D-branes, Matrix Theory and K-homology

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§1 Introduction

Our purpose:

1) Constructing a matrix theory, in which the creation and annihilation process of D-branes is incorporated.

(a) It is important to find a non-perturbative formulation of string theory $\Rightarrow$ matrix theory.

(b) The creation and annihilation process of non-BPS D-branes or D-$\overline{\text{D}}$ pairs plays an important role $\Rightarrow$ K-theory.
non-BPS D9-branes(IIA) and D9-$\overline{\text{D9}}$ system(IIB).
$\Rightarrow$ lower dim. D-branes are constructed from them.

However,

(a) $\Rightarrow$ the K-theory structure is not clear in the framework of existing matrix theories.

(b) $\Rightarrow$ since a 10-dim. gauge theory is non-renormalizable, it is hard to consider it as a fundamental theory.

We propose a new matrix theory based on

$$\begin{cases} \text{non-BPS D-instantons in type IIA,} \\ \text{D-instanton/anti-D-instanton system in type IIB.} \end{cases}$$

$\Longrightarrow$ We call it as K-matrix theory

2) Another classification of D-branes.

- The D-brane charge is defined by the behavior of RR-fields on the spacetime X. Therefore, D-branes should be naively classified by cohomology.

  $\Downarrow$ they have gauge theory on them.

  K-theory (refined cohomology theory).

- The D-brane worldvolume is naively thought of as a homology cycle in spacetime X.

  $\Downarrow$ it has gauge theory on it.

  K-homology (refined homology theory).

We propose that

D-branes are classified by the K-homology group.

- it is dual to the K-theory group.

Moreover, we see that

K-matrix theory is the natural framework for K-homology.

Outline

§1 Introduction

§2 K-matrix theory

- basic structure of the K-matrix theory
- configurations with finite action

§3 Spectral triples and D-branes

- geometric interpretation of the configurations
- physical interpretation of the configurations

§4 D-branes and K-homology

- D-branes embedded in spacetime
- analytic K-homology
- topological K-homology

§5 Conclusion and Discussion

§2 K-matrix theory

Type IIA K-matrix theory

The theory of N non-BPS D-instantons:

- gauge group is $U(N)$.
- the bosonic fields consist of

  $$\begin{cases} \Phi^\mu \ (\mu = 0, \ldots, 9) : \text{scalar fields} \\ T : \text{tachyon}, \end{cases}$$

  which are self-adjoint (Hermitian) $N \times N$ matrices and belong to the adjoint repr.

In order to create arbitrary numbers of non-BPS D-instantons, we must take $N = \infty$.

$$\Downarrow$$

We assume:

- $\mathcal{H}$: ∞ dim. separable Hilbert space.
- Φ^μ and T: linear operators acting on $\mathcal{H}$.

Note:

- There is a tachyon T.
  $\Rightarrow$ matrix model + tachyon system.
- ${}^\forall\mathcal{H} \simeq l^2(\mathrm{N})$: the space of Chan-Paton indices.
  $\Rightarrow$ we should also choose $\mathcal{H}$.
- We consider the bosonic part only.

The action is roughly given as

$$S \sim \mathrm{Tr}_{\mathcal{H}} \left(e^{-T^2} [\Phi^\mu, \Phi^\nu]^2 + e^{-T^2} [\Phi^\mu, T]^2 + e^{-T^2} + \cdots \right),$$

which is estimated by

$$|S| \leq \mathrm{Tr}_{\mathcal{H}}\, e^{-T^2} \left(\| [\Phi^\mu, \Phi^\nu] \|^2 + \| [\Phi^\mu, T] \|^2 + 1 \right) + \cdots .$$

<u>Finite action configurations</u>:

$$\mathrm{Tr}_{\mathcal{H}}\, e^{-T^2} < \infty, \quad \| [\Phi^\mu, \Phi^\nu] \| < \infty, \quad \| [\Phi^\mu, T] \| < \infty.$$

$$\Downarrow$$

We deal with the configurations satisfying

$$\boxed{\begin{array}{l} [\Phi^\mu, \Phi^\nu],\ [\Phi^\mu, T] \in \mathrm{B}(\mathcal{H}) \quad \text{for} \quad \mu, \nu = 0, 1, \ldots, 9, \\ (T - \lambda)^{-1} \in \mathrm{K}(\mathcal{H}) \quad \text{for} \quad {}^\forall \lambda \notin \mathrm{R}. \end{array}}$$

$\mathrm{B}(\mathcal{H})$: the algebra of bounded linear operators on $\mathcal{H}$.
$\mathrm{K}(\mathcal{H})$: the set of compact operators on $\mathcal{H}$.

