Classical Solutions in Cubic Open String Field Theory

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Contents

T. Takahashi and S. Tanimoto, Prog.Theor.Phys. 106 (2001) 863 (hep-th/0107046)
T. Takahashi and S. Tanimoto, hep-th/0112124
T. Takahashi and S. Tanimoto, in preparation

1 Introduction

1. 弦理論は「重力の量子化」「大統一理論」の構成にとって魅力的である

2. 弦理論によって自然を記述するためには、弦理論の非摂動的な解析が不可欠である

3. 近年、弦理論の非摂動的効果である「タキオン凝縮」が、D-brane の解析によって解明されてきている

A. Sen

4. 弦の場の理論は「タキオン凝縮」の解析に有用である

$V(T_0) = -(\text{brane tension})$ など

A. Sen and B. Zwiebach

5. Cubic Open String Field Theory では

- Feynman-Siegel gauge
- level truncation scheme

を用いて解析している (近似計算)

6. 「タキオン凝縮」に対応する弦の場の理論の厳密解が求められれば

- “Sen の予想” の証明ができる (はず)
- ‘Polynomial Closed String Field Theory’ の構成ができる (はず)
- $\cdots$

$\Longrightarrow$ ‘厳密解’ を求めたい

cf. Vacuum String Field Theory

2 Splitting Property of Delta Function

We define a delta function as follows,

$$\delta(w, w') = \sum_{n=-\infty}^{\infty} w^{-n} w'^{n-1} = \sum_{n=-\infty}^{\infty} w'^{-n} w^{n-1}, \tag{2.1}$$

where w and w' are complex coordinates. If $f(w)$ has no pole without the origin, the delta function satisfies

$$f(w) = \oint_{C_0} \frac{dw'}{2\pi i} \delta(w, w') f(w'), \tag{2.2}$$

where C_0 denotes a contour which encircles the origin along the unit circle.

Let us consider the integration along a left unit half circle, the path of which are depicted in Fig. 1. We find that

$$\int_{C_{\mathrm{L}}} \frac{dw'}{2\pi i} w'^n \delta(w, w') = \frac{1}{2} w^n + \frac{1}{\pi} \sum_{k \neq n} \frac{1}{n-k} w^k \sin\left(\frac{(n-k)\pi}{2}\right). \tag{2.3}$$

This implies that the delta function of Eq. (2.1) is not a delta function if the integration path is the left half only.

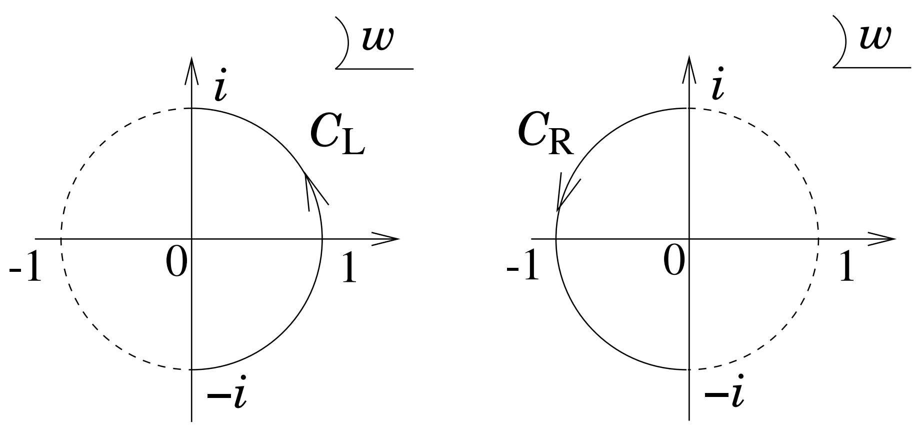

Figure 1:

We perform the left integration of Eq. (2.3) once more. We find that, if $m+n \neq 0$,

$$\begin{aligned}
&\int_{C_L} \frac{dw}{2\pi i} w^{m-1} \int_{C_L} \frac{dw'}{2\pi i} w'^n \delta(w,w') \\
&\quad = \frac{1}{\pi} \frac{1}{m+n} \sin\left(\frac{(m+n)\pi}{2}\right) \\
&\qquad + \frac{1}{\pi^2} \sum_{k\neq 0,\, m+n} \frac{1}{k(k-m-n)} \sin\left(\frac{k\pi}{2}\right) \sin\left(\frac{(k-m-n)\pi}{2}\right),
\end{aligned} \tag{2.4}$$

if $m+n=0$,

$$= \frac{1}{4} + \sum_{k\neq 0} \frac{1}{k^2} \sin^2\left(\frac{k\pi}{2}\right). \tag{2.5}$$

Using the formula

$$\sum_{m\neq 0, n} \frac{1}{m(m-n)} \sin\left(\frac{m\pi}{2}\right) \sin\left(\frac{(m-n)\pi}{2}\right) = \frac{\pi^2}{4}\delta_{n,0}, \tag{2.6}$$

we can calculate the infinite series and find the following equation,

$$\int_{C_L} \frac{dw}{2\pi i} w^{m-1} \int_{C_L} \frac{dw'}{2\pi i} w'^n \delta(w,w') = \int_{C_L} \frac{dw}{2\pi i} w^{m+n-1}. \tag{2.7}$$

