Chiral ring and twisting

$$Q_A := \bar{Q}_+ + Q_- , \qquad Q_B = \bar{Q}_+ + \bar{Q}_-$$

$Q_A^2 = Q_B^2 = 0$ There are four choices of a pair of mutually anti-com. SUSY gen.

$\Longrightarrow$ Q_A, Q_B, $\bar{Q}_A$, $\bar{Q}_B$ (Witten's origical choice was $\bar{Q}_A$ instead of Q_A)

<u>Def</u> An operator $\mathcal{O}$ is called chiral $\Longleftrightarrow$ $[Q_B, \mathcal{O}] = 0$

Similarly $\mathcal{O}$ is called twisted chiral $\Longleftrightarrow$ $[Q_A, \mathcal{O}] = 0$

Rmk The name comes from the fact that the lowest component ϕ of a chiral superfield Φ satisfies $[\overline{Q}_\pm, \phi] = 0$.

Similarly the lowest component ϑ of a twisted chiral superfield U satisfies $[\overline{Q}_+, \vartheta] = [Q_-, \vartheta] = 0$

Prop If $\mathcal{O}$ is (twisted) chiral operator the world sheet derivatives of $\mathcal{O}$ is Q_B (or Q_A)-exact.

(∵) $\partial_+ \mathcal{O} \sim [H+P, \mathcal{O}] = [\{Q_+, \overline{Q}_+\}, \mathcal{O}]$

Jacobi $= \{[Q_+, \mathcal{O}], \overline{Q}_+\} + \{Q_+, [\overline{Q}_+, \mathcal{O}]\}$

$[Q_B, \mathcal{O}]=0$ $= \{[Q_+, \mathcal{O}], \overline{Q}_+\} - \{Q_+, [\overline{Q}_-, \mathcal{O}]\}$

$\{Q_+, \overline{Q}_-\}=0$ $= \{\overline{Q}_+, [Q_+, \mathcal{O}]\} + \{\overline{Q}_-, [Q_+, \mathcal{O}]\}$

$= \{Q_B, [Q_+, \mathcal{O}]\}$

$$
\begin{aligned}
\partial_- \mathcal{O} \sim [H-P, \mathcal{O}] &= [\{Q_-, \overline{Q}_-\}, \mathcal{O}] \\
&= \{[Q_-, \mathcal{O}], \overline{Q}_-\} + \{Q_- [\overline{Q}_-, \mathcal{O}]\} \\
&= \{\overline{Q}_-, [Q_-, \mathcal{O}]\} - \{Q_- [\overline{Q}_+, \mathcal{O}]\} \\
&= \{\overline{Q}_-, [Q_-, \mathcal{O}]\} + \{\overline{Q}_-, [Q_-, \mathcal{O}]\} \\
&= \{Q_B, [Q_-, \mathcal{O}]\}
\end{aligned}
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Hence, Q_B-cohomology (Q_A-cohomology) class of a (twisted) chiral op. does not change under the translation on the worldsheet.

We can write $\mathcal{O} = [\mathcal{O}(z, \bar{z})] \leftarrow$ cohomology class.

<u>Prop</u> If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two (twisted) chiral operators then the product $\mathcal{O}_1 \cdot \mathcal{O}_2$ is also a (twisted) chiral op.

The Q-cohomology classes of (twisted) chiral operators form a ring. We call it chiral ring or twisted chiral ring.

$$\{\phi_i\}_{i\in I} \qquad \phi_i \cdot \phi_j = \phi_k C_{ij}{}^k + [Q, \Lambda]$$

(The singularity of $\lim_{w\to z} \phi_i(w)\phi_j(z)$ is Q-exact.)

Associativity of the operator product implies $C_{il}{}^m C_{jk}^l = C_{lk}^m C_{ij}^l$

The structure constants $C_{ij}{}^k$ of the chiral ring are captured by three point functions of topological theory on the Riemann sphere (genus 0 curve).

Up to now we assumed that our world sheet Σ is "flat".

In principle there is no obstruction to formulating susy theory on a curved Riemann surface Σ. by taking care of spin structure on Σ.

However, the action is not necessarily supersymmetric, since

$$\delta S = \int_\Sigma \sqrt{h}\, d^2x \left(\nabla_\mu \varepsilon_+ G_-^\mu - \nabla_\mu \varepsilon_- G_+^\mu - \nabla_\mu \bar{\varepsilon}_+ \bar{G}_-^\mu + \nabla_\mu \varepsilon_- \bar{G}_+^\mu \right)$$

$\varepsilon_\pm$, $\bar{\varepsilon}_\pm$: spinors on Σ that parametrize SUSY transformations

$G_\pm^\mu$, $\bar{G}_\pm^\mu$: Neother currents for SUSY.

