Field Theories on the Lattice

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第 I 部

入門編

1　格子上の場の理論とは？

Quantum field theory

⇕

Quantum mechanics with infinitely many degrees of freedom

Definition ?

1. Pertuebation theory (← expansion)
   - divergence → renormalization
   - gauge volume → gauge fixing
   - path-integral quantization, canonical quantization
   - regularization
2. Lattice (quantum) field theory
   - not rely on expansion
   - lattice spacing a gives natural cut-off → no divergence
   - continuum($a \to 0$) limit has to be taken
   - path-integral in Euclidean space
   - Strong coupling expansion, Weak coupling expansion, Monte Carlo simulation

2 Continuum limit in lattice Field theories

ex. $\lambda\phi^4$ theory

2.1 Determination of lattice action

指導原理

(1) $\lim_{a\to 0} S_{lat} = S_{cont}$

(2) symmetry of $S_{cont} \Rightarrow$ symmetry of S_{lat}

$$S_{cont} = \int d^4x \left[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right]$$

$$\Downarrow$$

$$\begin{aligned} S_{lat} &= a^4 \sum_n \left[-\frac{1}{2}\phi(na)\sum_\mu \frac{\phi(na+\mu a)+\phi(na-\mu a)-2\phi(na)}{a^2}\right. \\ &+ \left.\frac{1}{2}m^2\phi(na)^2 + \frac{\lambda}{4!}\phi(na)^4\right] \end{aligned}$$

$x = na = (n_1, n_2, n_3, n_4)a$

$$\Downarrow$$

$$\begin{aligned} S_{lat} &= \sum_n \left[-\frac{1}{2}\phi_L(n)\sum_\mu\{\phi_L(n+\mu)+\phi_L(n-\mu)-2\phi_L(n)\}\right. \\ &+ \left.\frac{1}{2}M^2\phi_L(n)^2 + \frac{\lambda}{4!}\phi_L(n)^4\right] \end{aligned}$$

$\phi_L(n) \equiv a\phi(na)$ and $M \equiv ma$ (dimensionless)

(1) is OK

(2) Z_2 invariance ($\phi \to -\phi$) $\Rightarrow \phi^3, \phi^5, \cdots$ are absent

Remark: Improved derivative

$$\nabla_\mu^2\phi(na) \equiv \frac{\phi(na+\mu a)+\phi(na-\mu a)-2\phi(na)}{a^2} = \partial_\mu^2\phi + a^2\frac{2}{4!}\partial^4\phi + O(a^4)$$

$$\Downarrow$$

$$\nabla_\mu^2\phi(na) - a^2\frac{2}{4!}\nabla_\mu^4\phi = \partial_\mu^2\phi + O(a^4)$$

(Classical) improvement of action

2.2 Continuum limit

2-points function at zero momentum

$$\lim_{|n_4-m_4|\to\infty}\sum_{\vec{m}}\langle\phi(\vec{n},n_4)\phi(\vec{m},m_4)\rangle_c \simeq Z\exp[-|n_4-m_4|/\xi]$$
$$= Z\exp[-t\ m_{\rm phy}], \qquad t = |n_4-m_4|a$$

ξ: correlation length, $m_{\rm phy}a = 1/\xi$

$$\Downarrow$$

$$\lim_{a\to 0}\xi(a) = \lim_{a\to 0}\frac{1}{m_{\rm phy}(a)a} \to \infty$$

"Continuum limit should be taken at $\xi\to\infty$"

$\xi\to\infty \Leftrightarrow$ 2nd(or more) order phase transition point

$a \neq 0$ but $a \ll 1$ $(1 \ll \xi \neq \infty)$

$$m_{\rm phy}(a) = m_{\rm phy}(0) + \underbrace{O(a)}_{\text{lattice spacing error}}$$

2.3 Phase structure of $\lambda\phi^4$ and continuum limit

$\xi = \infty \Leftrightarrow$ phase transition line in parameter space (M^2, λ)

Classical argument

$$V(\phi) = \frac{\lambda}{4!}(\phi^2 + \frac{3!M^2}{\lambda})^2 - \frac{3!}{4\lambda}M^4$$

- $M^2 > 0 \Rightarrow \langle\phi\rangle = 0$ (symmetric)

- $M^2 = 0 \Rightarrow \langle\phi\rangle = 0$ and $M_\phi = 0$ (critical)

- $M^2 < 0 \Rightarrow \langle\phi\rangle \neq 0$ (broken)

$\langle\phi\rangle$: order parameter

$$\begin{cases} \langle\phi\rangle = 0 & Z_2 \text{ symmetric phase} \\ \langle\phi\rangle \neq 0 & Z_2 \text{ broken phase} \end{cases}$$

Quantum effect

$$\langle \phi_n \phi_m \rangle \sim Z \exp[-M_\phi |n-m|]$$

$m_\phi a = M_\phi \simeq C|M^2 - M_c^2|^\nu$, ν : critical exponent

- $\nu = \frac{1}{2}$ (mean-field value) in both phases
- $C_{broken} \neq C_{symmetric}$ in general

continuum limit

$(M^2, \lambda) \to (M_c^2, \lambda_c)$ or $M^2 \to M_c^2(\lambda)^\pm$ for fixed λ

finite size effect

finite volume($V \neq \infty$) $\Rightarrow \langle \phi \rangle = 0$

add an external field H

$$\langle \phi \rangle_{H,V} = \frac{1}{Z(H,V)} \int \mathcal{D}\phi e^{-S(\phi)+H\sum_n \phi(n)} \phi = \frac{\partial \log Z(H,V)}{\partial H}$$

$$\langle \phi \rangle = \lim_{H \to 0} \lim_{V \to \infty} \langle \phi \rangle_{H,V}$$

- two limits don't commute
- finite size scaling from finite H and V to extrapolate $V \to \infty$ and $H \to 0$

3 Renormalization group and continuum limit

3.1 Renormalization Group (RG)

$$e^{-S'(\phi')} = \int \mathcal{D}\phi \delta(\sum_{m \in b_n} \phi_m - c\phi'_n) e^{-S(\phi)}$$

$(\mathrm{RG})_2$ を 1 回やると

$$\begin{aligned} S(\phi) &\to S'(\phi') \\ (M^2, \lambda) &\to (M^2_{(1)}, \lambda_{(1)}) \\ a &\to b \Leftrightarrow 2a \\ \xi &\to \xi' \Leftrightarrow 2\xi \end{aligned}$$

$(\mathrm{RG})_2$ を s 回やると

$$\begin{aligned} (M^2, \lambda) &\to (M^2_{(s)}, \lambda_{(s)}) \\ a &\to a_s \Leftrightarrow 2^s a \\ \xi &\to \xi' \Leftrightarrow 2^s \xi \\ \text{Micro (high energy)} &\to \text{Macro (low energy)} \end{aligned}$$

example: RG flow of $\lambda\phi^4$ in 4-dim.

3.2 Continuum limit via RG

$$
\begin{aligned}
(M_0^2, \lambda_0, \xi) &\rightarrow (M_{(s)}^2, \lambda_{(s)}, \xi_{(s)}) \\
a_0 &\rightarrow a_s \Leftrightarrow 2^s a_0
\end{aligned}
$$

Continuum limit

$$
\lim_{s\to\infty} a_s = a(\text{ fixed}) \Leftrightarrow a_0 = \lim_{s\to\infty} \frac{a}{2^s} \rightarrow 0
$$

$$
\Downarrow
$$

$$
(M_0^2, \lambda_0) \rightarrow (M_c^2, \lambda_c) \Leftrightarrow \lim_{s\to\infty} (M_s^2, \lambda_s) = (M_R^2, \lambda_R)
$$

RGの行き先の bare parameters $\simeq$ low energy parameters in the continuum limit

$$\langle \phi_0(0)\phi_0(2^s na)\rangle_0 \simeq \langle \phi_s(0)\phi_s(na)\rangle_s$$

3.3 Examples

3.3.1 $\lambda\phi^4$ in $d = 4$

$$\lambda_s = \frac{\lambda_0}{1 + A \cdot s} \to 0 \text{ as } s \to \infty$$

for $\lambda \ll 1$. Even non-perturbatively,

$\lim_{s\to\infty}(\mathrm{RG})_s\lambda = 0$

$\Downarrow$

$\lambda_R \to 0$ in $a \to 0$ "Triviality of $\lambda\phi^4$ theory"

($\lambda = 0$ is an infra-red fixed point)

3.3.2 $\lambda\phi^4$ in $2 < d < 4$

- ultra-violate fixed point: $(M^2, \lambda) = (0, 0)$
  non-trivial continuum limit can be defined around this point
- infra-red fixed point: $(M^2, \lambda) = (M_f^2, \lambda_f)$
  continuum limit toward this point $\rightarrow \lambda_R = \lambda_f$

3.4 Universality

continuum limit $\Leftrightarrow$ RG fixed point **の分類とその性質**
RG **によって** parameter **が**

増える	relevant	M^2
減る	irrelevant	
変わらない	marginal	$\rightarrow$ relevant or irrelevant (λ)

Fixed point **の分類** $\Leftarrow$ relevant parameters

Universality
symmetry of the system $\Rightarrow$ relevant parameters $\Rightarrow$ fixed point/critical phenomina $\Rightarrow$ continuum field theory

4 Pure gauge theories on the lattice

4.1 Continuum QCD

$$L = \bar{\psi}\gamma_\mu(\partial_\mu + igA_\mu)\psi + \frac{1}{2}\mathrm{Tr}F_{\mu\nu}^2$$

