- $\mathcal{N} = (2, 2)$ SUSY in two dimensions

four SUSY's

two with positive and two with negative chirality

[Ref: Hori et. al. Chap. 12]

One of the systematic ways to obtain SUSY Lagrangian is to introduce superspace and superfields (cf: Wess-Bagger)

$\mathbb{R}^2$ with coordinates $x^0 = t$, $x^1 = s$

flat Minkowski metric $\eta_{00} = -1$, $\eta_{11} = 1$

four complex fermionic (Grassmannian) coordinates

$$\theta^{\pm}, \quad \bar{\theta}^{\pm} \qquad \text{with} \qquad (\theta^{\pm})^* = \bar{\theta}^{\pm}$$

complex conjugate

the index $\pm$ refers to the 2D spin (or the chirality)

the local Lorentz transformation

$$\begin{pmatrix} x^0 \\ x^1 \end{pmatrix}' = \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

acts on fermionic coordinates

as $\theta^{\pm} \longrightarrow e^{\pm\frac{\gamma}{2}}\,\theta^{\pm}$, $\quad \bar{\theta}^{\pm} \longrightarrow e^{\pm\frac{\gamma}{2}}\,\bar{\theta}^{\pm}$

- the fermionic coordinates anti-commute each other

$$\theta^\alpha\theta^\beta = -\theta^\beta\theta^\alpha, \quad \bar{\theta}^\alpha\bar{\theta}^\beta = -\bar{\theta}^\beta\bar{\theta}^\alpha, \quad \theta^\alpha\bar{\theta}^\beta = -\bar{\theta}^\beta\theta^\alpha$$

$\alpha, \beta = \pm$

- Under the complex conjugation

$$(\theta_1\theta_2)^* = \theta_2^*\theta_1^* = -\theta_1^*\theta_2^*, \quad \text{e.g. } (\theta^+\bar{\theta}^+)^* = \theta^+\bar{\theta}^+$$

$\theta^+\bar{\theta}^+$ is real.

The (2, 2) superspace is the space with coordinates

$$(x^0, x^1, \theta^\pm, \bar{\theta}^\pm)$$

The fields defined on the superspace are called superfields.

By the Taylor expansion in fermionic coordinates we obtain $2^4 = 16$ component fields;

$$\mathcal{F} = f_0(x^0, x^1) + \theta^+ f_+(x^0, x^1) + \theta^- f_-(x^0, x^1)$$
$$+ \bar{\theta}^+ f'_+(x^0, x^1) + \bar{\theta}^- f'_-(x^0, x^1) + \cdots$$

Let us introduce differential operators on the superspace

$$Q_\pm = \frac{\partial}{\partial \theta^\pm} + i \bar{\theta}^\pm \partial_\pm , \qquad \bar{Q}_\pm = -\frac{\partial}{\partial \bar{\theta}^\pm} - i \theta^\pm \partial_\pm$$

$$\partial_\pm = \frac{\partial}{\partial x^\pm} = \frac{1}{2}\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right) ; \qquad x^\pm = x^0 \pm x^1$$

$Q_\pm$, $\bar{Q}_\pm$ are generators of supertranslation (SUSY) ↳ on the superfields.

They satisfy $\{Q_\pm, \bar{Q}_\pm\} = -2i\partial_\pm$ $= P_\pm$ (translation)

with all other anti-commutation relations vanish.

Together with bosonic generators $P_\pm$, M, L

they form $N=(2,2)$ SUSY algebra, M: SO(2) rot, L: Lorentz

We can define another set of differential operators

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - i\bar{\theta}^\pm\partial_\pm, \quad \bar{D}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} + i\theta^\pm\partial_\pm$$

which anti-commute with $Q_\pm$ and $\bar{Q}_\pm$ completely.

$D_\pm$, $\bar{D}_\pm$ satisfy $\{D_\pm, \bar{D}_\pm\} = 2i\partial_\pm$

with all other anti-commutators vanish.

Since $D_\pm$, $\bar{D}_\pm$ anti-commute with all the generators of SUSY,

we can impose a constraint such as $D_+ \mathcal{F} = 0$.

" Ker D_+ is an invariant subspace of SUSY "

In general a superfield $\mathcal{F}$ gives a reducible representation.

<u>Def</u> Φ is a chiral superfield $\Longleftrightarrow$ $\bar{D}_\pm \Phi = 0$

U is a twisted chiral superfield $\Longleftrightarrow$ $\bar{D}_+ U = D_- U = 0$

The complex conjugate of Φ satisfies $D_\pm \bar{\Phi} = 0$ anti-chiral

The complex conjugate of U satisfies $D_+ \bar{U} = \bar{D}_- \bar{U} = 0$

twisted anti-chiral

There are four possibilities of choosing a pair of constraints.