$\delta S = 0$ only for a covariantly const. spinors

$$\nabla_\mu \varepsilon_\pm = \nabla_\mu \bar{\varepsilon}_\pm = 0 \qquad (\exists \text{ covariantly const spinor} \Longleftrightarrow \text{Ricci-flat})$$

Hence, $\delta S \neq 0$ unless $\Sigma = T^2$ (torus)

One can make a modefication of the theory, called twisting.
to preserve (half of) SUSY. $\longrightarrow$ Topological sigma model.

Topological theory coincides with the original theory on flat space-time, but they are different on curved space-time.
(space-time = world sheet)

We make use of R symmetry of $N=(2,2)$ theory which acts on the superfield $\mathcal{F}(x, \theta^{\pm}, \bar{\theta}^{\pm})$ as follows

$$U(1)_V : e^{i\alpha F_V} \cdot \mathcal{F} = e^{i\alpha q_V} \mathcal{F}(x, e^{-i\alpha}\theta_{\pm}, e^{+i\alpha}\bar{\theta}^{\pm})$$

← R charge of $\mathcal{F}$

$$U(1)_A : e^{i\beta F_A} \cdot \mathcal{F} = e^{i\beta q_A} \mathcal{F}(x, e^{\mp i\beta}\theta_{\pm}, e^{\pm i\beta}\bar{\theta}_{\pm})$$

$\int d^2x\, d^4\theta\, K(\Phi^I, \bar{\Phi}^{\bar{J}})$ has $U(1)_V \times U(1)_A$ symmetry

for any R-charge assignment of Φ^I.

In the following we assume Φ^I has vanishing R-charge.

$U(1)_V$: $\delta\psi_\pm^I = -i\alpha\psi_\pm^I$

infinitesimal trf.

$U(1)_A$: $\delta\psi_\pm^I = \mp i\alpha\psi_\pm^I$

We will consider the Euclidean version of the theory by $x^0 = -ix^2$

then $z = x^1 + ix^2$ is a complex coordinate of Σ.

After the Wick rotation, the Lorentz group $SO(1,1)$ becomes $SO(2)_E = U(1)_E$

$SO($

From the viewpoint of algebra, the twisting is done by "redefing"

$$\widetilde{U(1)}_E = U(1)_E \times U(1)_R$$ where we choose

$U(1)_R = U(1)_V$ for A-twist and $U(1)_R = U(1)_A$ for B-twist

At Lagrangican level, this is regarded as a gauging of R-sym. by introducing a coupling of the Neother current with the spin connection (= the gauge field for $U(1)_E$) on Σ.

On a curved world sheet the covariant derivative are

$$D_{\bar{z}} \psi_+^I = \partial_{\bar{z}} \psi_+^I - \frac{i}{2} \omega_{\bar{z}} \psi_+^I + \Gamma^I_{KL} \partial_{\bar{z}} \phi^K \psi_+^L$$

spin connection

$$D_z \psi_+^I = \partial_z \psi_+^I + \frac{i}{2} \omega_z \psi_+^I + \Gamma^I_{KL} \partial_z \phi^K \psi_+^L$$

Noether currents of $U(1)_V$ and $U(1)_A$

$U(1)_V$ $\quad j_V^z = g_{I\bar{J}}(\phi)\,\overline{\psi}_+^{\bar{J}}\,\psi_+^I\,, \quad j_V^{\bar{z}} = g_{I\bar{J}}(\phi)\,\overline{\psi}_-^{\bar{J}}\,\psi_-^I$

$U(1)_A$ $\quad j_A^z = g_{I\bar{J}}(\phi)\,\overline{\psi}_+^{\bar{J}}\psi_+^I\,, \quad j_A^{\bar{z}} = -g_{I\bar{J}}(\phi)\,\overline{\psi}_-^{\bar{J}}\,\psi_-^I$

the kinetic term of fermions (after the Wick rotation)

$$S_{f,kin} = \int_{\Sigma_g} d^2z\, g_{I\bar{J}}(\phi)\left[\overline{\psi}_+^{\bar{J}}\, D_{\bar{z}}\,\psi_+^I + \overline{\psi}_-^{\bar{J}}\, D_z\,\psi_-^I\right]$$