<u>Note</u>:

- $\mathrm{K}(\mathcal{H}) \simeq M_\infty(\mathrm{C})$ (naive large N).
- The tachyon T is not a bounded operator.
  $\Leftrightarrow$ eigenvalues of T^2 accumulate to the min. of the potential.

It is sometimes convenient to use

$$T_b = \frac{T}{\sqrt{1+T^2}} \in \mathrm{B}(\mathcal{H})$$

normalized such that $T_b^2 = 1$ is the min. of the potential.

$$\Downarrow$$

$$\boxed{\begin{array}{l} [\Phi^\mu, \Phi^\nu] \in \mathrm{B}(\mathcal{H}) \quad (\mu, \nu = 0, 1, \ldots, 9), \\ T_b \in \mathrm{B}(\mathcal{H}), \quad T_b^2 - 1 \in \mathrm{K}(\mathcal{H}), \quad [\Phi^\mu, T_b] \in \mathrm{K}(\mathcal{H}). \end{array}}$$

Note:

An op. K on $\mathcal{H}$ is said to be compact if it has an expansion

$$K = \sum_{n\geq 0} \mu_n \,|\, \psi_n \,\rangle \,\langle\, \phi_n \,|$$

with $\mu_n \to 0$ as $n \to \infty$,
where $\{\psi_n\}_{n\in\mathrm{N}}$ and $\{\phi_n\}_{n\in\mathrm{N}}$ are orthonormal sets.

Type IIB K-matrix theory

This is based on the D-instanton/anti-D-instanton system.

The theory of N D(-1) and M $\overline{\text{D(-1)}}$ has $U(N) \times U(M)$ gauge group and

$$\begin{cases} \Phi^\mu \in (\text{adj.}, 1) \ (\mu = 0, \ldots, 9) : \text{scalar fields on D(-1)} \\ \overline{\Phi}^\mu \in (1, \text{adj.}) \ (\mu = 0, \ldots, 9) : \text{scalar fields on } \overline{\text{D(-1)}} \\ T \in (N, M) : \text{a complex tachyon} \end{cases}$$

We take both $N = M = \infty$.

$$\Downarrow$$

- $\widehat{\mathcal{H}} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)}$: the Chan-Paton Hilbert space.
- Φ^μ: operators acting on $\mathcal{H}^{(0)}$.
- $\overline{\Phi}^\mu$: operators acting on $\mathcal{H}^{(1)}$.
- T: an operator from $\mathcal{H}^{(0)}$ to $\mathcal{H}^{(1)}$.

Finite action configurations:

In terms of the normalized tachyon T_b, such that the min. of the potential is $T_b^* T_b = T_b T_b^* = 1$, and using matrix representation:

$$\widehat{\mathcal{H}} = \begin{pmatrix} \mathcal{H}^{(0)} \\ \mathcal{H}^{(1)} \end{pmatrix}, \quad \widehat{\Phi}^\mu = \begin{pmatrix} \Phi^\mu & 0 \\ 0 & \overline{\Phi}^\mu \end{pmatrix}, \quad \widehat{F} = \begin{pmatrix} 0 & T_b^* \\ T_b & 0 \end{pmatrix},$$

$$\Downarrow$$

$$[\widehat{\Phi}^\mu, \widehat{\Phi}^\nu] \in \mathrm{B}(\widehat{\mathcal{H}}) \quad (\mu, \nu = 0, 1, \ldots, 9),$$
$$\widehat{F} \in \mathrm{B}(\widehat{\mathcal{H}}), \quad \widehat{F}^2 - 1 \in \mathrm{K}(\widehat{\mathcal{H}}), \quad [\widehat{\Phi}^\mu, \widehat{F}] \in \mathrm{K}(\widehat{\mathcal{H}}).$$