Gauge invariance $\Omega(x) \in G$

$$\begin{cases} \psi(x)' &= \Omega(x)\psi(x) \\ \bar{\psi}(x)' &= \bar{\psi}(x)\Omega(x)^\dagger \\ A_\mu(x)' &= \dfrac{\Omega(x)\partial_\mu\Omega(x)^\dagger}{ig} + \Omega(x)A_\mu(x)\Omega(x)^\dagger \end{cases}$$

$\bar{\psi}(x)\psi(x+\Delta x)$を gauge invariant にするには？

$$\begin{aligned} \bar{\psi}(x)[1 + ig\Delta x^\mu A_\mu(x)]\psi(x+\Delta x) &= \bar{\psi}(x)[\psi(x) + \Delta x^\mu(\partial_\mu + igA_\mu)\psi(x)] \\ &+ O((\Delta x)^2) \end{aligned}$$

Gauge invariant at $O(\Delta x)$

$\Downarrow$: finite Δx

$$\bar{\psi}(x)\underbrace{P\exp[ig\int_x^{x+\Delta x} A_\mu(y)dy]}_{\equiv U(x,\ x+\Delta x)}\psi(x+\Delta x) \simeq \bar{\psi}(x)e^{ig\Delta x A_\mu(x+\Delta/2)}\psi(x+\Delta x)$$

P: path ordered product

$U(x, x+\Delta x)$の変換性

Abelian case: $\Omega(x) = e^{i\theta(x)}$, $A'_\mu = A_\mu - \partial_\mu\theta/g$

$$\begin{aligned} U(x,x+\Delta x)' &= \exp ig[-\int_x^{x+\Delta x}\frac{\partial_\mu\theta}{g}dy^\mu]U = e^{i\theta(x)}Ue^{-i\theta(x+\Delta x)} \\ &= \Omega(x)U(x,x+\Delta x)\Omega(x+\Delta x)^\dagger \end{aligned}$$

Same for non-abelian cases.

4.2 Lattice action for gauge fields

4.2.1 Gauge fields and link variables

scalar field $\rightarrow$ point n
vector field $\rightarrow$ link (n, μ)

$A_\mu = A_\mu^a T^a \in \mathcal{SU}(\mathrm{N})$, T^a: traceless, hermite

$$trT^aT^b = \frac{1}{2}\delta_{ab}, \quad [T^a, T^b] = if^{abc}T^c$$

But A_μ is NOT covariant.

<u>link variable</u>

$$\begin{aligned} U_{n,\mu} &= e^{igaA_\mu(n)} \in \mathrm{SU(N)} \\ U_{n+\mu,-\mu} &\equiv U_{n,\mu}^\dagger \end{aligned}$$

<u>Gauge transformation</u> $g_n \in \mathrm{SU(N)}$

- $U_{n,\mu} \rightarrow U_{n,\mu}^g = g_n\, U_{n,\mu}\, g_{n+\mu}^\dagger$
- product of U's

$$\prod U \equiv U_{n,\mu_1} U_{n+\mu_1,\mu_2} \cdots U_{m-\mu_k,\mu_k} \rightarrow g_n \prod U\, g_m^\dagger$$

  $\Downarrow$: closed loop C at n

$$\prod_C U \rightarrow g_n \prod_C U\, g_n^\dagger$$

- tr $\Pi_C\, U$ is gauge invariant

4.2.2 Lattice action

- gauge invariance

$$S = \sum_n \sum_{\Gamma \in \text{closed loop}} \beta_\Gamma \text{tr} O_\Gamma(n), O_\Gamma = \prod_{i \in \Gamma} U_i$$

$$\Downarrow \text{ simplest loop (plaquette)}$$

$$S = \sum_{n,\mu \neq \nu} \beta \text{tr}[U_{n,\mu} U_{n+\mu,\nu} U^\dagger_{n+\nu,\mu} U^\dagger_{n,\nu}] = \sum_{n,\mu<\nu} \beta \text{Re tr}[U_{\mu\nu}(n)]$$

- naive $a \to 0$ limit

$$U_{n,\mu} = e^{iagA_\mu(n)}$$

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots} \quad \text{Hausdorff formula}$$

$$\begin{aligned}
\text{tr}\, U_{\mu\nu}(n) &= \text{tr exp}[iag\{A_\mu(n) + A_\nu(n+\mu) + \frac{iag}{2}[A_\mu, A_\nu] + \cdots\}] \\
&\times \text{exp}[-iag\{A_\mu(n+\nu) + A_\nu(n) - \frac{iag}{2}[A_\mu, A_\nu] + \cdots\}] \\
&= \text{tr exp } iag\,[\{A_\nu(n+\mu) - A_\nu(n)\} - \{A_\mu(n+\nu) \\
&\quad -A_\mu(n)\} + iag[A_\mu, A_\nu] + O(a^2)\big] \\
&= \text{tr}\{1 + i \underbrace{a^2 g F_{\mu\nu}}_{\sim 0} - \frac{a^4 g^2}{2} F^2_{\mu\nu} + O(a^6)\}
\end{aligned}$$

$$\lim_{a\to 0} S = \sum_{n,\mu\neq\nu} \beta[\text{tr}1 - \frac{a^4 g^2}{2} \text{tr} F^2_{\mu\nu}]$$

- redefine

$$\begin{aligned}
S_G &= \sum_{n,\mu\neq\nu} \beta \text{tr}[U_{\mu\nu}(n) - 1] \\
&\to -a^4 \sum_n \sum_{\mu<\nu} \beta g^2 \text{tr} F^2_{\mu\nu} = -\int d^4x \sum_{\mu<\nu} \text{tr} F^2_{\mu\nu}
\end{aligned}$$

with $\beta = \dfrac{1}{g^2}$

4.2.3 Path integral

$$Z = \int \prod_{n,\mu} d\,U_{n,\mu}\, e^{\,S_G} = \int \prod_{n,\mu} d\,U_{n,\mu}\, \exp \frac{1}{g^2} \sum_{n,\mu\neq\nu} \mathrm{tr}(U_{\mu\nu}(n) - 1)$$

$d\,U$: haar measure

$$d\,(VU) = d\,(UV) = d\,U \qquad ({}^{\forall}V \in \mathrm{SU(N)})$$

Remark

Re tr $(U_{\mu\nu} - 1) \leq 0 \Rightarrow {}^{\forall}U_{\mu\nu} = 1$ as $g^2 \to 0$ (weak coupling limit)

For volume $V = \infty$

$${}^{\forall}U_{\mu\nu} = 1 \leftrightarrow {}^{\forall}U_{n,\mu} = \omega(n)\omega(n+\mu)^{\dagger}$$

Then, by the gauge transformation $g_n = \omega(n)^{\dagger}$, one can set

$${}^{\forall}U_{n,\mu} = 1.$$

Therefore $U_{n,\mu} = e^{iagA_\mu(n)}$ is justified.

4.3 Elitzur's theorem and observables

4.3.1 Elitzur's theorem

$\langle$ Gauge non-invariant operator $\rangle = 0$

"Local Gauge symmetry can not be broken spontaneously."

All observables = gauge invariant quantities = any functions of tr O_Γ for closed loop Γ

Global vs. Local

- Global symmetry: $\forall n\ \sigma_n \to -\sigma_n$

  $$\begin{aligned}\langle\sigma\rangle_H &= \int \mathcal{D}\sigma e^{-S+H\sum_n \sigma_n}\sigma = \frac{1}{2}\int \mathcal{D}\sigma e^{-S+H\sum_n \sigma_n}\sigma \\ &- \frac{1}{2}\int \mathcal{D}\sigma e^{-S-H\sum_n \sigma_n}\sigma = \frac{1}{2}\int \mathcal{D}\sigma e^{-S+H\sum_n \sigma_n}\sigma(1 - e^{-2H\sum_n \sigma_n}) \neq 0\end{aligned}$$

  as $V \to \infty$ for $\sum_n \sigma_n = O(V)$, then $H \to 0$.

- Local symmetry: $\sigma_n \to -\sigma_n$, $\sigma_{m\neq n} \to \sigma_m$

$$\begin{aligned}\lim_{V\to\infty}\langle\sigma_n\rangle_H &= \int \mathcal{D}\sigma e^{-S+H\sum_n \sigma_n}\sigma_n \\ &= \frac{1}{2}\int \mathcal{D}\sigma e^{-S+H\sum_n \sigma_n}\sigma(1-e^{-2H\sigma_n}) \to 0\end{aligned}$$

as $H \to 0$.

4.3.2 Observables in pure gauge theories

- Wilson loop

$$W(C = L\times T) = \mathrm{tr}\prod_c U$$

physical meaning

$$\langle W(C)\rangle \to \exp[-TV(L)] \text{ as } T\to\infty$$

$V(L)$: quark-antiquark static potential

  - area-law

$$\langle W(C)\rangle \to \exp[-cLT]$$

$V(L) = cL$ linearly raising potential $\Rightarrow$ quark confinement
$c = \sigma a^2$, σ: string tension

  - perimeter-law

$$\langle W(C)\rangle \to \exp[-c'(L+T)]$$

$$V(L) = c'(1+\frac{L}{T}) \sim c' \Rightarrow \text{deconfine}$$

- Glue-ball J^{PC}

$$\mathcal{O}_{\mu\nu}(n) = \mathrm{tr}U_{\mu\nu}(n)$$

$$\sum_{\vec{n}}\langle\mathcal{O}_{\mu\nu}(\vec{0},0)\mathcal{O}_{\mu\nu}(\vec{n},t)\rangle \sim Ze^{-m_G t}$$

m_G: mass of glue-ball
"Glue-ball has not been observed yet."