ω-dependent term $\sim \frac{i}{2}\int_{\Sigma_g} d^2z\, g_{I\bar{J}}(\phi)\left[\overline{\psi}_+^{\bar{J}}\,\omega_{\bar{z}}\,\psi_+^I - \overline{\psi}_-^{\bar{J}}\,\omega_z\,\psi_-^I\right]$

$$S_{f,kin} + \frac{i}{2}\int_{\Sigma_g} d^2z\,\omega_\mu\, j^\mu{}_V \sim i\int_{\Sigma_g} d^2z\, g_{I\bar{J}}(\phi)\,\overline{\psi}_+^{\bar{J}}\,\omega_{\bar{z}}\,\psi_+^I$$

$$S_{f,kin} + \frac{i}{2}\int_{\Sigma_g} d^2z\,\omega_\mu\, j^\mu{}_A \sim i\int_{\Sigma_g} d^2z\, g_{I\bar{J}}\left(\overline{\psi}_+^{\bar{J}}\,\omega_{\bar{z}}\,\psi_+^I - \overline{\psi}_-^{\bar{J}}\,\omega_z\,\psi_-^I\right)$$

Hence

A-twist	$\psi_+^I \longrightarrow$	(0, 1)-form	$\overline{\psi}_+^{\bar{J}} \longrightarrow$	scalar
	$\psi_-^I \longrightarrow$	scalar	$\overline{\psi}_-^{\bar{J}} \longrightarrow$	(1, 0)-form
B-twist	$\psi_+^I \longrightarrow$	(0, 1)-form	$\overline{\psi}_+^{\bar{J}} \longrightarrow$	scalar
	$\psi_-^I \longrightarrow$	(1, 0)-form	$\overline{\psi}_-^{\bar{J}} \longrightarrow$	scalar

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	$U(1)_E$	$U(1)_V$	$U(1)_A$	A-twist	B-twist
ψ_+	$-1/2$	$-1/2$	$-1/2$	-1	-1
ψ_-	$+1/2$	$-1/2$	$+1/2$	0	1
$\overline{\psi}_+$	$-1/2$	$+1/2$	$+1/2$	0	0
$\overline{\psi}_-$	$+1/2$	$+1/2$	$-1/2$	1	0

N = (2, 2) SUSY transformation

$$\delta\phi^{I} = \varepsilon_{+}\psi_{-}^{I} - \varepsilon_{-}\psi_{+}^{I}, \quad \delta\bar{\phi}^{\bar{I}} = -\bar{\varepsilon}_{+}\bar{\psi}_{-}^{\bar{I}} + \bar{\varepsilon}_{-}\bar{\psi}_{+}^{\bar{I}}$$

$$\delta\psi_{+}^{I} = 2i\,\bar{\varepsilon}_{-}\partial_{+}\phi^{I} + \varepsilon_{+}F^{I} \quad \delta\bar{\psi}_{+}^{\bar{I}} = -2i\,\varepsilon_{-}\partial_{+}\bar{\phi}^{\bar{I}} + \bar{\varepsilon}_{+}\bar{F}^{\bar{I}}$$

$$\delta\psi_{-}^{I} = -2i\,\bar{\varepsilon}_{+}\partial_{-}\phi^{I} + \varepsilon_{-}F^{I} \quad \delta\bar{\psi}_{-}^{\bar{I}} = 2i\,\varepsilon_{+}\partial_{-}\bar{\phi}^{\bar{I}} + \bar{\varepsilon}_{-}\bar{F}^{\bar{I}}$$

On shell condition $F^{I} = \Gamma^{I}_{JK}\,\psi_{+}^{J}\psi_{-}^{K}$, $\bar{F}^{\bar{I}} = \Gamma^{\bar{I}}_{\bar{J}\bar{K}}\,\bar{\psi}_{-}^{\bar{J}}\bar{\psi}_{+}^{\bar{K}}$ ordering !

$$\delta = \varepsilon_{+}Q_{-} - \varepsilon_{-}Q_{+} - \bar{\varepsilon}_{+}\bar{Q}_{-} + \bar{\varepsilon}_{-}\bar{Q}_{+}$$

Neother currents

$$G^{0}_{\pm} = 2g_{I\bar{J}}\,\partial_{\pm}\bar{\phi}^{\bar{J}}\psi^{I}_{\pm}, \quad G^{1}_{\pm} = \mp 2g_{I\bar{J}}\,\partial_{\pm}\bar{\phi}^{\bar{J}}\psi^{I}_{\pm}$$

$$\bar{G}^{0}_{\pm} = 2g_{I\bar{J}}\,\bar{\psi}^{\bar{J}}_{\pm}\partial_{\pm}\phi^{I} \quad G^{1}_{\pm} = \mp 2g_{I\bar{J}}\,\bar{\psi}^{\bar{J}}_{\pm}\partial_{\pm}\phi^{I}$$

A-twist

Change the notation according to $U(1)'_E$-spin

$$\chi^I = \psi_-^I \,,\quad \bar{\chi}^{\bar{I}} = \bar{\psi}_+^{\bar{I}} \,,\quad \rho_z^{\bar{I}} = \bar{\psi}_-^{\bar{I}} \,,\quad \rho_{\bar{z}}^I = \psi_+^I$$

$Q_A = \bar{Q}_+ + Q_-$ becomes a scalar supercharge "BRST"-op.