Chern-Simons terms

Chern-Simons terms for N non-BPS D-instantons:

$$S_{\rm CS} \;=\; \mathrm{SymTr}_N \mathrm{Tr}_2 \left(\sigma^1 \mathrm{Tr}_\psi \left(\widehat{C} e^{\left(-T^2+\frac{1}{2}[\Phi^\mu,\Phi^\nu]\psi_2^\mu\psi_2^\nu+i[\Phi^\mu,T]\psi_2^\mu\sigma^1\right)}\right)\right),$$

where

$$\widehat{C} = \sum_n C_{\mu_1\cdots\mu_n}(\Phi)\,\psi_1^{\mu_1}\cdots\psi_1^{\mu_n},$$

$C_{\mu_1\cdots\mu_n}(\Phi)$: **symmetric function in** Φ^μ

$$\{\psi_1^\mu,\psi_2^\nu\} = \delta^{\mu\nu},\quad \{\psi_1^\mu,\psi_1^\nu\} = \{\psi_2^\mu,\psi_2^\nu\} = 0.$$

$SO(10,10)$ **gamma matrices.**

Taking the limit $N\to\infty$, we obtain the Chern-Simons term for the IIA K-matrix theory. (Tr_N is replaced by $\mathrm{Tr}_{\mathcal{H}}$.)

It is estimated by

$$|S_{\rm CS}| \le \sum_n \frac{1}{n!} \parallel C_{\mu_1\ldots\mu_n}(\Phi) \parallel \mathrm{Tr}_{\mathcal{H}}\left(e^{-T^2}\right) \times \prod_{\{\mu_k\}} \parallel [T,\Phi^{\mu_k}] \parallel \text{ or } \parallel [\Phi^{\mu_k},\Phi^{\nu_l}] \parallel .$$

Thus the CS-term is finite if $\parallel C_{\mu_1\ldots\mu_n}(\Phi) \parallel < \infty$.

Basic example

BPS D($2m$)-brane configuration:

$$\mathcal{H} = L^2(\mathrm{R}^{2m+1}) \otimes S$$

where S: 2^m dim. $SO(2m+1)$ spinors.

$$T = u\,D = u \sum_{\alpha=0}^{2m} \widehat{p}_\alpha \otimes \gamma^\alpha,$$

$$\Phi^\alpha = \widehat{x}^\alpha \otimes 1 \quad (\alpha = 0, \cdots, 2m),$$

$$\Phi^i = 0 \quad (i = 2m+1, \cdots, 9).$$

Inserting this into the CS-term,

$$\begin{aligned} S_{\mathrm{CS}} &= u^{2m+1} C_{01\cdots 2m} \mathrm{Tr}_{\mathcal{H}} \left(e^{-T^2} \right) \\ &= u^{2m+1} C_{01\cdots 2m} 2^m \int d^{2m+1}k \, \langle k | e^{-u^2 k^2} | k \rangle \\ &= \mu_{2m} C_{01\cdots 2m} \int d^{2m+1}x, \end{aligned}$$

where $\mu_{2m} = 1/(2^{m+1}\sqrt{\pi}^{2m+1})$: a numerical constant.

$$\Downarrow$$

- The tachyon T is a Dirac operator.
  $\Rightarrow \mathrm{Tr}_{\mathcal{H}} \left(e^{-T^2} \right) \propto$ volume.
  $\Rightarrow$ for noncompact space, we can relax $\mathrm{Tr}_{\mathcal{H}} \left(e^{-T^2} \right) < \infty$.
- The correct coupling between D($2m$)-brane and RR ($2m+1$)-form.
  $\Rightarrow$ BPS commutative D-branes are constructed.

§3 Spectral triples and D-branes

Extracting geometric information from $(\mathcal{H}, \{\Phi^\mu\}, T)$,

$$\Downarrow$$

Each configuration in the K-matrix theory defines a spectral triple, and interpreted as a higher dim. D-brane.

Geometry of Φ^μ

Let $\widehat{\mathcal{A}} = \{\Phi^\mu\}$ be the algebra generated by the operators Φ^μ $(\mu = 0, \cdots, 9)$ for a fixed configuration.

$[\Phi^\mu, \Phi^\nu], C(\Phi) \in \mathrm{B}(\mathcal{H})$

$\Rightarrow$ We can assume that $\widehat{\mathcal{A}}$ is a C^*-algebra.

Def: a C^*-algebra is a norm closed self-adjoint subalgebra of the bounded operator algebra $\mathrm{B}(\mathcal{H})$.