Therefore, if $f(w)$ and $g(w)$ have the Laurent expansions within the unit circle around the origin, we find

$$\int_{C_L} \frac{dw}{2\pi i} \int_{C_L} \frac{dw'}{2\pi i} f(w) g(w') \delta(w,w') = \int_{C_L} \frac{dw}{2\pi i} f(w) g(w). \tag{2.8}$$

Thus, the delta function of Eq. (2.1) behaves as a delta function in the double left integrations. From Eqs. (2.1) and (2.8), other formulas are given by

$$\begin{aligned}
&\int_{C_R} \frac{dw}{2\pi i} \int_{C_R} \frac{dw'}{2\pi i} f(w) g(w') \delta(w,w') = \int_{C_R} \frac{dw}{2\pi i} f(w) g(w), \\
&\int_{C_L} \frac{dw}{2\pi i} \int_{C_R} \frac{dw'}{2\pi i} f(w) g(w') \delta(w,w') = 0.
\end{aligned} \tag{2.9}$$

These formula of Eqs. (2.8) and (2.9) are generalizations of the formulas for the delta function with the Neumann boundary condition [Takahashi and Tanimoto].

3 Marginal Solutions

We consider the ghost field $c(w)$, and a $U(1)$ current $J(w)$, namely a dimension one primary field. These fields are expanded by

$$c(w)=\sum_n c_n w^{-n+1},\quad J(w)=\sum_n j_n w^{-n-1}. \tag{3.1}$$

The commutation relations of the components j_n are given by

$$[j_m,\, j_n]=m\delta_{m+n}. \tag{3.2}$$

From Eq. (3.2), the commutation relation of $J(w)$ becomes

$$[J(w),\, J(w')]=-\partial_w\delta(w,w'). \tag{3.3}$$

Here, we introduce the following operators,

$$\begin{aligned}
V_{\mathrm{L}}(f)&=\int_{C_{\mathrm{L}}}\frac{dw}{2\pi i}f(w)c\,J(w),\quad V_{\mathrm{R}}(f)=\int_{C_{\mathrm{R}}}\frac{dw}{2\pi i}f(w)c\,J(w),\\
C_{\mathrm{L}}(f)&=\int_{C_{\mathrm{L}}}\frac{dw}{2\pi i}f(w)c(w),\quad C_{\mathrm{R}}(f)=\int_{C_{\mathrm{R}}}\frac{dw}{2\pi i}f(w)c(w),
\end{aligned} \tag{3.4}$$

where $f(w)$ is a holomorphic function within the unit circle except the origin, and $f(\pm i)=0$.

From the delta function formulas, it follows that

$$\begin{aligned}
\{V_{\mathrm{L}}(f),V_{\mathrm{L}}(g)\} &= -\int_{C_{\mathrm{L}}}\frac{dw}{2\pi i}\int_{C_{\mathrm{L}}}\frac{dw'}{2\pi i}f(w)g(w)c(w)c(w')\partial_w\delta(w,w')\\
&= -\int_{C_{\mathrm{L}}}\frac{dw}{2\pi i}f(w)g(w)c\partial c(w)\\
&= -\{Q_{\mathrm{B}},\, C_{\mathrm{L}}(fg)\}\,.
\end{aligned} \tag{3.5}$$

Similarly, we find that

$$\{V_{\mathrm{R}}(f),V_{\mathrm{R}}(g)\}=-\{Q_{\mathrm{B}},\, C_{\mathrm{R}}(fg)\}\,, \tag{3.6}$$

$$\{V_{\mathrm{L}}(f),V_{\mathrm{R}}(g)\}=0. \tag{3.7}$$

Let us consider the fields defined by $u_n^{(h)}(w) = w^{n-h} - (-)^{n-h} w^{-n-h}$ and $v_n^{(h)}(w) = w^{n-h} + (-1)^{n-h} w^{-n-h}$, where n is an integer. If w changes $w' = -1/w$, the fields is transformed into

$$\begin{aligned} u_n^{(h)}(w') &= -\left(\frac{dw}{dw'}\right)^h u_n^{(h)}(w), \\ v_n^{(h)}(w') &= \left(\frac{dw}{dw'}\right)^h v_n^{(h)}(w). \end{aligned} \tag{3.8}$$

The N-string vertex is defined by gluing the boundaries $|w_i| = 1$ ($i = 1, 2, \cdots, N$) of N unit disks with the identifications,

$$w_i\, w_{i+1} = -1, \quad \text{for}\, |z_i|\,, \mathrm{Re} z_i \leq 1, \tag{3.9}$$

where w_{N+1} denotes w_1. If $\phi(w)$ is a dimension h primary field, one forms of $dwu_n^{(-h+1)}\phi$ and $dwv_n^{(-h+1)}\phi$ are transformed into

$$\begin{aligned} &dw_{i+1} u_n^{(-h+1)}(w_{i+1})\phi(w_{i+1}) = -dw_i u_n^{(-h+1)}(w_i)\phi(w_i), \\ &dw_{i+1} v_n^{(-h+1)}(w_{i+1})\phi(w_{i+1}) = dw_i v_n^{(-h+1)}(w_i)\phi(w_i). \end{aligned} \tag{3.10}$$