Q_A-variation (BRST transformation) $\begin{cases} \varepsilon_- = \bar{\varepsilon}_+ = 0 \\ \varepsilon_+ = \bar{\varepsilon}_- \end{cases}$

$$\delta\phi^I = \varepsilon\chi^I \,,\quad \delta\bar{\phi}^{\bar{I}} = \varepsilon\bar{\chi}^{\bar{I}}$$

$$\delta\rho_z^{\bar{I}} = \varepsilon\left(2i\partial_z\bar{\phi}^{\bar{I}} + \bar{F}_z^{\bar{I}}\right) = \varepsilon\left(2i\partial_z\bar{\phi}^{\bar{I}} + \Gamma^{\bar{I}}_{\bar{J}\bar{K}}\,\bar{\chi}^{\bar{J}}\rho_z^{\bar{I}}\right)$$

$$\delta\rho_{\bar{z}}^I = \varepsilon\left(-2i\partial_{\bar{z}}\phi^I + F_{\bar{z}}^I\right) = \varepsilon\left(-2i\partial_{\bar{z}}\phi^I + \Gamma^I_{JK}\,\chi^J\rho_{\bar{z}}^K\right)$$

(on-shell)

Twisted chiral ring $=$ BRST cohomology of A-twisted model

B-twist

Change of the notation

$$\rho_z^I = \cancel{2}\,\tfrac{1}{2}\psi_-^I, \quad \rho_{\bar z}^I = \cancel{2}\,\tfrac{1}{2}\psi_+^I, \quad \chi_1^{\bar I} = \bar\psi_+^{\bar I}, \quad \chi_2^{\bar I} = \bar\psi_-^{\bar I}$$

more convenient to take

$$\bar\eta^{\bar I} = \chi_1^{\bar I} + \chi_2^{\bar I}, \quad \underline{\theta_I = g_{I\bar J}(\chi_1^{\bar J} - \chi_2^{\bar J})}$$

$Q_B = \bar Q_+ + \bar Q_-$ becomes a scalar supercharge

Q_B-variation $\quad (\varepsilon_+ = \varepsilon_- = 0, \quad \bar\varepsilon_- = -\bar\varepsilon_+ = \varepsilon)$

$$\delta\phi^I = 0, \quad \delta\bar\phi^{\bar I} = \bar\eta^{\bar I}, \quad \delta\rho_z^I = i\,\partial_z\phi^I, \quad \delta\rho_{\bar z}^I = i\,\partial_{\bar z}\phi^I$$

$$\delta\theta_I = \delta g_{I\bar J}(\chi_1^{\bar J} - \chi_2^{\bar J}) - g_{I\bar J}(2\bar F^{\bar J}) \qquad \bar F^{\bar J} = \Gamma^{\bar J}_{\bar K\bar L}\chi_2^{\bar K}\chi_1^{\bar L}$$

$$= g_{I\bar J}\left(\Gamma^{\bar J}_{\bar K\bar L}\,\eta^{\bar K}(\chi_1^{\bar L} - \chi_2^{\bar L}) - 2\bar F^{\bar J}\right) = 0 \quad \text{(on shell)}$$

Chiral ring = BRST cohomology of B-twisted model

In quantum theory R-sym has a chiral anomaly
due to fermion zero modes.

$U_V(1)$ has no anomaly (fermion zero modes balance)

$U_A(1)$ has anomaly proportional to $c_1(M)$

1st Chern class of M

(the difference of the number of zero modes
$\propto c_1(M)$

$\sim \mathrm{Tr}\left(\frac{i}{2\pi} R_{ij}\right)$

Ricci 2-form.

Hence, the B-model is well-defined only for
the Calabi-Yau target space.

canonical line bundle

$c_1(M) \longrightarrow$ M has $SU(n)$ holonomy $\longrightarrow K_M$ is trivial

$K_M^{-1} \sim \det TM$ $\longrightarrow$ $\exists\,\Omega$: non-vanishing hol. n-form