If $\Phi^\mu(\mu = 0, 1, \ldots, 9)$ are mutually commuting operators, $\widehat{\mathcal{A}}$ is a commutative C^*-algebra.

Example:

- If $\widehat{\mathcal{A}}$ is the algebra generated by $\Phi^\mu = \widehat{x}^\mu$ $(\mu = 0, \ldots, n)$ with a relation $\Sigma_{\mu=0}^n (\Phi^\mu)^2 = R^2$, then $\widehat{\mathcal{A}} = C(S^n)$.
- $\Phi^\mu = \widehat{x}^\mu$ $(\mu = 0, \ldots, n)$ and if all elements vanish at infinity, then $\widehat{\mathcal{A}} = C_0(\mathrm{R}^n)$.

Recall the correspondence between space and algebra.

- A space $M \longleftrightarrow$ an algebra $\widehat{\mathcal{A}} = C_0(M)$.
- A point $p \in M \longleftrightarrow$ a character ϕ_p of $\widehat{\mathcal{A}}$.

In our case $\widehat{\mathcal{A}} = \{\Phi^\mu\}$,

- The character ϕ_p is determined by $\phi_p(\Phi^\mu)$, which is given by picking up one spectrum of Φ^μ.
  $\Rightarrow$ a point p is given by $\phi_p(\vec{\Phi}) = (\phi_p(\Phi^0), \cdots \phi_p(\Phi^9))$.
  $\Rightarrow$ This agrees with the standard interpretation that the eigenvalues of the matrix Φ^μ represents the position of the non-BPS D-instantons.

- The whole set of spectrum of $(\Phi^0, \Phi^1, \ldots, \Phi^9)$ should agree with some space M (i.e. a set of ∞ number of points).
  $\Rightarrow$ M is interpreted as the world-volume of higher dimensional D-branes made from infinite number of non-BPS D-instantons.

If $\widehat{\mathcal{A}} = \{\Phi^\mu\}$ is noncommutative, corresponding space becomes noncommutative.

Geometry of T

Def: A spectral triple is a triple $(\mathcal{H}, \widehat{\mathcal{A}}, T)$, where

- $\mathcal{H}$: a Hilbert space.
- $\widehat{\mathcal{A}}$: a C^*-algebra acting on $\mathcal{H}$.
- T: a self-adjoint operator on $\mathcal{H}$, satisfying

$$(T-\lambda)^{-1} \in \mathrm{K}(\mathcal{H}) \quad \text{for } {}^\forall\lambda \notin \mathrm{R}, \qquad [\widehat{a}, T] \in \mathrm{B}(\mathcal{H}) \quad \text{for } {}^\forall\widehat{a} \in \widehat{\mathcal{A}}.$$

This agrees with the configuration of the K-matrix theory.

- $\mathcal{H}$: the Chan-Paton Hilbert space.
- $\widehat{\mathcal{A}} = \{\Phi^\mu\}$.
- T: (unbounded) tachyon.

Note: A spectral triple is the basic ingredient for noncommutative geometry. In particular, T carries the additional information, mertic and gauge field etc. on $\widehat{\mathcal{A}}$.

Canonical triples

$$(\mathcal{H}, \widehat{\mathcal{A}}, T) = (L^2(M, S), C^\infty(M), D)$$

- M: a closed Riemannian spin manifold,
- $L^2(M, S)$: the Hilbert space of square integrable sections of the spinor bundle S on M,
- D: the Dirac operator associated with the Levi-Civita connection on S.

Mertic aspects

In general, the distance between two states is

$$d(\phi_1, \phi_2) = \sup_{a \in \widehat{\mathcal{A}}} \{ \ |\phi_1(a) - \phi_2(a)| \ | \quad \| \ [T, a] \ \| \leq 1 \ \},$$

where states ϕ_i $(i = 1, 2)$ are linear functions $\phi_i : \widehat{\mathcal{A}} \to \mathrm{C}$ such that $\phi_i(a^* a) \geq 0$ for $^\forall a \in \widehat{\mathcal{A}}$ and normalized as $\phi_i(1) = 1$.

$\Rightarrow$ agrees with the geodesic distance for the canonical triple.

$\Rightarrow$ roughly, $ds \approx 1/|T|$.

