Brief review on complex manifolds and Kähler geometry

[Ref] P. Candelas, in Superstrings '87 (Trieste Spring School)

<u>Def</u> A complex manifold is a topological space M with a holomorphic atlas

$\exists$ (U_i, z_i) a collection of charts $M = \bigcup_{i\in I} U_i$

$z_i : U_i \longrightarrow V_i \subset \mathbb{C}^n$ one to one

$U_j \cap U_k \neq \phi$ $\quad V_{kj} = \{ z_k(p) \mid p \in U_j \cap U_k \} \subset V_k$

$V_{jk} = \{ z_j(p) \mid p \in U_j \cap U_k \} \subset V_j$

$\tau_{jk} := z_j \cdot z_k^{-1} : V_{kj} \longrightarrow V_{jk}$ biholomorphic

(双正则同相写像)

$\dim_{\mathbb{C}} M = n$

Example Complex projective space $\mathbb{C}P^n$ (n次元複素射影空間)

$$M = \mathbb{C}^{n+1} \setminus \{0\} / \sim \quad : \text{ the set of complex lines through the origin}$$

$$\zeta = \{ (\lambda \zeta_0, \lambda \zeta_1, \cdots, \lambda \zeta_n) ; \lambda \in \mathbb{C} \} \quad \text{complex line}$$

$$(\zeta_0, \zeta_1, \cdots, \zeta_n) \quad \text{homogeneous coordinates of } \zeta$$

$$(\zeta'_0, \cdots \zeta'_n) \sim (\zeta_0, \cdots, \zeta_n) \iff {}^\exists \lambda \in \mathbb{C}^\times \quad \zeta'_i = \lambda \zeta_i$$

$$U_j = \{ \zeta \mid \zeta_j \neq 0 \} \qquad M = \bigcup_{j=0}^{n} U_j$$

$$\zeta \in U_0 \qquad \zeta = (1, z^1, \cdots, z^n) \qquad z^\mu = \frac{\zeta_\mu}{\zeta_0}$$

inhomogeneous coordinates of ζ

$$z_j : \zeta \longrightarrow z_j(\zeta) = (z_j^0, \cdots z_j^{j-1}, z_j^{j+1}, \cdots z_j^n)$$

$$U_j \cap U_k \neq 0 \quad \Rightarrow \quad z_j^{\ k} = \frac{1}{z_k^{\ j}} \qquad z_j^{\ \mu} = \frac{z_k^{\ \mu}}{z_k^{\ j}} \quad (\mu \neq j, k)$$

$\tau_{jk} : z_k \longrightarrow z_j$ is biholomorphic "

<u>Def</u> M : an n-dim complex manifold

$\{ z^\mu \}$ local coordinates on a chart U

Define $(1,1)$-tensor J by

$$J = i \, dz^\mu \otimes \frac{\partial}{\partial z^\mu} - i \, dz^{\bar{\mu}} \otimes \frac{\partial}{\partial z^{\bar{\mu}}} \qquad \left(z^{\bar{\mu}} = \overline{z^\mu} \right)$$

J is called almost complex structure.

<u>Rmk</u> If we introduce real coordinates by $z^\mu = x^\mu + i y^\mu$

$$J = dx^\mu \otimes \frac{\partial}{\partial y^\mu} - dy^\mu \otimes \frac{\partial}{\partial x^\mu}$$: manifestly real tensor

$$J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ in } (z, \bar{z}) , \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ in } (x, y)$$

With an almost complex structure J, $(J^2 = -1)$
we can define two projections

$$P_m{}^n = \frac{1}{2}\left(\delta_m{}^n - i J_m{}^n\right) , \quad Q_m{}^n = \frac{1}{2}\left(\delta_m{}^n + i J_m{}^n\right)$$

$$P^2 = P , \quad Q^2 = Q , \quad PQ = 0 , \quad P + Q = 1$$

In the frame which diagonalizes J, $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

By P and Q, we can define projections to the holomorphic and the anti-holomorphic components of tensors

We can regard $J \in \mathrm{End}(T_x M)$, $J\left(\frac{\partial}{\partial x^m}\right) = J_m{}^n \frac{\partial}{\partial x^n}$ //

(1, 1)-tensor

Def A complex manifold M is called Hermitian if it is endowed with a metric which satisfies

$$g_{mn} = J_m{}^k J_n{}^l g_{kl} \Longrightarrow \begin{cases} ds^2 = g_{\mu\bar{\nu}}\, dz^\mu \otimes d\bar{z}^{\bar{\nu}} \\ g_{\mu\nu} = g_{\bar{\mu}\bar{\nu}} = 0 \end{cases}$$