Then, considering $J(w)$ and $c(w)$ as the primary fields, we obtain the following equations related to $*$ product of string fields:

$$(V_{\mathrm{R}}(F_-^{(1)})A) * B = (-)^{|A|} A * (V_{\mathrm{L}}(F_-^{(1)})B), \tag{3.11}$$

$$(C_{\mathrm{R}}(F_+^{(2)})A) * B = -(-)^{|A|} A * (C_{\mathrm{L}}(F_+^{(2)})B), \tag{3.12}$$

where A and B are arbitrary string fields and $|A|$ is 0 if A is Grassmann even and 1 if it is odd, and $F_\pm^{(h)}(w)$ is defined by

$$\begin{aligned} F_-^{(h)}(w) &= \sum_n a_n u_n^{(h)}(w), \\ F_+^{(h)}(w) &= \sum_n b_n v_n^{(h)}(w). \end{aligned} \tag{3.13}$$

Similarly, we find that

$$V_{\mathrm{L}}(F_-^{(1)})I = V_{\mathrm{R}}(F_-^{(1)})I, \quad C_{\mathrm{L}}(F_+^{(2)})I = -C_{\mathrm{R}}(F_+^{(2)})I, \tag{3.14}$$

where I denotes the identity string field. Here, the function $F_+^{(2)}(w)$ must satisfy $F_+^{(2)}(\pm i) = 0$, because the ghost $c(w)$ has the midpoint singularities on the identity string field, which are evaluated by the oscillator expressions as

$$c(w)\,|I\rangle \;=\; \left[-c_0\frac{w-w^3}{1+w^2}+c_1\frac{w^2}{1+w^2}+c_{-1}\frac{1+w^2+w^4}{1+w^2}+\sum_{n\geq 2}c_{-n}v_n^{(-1)}(w)\right]|I\rangle\,.$$

Now, we obtain a classical solution for the $U(1)$ current J,

$$\Psi_{\rm m} = {\rm diag}\left(a_i V_{\rm L}(F_-^{(1)}) + \frac{1}{2}a_i^2\, C_{\rm L}(F_-^{(1)^2})\right) I, \tag{3.15}$$

where a_i are parameters and i corresponds to the Chan-Paton indices. The function $F^{(0)}(w)$ satisfies $F^{(0)}(\pm i) = 0$. From Eqs. (3.5), (3.11), (3.12) and (3.14), we find that the classical solution satisfies the equation of motion, $Q_{\rm B}\Psi_{\rm m} + \Psi_{\rm m} * \Psi_{\rm m} = 0$. The solution for $F_-^{(1)}(w) = u_0^{(1)} + u_2^{(1)}$ are give by Takahashi and Tanimoto. Indeed, we find that $F_-^{(0)}(\pm i) = 0$ and

$$F_-^{(1)}(w)^2 = 3v_0^{(2)} + 4v_2^{(2)}(w) + v_4^{(2)}(w). \tag{3.16}$$

Since the classical solution of Eq. (3.15) dose not refer to any boundary condition, it is a generalization of the solution given by Ref. [?].

If we expand the string field around the classical solution, the BRS charge becomes

$$Q'_{\rm B} = Q_{\rm B} + a_i V_{\rm L}(F_-^{(1)}) - a_j V_{\rm R}(F_-^{(1)}) + \frac{1}{2}a_i^2 C_{\rm L}(F_-^{(1)^2}) + \frac{1}{2}a_j^2 C_{\rm R}(F_-^{(1)^2}) \tag{3.17}$$

Supposed that there is the dimension 'zero' field $\varphi(w)$ which satisfies

$$[\varphi(w),\, J(w')] = i\,\delta(w,w'), \quad [Q_{\rm B},\, \varphi(w)] = -ic\, J(w). \tag{3.18}$$

We introduce the left and right integrated operators for φ

$$\Phi_{\rm L}(f) = \int_{C_{\rm L}} \frac{dw}{2\pi i} f(w)\varphi(w), \quad \Phi_{\rm R}(f) = \int_{C_{\rm R}} \frac{dw}{2\pi i} f(w)\varphi(w). \tag{3.19}$$

From Eq. (3.18), it follows that

$$[\Phi_{\rm L}(f),\, Q_{\rm B}] = -iV_{\rm L}(f), \quad [\Phi_{\rm L}(f),\, V_{\rm L}(f)] = iC_{\rm L}(fg). \tag{3.20}$$

The shifted BRS charge can be written by

$$Q'_{\rm B} = e^{-B(F_-^{(1)})} Q_{\rm B} e^{B(F_-^{(1)})}, \tag{3.21}$$

where B is defined by

$$B(F_-^{(1)}) = ia_i\Phi_{\rm L}(F_-^{(1)}) - ia_j\Phi_{\rm R}(F_-^{(1)}). \tag{3.22}$$

Therefore, it seems that the shifted theory transformed into the original theory by the redefinition of the string field and the classical solution is trivial. However, it is true only if the field $\varphi(w)$ exists and the redefinition is well-defined for the zero-mode part of φ. For example, we consider $J \sim i\partial X$ and $\varphi \sim X$. In this case, if the direction X is compactified, the redefinition is generally ill-defined for the zero-mode of X. Indeed, the shifted theory represents strings in the Wilson lines background, and the classical solution corresponds to the condensation of the gauge fields.