More explicitly, for the canonical triple, the heat kernel expansion gives

$$\mathrm{Tr}_{\mathcal{H}} \left(e^{-tT^2} \right) \sim \frac{2^{[n/2]}}{(4\pi t)^{n/2}} \int_M d^n x \sqrt{g} \left(1 + \frac{t}{12} R + O(t^2) \right),$$

$\Rightarrow$ we can measure the volume of M.

Note: the metric here is not the induced metric from the background, but the worldvolume metric.

Dimension

The dimension spectrum is defined by a subset $\Sigma \subset \mathrm{C}$ of the singularities of the analytic function

$$\zeta_T(z) = \mathrm{Tr}_{\mathcal{H}}(|T|^{-z}).$$

$\Rightarrow$ gives the dimension n for the canonical triple.

Diffeomorphism

In general, the automorphism of $\widehat{\mathcal{A}}$ generated by unitary operators U($\mathcal{H}$) in B($\mathcal{H}$) can be interpreted as

$$\mathrm{Aut}(\widehat{\mathcal{A}}) = \{\text{local gauge transf.}\} \times \{\text{diffeo.}\}.$$

A curved D($2m$)-brane:

$$\begin{aligned} \Phi^\mu &= f^\mu(\hat{x}^i) \quad (i = 0, \ldots, 2m) \\ T &= \frac{1}{2}\left\{\gamma^a e^i_{\ a}(\hat{x}), \left(\hat{p}_i + w_i^{ab}(\hat{x})\gamma_{ab}\right)\right\}, \end{aligned}$$

where $[\hat{x}^i, \hat{x}^j] = 0$, $[\hat{x}^i, \hat{p}_j] = i\delta^i_j$ and $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$, and

- $f^\mu(\hat{x})$: embedding function,
- $e^i_{\ a}(\hat{x})$: vielbein,
- $w_i^{ab}(\hat{x})$: spin connection.

The unitary operators $u_d = \exp(i\frac{1}{2}\{\hat{p}_i, \epsilon^i(\hat{x})\})$ corresponds to the diffeomorphism of the world-volume.

For infinitesimal trf. $\hat{y}^i = \hat{x}^i + \epsilon^i(\hat{x})$,

$$\begin{aligned} &u_d \Phi^\mu u_d^{-1} \sim f^\mu(\hat{y}^i), \\ &u_d T u_d^{-1} = \frac{1}{2}\left\{\gamma^a e'^i_{\ a}(\hat{y}^i), \left(\hat{p}'_i + w'^{ab}_i(\hat{y}^i)\gamma_{ab}\right)\right\}. \end{aligned}$$

Inserting them into the K-matrix action, we obtain a Polyakov type action

$$S \sim \int dx^{2m+1}\sqrt{g}\left(1 + 2\log 2 G_{\mu\nu}\partial_i f^\mu \partial_j f^\nu g^{ij} + \cdots\right),$$

where $g^{ij} = e^i_{\ a}\eta^{ab}e^j_{\ b}$ is the world-volume metric and $G_{\mu\nu} = \eta_{\mu\nu}$ is the background metric.

$\Rightarrow$ invariant under the diffeo.

§4 D-branes and K-homology

Embedding of D-branes

We fix a spacetime manifold X
and consider a D-brane world-volume M embedded in X.
$\Rightarrow$ the inclusion $i : M \to X$ is a proper map.

$$\Downarrow$$

In algebraic description,
$\mathcal{A} = C_0(X)$ and $\widehat{\mathcal{A}} = C_0(M)$: the corresponding algebras.
A *-homomorphism $i^* : \mathcal{A} \to \widehat{\mathcal{A}} \simeq \mathcal{A}/J_M$.
$\Rightarrow$ generalized to the noncommutative cases.

- $\mathcal{A}$: C^*-algebra of the fixed spacetime manifold, which could be noncommutative.
- $\phi : \mathcal{A} \to \mathrm{B}(\mathcal{H})$: a *-homomorphism
$\Rightarrow$ $\widehat{\mathcal{A}} = \mathrm{Image}\,\phi \cong \mathcal{A}/\ker\phi$ gives the world-volume of the D-brane embedded in $\mathcal{A}$.

Note: We do not apriori have the notion of spacetime in matrix theory.