(The differential operators that define constrants should anti-commute.)

<u>Prop</u> A general chiral superfield has the expansion

$$\Phi(x^\pm, \theta^\pm, \bar{\theta}^\pm) = \phi(y^\pm) + \theta^\alpha \psi_\alpha(y^\pm) + \theta^+ \theta^- F(y^\pm)$$

where $$y^\pm = x^\pm - i\theta^\pm \bar{\theta}^\pm$$

Similarly, a general twisted chiral field has the expansion

$$U(x^\pm, \theta^\pm, \bar{\theta}^\pm) = \vartheta(\tilde{y}^\pm) + \theta^+ \bar{\chi}_+(\tilde{y}^\pm) + \bar{\theta}^- \chi_-(\tilde{y}^\pm) + \theta^+ \bar{\theta}^- E(\tilde{y}^\pm)$$

where $$\tilde{y}^\pm = x^\pm \mp i\theta^\pm \bar{\theta}^\pm$$

$$(\tilde{y}^+ = y^+, \quad \tilde{y}^- = (y^-)^*)$$

(∵) Let us consider a coordinate change

$$(x^\pm, \theta^\pm, \bar\theta^\pm) \longrightarrow (y^\pm, \theta^\pm, \bar\theta^\pm)$$

$$\frac{\partial}{\partial X^\mu} \to \frac{\partial Y^\nu}{\partial X^\mu}\frac{\partial}{\partial Y^\nu} : \quad \frac{\partial}{\partial x^\pm} \to \frac{\partial}{\partial y^\pm}, \quad \frac{\partial}{\partial \theta^\pm} \longrightarrow -i\bar\theta^\pm \frac{\partial}{\partial y^\pm} + \frac{\partial}{\partial \theta^\pm}$$

$$\frac{\partial}{\partial \bar\theta^\pm} \longrightarrow +i\theta^\pm \frac{\partial}{\partial y^\pm} + \frac{\partial}{\partial \bar\theta^\pm}$$

$$\therefore \quad \bar D_\pm \longrightarrow -\frac{\partial}{\partial \bar\theta^\pm} - i\theta^\pm \frac{\partial}{\partial y^\pm} + i\theta^\pm \frac{\partial}{\partial y^\pm} = -\frac{\partial}{\partial \bar\theta^\pm}$$

Similarly under a coordinate change

$$(x^\pm, \theta^\pm, \bar\theta^\pm) \longrightarrow ((y^\pm)^*, \theta^\pm, \bar\theta^\pm)$$

$$\frac{\partial}{\partial x^\pm} \to \frac{\partial}{\partial y^\pm}, \quad \frac{\partial}{\partial \theta^\pm} \longrightarrow +i\bar\theta^\pm \frac{\partial}{\partial y^\pm} + \frac{\partial}{\partial \theta^\pm}$$

$$\frac{\partial}{\partial \bar\theta^\pm} \longrightarrow -i\theta^\pm \frac{\partial}{\partial y^\pm} + \frac{\partial}{\partial \bar\theta^\pm}, \quad D_\pm \to \frac{\partial}{\partial \theta^\pm}$$

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- 2 dim $N=(2,2)$ non-linear sigma model.

$$\begin{cases} \Phi^1, \cdots, \Phi^n & : \text{chiral superfields} \\ \bar{\Phi}^{\bar{1}}, \cdots, \bar{\Phi}^{\bar{n}} & : \text{anti-chiral superfields} \end{cases}$$

$K(\Phi^i, \bar{\Phi}^{\bar{j}})$ a real function (Kähler potential)

$$\mathcal{L} = \int d^4\theta \, K(\Phi^i, \bar{\Phi}^{\bar{j}}) = K(\Phi^i, \bar{\Phi}^{\bar{j}})\Big|_{\theta^+\bar{\theta}^+\theta^-\bar{\theta}^-}$$