Using the operator $\Phi_{\rm L}$, the classical solution can be rewritten by

$$\Psi_{\rm m} = \exp(-i\, a_i \Phi_{\rm L}(F_-^{(1)})I) * Q_{\rm B} \exp(i\, a_i \Phi_{\rm L}(F_-^{(1)})I). \tag{3.23}$$

It implies that the classical solution is the gauge transformation from zero string field, namely pure gauge. However, as well as the string field redefinition, there are cases in which the gauge transformation is ill-defined because of the zero-mode of Φ_L. For the Wilson lines solution, the classical solution is locally pure gauge, but it is globally non-trivial configuration, analogously to field theoretical situations.

Let us consider the potential height $S[\Psi_{\rm m}]$ at the classical solution. Since the classical solution has parameters a_i, it follows that

$$\frac{d}{da_i} S[\Psi_{\rm m}] = \frac{2}{g} \int (Q_{\rm B}\Psi_{\rm m} + \Psi_{\rm m} * \Psi_{\rm m}) * \frac{d\Psi_{\rm m}}{da_i} = 0. \tag{3.24}$$

Then, we find that $S[\Psi_{\rm m}(a_i)] = S[\Psi_{\rm m}(a_i = 0)] = 0$.

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● level truncation, Feynman-Siegel gauge による解析

level (1,2)

$$|\Phi\rangle = t\, c_1\, |0\rangle + a\, c_1 \alpha_{-1}\, |0\rangle$$

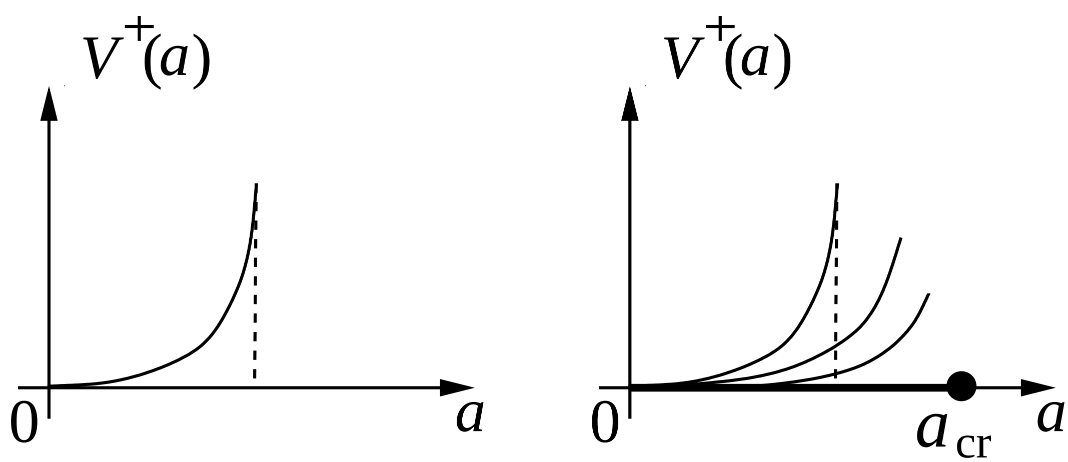
ポテンシャルは

$$V(t,a) = -\frac{1}{2}t^2 + \frac{1}{3}K^3 t^3 + Kta^2, \quad (K = 3\sqrt{3}/4)$$

タキオン場 t を積分すると

$$V^{\pm}(a) = \frac{2}{59049}\left(-512 + 874a^2 \pm (64 - 729a^2)^{\frac{3}{2}}\right)$$

「$+$」は $t(a=0)=0$,「$-$」は $t(a=0) = (3/2K^3)^{1/2}$(タキオン凝縮)に対応



level(1,2) $\Longrightarrow$ level をあげる

4 Scalar Solution

The BRS current is defined by

$$J_{\mathrm{B}}(w) = c\left(T_X + \frac{1}{2}T_{\mathrm{gh}}\right)(w) + \frac{3}{2}\partial^2 c(w), \tag{4.1}$$

where $c(w)$ is a ghost field and $T_X(w)$ and $T_{\mathrm{gh}}(w)$ denote the energy momentum tensors of string coordinates and reparametrization ghosts, respectively. The operator product expansions (OPEs) of the BRS current and the ghost field are given by