RmK For any metric h_{mn}

$$g_{mn} = h_{mn} + J_m{}^k J_n{}^l h_{kl} \quad \text{is hermitian.}$$

$$(\because) \quad J_m{}^k J_n{}^l \left(h_{kl} + J_k{}^i J_l{}^j h_{ij} \right)$$

$$= J_m{}^k J_n{}^l h_{kl} + (-1)^2 h_{mn} \qquad //$$

Prop If g_{mn} is hermitian $(J_m{}^k g_{kn} = - J_n{}^j g_{mj})$

$$J_{mn} := J_m{}^k g_{kn} \quad \text{is anti-symmetric}$$

On a hermitian manifold

the almost complex structure defines a natural 2-form

$$J = \frac{1}{2} J_{mn}\, dx^m \wedge dx^n = \frac{i}{2}\left(g_{\mu\bar{\nu}}\, dz^\mu \wedge d\bar{z}^{\bar{\nu}} - g_{\bar{\nu}\mu}\, d\bar{z}^{\bar{\nu}} \wedge dz^\mu\right)$$

$$= i\, g_{\mu\bar{\nu}}\, dz^\mu \wedge d\bar{z}^{\bar{\nu}}$$

Up to now, we started with a complex manifold.

Now let us ask a question ; if a real 2n-dim manifold M admits a globally defined (1, 1) tensor $J_m{}^n$ s.t $J^2 = -1$, M is a complex manifold ?

Def If a real manifold M admits a globally defined $(1,1)$-tensor $J_m{}^n$ with $J^2 = -1$, M is called almost complex manifold. If in addition M admits a metric g_{mn} with $g_{mn} = J_m{}^k J_n{}^l g_{kl}$ (or $J_{mn} = -J_{nm}$) M is called almost hermitian manifold.

Thm (Newlander-Nirenberg)

An almost complex str. $J_m{}^n$ defines a complex structure

$\Updownarrow$

the almost complex str. comes from a complex manifold

$\Longleftrightarrow$ the Nijenhuis tensor $N_{ij}{}^k(J) = 0$ vanishes

Def $N_{ij}{}^k(J) = 0 \Longleftrightarrow J$ is integlable.

<u>Def</u> If an almost complex str J is integlable
we call J complex str.

<u>Rmk</u> The almost complex str. of a complex manifold is integlable $\Rightarrow$ J is a complex str.

<u>Thm</u> On a hermitian manifold, there is a unique connection with the properties

(1) The covariant derivatives of g_{mn}, $J_m{}^n$ vanish.

$\Rightarrow$ The projections P and Q are covariantly const.
(compatible with pallarel transport)

(2) The mixed components of the torsion $\Gamma^k_{[mn]} = \Gamma_{mn}{}^k - \Gamma_{nm}{}^k$ vanish.

We call it hermitian connection.

<u>Prop</u> The hermitian connection Γ is "pure" in its indices, namely only non-vanishing components are $\Gamma_{\mu\nu}{}^{\kappa}$ and $\Gamma_{\bar{\mu}\bar{\nu}}{}^{\bar{\kappa}}$.

(∵) Since P and Q are covariantly constants

$$\Gamma_{\mu\nu}{}^{\bar{k}} = \Gamma_{\mu\bar{\nu}}{}^{\kappa} = \Gamma_{\bar{\mu}\nu}{}^{\bar{\kappa}} = \Gamma_{\bar{\mu}\bar{\nu}}{}^{\kappa} = 0$$

Since the mixed components of the torsion vanish,

$$\Gamma_{\bar{\nu}\mu}{}^{\kappa} = \Gamma_{\nu\bar{\mu}}{}^{\bar{\kappa}} = 0 \qquad //$$

Recall that $\quad \nabla_m g_{nr} = \partial_m g_{nr} - \Gamma_{mn}^{K} g_{Kr} - \Gamma_{mr}^{K} g_{nK} = 0$

Take $(m, n, r) = (\mu, \nu, \bar{\rho}) \qquad \partial_\mu g_{\nu\bar{\rho}} = \Gamma_{\mu\nu}^{\kappa} g_{\kappa\bar{\rho}}$

$$\therefore \quad \Gamma_{\mu\nu}^{\kappa} = g^{\kappa\bar{\rho}} \partial_\mu g_{\nu\bar{\rho}} \qquad //$$

For a hermitian connection the structure of Riemann curvature tensor is also much simplified.