- Polyakov loop

  $L^3 \times T$ lattice with periodic b.c. in "time"

$$P(\vec{n}, T) = \mathrm{tr} U_4(\vec{n}, 0) U_4(\vec{n}, 1) \cdots U_4(\vec{n}, T)$$

$$P(T) = \langle P(\vec{n}, T) \rangle \sim e^{-F_q T}$$

  F_q: free energy for isolated quark

$$\begin{cases} P(T) = 0 \;\Rightarrow\; F_q = \infty \;\Rightarrow\; \text{quark confinement} \\ P(T) \neq 0 \;\Rightarrow\; F_q \neq \infty \;\Rightarrow\; \text{deconfine} \end{cases}$$

  Global Z_N symmetry

  $^{\exists}t$, $^{\forall}\vec{n}$

$$U_4(\vec{n}, t) \to \Omega_t U_4(\vec{n}, t), \quad \Omega_t \in \text{center of SU(N)} = Z_N$$

  S_G is invariant under Z_N,
  but $P(\vec{n}, T) \to \Omega_t P(\vec{n}, T)$

$$P(T) \neq 0 \Leftrightarrow \text{spontaneous breakdwon of } Z_N$$

- Polyakov loop correlation

$$\langle P(\vec{n}, T) P(\vec{m}, T)^\dagger \rangle \sim e^{-\mu(T)|\vec{n} - \vec{m}|}$$

  $\mu(T)$: torelon mass,
  $\sigma(T) = \dfrac{\mu(T)}{T}$: string tension for $T \neq \infty$

$$\sigma \simeq \sigma(T) + \frac{\pi}{3T^2}, \quad T \to \infty$$

4.4 Strong Coupling Expansion (SCE) and Confinement

4.4.1 SU(N) group integral

$$\int dU\ 1 = 1$$

$$\int dU\ U_{ab} = 0$$

$$\int dU\ U_{ab}U^{\dagger}_{kl} = \frac{1}{N}\delta_{al}\delta_{bk}$$

$$\int dU\ U_{a_1,b_1}U_{a_2,b_2}\cdots U_{a_N,b_N} = \frac{1}{N!}\epsilon_{a_1a_2\cdots a_N}\epsilon_{b_1b_2\cdots b_N}$$

$$\begin{aligned}\int dU\ U_{ab}U_{cd}U^{\dagger}_{ij}U^{\dagger}_{kl} &= \frac{1}{N^2-1}\left[\delta_{aj}\delta_{bi}\delta_{cl}\delta_{dk} + \delta_{al}\delta_{bk}\delta_{cj}\delta_{di}\right. \\ &- \left.\frac{1}{N}(\delta_{aj}\delta_{bk}\delta_{cl}\delta_{di} + \delta_{al}\delta_{bi}\delta_{cj}\delta_{dk})\right]\end{aligned}$$

4.4.2 Strong Coupling Expansion (SCE)

$$S = \frac{1}{g^2}\sum_{n,p}\mathrm{tr}U_p(n)$$

As $g^2 \to \infty$ (SCE)

$$e^S = 1 + \frac{1}{g^2}\sum_{n,p}U_p(n) + \cdots$$

"gauge field is random" $\int dU\ U = 0$

Wilson loop

$$W(C) = \mathrm{tr}U_1(n)\overbrace{U_1(n+1)\cdots}^{\hat{W}}U_4^{\dagger}(n)$$

$$\frac{1}{g^2}\int dU_1(n)dU_4(n)\mathrm{tr}U_4(n)U_1(n+4)U_4^{\dagger}(n+1)U_1^{\dagger}(n)$$

$$
\begin{aligned}
&\times \ \mathrm{tr}U_1(n)\hat{W}U_4^\dagger(n) = \frac{1}{g^2}\int dU_4(n)\{U_4(n)U_1(n+4)U_4^\dagger(n+1)\}_{lk} \\
&\times \ \{U_1^\dagger(n)\}_{kl}\{U_1(n)\}_{ab}dU_1(n)\{\hat{W}U_4^\dagger(n)\}_{ba} \\
&= \ \frac{1}{g^2N}\int dU_4(n)\mathrm{tr}U_4(n)U_1(n+4)U_4^\dagger(n+1)\hat{W}U_4^\dagger(n) \\
&= \ \frac{1}{g^2N}\mathrm{tr}U_1(n+4)U_4^\dagger(n+1)\hat{W}
\end{aligned}
$$

Repeating this we obtain

$$
\begin{aligned}
\langle W(T\times L)\rangle &= (\frac{1}{g^2N})^L\langle W((T-1)\times L)\rangle = (\frac{1}{g^2N})^{LT-1}\langle W(1\times 1) \\
&= (\frac{1}{g^2N})^{LT}\mathrm{tr}1 = N(\frac{1}{g^2N})^{LT}
\end{aligned}
$$

Therefore

$$
\langle W(T\times L)\rangle = Ne^{-TL\log(g^2N)} \qquad \text{Area-law}
$$

$$
\Downarrow
$$

$$
V(L) = L\log(g^2N) \qquad \text{confinement}
$$

string tension $\sigma = \frac{1}{a^2}\log(g^2N)$

4.4.3 Character expansion

$U \in$ SU(N) or U(N)

$U(R)$: Rep. of U in irreducible representation R

$$
\exp[\frac{1}{g^2}\mathrm{tr}(U_p + U_p^\dagger)] = \sum_R d_R\lambda_R(1/g^2)\chi_R(U_p)
$$

$$
d_R = dimR = \begin{cases} 1 & R = 1 \\ N & R = F \\ N^2-1 & R = A \end{cases}
$$

$$
\begin{aligned}
\chi_R(U) &= \mathrm{tr}U(R) \qquad \text{character of } R, \qquad d_R = \chi_R(1) \\
\lambda_R(1/g^2) &= \frac{1}{d_R}\int dU \chi_R(U) \exp[\frac{1}{g^2}\mathrm{tr}(U+U^\dagger)] \\
\int dU U_{ab}(R) U^\dagger_{kl}(R') &= \frac{1}{d_R}\delta_{RR'}\delta_{al}\delta_{bk} \\
\int d\Omega \chi_R(U\Omega)\chi_{R'}(\Omega^\dagger W) &= \frac{1}{d_R}\delta_{RR'}\chi_R(UW) \\
\int d\Omega \chi_R(A\Omega B\Omega^\dagger) &= \frac{1}{d_R}\chi_R(A)\chi_R(B)
\end{aligned}
$$

2-dimensional pure Yang-Mills theory

Loop C: $\Gamma = \Pi_{i\in C} U_i$, **ループの内部を積分**

$$
K_A(\Gamma) = \int \prod_{i\neq C} dU_i e^S
$$

A: **ループ** C **の面積**

$U_{p_1} = U_1\Omega$, $U_{p_1 2} = \Omega^\dagger U_2$,

$$
\begin{aligned}
K_{p_1 p_2} &= \int d\Omega K_{p_1} K_{p_2} = \sum_{R,R'} d_R d_{R'} \lambda_R \lambda_{R'} \chi_R(U_1\Omega)\chi_{R'}(\Omega^\dagger U_2) d\Omega \\
&= \sum_R d_R \lambda_R^2 \chi_R(U_1 U_2)
\end{aligned}
$$

これを繰り返すと

$$
K_A(\Gamma) = \sum_R d_R \lambda_R^A \chi_R(\Gamma)
$$

Periodic b.c.

$$
\chi_R(\Gamma) = \chi_R(ABA^\dagger B^\dagger) = \chi_R(U_A \hat{A} B \hat{A}^\dagger U_A^\dagger B^\dagger)
$$

$$
\int dU_A \chi(\Gamma) = \frac{1}{d_R}\chi_R(\hat{A}B\hat{A}^\dagger)\chi_R(B^\dagger) = \frac{1}{d_R}\chi_R(B)\chi_R(B^\dagger)
$$

$$
\begin{aligned}
Z &= \sum_R d_R \lambda_R^A \frac{1}{d_R}\int dB \chi_R(B)\chi_R(B^\dagger) \\
&= \sum_R \lambda_R^A \int dU_B \chi_R(U_B\hat{B})\chi_R(\hat{B}^\dagger U_B^\dagger) = \sum_R \frac{\lambda_R^A}{d_R}\chi_R(\hat{B}\hat{B}^\dagger) = \sum_R \lambda_R^A
\end{aligned}
$$

4.5 Scaling behaviour in the continuum limit

4.5.1 β function and lattice spacing

$$\beta(g) = \mu\frac{d\ g}{d\ \mu} = -b_0 g^3 - b_1 g^5 + \cdots, \quad b_0 = \frac{11}{16\pi^2},\ b_1 = \frac{102}{16\pi^2}\,(N=3)$$

$$\int_{1/a}^{\Lambda_L} \frac{d\ \mu}{\mu} = -\frac{1}{2b_0}\int_{g_0^2}^{\infty} \frac{d\ g^2}{g^4(1+\frac{b_1}{b_0}g^2)}$$

$$\Downarrow$$

$$-\frac{1}{2b_0 g_0^2} - \frac{b_1}{2b_0^2}\log(b_0 g_0^2) = \log(\Lambda_L a)$$

g_0: bare lattice coupling, Λ_L: lattice Λ parameter

$$\Lambda_L a = (b_0 g_0^2)^{-\frac{b_1}{2b_0^2}} e^{-\frac{1}{2b_0 g_0^2}}$$

したがって、

$$a \to 0 \Longleftrightarrow g_0^2 \to 0$$

all dimensionfull quantities K with dimension d_K should behave on the lattice as

$$K a^{d_K} = C(\Lambda_L a)^{d_K} = C e^{-\frac{d_K}{2b_0 g_0^2}} + \cdots$$

4.6 Continuum limit

$g_0^2 \to 0 \Longleftrightarrow$ continuum limit

confinement(strong coupling) $\longleftrightarrow$ continuum limit(asymptotic free)