$$\begin{aligned} J_{\mathrm{B}}(w)J_{\mathrm{B}}(w') &= \frac{-(d-18)/2}{(w-w')^3}c\partial c(w') + \frac{-(d-18)/4}{(w-w')^2}c\partial^2 c(w') - \frac{(d-26)/12}{w-w'}c\partial^3 c(w') + \cdots \\ &= \frac{-4}{(w-w')^3}c\partial c(w') + \frac{-2}{(w-w')^2}c\partial^2 c(w') + \cdots, \\ J_{\mathrm{B}}(w)c(w') &= \frac{1}{w-w'}c\partial c(w') + \cdots, \end{aligned} \tag{4.2}$$

where $d = 26$ is the matter central charge of the conformal field theory. We can expand the BRS current and the ghost field using oscillation modes,

$$\begin{aligned} J_{\mathrm{B}}(w) &= \sum_{n=-\infty}^{\infty} Q_n w^{-n-1}, \\ c(w) &= \sum_{n=-\infty}^{\infty} c_n w^{-n+1}. \end{aligned} \tag{4.3}$$

Since $\{Q_{\mathrm{B}},\, c(w)\} = c\partial c(w)$, the OPEs of Eq. (4.2) can be rewritten in the form of anti-commutation relations of these oscillators,

$$\{Q_m,\, Q_n\} = 2mn\{Q_{\mathrm{B}},\, c_{m+n}\}, \quad \{Q_m,\, c_n\} = \{Q_{\mathrm{B}},\, c_{m+n}\}. \tag{4.4}$$

From Eq. (4.4), we find the anti-commutation relation of the BRS current and the ghost,

$$\begin{aligned} \{J_{\mathrm{B}}(w),\, J_{\mathrm{B}}(w')\} &= \{Q_{\mathrm{B}},\, 2\,\partial_w \partial_{w'}\left(c(w)\delta(w,\, w')\right)\}, \\ \{J_{\mathrm{B}}(w),\, c(w')\} &= \{Q_{\mathrm{B}},\, c(w)\delta(w,\, w')\}. \end{aligned} \tag{4.5}$$

We now define the following operators,

$$Q_{\rm L}(f)=\int_{C_{\rm L}}\frac{dw}{2\pi i}f(w)J_{\rm B}(w),\quad Q_{\rm R}(f)=\int_{C_{\rm R}}\frac{dw}{2\pi i}f(w)J_{\rm B}(w),\tag{4.6}$$

where $f(w)$ is a holomorphic function within the unit circle except the origin, and, in addition, its values at the midpoint is zero, $f(\pm i)=0$. From Eq. (4.5), we can calculate the anti-commutation relation of the operators as follows,

$$\begin{aligned}
&\{Q_{\rm L}(f),\,Q_{\rm L}(g)\}\\
=&\left\{Q_{\rm B},\,\int_{C_{\rm L}}\frac{dw}{2\pi i}\int_{C_{\rm L}}\frac{dw'}{2\pi i}f(w)g(w')2\,\partial_w\partial_{w'}\left(c(w)\delta(w,\,w')\right)\right\}\\
=&\ 2\left\{Q_{\rm B},\,\int_{C_{\rm L}}\frac{dw}{2\pi i}\int_{C_{\rm L}}\frac{dw'}{2\pi i}\partial f(w)\partial g(w')c(w)\delta(w,\,w')\right\},
\end{aligned}\tag{4.7}$$

where surface terms are vanished due to $f(\pm i)=g(\pm i)=0$. Using the delta function formula, we find that

$$\{Q_{\rm L}(f),\,Q_{\rm L}(g)\}=2\{Q_{\rm B},\,C_{\rm L}(\partial f\partial g)\}.\tag{4.8}$$

Similarly, other anti-commutation relations are given by

$$\begin{aligned}
&\{Q_{\rm R}(f),\,Q_{\rm R}(g)\}=2\left\{Q_{\rm B},\,C_{\rm R}(\partial f\partial g)\right\},\\
&\{Q_{\rm L}(f),\,C_{\rm L}(g)\}=\left\{Q_{\rm B},\,C_{\rm L}(fg)\right\},\\
&\{Q_{\rm R}(f),\,C_{\rm R}(g)\}=\left\{Q_{\rm B},\,C_{\rm R}(fg)\right\},\\
&\{Q_{\rm L}(f),\,Q_{\rm R}(g)\}=\{Q_{\rm L}(f),\,C_{\rm R}(g)\}=\{Q_{\rm R}(f),\,C_{\rm L}(g)\}=0.
\end{aligned}\tag{4.9}$$

We consider the properties of the $Q_{\rm L(R)}$ related to $*$ product and the identity string field. Since $dw_i F_+^{(0)}(w_i) J_{\rm B}(w_i)$ is a globally defined one form on the gluing N-string surface, we obtain a similar equation to Eqs. (3.11) and (3.12),

$$\left(Q_{\rm R}(F_+^{(0)})\right) A * B = -(-)^{|A|} A * \left(Q_{\rm L}(F_+^{(0)}) B\right). \tag{4.10}$$

