Construction of (0-form) observables (Q-cohomology classes)

A-twist case

$$\omega = \omega_{I_1\cdots I_p;\bar J_1\cdots \bar J_q}(z,\bar z)\, dz^{I_1}\wedge\cdots\wedge dz^{I_p}\wedge d\bar z^{\bar J_1}\wedge\cdots\wedge d\bar z^{\bar J_q}$$

(p,q)-form on M

$$\mathcal{O}_\omega := \omega_{I_1\cdots I_p \bar J_1\cdots \bar J_q}(\phi,\bar\phi)\,\chi^{I_1}\cdots\chi^{I_p}\,\bar\chi^{\bar J_1}\cdots\bar\chi^{\bar J_q}$$

$$Q_A = \bar Q_+ + Q_-$$

$$[\bar Q_+, \bar\phi^{\bar I}] = \psi_+^{\bar I} \Rightarrow \bar\chi^{\bar I}, \quad [Q_-, \phi^I] = \psi_-^I \Rightarrow \chi^I,$$

$$Q_- \longleftrightarrow \partial, \quad \bar Q_+ \longleftrightarrow \bar\partial$$

$$[Q_A, \mathcal{O}_\omega] = \mathcal{O}_{(\partial+\bar\partial)\omega}$$

$$= \mathcal{O}_{d\omega}$$

Q_A-cohomology $\Leftrightarrow$ de Rham cohomology of M

B-twist case

$$F = F_{\bar{I}_1 \cdots \bar{I}_p}{}^{J_1 \cdots J_q}(z, \bar{z})\, d\bar{z}^{I_1} \wedge \cdots \wedge d\bar{z}^{I_p} \otimes \frac{\partial}{\partial z^{J_1}} \wedge \cdots \wedge \frac{\partial}{\partial z^{J_q}}$$

q- polyvector field valued $(0, p)$-form on M

$$\mathcal{O}_F = F_{\bar{I}_1 \cdots \bar{I}_p}{}^{J_1 \cdots J_q}(\phi, \bar{\phi})\, \bar{\eta}^{\bar{I}_1} \cdots \bar{\eta}^{\bar{I}_p}\, \theta_{J_1} \cdots \theta_{J_q}$$

$$Q_B = \bar{Q}_+ + \bar{Q}_-$$

$$[\bar{Q}_\pm, \phi^I] = 0, \quad [\bar{Q}_\pm, \bar{\phi}^{\bar{I}}] = \frac{1}{2}(\bar{\eta}^{\bar{I}} \pm g^{J\bar{I}}\theta_J),$$

$$[Q_B, \mathcal{O}_F] = \mathcal{O}_{\bar{\partial} F} \qquad (\delta\theta = 0)$$

Q_B-cohomology $\Longleftrightarrow$ Dolbeault cohomology $H^{(0,*)}_{\bar{\partial}}(M, \wedge^q TM)$

Selection rule for correlation functions

$\langle \mathcal{O}_1, \mathcal{O}_2 \cdots \mathcal{O}_5 \rangle_{\Sigma_g} \neq 0$: # of fermionic zero modes should match. in the path integral.

Index theorem (GRR theorem) index for $\bar{\partial}$ operator coupled with $\phi^*(TM)$

$$\dim_{\mathbb{C}} H^0(\phi^* TM) - \dim_{\mathbb{C}} H^1(\phi^* TM) = (1-g)\dim_{\mathbb{C}} M + \int_{\Sigma_g} \phi^*(c_1(M))$$

$\dim_{\mathbb{C}} H^0(\phi^* TM)$ = # of zero modes of scalars = "ghost"

$\dim_{\mathbb{C}} H^1(\phi^* TM)$ = # of zero modes of one-forms = "anti-ghost" (after twist)

"ghost"	"anti-ghost"	
$(\chi^I, \bar{\chi}^{\bar{I}})$	$(\rho_z{}^{\bar{I}}, \rho_{\bar{z}}{}^I)$	A-model
$(\eta^{\bar{I}}, \theta_I)$	$(\rho_z{}^I, \rho_{\bar{z}}{}^I)$	B-model

Note that the observables involve no anti-ghost!