- No phase transition from $g_0^2 = \infty$ to $g_0^2 = 0 \Rightarrow$ confinement in the continuum limit
- Phase transition at some $g_0^2 \neq 0 \Rightarrow$ deconfine in the continuum limit

5 Lattice fermions

5.1 Naive fermion and doublers

5.1.1 Naive fermion

$$S_F = \int d^4x \bar{\psi}(\gamma_\mu D_\mu + m)\psi$$

$$\Downarrow$$

$$S_F = a^4 \sum_n \bar{\psi}_n \left[\sum_\mu \gamma_\mu \frac{U_{n,\mu}\psi_{n+\mu} - U^\dagger_{n-\mu,\mu}\psi_{n-\mu}}{2a} + m\psi_n \right]$$

$\psi' = a^{3/2}\psi$, $ma = M$

$$S_F = \frac{1}{2} \sum_{n,\mu} [\bar{\psi}_n \gamma_\mu U_{n,\mu}\psi_{n+\mu} - \bar{\psi}_{n+\mu}\gamma_\mu U^\dagger_{n,\mu}\psi_n] + M \sum_n \bar{\psi}_n\psi_n$$

5.1.2 Fermion doubling

$g^2 \to 0$ ($^\forall U_{n,\mu} = 1$)

$$S_F = \int \frac{d^4p}{(2\pi)^4} \bar{\psi}(-p)[i\gamma_\mu \sin p_\mu a + M]\psi(p)$$

propagator

$$G_F(p) = \frac{1}{i\not{s} + M} = \frac{-i\not{s} + M}{s^2 + M^2}, \qquad \not{s} = \gamma_\mu s_\mu(p) = \gamma_\mu \sin p_\mu a$$

pole of $G_F(p)$ in $a \to 0$ ($\hat{p}_\mu \ll 1/a$)

$$\sin p_\mu a = \begin{cases} \hat{p}_\mu a & p_\mu = \hat{p}_\mu \\ -\hat{p}_\mu a & p_\mu = \pi/a + \hat{p}_\mu \end{cases}$$

$$\lim_{a\to 0} G_F(p) = \sum_{p_\mu = 0, \pi/a} \frac{1}{a} \frac{-i\not{\hat{p}}(-1)^\delta + m}{\hat{p}^2 + m^2}$$

with $\delta_\mu = 0$ for $p_\mu = 0$ or 1 for $p_\mu = \pi/a$.

1 lattice fermion field $\Rightarrow 2^d = 16$ particles ("doubling problem")

$$2^d \to \begin{cases} 2^{d-1} & \text{chirality +} \quad (|\delta| = \text{even}) \\ 2^{d-1} & \text{chirality -} \quad (|\delta| = \text{odd}) \end{cases}$$

5.1.3 Nielsen-Ninomiya's theorem

Assumption

(a) Translational invariance

(b) Chiral symmetry

(c) Hermiticity

(d) Fermion bilinear

(e) Locality

$\Downarrow$

# of chirality + poles = # of chirality - poles (doubling problem exists)

Sketch of the proof

$$S_F = \overbrace{i\bar{\psi} \underbrace{\gamma_\mu}_{\text{(b)}} F^\mu \underbrace{(p)}_{\text{(a)}} \psi}^{\text{(d)}}$$

where $F^\mu(p)$ is real (from (c)), continious at $p = 0$ (from (e)).
$F^\mu(p)$ is real vector field on tourous T^d.

Poincare-Hopf's theorem

Σ zero point of $F^\mu(p)$ with index = Euler number of $T^d = 0$

If $F^\mu(p_0) = 0$ then

$$F^\mu(p) \simeq A_\mu (p - p_0)_\mu$$

index = sign of $\prod_\mu A_\mu$

$$\text{index +/-} \Longleftrightarrow \text{chirality +/-}$$

Example: $A_\mu = (-1, 1, 1, 1)$

$$\bar{\psi}(1 - 2\delta_{\mu 1})\gamma_\mu (p - p_0)_\mu \psi = \bar{\psi}\gamma_5\gamma_1\gamma_\mu (p - p_0)_\mu \gamma_1\gamma_5\psi$$

$\gamma_1\gamma_5\psi$ has an opposite chirality of ψ.

5.2 Solution to the doubling problem: Wilson fermion

5.2.1 Wilson term and decoupling of doublers

Add $O(a)$ term $\Rightarrow$ mass to doublers($^\exists p_\mu = \pi/a$)

$$\begin{aligned}
S_W &= -ar \int d^4x \bar{\psi} D^2 \psi \\
&\to -\frac{r}{2} \sum_{n,\mu} [\bar{\psi}_n U_{n,\mu} \psi_{n+\mu} + \bar{\psi}_{n+\mu} U^\dagger_{n,\mu} \psi_n - 2\bar{\psi}_n \psi_n] \quad \text{“Wilosn term”} \\
S_F &= S_F^0 + S_W \longrightarrow \bar{\psi}(\not{D} + m)\psi \qquad (a \to 0)
\end{aligned}$$

$g^2 = 0$ ($U_{n,\mu} = 1$)

$$S_F = \bar{\psi}(-p)[i\not{s} + \overbrace{M + r \sum_\mu (1 - \cos p_\mu a)}^{M(p)}]\psi(p)$$

$$G_F(p) = \frac{-i\not{s} + M(p)}{s^2 + M(p)^2}$$

$a \to 0$

$$\begin{aligned}
M(p) &= \begin{cases} ma & \text{for physical pole} \\ ma + 2r|\delta| & \text{for doublers} \end{cases} \\
m_{\text{phys}} &= m \\
m_{\text{doubler}} &= m + \frac{2r}{a}|\delta| \to \infty
\end{aligned}$$

“decoupling of doublers at low energy”

5.2.2 Chiral symmetry

- QCD action at $m = 0$, $\bar{\psi}\not{D}\psi$, is invariant under chiral tr.

$$\begin{aligned} \psi &\rightarrow \psi' = e^{i\theta\gamma_5}\psi \\ \bar{\psi} &\rightarrow \bar{\psi}' = \bar{\psi}e^{i\theta\gamma_5} \end{aligned}$$

- this symmetry is spontaneously broken

$$\lim_{m\to 0}\langle\bar{\psi}\psi\rangle \neq 0 \Longrightarrow \text{NG bosons } \simeq \text{ pions}$$

- Wrad-Takahashi identity ($m \neq 0$)

$$\langle\sum_y(\partial_\mu A_\mu + 2mP)(y)P(x) + 2S(x)\rangle = 0$$

A_μ: axial-vector current, S/P: scalar/pseudo-scalar densities
sum over x ($f_\pi = \langle 0|P|\pi\rangle$)

$$2mV\frac{f_\pi^2}{m_\pi^2} + 2V\langle\bar{\psi}\psi\rangle = 0 \Longrightarrow m_\pi^2 = \frac{f_\pi^2}{\langle\bar{\psi}\psi\rangle}\cdot m \qquad \text{(PCAC relation)}$$

- Wilson fermion explicitly breaks chiral symmetry at $m = 0$
($\Leftarrow$ Nielsen-Ninomiya's Th), $\langle\bar{\psi}\psi\rangle$? m_π^2 ? (later sections)

5.2.3 Pole mass at finite a

$$G(\vec{p} = 0, p_0) = \frac{-i\gamma_0 s_0^2 + M(p_0)^2}{s_0^2 + M(p_0)^2}$$

Pole mass: $p_0 = im_p$ from $s_0^2 + M(p_0)^2 = 0$

$$\sinh(m_p a) = M + r(1 - \cosh(m_p a))$$

$$e^{m_p a} = \frac{(M + r) \pm \sqrt{M^2 + 2Mr + 1}}{1 + r}$$

At $r = 1$

$$m_p = \frac{1}{a}\log(1 + ma) = m + O(a)$$

6 Lattice QCD with Wilson fermion

6.1 Strong coupling limit and hopping parameter expansion

6.1.1 Action and Feynmann rules

$$S = S_G(U) + S_F(\psi, \bar{\psi}U)$$

$$S_F = (M+4r)\sum_n \bar{\psi}_n\psi_n - \frac{1}{2}\sum_{n,\mu}\bar{\psi}_n[(r-\gamma_\mu)U_{n,\mu}\psi_{n+\mu}+(r+\gamma_\mu)U^\dagger_{n-\mu,\mu}\psi_{n-\mu}]$$

$$\Downarrow \text{ rescale } K = \frac{1}{2(M+4r)},\ \psi = \sqrt{2K}\psi'$$

$$S_F = \sum_n \bar{\psi}_n\psi_n - K\sum_{n,\mu}\bar{\psi}_n[(r-\gamma_\mu)U_{n,\mu}\psi_{n+\mu} + (r+\gamma_\mu)U^\dagger_{n-\mu,\mu}\psi_{n-\mu}]$$

K: hopping parameter, small $K \Leftrightarrow$ heavy fermion
K expansion (hopping parameter expansion)