"D-term"

$$\varphi = \Phi - \phi(x) = \theta^+\psi_+(x) + \theta^-\psi_-(x) - i\theta^+\bar{\theta}^+\partial_+\phi(x)$$
$$-i\theta^-\bar{\theta}^-\partial_-\phi(x) + \theta^+\theta^- F(x)$$
$$-i\theta^+\bar{\theta}^+\theta^-\partial_+\psi_-(x) - i\theta^-\bar{\theta}^-\theta^+\partial_-\psi_+(x) - \theta^+\bar{\theta}^+\theta^-\bar{\theta}^-\partial_+\partial_-\phi(x)$$

$$\bar{\varphi} = \bar{\Phi} - \bar{\phi}(x) = -\bar{\theta}^{+}\bar{\psi}_{+}(x) - \bar{\theta}^{-}\bar{\psi}_{-}(x) + i\theta^{+}\bar{\theta}^{+}\partial_{+}\bar{\phi}(x)$$
$$+ i\theta^{-}\bar{\theta}^{-}\partial_{-}\bar{\phi}(x) - \bar{\theta}^{+}\bar{\theta}^{-}\bar{F}(x)$$
$$- i\theta^{+}\bar{\theta}^{+}\bar{\theta}^{-}\partial_{+}\bar{\psi}(x) - i\theta^{-}\bar{\theta}^{-}\bar{\theta}^{+}\partial_{-}\bar{\psi}_{+}(x) - \theta^{+}\bar{\theta}^{+}\theta^{-}\bar{\theta}^{-}\partial_{+}\partial_{-}\bar{\phi}(x)$$

$$K(\Phi^{I}, \bar{\Phi}^{\bar{I}}) \sim \partial_{I}K(\phi)\varphi^{I} + \bar{\partial}_{\bar{I}}K(\phi)\bar{\varphi}^{\bar{I}}$$
$$+ \frac{1}{2}\left(\partial_{I}\partial_{J}K(\phi)\varphi^{I}\varphi^{J} + 2\partial_{I}\bar{\partial}_{\bar{J}}K(\phi)\varphi^{I}\bar{\varphi}^{\bar{J}} + \bar{\partial}_{\bar{I}}\bar{\partial}_{\bar{J}}K(\phi)\bar{\varphi}^{\bar{I}}\bar{\varphi}^{\bar{J}}\right)$$
$$+ \frac{3}{3!}\left(\partial_{I}\partial_{J}\bar{\partial}_{\bar{K}}K(\phi)\varphi^{I}\varphi^{J}\bar{\varphi}^{\bar{K}} + \partial_{I}\bar{\partial}_{\bar{J}}\bar{\partial}_{\bar{K}}K(\phi)\varphi^{I}\bar{\varphi}^{\bar{J}}\bar{\varphi}^{\bar{K}}\right)$$
$$+ \frac{6}{4!}\partial_{I}\partial_{J}\bar{\partial}_{\bar{K}}\bar{\partial}_{\bar{L}}K(\phi)\varphi^{I}\varphi^{J}\bar{\varphi}^{\bar{K}}\bar{\varphi}^{\bar{L}}$$

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linear terms

$$\partial_I K(\phi)\, \varphi^I + \bar{\partial}_{\bar{I}} K(\phi)\, \bar{\varphi}^{\bar{I}} \Big|_{\theta^+\bar{\theta}^+\theta^-\bar{\theta}^-}$$

$$= -\partial_I K\, \partial_+\partial_-\phi^I - \partial_{\bar{I}} K\, \partial_+\partial_-\bar{\phi}^{\bar{I}}$$

partial integration

$$= \partial_+\phi^J (\partial_I\partial_J K)\, \partial_-\phi^I + \partial_+\bar{\phi}^{\bar{J}} (\partial_I\bar{\partial}_{\bar{J}} K)\, \partial_-\phi^I$$

$$+ \partial_+\phi^J (\partial_J\bar{\partial}_{\bar{I}} K)\, \partial_-\bar{\phi}^{\bar{I}} + \partial_+\bar{\phi}^{\bar{J}} (\bar{\partial}_{\bar{I}}\bar{\partial}_{\bar{J}} K)\, \partial_-\bar{\phi}^{\bar{I}} \quad //$$

quadratic terms

$$\frac{1}{2}(\partial_I\partial_J K)\, \varphi^I\varphi^J \Big|_{\theta^+\bar{\theta}^+\theta^-\bar{\theta}^-} = -\frac{1}{2}\partial_I\partial_J K \left(\partial_+\phi^I\partial_-\phi^J + \partial_-\phi^I\partial_+\phi^J\right)$$

$$= -(\partial_I\partial_J K)\, \partial_+\phi^I\partial_-\phi^J$$

$$\frac{1}{2}(\bar{\partial}_{\bar{I}}\bar{\partial}_{\bar{J}} K)\, \bar{\varphi}^{\bar{I}}\bar{\varphi}^{\bar{J}} \Big|_{\theta^4}$$ cancels the 4-th $\}$ cancels the 1st term

$$\partial_I \partial_{\bar J} K(\phi)\, \varphi^I \bar\varphi^{\bar J} \Big|_{\theta^4}$$

$$= (\partial_I \partial_{\bar J} K)\Big[\partial_+ \phi^I \partial_- \bar\phi^{\bar J} + \partial_- \phi^I \partial_+ \bar\phi^{\bar J} + F^I \bar F^{\bar J}$$
$$+ i \psi_+^I \partial_- \bar\psi_+^{\bar J} + i \psi_-^I \partial_+ \bar\psi^{\bar J}$$
$$- i\, \partial_+ \psi_-^I \bar\psi_-^{\bar J} - i\, \partial_- \psi_+^I \bar\psi_+^{\bar J} \Big]$$