Similarly, we find that

$$Q_{\rm L}(F_+^{(0)}) I = -Q_{\rm R}(F_+^{(0)}) I. \tag{4.11}$$

Now, we can show that a classical solution is given by

$$\Psi_0 = Q_{\rm L}(F_+^{(0)})I + C_{\rm L}(G_+^{(2)})I, \tag{4.12}$$

where $G_+^{(2)}(w)$ is

$$G_+^{(2)}(w) = -\frac{\left(\partial F_+^{(0)}(w)\right)^2}{1+F_+^{(0)}(w)}. \tag{4.13}$$

Here, $G_+^{(2)}(\pm i)$ must be zero in order to cancel the midpoint singularity of the ghost on the identity, as in the case of the marginal solutions. Indeed, from Eqs. (3.12), (4.8), (4.9), (4.10) and (4.11), we find that the classical equation satisfies the equation of motion:

$$\begin{aligned} &Q_{\rm B}\Psi_0 = \{Q_{\rm B},\, C_{\rm L}(G_+^{(2)})\}I, \\ &\Psi_0 * \Psi_0 = \{Q_{\rm B},\, C_{\rm L}((\partial F_+^{(0)})^2 + F_+^{(0)}G_+^{(2)})\}I. \end{aligned} \tag{4.14}$$

Then, it follows that $Q_{\rm B}\Psi_0 + \Psi_0 * \Psi_0 = 0$.

If we expand the string field around the classical solution, the shifted theory has the following BRS charge,

$$Q'_{\rm B} = Q_{\rm B} + Q(F_+^{(0)}) + C(G_+^{(2)}), \tag{4.15}$$

where we define

$$Q(f) = Q_{\rm L}(f) + Q_{\rm R}(f), \quad C(f) = C_{\rm L}(f) + C_{\rm R}(f). \tag{4.16}$$

If we take $F_+^{(0)}(w) = \exp(h(w)) - 1$, the classical solution and the shifted BRS charge are rewritten by

$$\Psi_0 = Q_{\rm L}(e^h - 1)I - C_{\rm L}\left((\partial h)^2 e^h\right)I, \tag{4.17}$$

$$Q'_{\rm B} = Q(e^h) - C\left((\partial h)^2 e^h\right). \tag{4.18}$$

Let us consider the redefinition of the string field. The ghost number currents are given by

$$J_{\rm gh}(w) = cb(w), \tag{4.19}$$

where $c(w)$ and $b(w)$ are ghost and anti-ghost fields, respectively. The OPEs of the ghost current with the BRS current and with the ghost field are given by

$$\begin{aligned} J_{\rm gh}(w)J_{\rm B}(w') &= \frac{4}{(w-w')^3}c(w') + \frac{2}{(w-w')^2}\partial c(w') + \frac{1}{(w-w')^2}J_{\rm B}(w') \qquad (4.20) \\ J_{\rm gh}(w)c(w') &= \frac{1}{w-w'}c(w') + \cdots . \qquad (4.21) \end{aligned}$$

We introduce three operators for an holomorphic function $f(w)$ within the unit circle without the origin,

$$q(f) = \oint \frac{dw}{2\pi i} f(w) J_{\rm gh}(w). \tag{4.22}$$

The OPEs of Eqs. (4.20) and (4.21) give the following commutation relations,

$$\begin{aligned} [q(f),\, Q(g)] &= Q(fg) - 2\, C(\partial f \partial g), \qquad (4.23) \\ [q(f),\, C(g)] &= C(fg). \qquad (4.24) \end{aligned}$$

From the commutation relations of Eqs. (4.23) and (4.24), we find that, through the transformation generated by $q(f)$, the BRS charge becomes

$$\begin{aligned} e^{q(f)}\, Q_{\rm B}\, e^{-q(f)} &= Q_{\rm B} + [q(f),\, Q_{\rm B}] + \frac{1}{2!}[q(f),\, [q(f),\, Q_{\rm B}]] + \cdots \\ &= Q_{\rm B} + Q(f) + \frac{1}{2!}\left\{Q(f^2) - 2C((\partial f)^2)\right\} + \cdots \\ &= Q(e^f) - C\left((\partial f)^2 e^f\right). \qquad (4.25) \end{aligned}$$

Therefore, if the string field Ψ is redefined as $\Psi = e^{q(h)}\Psi'$, the shifted BRS charge is transformed into the original BRS charge.