Feynmann rules

$$\begin{aligned}
(n,a,\alpha) \quad \bullet \quad (m,b,\beta) = \langle \psi^{a\alpha}_n \bar{\psi}^{b\beta}_m \rangle_0 &= -\delta_{mn}\delta_{ab}\delta_{\alpha\beta} \\
(n,a,\alpha) \longrightarrow (n+\mu,b,\beta) &= -K(r-\gamma_\mu)_{\alpha\beta}(U_{n,\mu})^{ab} \\
(n,b,\beta) \longleftarrow (n+\mu,a,\alpha) &= -K(r+\gamma_\mu)_{\alpha\beta}(U^\dagger_{n,\mu})^{ab}
\end{aligned}$$

6.1.2 VEV (1 point function)

$$\begin{aligned}
\langle \psi^{a\beta}_n \bar{\psi}^{b\alpha}_n \rangle &\equiv -\delta_{ab}\Sigma_{\beta\alpha} \\
&= \langle \psi^\beta_n \bar{\psi}^\alpha_n \rangle_0 + K^2 \sum_\mu \langle \psi^\beta_n \bar{\psi}_n \rangle_0 (r \mp \gamma_\mu) U_{n,\mu} \langle \psi_{n+\mu}\bar{\psi}_{n+\mu}\rangle \\
&\times\ (r \pm \gamma_\mu) U^\dagger_{n,\mu} \langle \psi_n \bar{\psi}^\alpha_n \rangle \\
-\Sigma^{\beta\alpha} &= -\delta_{\beta\alpha} - K^2 \sum_\mu [(r-\gamma_\mu)\Sigma(r+\gamma_\mu)\Sigma + (r+\gamma_\mu)\Sigma(r-\gamma_\mu)\Sigma]^{\beta\alpha}
\end{aligned}$$

Gap equation ($r = 1$)

$$\Sigma = \mathbf{1} + K^2 \sum_{\mu} [(1-\gamma_\mu)\Sigma(1+\gamma_\mu)\Sigma + (1+\gamma_\mu)\Sigma(1-\gamma_\mu)\Sigma]^{\beta\alpha}$$

Solutions

- $\Sigma = \mathbf{1}\Sigma_0 \Longrightarrow \Sigma_0 = 1$

$$\langle \bar{\psi}\psi \rangle = N_c C(2K) = \frac{N_c C}{M+4} \neq 0 \text{ for all} M$$

$C = 2^{[d/2]} = 4$ for $d = 4$.

- $\Sigma = \mathbf{1}\Sigma_0 + i\gamma_5\Sigma_5$ $(r = 1)$

$$\begin{aligned} \Sigma_0 + i\gamma_5\Sigma_5 &= 1 + K^2 4d i\gamma_5\Sigma_5(\Sigma_0 + i\gamma_5\Sigma_5) \\ \Sigma_0 &= 1 - K^2 4d\Sigma_5^2 \\ \Sigma_5 &= K^2 4d\Sigma_5\Sigma_0 \end{aligned}$$

$$\Downarrow (d = 4)$$

$$\begin{aligned} \langle \bar{\psi}\psi \rangle = N_c C(2K) \quad & \langle \bar{\psi} i\gamma_5 \psi \rangle = 0 \quad |K| \leq 1/4 \\ \langle \bar{\psi}\psi \rangle = \frac{N_c C}{8K}, \quad & \langle \bar{\psi} i\gamma_5 \psi \rangle = \pm \frac{\sqrt{16K^2 - 1}}{8K} \quad |K| \geq 1/4 \end{aligned}$$

meaning of this solution ? (See later.)

6.1.3 2 point functions

Meson Operators:

$$M_A(n) = \bar{\psi}^a_n \Gamma_A \psi^a_n, \quad \Gamma_A = 1(S), \gamma_5(P), \gamma_\mu(V), \gamma_5\gamma_\mu(A), \gamma_\mu\gamma_\nu(T)$$

$$\begin{aligned} S_{AB}(0,n) &\equiv \langle M_A(0) M_B(n)\rangle = \langle \bar{\psi}^a_0 \Gamma_A \psi^a_0 \bar{\psi}^b_n \Gamma_B \psi^b_n\rangle \\ &= -\delta_{0n} N_c \mathrm{tr}\, \Gamma_A\Gamma_B + K^2 \langle \bar{\psi}^a_0 \Gamma_A \psi^a_0 \{\bar{\psi}^c_0 (1-\gamma_\mu) U^{cd}_{0,u} \psi^d_\mu \bar{\psi}^e_\mu (1+\gamma_\mu) \\ &\times \ (U^\dagger_{0,u})^{ef} \psi^f_0 + (+, U) \leftrightarrow (-, U^\dagger)\} \bar{\psi}^b_n \Gamma_B \psi^b_n\rangle \\ &= -\delta_{n0} N_c \mathrm{tr}\, \Gamma_A\Gamma_B \\ &- \ K^2 \sum_{\pm\mu} \langle \mathrm{tr}\,[\Gamma_A(1-\gamma_\mu)\psi_\mu \bar{\psi}^c_\mu (1+\gamma_\mu)] \bar{\psi}^b_n \Gamma_B \psi^b_n\rangle \end{aligned}$$

ここで、$\bar{\psi}^\alpha \psi^\beta = \dfrac{1}{4}\sum_C (\bar{\psi}\Gamma_C\psi)(\Gamma^\dagger_C)^{\alpha\beta}$ を使うと

$$S_{AB}(0,n) = -\delta_{n0} 4 N_c \delta_{AB} + K^2 \sum_{C,\mu} [T^{+\mu}_{AC} S_{CB}(\mu, n) + T^{-\mu}_{AC} S_{CB}(-\mu, n)]$$

where

$$T^{\pm\mu}_{AC} = \frac{1}{4}\mathrm{tr}\, \Gamma_A (1 \mp \gamma_\mu)\Gamma^\dagger_C (1 \pm \gamma_\mu)$$

$$S_{AB}(p) = -4N_c\delta_{AB} + K^2 [T^{+\mu}_{AC} e^{-ip_\mu} + T^{-\mu}_{AC} e^{ip_\mu}] S_{CB}(p)$$

P-A sector with $\vec{p} = 0$

$$\begin{aligned} S_{PP} &= -4N_c + 4K^2[(3+c_0)S_{PP} - i s_0 S_{A_0P}] \\ S_{A_0P} &= 4K^2[-i s_0 S_{PP} + c_0 S_{A_0P}] \end{aligned}$$

$$\overbrace{\begin{pmatrix} 1-4K^2(3+c_0) & 4K^2 i s_0 \\ 4K^2 i s_0 & 1-4K^2 c_0 \end{pmatrix}}^{D(p_0)} \begin{pmatrix} S_{PP} \\ S_{A_0P} \end{pmatrix} = -4N_c \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where $c_0 = \cos p_0$, $s_0 = \sin p_0$.

On-shell condition

$$\det D(p_0) = 1 - 12K^2 + 16K^4 - 8K^2c_0 + 48K^4c_0 = 0$$

$$\cosh(m_\pi) = 1 + \frac{1 - 16K^2}{8K^2(1 - 6K^2)}(1 - 4K^2)$$

with $p_0 = im_\pi$.

$$m_\pi = 0 \text{ at } K = \pm\frac{1}{4}$$
$$m_\pi^2 < 0 \text{ ! for } |K| > 1/4 \text{ !}$$

- $m_\pi^2 = 0$ となる K を K_c と書き、 critical hopping parameter と呼ぶ。
- $K_c(\beta = 0) = 1/4$, $K_c(\beta = \infty) = 1/8$
- $m_\pi^2 \propto 1/K_c - 1/K = 2m_q$ (PCAC)
  $K = K_c$ で chiral symmetry はほとんど回復？

6.2 Phase structure of QCD with Wilosn fermion

6.2.1 Massless pion from Wilson fermion

Why $m_\pi = 0$ at $K = K_c$ $(m_q = m_q^c)$?

$(N_f = 2)$ $\pi(x)^a = \bar{\psi} i\gamma_5 \tau^a \psi(x)$

$$\sum_{\vec{x}} \langle \pi^a(\vec{0}, 0)\pi^a(\vec{x}, t)\rangle \quad \begin{array}{ll} \simeq Ze^{-m_\pi t} & m_q > m_q^c \\ \simeq Z|t|^{-\alpha} & m_q = m_q^c \\ \simeq \langle \pi^a \rangle^2 + Z'e^{-m_\pi t} & m_q < m_q^c \end{array}$$

$\langle \pi^a \rangle = \langle \bar{\psi} i\gamma_5 \tau^a \psi \rangle \neq 0 \Rightarrow$ spontaneous breakdown of parity and flavor symmetry

π meson is a massless mode associated with this 2nd order phase transition

$$m_\pi = c(m_q - m_q^c)^\nu$$

ν: critical exponent

$\nu = 1/2$ for Mean-Field approximation in 4-dim.

$$m_\pi^2 \propto |m_q - m_q^c| \qquad (\text{ PCAC relation})$$

$$\begin{aligned} m_\pi^0 &= 0 \quad \text{at} m_q = m_q^c, & \pi^0 &= \bar{\psi} i\gamma_5 \tau^3 \psi \\ m_\pi^\pm &= 0 \quad \text{at} m_q \leq m_q^c, & \pi^\pm &= \bar{\psi} i\gamma_5 \tau^\pm \psi \\ m_\eta &\neq 0 \quad \text{at} \forall m_q, & \eta &= \bar{\psi} i\gamma_5 1 \psi \end{aligned}$$

"U(1) problem is also solved"

6.2.2 Phase structure

6.2.3 Order parameter and eigen-values spectra of $\gamma_5 D_W = H_W$

$$S_F = \bar{\psi}(D_W + i\gamma_5\tau^3 H)\psi, \qquad H : \text{external field}$$

Note $D_W^\dagger = \gamma_5 D_W \gamma_5$, $H_W = \gamma_5 D_W$: Hermite Wilson-Dirac operator

$$\begin{aligned}
\lim_{H\to 0}\langle\bar{\psi} i\gamma_5\tau^3\psi\rangle &= -\mathrm{Tr} i\gamma_5\tau^3 \frac{1}{D_W + i\gamma_5\tau^3 H} \\
&= -\mathrm{tr} i\gamma_5 \left[\frac{1}{D_W + i\gamma_5 H} - \frac{1}{D_W - i\gamma_5 H}\right] \\
&= -i\mathrm{tr}\left[\frac{1}{H_W + iH} - \frac{1}{H_W - iH}\right] \\
&= -i\sum_\lambda \langle\lambda|\left[\frac{1}{\lambda + iH} - \frac{1}{\lambda - iH}\right]|\lambda\rangle \\
&= -i\int d\lambda\rho(\lambda)\left[\frac{1}{\lambda + iH} - \frac{1}{\lambda - iH}\right] = -2\pi\rho(0)
\end{aligned}$$

$\rho(\lambda) = \Sigma_{\lambda'}\,\delta(\lambda - \lambda')$: spectral density of H_W.