A-model $\quad \mathcal{O}_\omega$ has ghost number $(p+q)$ $\quad \omega : (p,q)$-form

B-model $\quad \mathcal{O}_F$ has ghost number $(p+q)$

$F \in PV^{q,p}(M)$ $\quad$ q-polyvector valued $(0,p)$-form

M : Calabi-Yau $\Rightarrow$ $c_1(M) = 0$

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle \neq 0 \iff \begin{array}{l} \sum_{i=1}^{s} p_i = \sum_{i=1}^{s} q_i \quad \sim U(1)_V \text{ symmetry} \\ \sum_{i=1}^{s} (p_i + q_i) = 2 \dim_{\mathbb{C}} M \, (1-g) \end{array}$$

$g > 1$: the selection rule is never satisfied

$\longrightarrow$ al the correlation functions vanish

$g = 1$: Only the partition fn $\langle 1 \rangle$ is non-vanishing

$\longrightarrow$ elliptic genus

World sheet formulation of topological string

$=$ coupling of twisted theory to 2D gravity

We can define coupling of twisted theory to gravity by integrating correlation functions of observables over the moduli space of Riemann surface.

(i.e we employ the same prescription as bosonic strings)

$$T_{zz}(z) = \{ Q_{BRST} , b_{zz}(z) \}$$ similarly for $\bar{z}\bar{z}$ - sector

$g > 1$ $\quad \dim_{\mathbb{C}} M_g = 3g - 3$ $\quad$ Beltrani differential $(\mu_K)_{\bar{z}}^{z}$

(basis of TM_g)

$$(b \cdot \mu_K) = \int_{\Sigma_g} d^2 z \; b_{zz} (\mu_K)_{\bar{z}}^{z}$$

genus g amplitude $\quad F_g = \int_{M_g} \left\langle \prod_{K=1}^{3g-3} (b \cdot \mu_K)(\bar{b} \cdot \bar{\mu}_K) \right\rangle$

Generating currents of $N=(2,2)$ SUSY

$$G_\pm \sim g_{I\bar{J}}\, \partial_\pm \bar{\phi}^{\bar{J}}\, \psi_\pm^I \qquad \bar{G}_\pm \sim g_{I\bar{J}}\, \bar{\chi}_\pm^{\bar{J}}\, \partial_\pm \phi^I$$

A-twist

$$\underbrace{G_+ \sim g_{I\bar{J}}\, \partial_+ \bar{\phi}^{\bar{J}}\, \rho_{\bar{z}}^I}_{G_{\bar{z}\bar{z}}},\quad G_- \sim g_{I\bar{J}}\, \partial_- \bar{\phi}^{\bar{J}}\, \chi^I,\quad \bar{G}_+ \sim g_{I\bar{J}}\, \bar{\chi}^{\bar{J}}\, \partial_+ \phi^I,\quad \underbrace{\bar{G}_- \sim g_{I\bar{J}}\, \rho_z^{\bar{J}}\, \partial_- \phi^I}_{G_{zz}}$$

B-twist

$$\underbrace{G_+ \sim g_{I\bar{J}}\, \partial_+ \bar{\phi}^{\bar{J}}\, \rho_{\bar{z}}^I}_{G_{\bar{z}\bar{z}}},\quad \underbrace{G_- \sim g_{I\bar{J}}\, \partial_- \bar{\phi}^{\bar{J}}\, \rho_z^I}_{G_{zz}},\quad \bar{G}_+ \sim g_{I\bar{J}}\, \bar{\chi}_1^{\bar{J}}\, \partial_+ \phi^I,\quad \bar{G}_- \sim g_{I\bar{J}}\, \bar{\chi}_2^{\bar{J}}\, \partial_- \phi^I$$

Replace & with G to define to define F_g of top. str.