Example: commutative D-branes embedded in R^{10}
$\Rightarrow$ $\mathcal{A} = C_0(\mathrm{R}^{10})$ and consider $\widehat{\mathcal{A}} = \{\Phi^\mu = \phi(x^\mu)\}$.

Analytic K-homology

Def: A **odd Fredholm module** over $\mathcal{A}$ is a triple $(\mathcal{H}, \phi, F)$, where

- $\mathcal{H}$ is a separable Hilbert space,
- $\phi : \mathcal{A} \to \mathrm{B}(\mathcal{H})$ is a *-homomorphism,
- F is a self adjoint operator in $\mathrm{B}(\mathcal{H})$, which satisfies

$$F^2 - 1 \in \mathrm{K}(\mathcal{H}), \quad [F, \phi(a)] \in \mathrm{K}(\mathcal{H}) \ \text{ for } {}^\forall a \in \mathcal{A}.$$

Note: The direct sum $(\mathcal{H}_0 \oplus \mathcal{H}_1, \phi_0 \oplus \phi_1, F_0 \oplus F_1)$ is again a Fredholm module.

A Fredholm module $(\mathcal{H}, \phi, F)$ describes a configurations of the IIA K-matrix theory, the D-branes embedded in the fixed space-time algebra $\mathcal{A}$.

- $\mathcal{H}$: the Chan-Paton Hilbert space.
- ϕ specifies the world-volume $\widehat{\mathcal{A}} = \mathrm{Image}\ \phi$.
- F: the normalized tachyon T_b.

$$\Downarrow$$

the classification of the D-brane configurations
= the classification of the Fredholm modules.

Def: **K-homology** $K^1(\mathcal{A})$ is defined by

$$K^1(\mathcal{A}) = \{(\mathcal{H}, \phi, F) : \text{odd Fredholm module}\}/\sim .$$

The equivalence relation $\sim$ is generated by

(a) unitary equivalence:
$(\mathcal{H}_i, \phi_i, F_i)$ $(i = 0, 1)$ are unitary equivalent if there is a unitary operator in $B(\mathcal{H}_0, \mathcal{H}_1)$ intertwining ϕ_i and F_i.

(b) operator homotopy:
They are operator homotopic if $\mathcal{H}_0 = \mathcal{H}_1$, $\phi_0 = \phi_1$ and there is a norm continuous path between F_0 and F_1.

(c) addition of a degenerate Fredholm module:
which satisfy $F^2 - 1 = [F, \phi(a)] = 0$.

Physical interpretations

(a) the gauge equivalence of Fredholm modules.

(b) the continuous deformation of the tachyon configuration.

(c) the addition of non-BPS D-instantons that would be annihilated by the tachyon condensation.

The K-homology $K^1(\mathcal{A})$ classifies the D-brane configurations in the IIA K-matrix theory.

The K-homology which classifies the D-brane configurations in the IIB K-matrix theory is $K^0(\mathcal{A})$.

Def: An even Fredholm module over $\mathcal{A}$ is a triple $(\widehat{\mathcal{H}}, \widehat{\phi}, \widehat{F})$, where

$$\widehat{\mathcal{H}} = \begin{pmatrix} \mathcal{H}^{(0)} \\ \mathcal{H}^{(1)} \end{pmatrix}, \quad \widehat{\phi} = \begin{pmatrix} \phi_0 & 0 \\ 0 & \phi_1 \end{pmatrix}, \ \widehat{F} = \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix},$$

- $\mathcal{H}^{(i)}$ are separable Hilbert spaces $(i = 0, 1)$,
- $\phi_i : \mathcal{A} \to \mathrm{B}(\mathcal{H}^{(i)})$ are *-homomorphisms $(i = 0, 1)$,
- $\widehat{F} \in \mathrm{B}(\widehat{\mathcal{H}})$ satisfies

$$\widehat{F}^2 - 1 \in \mathrm{K}(\widehat{\mathcal{H}}), \quad [\widehat{\phi}(a), \widehat{F}] \in \mathrm{K}(\widehat{\mathcal{H}}), \quad \text{for } {}^\forall a \in \mathcal{A}.$$

A Fredholm module $(\widehat{\mathcal{H}}, \widehat{\phi}, \widehat{F})$ describes a configurations of the IIB K-matrix theory.