In order to identify the redefined theory as $\Psi = e^{q(h)}\Psi'$ from the shifted theory, let us consider the conservation law of $q(h)$ on the N-strings vertex. The gluing N-strings surface can be transformed into the whole complex z-plane by the mapping

$$z = e^{\frac{2\pi(k-1)i}{N}} \left(\frac{1+iw_k}{1-iw_k}\right)^{\frac{2}{N}}, \quad (k = 1, \cdots, N). \tag{4.26}$$

Here, $\exp(2\pi(k-1)i/N)$ $(k = 1, \cdots, N)$ correspond to the N punctures in the z-plane, which represent N strings insertions, and the origin and the infinity in the z plane correspond to the midpoints of the N strings. Since $h(w)$ is an analytic scalar and $F_+^{(0)}(\pm i) = 0$, we find that $h(z=0) = h(z=\infty) = 0$. Therefore, the conservation law in the z plane is given by

$$\langle V_N| \sum_{k=1}^{N} \int_{C_k} \frac{dz}{2\pi i} h(z) J_{\text{gh}}(z) = 0, \tag{4.27}$$

where the contours C_k encircle the puncture at the k-string in the z plane. The anomalous terms at the infinity vanishes due to $h(\infty) = 0$. We can express the contour integral around the k-string's puncture in terms of the local coordinate w_k. Since the transformation low of the ghost number current J_{gh} is given by

$$\frac{dz}{dw} J_{\text{gh}}(z) = J_{\text{gh}}(w) + \frac{3}{2} \frac{d^2 z}{dw^2} \left(\frac{dz}{dw}\right)^{-1}, \tag{4.28}$$

we obtain the following identity,

$$\begin{aligned} \langle V_N| \sum_{k=1}^{N} \oint_{C_0} \frac{dw_k}{2\pi i} h(w_k)\, J_{\text{gh}}(w_k) &= \kappa_N(h)\, \langle V_N|\,, \\ \kappa_N(h) &= -\frac{3}{2} \sum_{k=1}^{N} \oint_{C_0} \frac{dw_k}{2\pi i} h(w_k)\, \frac{d^2 z}{d{w_k}^2} \left(\frac{dz}{dw_k}\right)^{-1}. \end{aligned} \tag{4.29}$$

From Eq. (4.26), we find that

$$\begin{aligned} \kappa_N(h) &= -\frac{3}{2} \sum_{k=1}^{N} \oint_{C_0} \frac{dw_k}{2\pi i} h(w_k) \left(\frac{4i}{N}\frac{1}{1+w_k^2} - \frac{2w_k}{1+w_k^2}\right) \\ &= -6i \oint_{C_0} \frac{dw}{2\pi i} h(w) \frac{1}{1+w^2} + 3N \oint_{C_0} \frac{dw}{2\pi i} h(w) \frac{2w}{1+w^2}. \end{aligned} \tag{4.30}$$

The action involves a reflector and a three strings vertex as constitutions, which are 2-strings and 3-strings vertex, respectively. Supposed that the first term of Eq. (4.30) has non-zero value. In this case, the shifted theory becomes the theory with the original BRS charge and the different coupling constant by the redefinition of the string field, $\Psi = e^{q(h)}\Psi'$. If the first term vanishes, the shifted theory might become the original one through the redefinition. However, each cases should be investigated more carefully. Because, there is a possibility that the string field redefinition itself is ill-defined.

Let us consider the classical solution for

$$h_a(w) \;=\; \log\left(1+\frac{a}{2}\left(w+\frac{1}{w}\right)^2\right), \tag{4.31}$$

where a is a real parameter which is larger than or equal to $-1/2$. For this $h_a(w)$, $F_+^{(0)}(w)$ and $G_+^{(2)}(w)$ are given by

$$\begin{aligned} F_+^{(0)}(w) &= e^{h(w)}-1=\frac{a}{2}\left(w+\frac{1}{w}\right)^2=\frac{a}{2}\left(v_0^{(0)}(w)+v_2^{(0)}(w)\right), \\ G_+^{(2)}(w) &= -(\partial h(w))^2 e^{h(w)}=-a^2 w^{-2}\frac{\left(w^2-\frac{1}{w^2}\right)^2}{1+\frac{a}{2}\left(w+\frac{1}{w}\right)^2}. \end{aligned} \tag{4.32}$$

Indeed, we find that $F_+^{(0)}(\pm i)=0$ and $G_+^{(0)}(\pm i)=0$, and this $h(w)$ gives the classical solution by Eq. (4.17). The function $h_a(w)$ has the Laurent expansion as follows,

$$\begin{aligned} h_a(w) &= -\log(1-Z(a))^2-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}Z(a)^n\left(w^{2n}+\frac{1}{w^{2n}}\right), \\ Z(a) &= \frac{1+a-\sqrt{1+2a}}{a}. \end{aligned} \tag{4.33}$$

Using this expansion of $h_a(w)$, we can evaluate $\kappa(h_a)$ as

$$\kappa_N(h_a) \;=\; 3N\log(1-Z(a)). \tag{4.34}$$

Here, the first term of Eq. (4.30) vanishes. Therefore, we may naively expect that the shifted action is reduced to the original one and the classical solution should be pure gauge.