7 New fermion formulation: Domain-wall fermions

7.1 Action

5 dimensional Massive Wilson fermion ($r = -1$)
+ 4 dimensional gauge fields
+ free boundary condition in 5-th direction

$$
\begin{aligned}
S_F &= \frac{1}{2}\sum_{n,s}\bar{\psi}_{n,s}[\sum_{\mu}\gamma_\mu(U_{n,\mu}\psi_{n+\mu,s} - U^\dagger_{n-\mu,\mu}\psi_{n-\mu,s}) + \gamma_5(\psi_{n,s+1} - \psi_{n,s-1})] \\
&+ M_0\sum_{n,s}\bar{\psi}_{n,s}\psi_{n,s} + \frac{1}{2}\sum_{n,s}\bar{\psi}_{n,s}[\sum_{\mu}(U_{n,\mu}\psi_{n+\mu,s} + U^\dagger_{n-\mu,\mu}\psi_{n-\mu,s} \\
&- 2\psi_{n,s}) + (\psi_{n,s+1} + \psi_{n,s-1} - 2\psi_{n,s})]
\end{aligned}
$$

$$\Downarrow \text{ 5-th cordinate } s \text{ as flavors } (P_{R/L} = \frac{1 \pm \gamma_5}{2})$$

$$
\begin{aligned}
S_F &= \sum_{n,\mu}\frac{1}{2}\bar{\psi}_{n,s}\gamma_\mu(U_{n,\mu}\psi_{n+\mu,s} - U^\dagger_{n-\mu,\mu}\psi_{n-\mu,s}) \\
&+ \sum_{n,m,s,t}\bar{\psi}_{n,s}[MP_R + M^\dagger P_L]^{nm}_{st}\psi_{m,t}
\end{aligned}
$$

$$
\begin{aligned}
(M)^{n,m}_{s,t}\psi_{m,t} &= \overbrace{\frac{1}{2}(U_{n,\mu}\psi_{n+\mu,s} + U^\dagger_{n-\mu,\mu}\psi_{n-\mu,s} - 2\psi_{n,s})}^{\nabla^2} \\
&+ \psi_{n,s+1} - \psi_{n,s} + M_0\psi_{n,s} \\
(M^\dagger)^{n,m}_{s,t}\psi_{m,t} &= (\nabla^2\psi)_{n,s} + \psi_{n,s-1} - \psi_{n,s} + M_0\psi_{n,s}
\end{aligned}
$$

Explicitly (6×6)

$$M = \begin{pmatrix} \overbrace{\nabla^2 + M_0 - 1}^{-W(U)} & 1 & 0 & 0 & 0 & 0 \\ 0 & -W(U) & 1 & 0 & 0 & 0 \\ 0 & 0 & -W(U) & 1 & 0 & 0 \\ 0 & 0 & 0 & -W(U) & 1 & 0 \\ 0 & 0 & 0 & 0 & -W(U) & 1 \\ 0 & 0 & 0 & 0 & 0 & -W(U) \end{pmatrix}$$

$$M^\dagger = \begin{pmatrix} -W(U) & 0 & 0 & 0 & 0 & 0 \\ 1 & -W(U) & 0 & 0 & 0 & 0 \\ 0 & 1 & -W(U) & 0 & 0 & 0 \\ 0 & 0 & 1 & -W(U) & 0 & 0 \\ 0 & 0 & 0 & 1 & -W(U) & 0 \\ 0 & 0 & 0 & 0 & 1 & -W(U) \end{pmatrix}$$

7.2 Massless modes in free theory

$g^2 = 0$ $(U_{n,\mu} = 1)$

$$S_F = \bar{\psi}_s(-p) i \not{s} \psi_s + \bar{\psi}_s [M P_R + M^\dagger P_L]_{st} \psi_t(p)$$

$W(U = 1) = 1 - M_0 + \Sigma_\mu (1 - \cos p_\mu)$

The eigen-value of M gives mass of the particle.

If M or $M^\dagger$ has zero eigen-value:

$$M u_R = 0, \quad M^\dagger u_L = 0$$

$$\psi = \underbrace{P_R u_R}_{\text{Right-handed zero mode}} + \overbrace{P_L u_L}^{\text{Left-handed zero mode}}$$

Solution to $M\phi = 0$ $s = 1, 2, \cdots, N_s$

$$M_{st}\phi_t = -W\phi_s + \phi_{s+1} = 0, \qquad -W\phi_{N_s} = 0$$

$$\phi_s = W^{s-1}\phi_1$$

but $\phi_{N_s} = W^{N_s-1}\phi_1 \neq 0$ unless

$$|W| < 1, \quad \text{and} \quad N_s \to \infty$$

Normalization

$$1 = \sum_{s=1}^{N_s} \phi_s^\dagger \phi_s = \phi_1^\dagger \phi_1 \frac{1 - W^{2N_s}}{1 - W^2} \to \phi_1^\dagger \phi_1 \frac{1}{1 - W^2}$$

$$\phi_s = c_0 W^{s-1}, \quad c_0 = \sqrt{1 - W^2}$$

Solution to $M^\dagger \phi = 0$

$$-W\phi_s + \phi_{s-1} = 0, \qquad -W\phi_1 = 0$$

$$\phi_s = c_0 W^{N_s - s}$$

Condition for $W(p)$

$$|W(p)| < 1 \Rightarrow 0 < M - \sum_{\mu}(1 - \cos p_\mu) < 2$$

p_μ	condition for M	# of poles
(0,0,0,0)	$0 < M < 2$	1
(π,0,0,0)	$2 < M < 4$	4
(π,π,0,0)	$4 < M < 6$	6
(π,π,π,0)	$6 < M < 8$	4
(π,π,π,π)	$8 < M < 10$	1

M controls # of poles.

7.3 Quark fields

Right-handed zero mode + Left-handed zero mode
$\Longrightarrow$ massless Dirac fermion = quark !
quark field

$$\begin{aligned} q_n &= P_R\psi_{n,1} + P_L\psi_{n,N_s} \\ \bar{q}_n &= \bar{\psi}_{n,1}P_L + \bar{\psi}_{n,N_s}P_R \end{aligned}$$

mass term

$$-m_f\bar{q}_n q_n = -m_f(\bar{\psi}_{n,1}P_L\psi_{n,N_s} + \bar{\psi}_{n,N_s}P_R\psi_{n,1})$$

$$M = \begin{pmatrix} -W & 1 & 0 & 0 & 0 & 0 \\ 0 & -W & 1 & 0 & 0 & 0 \\ 0 & 0 & -W & 1 & 0 & 0 \\ 0 & 0 & 0 & -W & 1 & 0 \\ 0 & 0 & 0 & 0 & -W & 1 \\ -m_f & 0 & 0 & 0 & 0 & -W \end{pmatrix}$$

($m_f = 1$: Anti-periodic b.c.)

From zero mode solution

$$\begin{aligned}\psi_{n,s} &\simeq W^{s-1}P_R q_n + W^{N_s-s}P_L q_n \\ \bar{\psi}_{n,s} &\simeq \bar{q}_n W^{s-1}P_L + \bar{q}_n W^{N_s-s}P_R\end{aligned}$$

Report

Using the above expression, rewrite

$$\sum_{s,t}\bar{\psi}_s(-p)[i\not{s} + MP_R + M^\dagger P_L]_{st}\psi_t(p) = \bar{q}(-p)[Zi\not{s} + B(p)]q(p)$$

then calculate the quark mass for $0 < M < 2$ as

$$m_{quark} = \lim_{p\to 0}\frac{B(p)}{Z}$$

QCD with DWF = Domain-Wall QCD (DWQCD)

7.4 Wrad-Takahashi identities

7.4.1 Axial W-T identity in the continuum

$$\langle(\partial_\mu A^a_\mu + 2mP^a)(x)\mathcal{O}\rangle + \langle\delta^a_x\mathcal{O}\rangle = 0$$

$$\begin{aligned}A^a_\mu(x) &= \bar{\psi}\gamma_5\gamma_\mu\tau^a\psi(x) \quad \text{axial-vector current} \\ P^a(x) &= \bar{\psi}\gamma_5\tau^a\psi(x) \quad \text{pseudo-scalar density}\end{aligned}$$

δ^a_x: local chiral tr., $\mathcal{O}$: arbitrary operator

7.4.2 Axial transformation in DWQCD

$$\begin{aligned}\delta\psi_{n,s} &= -i\mathrm{sign}(s-\frac{N_s+1}{2})\psi_{n,s}\\ \delta\bar{\psi}_{n,s} &= -i\bar{\psi}_{n,s}\mathrm{sign}(s-\frac{N_s+1}{2})\end{aligned}$$

Vector transformation at each s
$-i$ for $s=1,2,\cdots,N_s/2$ and $+i$ for $s=N_s/2+1,\cdots N_s$.
$\Longrightarrow$ Chiral transformation for q_n !