Recall before the coupling of 2D gravity the selection rule was

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_{\Sigma_g} \neq 0 \iff \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = \dim_{\mathbb{C}} M \, (1-g)$$

$c_1(M) = 0$

After the coupling of 2D gravity due to $(G \cdot \mu_k)$ insersion

the selection rule becomes $\displaystyle \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = (\dim_{\mathbb{C}} M - 3)(1-g)$

In particular when $\dim_{\mathbb{C}} M = 3$, the selection rule is indep. of genus g

$\longrightarrow$ makes sense of "all genus topological string amplitude"

$$Z = \exp F \qquad F = \sum_{g=0}^{\infty} \hbar^{2g-2} F_g$$

B-model $\quad F_g$ satisfies the holomorphic anomaly eq.

To obtain the observables with ghost number, we use the descent eq.

$$[Q, \mathcal{O}^{(0)}] = 0$$
$$d\mathcal{O}^{(0)} = [Q, \mathcal{O}^{(1)}]$$
$$d\mathcal{O}^{(1)} = [Q, \mathcal{O}^{(2)}]$$
$$d\mathcal{O}^{(2)} = 0$$

Recall for B-model we have seen

$$\partial_\pm \mathcal{O} \sim \{Q_B, [\underbrace{Q_\pm}_{G}, \mathcal{O}]\}$$

$\mathcal{O}^{(0)}$: ghost number two $\longrightarrow$ $\mathcal{O}^{(2)}$: (1,1) form with ghost number zero.

cf bosonic string case $c\bar{c}V \longrightarrow V$: phisical vertex op.

We can deform the topological action by $\int_{\Sigma_g} \mathcal{O}^{(2)}$

$$S(t) = S^{top} + \sum_{i=1}^{n} t_i \int_{\Sigma_g} \mathcal{O}_i^{(2)} \qquad \begin{cases} \text{A-model} & n = \dim H^{1,1}(M) \\ \text{B-model} & n = \dim H^{2,1}(M) \end{cases}$$

$F_g(t)$: the generating function of the topological correlation functions

$$C^{(g)}_{i_1 \cdots i_n} = \int_{M_g} \left\langle \int_{\Sigma_g} \mathcal{O}^{(2)}_{i_1} \cdots \int_{\Sigma_g} \mathcal{O}^{(2)}_{i_n} \prod_{k=1}^{3g-3} (G \cdot \mu_k)(\bar{G} \cdot \bar{\mu}_k) \right\rangle$$

$$Z(t) = \exp\left(\sum_{g=0}^{\infty} \hbar^{2g-2} F_g(t) \right)$$

All genus partition fn. of topological strings.

[Notes]

When $\dim_{\mathbb{C}} M \neq 3$, or $c_1(M) \neq 0$

we need other operator insersion, called "gravitational descendants"

associated with cohomology classes of M_g

↑ M-M-M class

Appendix Relevant index theorem Hori et al §3.5 ~ 6

E : a complex line bundle of rank r over M

$F = dA + A \wedge A$ the curvature 2-form of a connection A on E

total Chern class of E

$$c(E) = \det\left(1 + \frac{i}{2\pi} F\right) = 1 + c_1(E) + c_2(E) \cdots$$

$c_k(E) \in H^{2k}(M, \mathbb{R})$ c_k is a closed $2k$ form by Bianchi id.

$c_1(E) = \frac{i}{2\pi} \mathrm{Tr}\, F$ is also the 1st Chern class of the determinant line bundle $\det E = \Lambda^r E$.

M: Kähler $c_1(M) = c_1(TM)$ TM with Levi-Civita connection

$$c_1(M) = \frac{i}{2\pi} \mathrm{Tr}\, R^{\leftarrow \text{Ricci form}} = 0 \iff M: \text{Calabi-Yau}.$$

In general Kähler manifold has $U(n) \subset SO(2n)$ holonomy.

holom. anti holom subspaces are preserved under parallel transport.

Tr R represents $U(1)$ piece of $U(n)$ holonomy

$\Longrightarrow$ Calabi-Yau manifold has $SU(n)$ holonomy.