Def: K-homology $K^0(\mathcal{A})$ is defined by

$$K^0(\mathcal{A}) = \{(\widehat{\mathcal{H}}, \widehat{\phi}, \widehat{F}) : \text{even Fredholm module}\}/\sim .$$

K-homology vs K-theory

Note: For $\mathcal{A} = C_0(X)$: commutative, we also denote

$$K^i(C_0(X)) = K_i(X) \quad : \quad \text{K-homology,}$$
$$K_i(C_0(X)) = K^i(X) \quad : \quad \text{K-theory.}$$

Example: $\mathcal{A} = C_0(\mathrm{R}^n)$,

$$K_0(\mathrm{R}^n) = \begin{cases} \mathrm{Z} & (n : \text{even}) \\ 0 & (n : \text{odd}), \end{cases} \qquad K_1(\mathrm{R}^n) = \begin{cases} 0 & (n : \text{even}) \\ \mathrm{Z} & (n : \text{odd}). \end{cases}$$

$$\Downarrow$$

Flat Dp-brane is classified by
$K_1(\mathrm{R}^{p+1})$ ($K_0(\mathrm{R}^{p+1})$) in the IIA (IIB) K-matrix theory.

In K-theory, flat Dp-brane charges are classified by $K^i(\mathrm{R}^{9-p})$.
R^{9-p} is Poincaré-dual to R^{p+1} in the space-time R^{10}.

This comes from

- K-homology of X classifies the world-volume M of the D-brane embedded in the space-time manifold X.
- K-theory of X classifies the D-brane charge defined by RR-fields on the space transverse to M in X.

In general, for a n-dimensional compact manifold X,

$$\begin{array}{ccccc} \text{K-homology} & & \text{K-theory} & & \\ K_i(X) & \simeq & K^{n-i}(X) & : & \text{K-dual} \\ \updownarrow & & \updownarrow & & \\ H_i(X;\mathrm{Z}) & \simeq & H^{n-i}(X;\mathrm{Z}) & : & \text{Poincaré dual} \\ \text{homology} & & \text{cohomology} & & \end{array}$$

Physical interpretation: In type IIA ($i = 1$),

- $K_1(X)$ classifies the D-brane constructed by non-BPS D-instanton system.
- $K^{n-1}(X)$ classifies the D-brane constructed by non-BPS D$(n-1)$-brane system(n: even), or D$(n-1)$-brane - anti D$(n-1)$-brane system(n: odd).

The spectrum of the D-branes should not depend on how they are constructed, and hence $K_1(X) \simeq K^{n-1}(X)$.

Topological K-homology

When the algebra $\mathcal{A}$ is commutative, we have a topological definition of K-homology.

Def: An even(odd) K-cycle on X is defined to be a triple (M, E, φ), where

- M is an even(odd) dim. closed Spin^c manifold.
- E is a complex vector bundle on M.
- φ is a continuous map from M to X.

Note: the disjoint union $(M_0, E_0, \varphi_0) \cup (M_1, E_1, \varphi_1)$ is again a K-cycle.

Physical interpretation

K-cycle (M, E, φ) givess the world-volume geometry of the D-brane (massless modes of open strings):

- M: the world-volume of the D-brane.
- E: Chan-Paton gauge bundle E on M (gauge field).
- φ: the embedding of the D-brane to the space-time X (tranverse scalar fields).

Def: The topological K-homology $K_*^{top}(X)$ is defined by

$$K_0^{top}(X) = \{(M, E, \varphi) : \text{even K-cycle}\}/\sim .$$
$$K_1^{top}(X) = \{(M, E, \varphi) : \text{odd K-cycle}\}/\sim .$$

Note: $\{(M, \varphi)\}/\sim \quad \Rightarrow$ ordinary homology.

The equivalence relation $\sim$ is generated by

(a) Bordism :

$(M_0, E_0, \varphi_0) \sim (M_1, E_1, \varphi_1)$, if ${}^{\exists}(W, E, \varphi)$ s.t.
$(\partial W, E|_{\partial W}, \varphi|_{\partial W}) \simeq (M_0, E_0, \varphi_0) \cup (-M_1, E_1, \varphi_1)$.
Here $-M_1$ denotes M_1 with the reversed Spin^c str..

(b) Direct sum:

$(M, E_1 \oplus E_2, \varphi) \sim (M, E_1, \varphi) \cup (M, E_2, \varphi)$.