Together with linear terms ϕ - bilinear term is

$$2\, g_{I\bar J}(\phi, \bar\phi) \left(\partial_+ \phi^I \partial_- \bar\phi^{\bar J} + \partial_- \phi^I \partial_+ \bar\phi^{\bar J} \right)$$
$$= g_{I\bar J}(\phi, \bar\phi) \left(\frac{\partial \phi^I}{\partial x^0} \frac{\partial \bar\phi^{\bar J}}{\partial x^0} - \frac{\partial \phi^I}{\partial x^1} \frac{\partial \bar\phi^{\bar J}}{\partial x^1} \right)$$
$$= -g_{I\bar J}(\phi, \bar\phi)\, \partial_\mu \phi^I \partial^\mu \bar\phi^{\bar J}$$

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cubic terms

$$\frac{1}{2}(\partial_I\partial_J\bar\partial_{\bar L} K)\,\varphi^I\varphi^J\bar\varphi^{\bar L}\Big|_{\theta^4}$$

$$= (\partial_I\partial_J\bar\partial_{\bar L}K)\Big(-i\,\partial_+\phi^I\psi_-^J\bar\psi_-^{\bar L} - i\,\partial_-\phi^I\psi_+^J\bar\psi_+^{\bar L} + \frac{1}{2}(\psi_-^I\psi_+^J - \psi_+^I\psi_-^J)\,\bar F^{\bar L}\Big)$$

$$= g_{K\bar L}\,\Gamma^K_{IJ}\,(-i\,\partial_+\phi^I\psi_-^J\psi_-^{\bar L} - i\,\partial_-\phi^I\psi_+^J\bar\psi^{\bar L}) + g_{K\bar L}\,\Gamma^K_{IJ}\,\psi_-^I\psi_+^J\,\bar F^{\bar L}$$

Similarly

$$\frac{1}{2}(\partial_I\bar\partial_{\bar J}\bar\partial_{\bar L} K)\,\varphi^I\bar\varphi^{\bar J}\bar\varphi^{\bar L}\Big|_{\theta^4}$$

$$= g_{I\bar K}\,\Gamma^{\bar K}_{\bar J\bar L}\,(+i\,\partial_+\bar\phi^{\bar J}\psi_-^I\psi_-^{\bar L} + i\,\partial_-\bar\phi^{\bar J}\psi_+^I\psi_+^{\bar L}) + g_{I\bar K}\,\Gamma^{\bar K}_{\bar J\bar L}\,\bar\psi_+^{\bar J}\bar\psi_-^{\bar L}\,F^I$$

Together with fermion quadratic terms

$$g_{I\bar{J}}\left(i\psi_+^I(D_-\bar{\psi})^{\bar{J}}+i\psi_-^I(D_+\bar{\psi})^{\bar{J}}\right.$$
$$\left.-i(D_+\psi_-)^I\bar{\psi}_-^{\bar{J}}-i(D_-\psi_+)^I\bar{\psi}_+^{\bar{J}}\right)$$

where
$$\begin{cases}(D_\mu\psi_\pm)^I=\partial_\mu\psi_\pm^I+(\partial_\mu\phi^J)\Gamma^I_{JK}\psi_\pm^K\\(D_\mu\bar{\psi}_\pm)^{\bar{I}}=\partial_\mu\bar{\psi}_\pm^{\bar{I}}+(\partial_\mu\bar{\phi}^{\bar{J}})\Gamma^{\bar{I}}_{\bar{J}\bar{K}}\bar{\psi}_\pm^{\bar{K}}\end{cases}$$

<u>quartic term</u>

$$\frac{1}{4}(\partial_I\partial_J\bar{\partial}_{\bar{L}}\bar{\partial}_{\bar{M}}K)\,\varphi^I\varphi^J\bar{\varphi}^{\bar{L}}\bar{\varphi}^{\bar{M}}\Big|_{\theta^4}$$
$$=-(\partial_I\partial_J\bar{\partial}_{\bar{L}}\bar{\partial}_{\bar{M}}K)\,\psi_+^I\psi_-^J\bar{\psi}_+^{\bar{L}}\bar{\psi}_-^{\bar{M}}$$
$$=\left(R_{I\bar{L}J\bar{M}}-g_{K\bar{N}}\Gamma^K_{IJ}\Gamma^{\bar{N}}_{\bar{L}\bar{M}}\right)\psi_+^I\psi_-^J\bar{\psi}_+^{\bar{L}}\bar{\psi}_-^{\bar{M}}$$