$c(M) = 0 \Rightarrow$ The det line bundle of TM is trivial.

$\Rightarrow$ The canonical line bundle $K_M = (\det TM)^{-1}$ is also trivial.

$\Rightarrow$ K_M has non-vanishing global section $\Rightarrow$ holom. (3,0)-form Ω_M

$c(E) \Rightarrow \prod_{i=1}^{r} (1 + x_i)$ $\quad E \sim L_1 \oplus L_2 \oplus \cdots \oplus L_r$

$c_k(E)$: the k-th elementary symmetric polynomial $e_k(x)$ // $\quad x_i = c_1(L_i)$ $\quad$ (write exact sequence)

We define the Chern character by $\mathrm{ch}(E) = \sum_{i=1}^{r} e^{x_i}$

e^{x_i} : the $U(1)$ character of L_i

$E \curvearrowleft U(n) \longrightarrow U(1)^n \curvearrowright L_1 \oplus \cdots \oplus L_r$

$$\mathrm{ch}(E) = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots$$

(cf the power sum $p_k(x)$ as a polynomial in $e_k(x)$)

The Todd class ($\sim$ index of $\bar{\partial}$) is defined by

$$\mathrm{td}(E) = \prod_{i=1}^{r} \frac{x_i}{1 - e^{-x_i}} = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \cdots$$

The Euler number

$$\chi(E) = \sum_k (-1)^k \dim H^k(E) = \int_M \mathrm{ch}(E) \wedge \mathrm{td}(TX)$$

Σ_g : Riemann surface with genus g $\qquad T\Sigma_g$: rank 1 $\leftarrow$ holom. tangent

Grothendieck - Riemann - Roch th $\leftarrow$ index th.

$$\dim H^0(T\Sigma_g) - \dim H^1(T\Sigma_g) = \int_{\Sigma_g} ch(T\Sigma_g) \wedge td(T\Sigma_g)$$

$$= \int_{\Sigma_g} (1 + c_1(T\Sigma_g))(1 + \frac{1}{2} c_1(T\Sigma_g)) = \frac{3}{2} \int_{\Sigma_g} c_1(TX)$$

$$\int_{\Sigma_g} c_1(TX) = \int_{\Sigma_g} R \sqrt{h}\, d^2 z = 2(1-g) \qquad = 3(1-g)$$

M_g : the moduli sp. of Σ_g $\qquad \dim_{\mathbb{C}} M_g = \dim H^1(T\Sigma_g)$

$g > 1$ $\quad \dim H^0(T\Sigma_g) = 0$ $\qquad \dim M_g = 3g - 3 \quad (g > 1)$

$g = 1$ $\quad \dim H^0(T\Sigma_g) = 1$ $\qquad = 1 \quad (g = 1)$

$g = 0$ $\quad \dim H^0(T\Sigma_g) = 3 \leftarrow SL(2, \mathbb{C})$ $\qquad = 0 \quad (g = 0)$

G-R-R th. for $E = \phi^*(TM)$

$$\dim H^0(\phi^* TM) - \dim H^1(\phi^* TM) = \int_{\Sigma_g} ch(\phi^* TM) \wedge td(T\Sigma_g)$$

\# of zero modes of scalars

\# of zero modes of one-forms

$$= \int_{\Sigma_g} (\mathrm{rank}\, TM + \phi^*(c_1(M)))\left(1 + \frac{1}{2} c_1(T\Sigma_g)\right)$$

$$= -\int_{\Sigma_g} \phi^*(c_1(K_M)) + \frac{1}{2} \dim_{\mathbb{C}} M \int_{\Sigma_g} c_1(T\Sigma_g)$$

$$= (1-g) \dim_{\mathbb{C}} M - \deg(\phi^* K_M)$$

When K_M: trivial $\dim_{\mathbb{C}} M = 3$ (M is a Calabi-Yau 3-fold)

$$\dim H^0(\phi^* TM) - \dim H^1(\phi^* TM) + \dim \mathcal{M}_g = 0 \quad //$$