7.4.3 Axial WT identity of DWQCD

$$\langle\{\nabla^-_\mu A^a_\mu + 2m_f\bar{q}\gamma_5\tau^a q - \underbrace{2j^a_5}_{\text{Anomalous term}}\}\mathcal{O} + \delta^a\mathcal{O}\rangle = 0$$

$$\begin{aligned}A^a_\mu &= \frac{1}{2}\sum_{s=1}^{N_s}\mathrm{sign}(s-\frac{N_s+1}{2})[\bar{\psi}_{n+\mu,s}(1-\gamma_\mu)U^\dagger_{n,\mu}\tau^a\psi_{n,s}\\ &- \bar{\psi}_{n,s}(1+\gamma_\mu)U_{n,\mu}\tau^a\psi_{n+\mu,s}]\\ j^a_5 &= \frac{1}{2}[\bar{\psi}_{n,N_s/2+1}(1-\gamma_5)\tau^a\psi_{n,N_s/2} - \bar{\psi}_{n,N_s/2}(1+\gamma_5)\tau^a\psi_{n,N_s/2+1}]\\ \nabla^-_\mu f(n) &\equiv f(n)-f(n-\mu)\end{aligned}$$

If $\mathcal{O}$ contains only q and $\bar{q}$

$$\begin{aligned}\langle j^a_5\mathcal{O}\rangle &\to 0 \qquad \mathrm{tr}\,\tau^a = 0(\text{non-singlet})\\ &\propto \tilde{F}_{\mu\nu}\cdot F_{\mu\nu} \quad \tau^a = \mathbf{1}(\text{singlet})\end{aligned}$$

as $N_s\to\infty$.
Non-singlet Axial WT identities as well as chiral U(1) anomaly are correctly reproduced.

7.5 Pauli-Villars subtraction

DWQCD = 1 light Dirac fermion $\oplus$ $(N_s - 1)$ heavy fermions

$$logZ \propto O(N_s)$$

Pauli-Villars (PV) subtraction $\rightarrow$ define "finite theory" as $N_s \rightarrow \infty$

$$\begin{aligned} S_F^{DWF} &= \bar{\psi} D_F(m_f)\psi \quad \text{fermion} \\ S_B^{PV} &= \bar{\phi} D_F(m_f = 1)\phi \quad \text{boson} \end{aligned}$$

$$\begin{aligned} Z &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\phi\mathcal{D}\bar{\phi}\exp[\bar{\psi}D_F(m_f)\psi + \bar{\phi}D_F(m_f = 1)\phi] \\ &= \frac{\det D_F(m_f)}{\det D_F(m_f = 1)} \end{aligned}$$

8 Ginsparg-Wilson relation and chiral symmetry

8.1 GW relation

$$S_F = \bar{\psi} D \psi$$

$$D\gamma_5 + \gamma_5 D = 0 \quad \text{chiral symmetry}$$

$$\Downarrow$$

$$D\gamma_5 + \gamma_5 D = O(a) \quad \text{Lattice, optimal choice ?}$$

Ginsparg-Wlison (Free field analysis)
chiral symmetry(continuum) + non-chiral RG tr. $\rightarrow$ lattice action

$$D\gamma_5 + \gamma_5 D = aD\gamma_5 D \qquad \text{GW relation}$$

$$\gamma_5 D^{-1} + D^{-1}\gamma_5 = a\gamma_5$$

Chiral symmetry violation in propagator D^{-1} is $O(a)$ and local

8.2 "Chiral" symmetry

Chiral + $O(a)$ transformation

$$\begin{aligned} \delta\psi &= T^a \gamma_5 (1 - \frac{a}{2} D)\psi \\ \delta\bar{\psi} &= \bar{\psi}(1 - \frac{a}{2} D)\gamma_5 T^a \end{aligned}$$

$$\delta S = \bar{\psi} T^a (D\gamma_5 + \gamma_5 D - aD\gamma_5 D)\psi = 0 \quad \text{chiral symmetry}$$

if GW relation is satisfied.

- mass term ($m_f \bar{\psi}\psi$) breaks this symmetry $\Rightarrow$ Axial WT identity
- transformation is D dependent $\Rightarrow$ Jacobian is non-trivial
  $\longrightarrow$ chiral U(1) anomaly

$$-aTr\gamma_5 D = 2N_f \cdot \text{index}(D)$$

8.3 An explicit solution to GW relation

DWQCD with PV subtraction as $N_s \to \infty$

$\Longrightarrow$ Overlap formulation (Narayanann-Neuberger)

$\Downarrow$: explcit integration of PV fields

Overlap Dirac Operator

$$aD = 1 - \frac{A}{\sqrt{A^\dagger A}}, \qquad A = M - D_W$$

M: DW mass, D_W: 4-dim. Wilson-Dirac operator

- $\gamma_5 A \gamma_5 = A^\dagger$
- $V \equiv A(A^\dagger A)^{-1/2}$

  $$\gamma_5 V \gamma_5 = A^\dagger (AA^\dagger)^{-1/2} = \sum_n a_n A^\dagger (AA^\dagger)^n = (A^\dagger A)^{-1/2} A^\dagger = V^\dagger$$

- $VV^\dagger = A(A^\dagger A)^{-1/2}(A^\dagger A)^{-1/2} A^\dagger A (A^\dagger A)^{-1} A^\dagger = 1$

$$\begin{aligned}
\gamma_5 \frac{1}{aD} \gamma_5 &= \frac{1}{1 - \gamma_5 V \gamma_5} = \frac{1}{1 - V^\dagger} = \frac{V}{V - 1} \\
&= 1 - \frac{1}{1 - V} \\
\frac{1}{a} \gamma_5 D^{-1} \gamma_5 &= 1 - \frac{1}{a} D^{-1} \\
&\Downarrow \\
\gamma_5 D^{-1} + D^{-1} \gamma_5 &= a \gamma_5 \\
&\Downarrow \\
D \gamma_5 + \gamma_5 D &= a D \gamma_5 D \quad \text{(GW relation)}
\end{aligned}$$

(注) 一般に

$$D = \frac{1}{a}(1 - V), \gamma_5 V \gamma_5 = V^\dagger, VV^\dagger = 1$$

なら、 D **は** GW relation **を満たす。**

8.4 Relation between DWQCD and Overlap-Dirac operator

$$S_{DW} = \bar{\psi}[\frac{1}{2}\gamma_\mu(\nabla_\mu + \nabla_\mu^\dagger) + P_R M + P_L M^\dagger]\psi$$

$$\nabla_\mu f = \frac{1}{a}(U_{n,\mu} f(n+\mu) - f(n)), \quad \nabla_\mu^\dagger f = \frac{1}{a}(f(n) - U_{n-\mu,\mu}^\dagger f(n-\mu))$$

change of variables

$$\psi' = (PP_R + P_L)\psi, \quad \bar{\psi}' = \bar{\psi}(PP_L + P_R)$$

Parity tarnsformation in flavor space

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad s \to N_s + 1 - s$$

$$M^H = MP = PM^\dagger = \begin{pmatrix} 0 & 0 & -1 & B \\ 0 & -1 & B & 0 \\ -1 & B & 0 & 0 \\ B & 0 & 0 & 0 \end{pmatrix}, M^{PV} = \begin{pmatrix} 0 & 0 & -1 & B \\ 0 & -1 & B & 0 \\ -1 & B & 0 & 0 \\ B & 0 & 0 & 1 \end{pmatrix},$$

with $B = 1 - \frac{1}{2}\nabla_\mu \nabla_\mu^\dagger - M$.

$$q = \psi'_{N_s}, \quad \bar{q} = \bar{\psi}'_{N_s}$$

chiral transformation of DWQCD

$$\delta\psi_s = (\Gamma_5)_{st}\psi_t, \quad \Gamma_5 = \begin{pmatrix} -\gamma_5 & 0 & 0 & 0 \\ 0 & -\gamma_5 & 0 & 0 \\ 0 & 0 & \gamma_5 & 0 \\ 0 & 0 & 0 & \gamma_5 \end{pmatrix}$$

$$\delta q = \gamma_5 q, \quad \delta\bar{q} = \bar{q}\gamma_5$$

$\{\Gamma_5 D'_{DW} + D'_{DW}\Gamma_5\}_{st} = 2\gamma_5\delta_{s,N_s/2}\delta_{t,N_s/2}$ WT identity of DWQCD

$$\begin{aligned} D'_{DW} &= \frac{1}{2}\gamma_\nu(\nabla_\mu + \nabla_\mu\ \dagger) + M^H \\ D'_{PV} &= \frac{1}{2}\gamma_\nu(\nabla_\mu + \nabla_\mu\ \dagger) + M^{PV} \end{aligned}$$

If we integrate $(N_s - 1)$ heavy modes

$$\begin{aligned} S_{DW} &= \bar{q} a D_{eff} q \\ S_{PV} &= \bar{Q}(aD_{eff} + 1)Q. \\ \bar{Z}_{DW} &= Z_{DW}Z_{PV} = \det aD_{N_s} = \frac{\det aD_{eff}}{\det(aD_{eff}+1)} \end{aligned}$$