(c) Vector bundle modification:

$(M, E, \varphi) \sim (\widehat{M}, \widehat{H} \otimes \rho^*(E), \varphi \circ \rho)$, where

- $\rho : \widehat{M} \to M$: a sphere bundle on M with fiber $S_p = \rho^{-1}(p)$, even dim. sphere.
- $\widehat{H}$: a vector bundle on $\widehat{M}$, such that the restriction $\widehat{H}|_{S_p}$ is the generator of $\widetilde{K}(S_p) = \mathbb{Z}$.

Physical interpretation

(a) the deformations of the world-volume of the D-brane.

(b) the gauge symmetry enhancement for coincident D-branes.

(c) the descent relation of the D-branes:
{a spherical D-brane with a non-trivial gauge bundle}
$\sim$ {a lower dimensional D-brane}.

$\boxed{\text{The topological K-homology } K_1^{top}(X)\ (K_0^{top}(X)) \text{ classifies the stable D-brane configurations in type IIA (IIB) string theory.}}$

Isomorphism

The topological K-homology is isomorphic to the analytic K-homology:

$$\mu_i : K_i^{top}(X) \xrightarrow{\sim} K_i(X) \quad (i = 0, 1)$$
$$(M, E, \varphi) \mapsto (\mathcal{H}, \phi, D)$$

Let $(M, E, \varphi) \in K_1^{top}(X)$. Since M is an odd dimensional closed Spin^c manifold, we can define a spin bundle S on M.

- $\mathcal{H} = L^2(M, S \otimes E)$: the space of L^2-section of the vector bundle $S \otimes E$.
- We can define a Dirac operator D on $\mathcal{H}$ by choosing a connection on the bundle $S \otimes E$ as usual.
- $\phi : C(X) \to \mathrm{B}(\mathcal{H})$ is defined by the multiplication of the function $\phi(f(x)) \equiv f(\varphi(x))$ for $f(x) \in C(X)$.
  ($\Leftrightarrow$ Image $\phi = C(M)$.)

Note: This is a genralization of the canonical triple, including gauge fields.

We can always obtain a configuration in the K-matrix theory corresponding to a conventional world-volume description of the D-brane.

Chern character and Chern-Simons terms

There is a Chern-character map from the K-homology group to the ordinary homology group:

$$\begin{aligned} \mathrm{ch.} &: K_0(X) \to H_{even}(X;\mathrm{Q}), \\ \mathrm{ch.} &: K_1(X) \to H_{odd}(X;\mathrm{Q}). \end{aligned}$$

$$\mathrm{ch.}(M,E,\varphi) = \varphi_*(\mathrm{ch}(E) \cup \mathrm{Td}(TM) \cap [M]).$$

The coupling of homology to RR-fields are as usual

$$\begin{aligned} S_{\mathrm{CS}} &= \int_{\mathrm{ch.}(M,E,\varphi)} C \\ &= \int_M \varphi^* C \wedge \mathrm{ch}(E) \wedge \mathrm{Td}(TM), \end{aligned}$$

where $C \in \Omega(X)$ is the formal sum of RR-fields.

$\Longrightarrow$ This agrees with the CS-term for a BPS D-brane.

§5 Conclusion and Discussion

Conclusion

- We proposed the K-matrix theory, based on non-BPS D-instantons in type IIA string theory and D-instanton/anti-D-instanton system in type IIB string theory.
- The configurations with finite action are identified with spectral triples, which are algebraic description of the geometry on the world-volume of higher dimensional D-branes.
- We claimed that the configurations in the K-matrix theory are classified by K-homology.
  More ganerally, D-branes embedded in spacetime are classified by K-homology.

Omitted today

- KK-theory
- Boundary states

Discussion

- Chern-Simons term (Myers' term).
- Consistency of the theory as a quantum theory.
- K-matrix theory in general curved backgrounds.
  An ad hoc resolution: embedding the manifold to a higher dimensional Euclidean space R^N $(N \geq 10)$.
- Relation between the closed string background and the space-time algebra we can choose.
- The appearance of the closed strings.
  Unfortunately, K-homology is not powerful enough to classify the fundamental strings and NS5-branes.
- Applications to the formulation of M-theory.
- Relation to the description of D-branes as objects of the derived category of coherent sheaves.