Namely only non-vanishing components are those that are "mixed" in both the first and the second pairs of indices

$$R_{\mu\bar\nu\rho\bar\sigma}\,,\quad R_{\bar\nu\mu\rho\bar\sigma}\,,\quad R_{\mu\bar\nu\bar\sigma\rho}\,,\quad R_{\bar\nu\mu\bar\sigma\rho}$$

$$R_{\mu\bar\nu\bar\rho}{}^{\bar\sigma} = -R_{\bar\nu\mu\bar\rho}{}^{\bar\sigma} = \partial_\mu \Gamma_{\bar\nu\bar\rho}{}^{\bar\sigma}$$

$$\begin{aligned}
R_{\mu\bar\nu\kappa\bar\rho} &= -R_{\mu\bar\nu\bar\rho\kappa} = -g_{\kappa\bar\sigma}\,\partial_\mu \Gamma_{\bar\nu\bar\rho}{}^{\bar\sigma} \\
&= -g_{\kappa\bar\sigma}\,\partial_\mu\left(g^{\tau\bar\sigma}\,\partial_{\bar\nu}\, g_{\tau\bar\rho}\right) \\
&= -\partial_\mu\partial_{\bar\nu}\, g_{\kappa\bar\rho} - g_{\kappa\bar\sigma}\left(\partial_\mu g^{\tau\bar\sigma}\right) g_{\tau\bar\tau}\,\Gamma_{\bar\nu\bar\rho}{}^{\bar\tau} \\
&= -\partial_\mu\partial_{\bar\nu}\, g_{\kappa\bar\rho} + \left(\partial_\mu g_{\kappa\bar\tau}\right)\Gamma_{\bar\nu\bar\rho}{}^{\bar\tau} \rightarrow g_{\tau\bar\tau}\,\Gamma_{\mu\nu}{}^{\tau}\,\Gamma_{\bar\nu\bar\rho}{}^{\bar\tau}
\end{aligned}$$

Recall that a hermitian manifold has the natural two-form

$$J = \frac{1}{2} J_{mn}\, dx^m \wedge dx^n$$

<u>Def</u> A hermitian manifold is Kähler if $dJ = 0$

In this case J is called Kähler form.

<u>Rmk</u> All one-dimensional complex manifolds are Kähler,

since dJ is a three form and hence vanishes.

Since $dJ = \partial J + \bar{\partial} J = i\, \partial_\kappa g_{\mu\bar{\nu}}\, dz^\kappa \wedge dz^\mu \wedge dz^{\bar{\nu}}$

$$- i\, \partial_{\bar{\rho}} g_{\mu\bar{\nu}}\, dz^\mu \wedge dz^{\bar{\rho}} \wedge dz^{\bar{\nu}}$$

$\partial_\kappa g_{\mu\bar{\nu}} = \partial_\mu g_{\kappa\bar{\nu}}$, $\quad \partial_{\bar{\rho}} g_{\mu\bar{\nu}} = \partial_{\bar{\nu}} g_{\mu\bar{\rho}}$ for Kähler

which means on each chart U_j (namely locally)

$\exists\, \varphi_j$ (potential on U_j) s.t $\quad g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} \varphi_j$ or $J = i \partial \bar{\partial} \varphi_j$

<u>Rmk</u> The potential function φ_j on each U_j cannot come from a globally defined function on M, if M is compact. $\nexists\, \varphi$ on M s.t $\varphi_j = \varphi|_{U_j}$ //

$(\because)$ $$J \wedge \cdots \wedge J = i^n n! \sqrt{g}\, dz^1 d\bar{z}^{\bar{1}} \cdots dz^n d\bar{z}^n$$

M: compact $\quad \int_M J \wedge \cdots \wedge J \propto \mathrm{vol}(M)$

On the other hand, if $J = d\varphi$ (globally) $\quad J \wedge \cdots \wedge J = d(*)$

$$\therefore \int_M J \wedge \cdots \wedge J = 0 \quad \text{by Stokes thm.}$$ //

Since $\partial_\kappa g_{\mu\bar{\nu}} = \partial_\mu g_{\kappa\bar{\nu}}$, $\partial_{\bar{\rho}} g_{\mu\bar{\nu}} = \partial_{\bar{\nu}} g_{\mu\bar{\rho}}$ for Kähler manifold, the hermitian connection is torsion free $\Gamma_{\mu\nu}^{\kappa} = \Gamma_{\nu\mu}^{\kappa}$, $\Gamma_{\bar{\mu}\bar{\nu}}^{\bar{\kappa}} = \Gamma_{\bar{\nu}\bar{\mu}}^{\bar{\kappa}}$ and coincides with the Levi-Civita connection //