad \because$$

Hence $\quad (\bar\partial \varphi^\lambda)\, \partial_\lambda \zeta^\alpha_j = (\varphi^\mu \wedge \partial_\mu \varphi^\lambda)\, \partial_\lambda \zeta^\alpha_j$

$$\det(\partial_\lambda \zeta^\alpha_j) \neq 0 \implies \bar\partial \varphi^\lambda = \varphi^\mu \wedge \partial_\mu \varphi^\lambda \quad //$$

We define a bracket of vector valued $(0,p)$ form φ and $(0,q)$ form ψ

by $\quad [\varphi, \psi] = (\varphi^\mu \wedge \partial_\mu \psi^\lambda - (-1)^{p \cdot q}\, \psi^\mu \wedge \partial_\mu \varphi^\lambda)\, \partial_\lambda$

vector valued $(0, p+q)$ form //

Then, we can write

$$\bar\partial \varphi(t) = \frac{1}{2} [\varphi(t), \varphi(t)]$$

Kodaira - Spencer equation //

cf BCOV convention $\quad -\varphi \rightsquigarrow +A \qquad \bar\partial A + \frac{1}{2}[A, A] = 0$