We consider the string field redefinition in detail. From Eq. (4.33), the operator $q(h_a)$ can be expressed by using the mode expansion $J_{\mathrm{gh}}(w)=\sum_n q_n w^{-n-1}$ as

$$\begin{aligned} q(h_a) &= -q_0\log(1-Z(a))^2+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(q_{2n}+q_{-2n})Z(a)^n \\ &= -q_0\log\frac{Z(a)}{2a}+q^{(+)}(h_a)+q^{(-)}(h_a), \end{aligned} \tag{4.35}$$

where $q^{(+)}$ and $q^{(-)}$ denote the positive and negative modes part of q, namely

$$q^{(+)}(h_a) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} q_{2n} Z(a)^n, \quad q^{(-)}(h_a) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} q_{-2n} Z(a)^n. \tag{4.36}$$

From the OPE of the ghost number currents

$$J_{\rm gh}(w) J_{\rm gh}(w') = \frac{1}{(w-w')^2} + \cdots, \tag{4.37}$$

the oscillator q_n satisfies $[q_m,\, q_n] = m\delta_{m+n}$. Therefore, we find the commutation relation of $q^{(\pm)}$ as follows,

$$[q^{(+)}(h_a),\, q^{(-)}(h_a)] = 2\sum_{n=1}^{\infty} \frac{1}{n} Z(a)^{2n} = -2\log(1 - Z(a)^2). \tag{4.38}$$

Using Eq. (4.38), we can rewrite the operator $\exp q(\xi_a)$ by the 'normal ordered' form

$$e^{q(h_a)} \;=\; = \left(1 - Z(a)^2\right)^{-1} \exp\left(-q_0 \log(1 - Z(a))^2\right) e^{q^{(-)}(h_a)} e^{q^{(+)}(h_a)}. \tag{4.39}$$

This is a well-defined operator since $|Z(a)| < 1$ for $a > -1/2$. Therefore, the classical solution for $a > -1/2$ should be pure gauge solution. However, in the case of $Z(a = -1/2) = -1$, this operator $e^{q(h_a)}$ has a singularity and the string field redefinition is ill-defined. Thus, we can obtain a non-trivial classical solution for $a = -1/2$.

In the case of $a=-1/2$, the classical solution is given by

$$\Psi_0 \;=\; Q_{\mathrm{L}}\left(-\frac{1}{4}\left(w+\frac{1}{w}\right)^2\right) I + C_{\mathrm{L}}\left(w^{-2}\left(w+\frac{1}{w}\right)^2\right) I. \qquad (4.40)$$

Each term of Eq. (4.40) has a well-defined Fock space expression as follows,

$$Q_{\mathrm{L}}\left(-\frac{1}{4}\left(w+\frac{1}{w}\right)^2\right)|I\rangle = -\sum_{n=0}^{\infty}\frac{(-1)^n}{2\pi}\left(\frac{2}{2n+1}-\frac{1}{2n+3}-\frac{1}{2n-1}\right) Q_{-2n-1}\,|I\rangle\,,$$

$$C_{\mathrm{L}}\left(w^{-2}\left(w+\frac{1}{w}\right)^2\right)|I\rangle$$
$$= \left[\frac{2}{\pi}c_1 + \frac{10}{3\pi}c_{-1} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{\pi}\left(\frac{2}{2n+1}-\frac{1}{2n+3}-\frac{1}{2n-1}\right)c_{-2n-1}\right]|I\rangle\,,$$

If we expand the string field around the classical solution, the shifted BRS charge is given by

$$Q'_{\mathrm{B}} = \frac{1}{2}Q_{\mathrm{B}} - \frac{1}{4}\left(Q_2 + Q_{-2}\right) + 2c_0 + c_2 + c_{-2}. \qquad (4.41)$$

We propose that the classical solution of Eq. (4.40) represents the condensation of the tachyon. The reasons are the following. First, it is impossible to connect the shifted theory to the original one by the string field redefinition, and so, the classical solution represents a non-trivial solution. Secondly, the classical solution is scalar. Thirdly, the physical states of the original theory are no longer physical in the expanded theory around the classical solution. Indeed, we can find that $Q'_{\rm B}\,|{\rm phys}\rangle \neq 0$ for all states $|{\rm phys}\rangle$ such that $Q_{\rm B}\,|{\rm phys}\rangle = 0$.

Of course, in order to clarify this conjecture, we must prove at least two propositions. First, there exists no BRS singlet in the Hilbert space. Secondly, the potential height $S[\Psi_0]$ is equal to the D-brane tension. At present, we can not deny the possibilities to prove two propositions. It should be noted about the latter proposition. As in the case of the marginal solutions, we find that

$$\frac{d}{da}S[\Psi_0] = \int (Q_{\rm B}\Psi_0 + \Psi_0 * \Psi_0) * \frac{d\Psi_0}{da} = 0. \tag{4.42}$$

So, the potential height $S[\Psi_0]$ is equal to zero for $a > -1/2$. However, it may become non-zero value at $a = -1/2$, because the classical solution is ill-defined for $a < -1/2$ and so $d\Psi_0/da$ may have a discontinuity at $a = -1/2$.

5 Summary

1. We proved the splitting property of the delta function.

2. We constructed the marginal solutions with well-defined Fock space expressions related to the $U(1)$ current in CSFT, and showed that the critical value in the level-truncated solution given by Sen and Zwiebach should be gauge artifact.

3. We constructed the scalar solution with a well-defined Fock space expression in CSFT, and proposed that it should be the classical solution corresponding to the tachyon condensation.