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In summary we obtain the following action density;

$$
\begin{aligned}
\mathcal{L} = & -g_{I\bar{J}}(\phi,\bar{\phi})\Big[\partial_\mu \phi^I \partial^\mu \bar{\phi}^{\bar{J}} + F^I \bar{F}^{\bar{J}} \\
& + i\psi_+^I (D_-\bar{\psi})^{\bar{J}} + i\psi_-^I (D_+\bar{\psi})^{\bar{J}} - i(D_+\psi_-)^I \bar{\psi}_-^{\bar{J}} - i(D_-\psi_+)^I \bar{\psi}_+^{\bar{J}}\Big] \\
& - g_{I\bar{J}} \Gamma^I_{KL} \psi_+^K \psi_-^L \bar{F}^{\bar{J}} + g_{I\bar{J}} \Gamma^{\bar{J}}_{\bar{K}\bar{L}} \bar{\psi}_+^{\bar{K}} \bar{\psi}_-^{\bar{L}} F^I \\
& + (R_{I\bar{J}K\bar{L}} - g_{M\bar{N}} \Gamma^M_{IK} \Gamma^{\bar{N}}_{\bar{J}\bar{L}}) \psi_+^I \psi_-^K \bar{\psi}_+^{\bar{J}} \bar{\psi}_-^{\bar{L}} \\
= & -g_{I\bar{J}}(\phi,\bar{\phi})\Big[\partial_\mu \phi^I \partial^\mu \bar{\phi}^{\bar{J}} + 2i\bar{\psi}_-^{\bar{J}} (D_+\psi_-)^I + 2i\bar{\psi}_+^{\bar{J}} (D_-\psi_+)^I\Big] \\
& + R_{I\bar{J}K\bar{L}} \psi_+^I \psi_-^K \bar{\psi}_+^{\bar{J}} \bar{\psi}_-^{\bar{L}} \\
& + g_{I\bar{J}} (F^I - \Gamma^I_{KL} \psi_+^K \psi_-^L)(\bar{F}^{\bar{J}} + \Gamma^{\bar{J}}_{\bar{K}\bar{L}} \bar{\psi}_+^{\bar{K}} \bar{\psi}_-^{\bar{L}})
\end{aligned}
$$

up to total deriv. $D_\pm g = 0$

$2D_\pm = D_0 \pm D_1$

Except the last term, $\mathcal{L}$ is (manifestly) invariant under holomorphic coordinate transformation of ϕ^I.

$\mathcal{L}$ is also invariant under

holom.

$$K(\Phi, \bar{\Phi}) \longrightarrow K(\Phi, \bar{\Phi}) + f(\Phi) + \bar{f}(\bar{\Phi})$$

since the metric $g_{I\bar{J}}$ does not change.

By EOM of F, $\bar{F}$ we can eliminate the last term $\rightarrow$ "on-shell" action

We can define an action for each coordinate patch of a Kähler manifold M and glue them together consistently, which leads to $\mathcal{N}=(2,2)$ non-linear sigma model for a map

$$\phi : \underset{\text{(world sheet)}}{\Sigma} \longrightarrow M \quad \text{(Kähler)}$$

<u>Rmk</u> From the geometry of the map $\phi : \Sigma \longrightarrow M$

the fermions are spinors on Σ with values in the pull-back

of the tangent bundle $TM = \underset{\text{holom.}}{TM^{(1,0)}} \oplus \underset{\text{anti-holom.}}{TM^{(0,1)}}$

$$\psi_\pm \in \Gamma(\Sigma, \phi^* TM^{(1,0)} \otimes S_\pm), \quad \overline{\psi}_\pm \in \Gamma(\Sigma, \phi^* TM^{(0,1)} \otimes S_\pm)$$

the connection in the definition of D_μ

$$D_\mu \psi_\pm^K = \partial_\mu \psi_\pm^K + \partial_\mu \phi^I \Gamma^K_{IJ}(\phi, \overline{\phi}) \psi_\pm^J$$

is nothing but the pull-back of the hermitian connection on M.

<u>Rmk</u> This formulation of SUSY sigma model is local,

in the sense we glue together a family of SUSY actions.

There is a global formulation called Gauged Linear Sigma Model.