$\Downarrow$

$$\begin{aligned} aD_{N_s} &= \frac{aD_{eff}}{aDeff + 1} \\ D_{N_s}^{-1} &= a + D_{eff}^{-1} \\ \lim_{N_s\to\infty} D_{N_s}^{-1}\gamma_5 + \gamma_5 D_{N_s}^{-1} &= a2\gamma_5 \quad \text{GW relation} \end{aligned}$$

Finally one obtain

$$\gamma_5 D_{eff} + D_{eff}^{-1}\gamma_5 = 0 \qquad \text{chiral symmetry}$$

but D_{eff} is non-local:

$$D_{eff} = \frac{aD}{1 - aD}, \quad D = \lim_{N_s\to\infty} D_{N_s}$$

DWF				GWF
q	$\Longleftrightarrow$	$(1-\frac{aD}{2})\psi$		
$\bar{q}$	$\Longleftrightarrow$	$\bar{\psi}$		
chiral	$\Longleftrightarrow$	chiral $+ O(a)$	$\longrightarrow$	chiral
$\delta q = \gamma_5 q$		$\delta\psi = \gamma_5(1-aD)\psi$		$\delta(1-\frac{aD}{2})\psi = \gamma_5(1-\frac{aD}{2})\psi$
$\delta\bar{q} = \bar{q}\gamma_5$		$\delta\bar{\psi} = \bar{\psi}\gamma_5$		$\delta\bar{\psi} = \bar{\psi}\gamma_5$
$m\bar{q}q$	$\Longleftrightarrow$	$m\bar{\psi}(1-\frac{aD}{2})\psi$		

8.5 Index theorem of GW fermion

8.5.1 Index theorems

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D$$

Continuum index theorem

$$\mathrm{Tr}\,\gamma_5 = n_+ - n_-$$

Lattice index theorem (GW)

$$\mathrm{Tr}\,\gamma_5(1-\frac{1}{2}aD) = n_+ - n_-$$

$n_\pm$: # of normalized zero modes

$$D\phi_n = 0, \quad \gamma_5\phi_n = \pm\gamma_5\phi_n$$

$$\mathrm{Tr}\,\gamma_5 = n_+ - n_- + N_+ - N_- = 0$$

$N_\pm$: # of modes (doublers) which satisfy

$$D\phi_n = \frac{2}{a}\phi_n, \quad \gamma_5\phi_n = \pm\phi_n$$

8.5.2 Eigen-vales of D

$$D\phi = \lambda\phi, \qquad \phi^\dagger D^\dagger = \phi^\dagger\bar{\lambda}$$

and $\gamma_5 D\gamma_5 = D^\dagger$. From GW relation

$$\begin{aligned}
\phi^\dagger(D+\gamma_5 D\gamma_5)\phi &= \phi^\dagger(a\gamma_5 D\gamma_5 D)\phi \\
\lambda+\bar{\lambda} &= a\bar{\lambda}\lambda, \lambda = x+iy \\
2x = a(x^2+y^2) &\rightarrow (x-\frac{1}{a})^2+y^2 = \frac{1}{a^2}
\end{aligned}$$

$\Gamma_5 = \gamma_5(1-\frac{a}{2}D)$ **とすると**

$\Gamma_5\gamma_5 D+\gamma_5 D\Gamma_5 = D-\frac{1}{2}a\gamma_5 D\gamma_5 D+\gamma_5 D\gamma_5-\frac{1}{2}a\gamma_5 D\gamma_5 D = 0$ GW relation

Set $H = \gamma_5 D = H^\dagger$ (hermite)

$$H\phi_n = \lambda_n\phi_n, \qquad (\phi_n,\phi_n) = 1, \qquad H(\Gamma_5\phi_n) = -\lambda_n(\Gamma_5\phi_n)$$

pair $(\lambda_n, -\lambda_n)$ for $\lambda_n \neq 0$ unless

$$(\Gamma_5\phi_n, \Gamma_5\phi_n) = 0$$

$(\phi_n, (\gamma_5-\frac{a}{2}H)(\gamma_5-\frac{a}{2}H)\phi_n) = (\phi_n, (1-\frac{a}{4}H^2)\phi_n) = (1-\frac{a}{2}\lambda_n)(1+\frac{a}{2}\lambda_n) = 0$

したがって、 $\lambda_n = \pm\frac{2}{a}$ **が例外。この時**

$$\gamma_5\phi_n = \frac{a}{2}H\phi_n = \pm\phi_n$$

固有値と固有関数の性質をまとめると

1. $\lambda_n = 0$

$$\gamma_5 D\phi_n = H\phi_n = 0, \quad \gamma_5\phi_n = \pm\phi$$

$$\left(H(\gamma_5\phi_n) = \gamma_5 D\gamma_5\phi_n = 0 \to \phi_n^{\pm} = \frac{1\pm\gamma_5}{2}\phi_n\right)$$

2. $0 < |\lambda_n| < \dfrac{2}{a}$

$$H\phi_n = \lambda_n\phi_n, \qquad H(\Gamma_5\phi_n) = -\lambda_n(\Gamma_5\phi_n)$$

$H\Gamma_5 + \Gamma_5 H = 0$ より

$$\phi_n^\dagger\Gamma_5\phi_n = 0, \qquad \phi_n^\dagger\gamma_5\phi_n = \frac{a}{2}\lambda_n$$

$$|\frac{a}{2}\lambda_n| = |\phi_n^\dagger\gamma_5\phi_n| \le |\phi^\dagger||\gamma_5\phi_n| = 1$$

3. $\lambda_n = \pm\dfrac{2}{a}$

$$\Gamma_5\phi_n = 0$$

$$\begin{aligned} H\phi_n &= +\frac{2}{a}\phi_n, \ \gamma_5\phi_n = \phi_n \\ H\phi_n &= -\frac{2}{a}\phi_n, \ \gamma_5\phi_n = -\phi_n \end{aligned}$$

8.5.3 Calculation of index

$$\begin{aligned} \mathrm{Tr}\,\gamma_5\left(1-\frac{a}{2}D\right)\mathrm{Tr}\,\Gamma_5 &= \sum_n \phi_n^\dagger\Gamma_5\phi_n \\ &= \sum_{\lambda_n=0}\phi_n^\dagger\Gamma_5\phi_n + + \sum_{0<|\lambda_n|<2/a}\underbrace{\phi_n^\dagger\Gamma_5\phi_n}_{\to 0} + \sum_{\lambda_n=\pm 2/a}\underbrace{\phi_n^\dagger\Gamma_5\phi_n}_{\to 0} \\ &= \sum_{\lambda_n=0}\phi_n^\dagger\gamma_5\phi_n = n_+ - n_- \end{aligned}$$

On the other hand,

$$\begin{aligned}\mathrm{Tr}\,(-\frac{1}{2}a\gamma_5 D) &= -\frac{a}{2}\sum_n \phi_n^\dagger \phi_n \lambda_n \\ &= -\frac{a}{2}\sum_{0<|\lambda_n|<2/a}(\lambda_n - \lambda_n) - \frac{a}{2}\sum_{\lambda_n=\pm 2/a}\lambda_n = -\underbrace{(N_+ - N_-)}_{\text{doublers}}\end{aligned}$$

Furthermore

$$\begin{aligned}\mathrm{Tr}\,\gamma_5 &= \sum_n \phi_n^\dagger \gamma_5 \phi_n = \sum_{\lambda_n=0}\phi_n^\dagger\gamma_5\phi_n + \sum_{0<|\lambda_n|<2/a}\underbrace{\phi_n^\dagger\gamma_5\phi_n}_{\to \frac{a\lambda_n}{2}} \\ &+ \sum_{\lambda_n=\pm 2/a}\frac{a\lambda_n}{2} = n_+ - n_- + N_+ - N_- = 0 \text{ (lattice)}\end{aligned}$$

結局、 $\mathrm{Tr}\,\Gamma_5 = \mathrm{Tr}\,(-\frac{1}{2}a\gamma_5 D)$

$\mathrm{Tr}\,\gamma_5 = 0 \Leftarrow$ cancellation among physical mode and doublers

$N_\pm$ **が** doubler **の数であることを見る**

$$\frac{1}{a}A = M_n - i\not{p}, \quad M_n = \frac{1}{a}(M_0 - 2nr) \quad (n = 0,1,2,3,4)$$

$0 < M_0 < 2$

$$D\psi = \frac{1}{a}(1 - \frac{A}{\sqrt{A^\dagger A}})\psi = \frac{1}{a}(1 - \frac{-i\not{p} + M_n}{\sqrt{\not{p}^2 + M_n^2}})\psi$$

$\not{p}\psi = 0$ (light modes) **に対しては**

$$= \frac{1}{a}(1 - \frac{M_n}{|M_n|})\psi = \begin{cases} 0 & n = 0 \\ \frac{2}{a}\psi & n \neq 0 \end{cases}$$

第II部

実践編: Hadron mass calculation by Monte-Carlo method

9 Hadrons

Hadron: bound state of quarks, u(up), d(down), s(strange) quarks
light quark mass $m_l = (m_u + m_d)/2$, strange quark mass m_s

Quarks	Pesudo-Scalar meson	Vector meson
$\bar{u}u - \bar{d}d$	π^0	ρ^0
$\bar{d}u$, $\bar{u}d$	$\pi^\pm$	$\rho^\pm$
$\bar{u}u + \bar{d}d$	η	ω
$\bar{s}d$, $\bar{d}s$	K^0,$\bar{K}^0$	$(K^*)^0$, $(\bar{K}^*)^0$
$\bar{s}u$, $\bar{u}s$	$K^\pm$	$(K^*)^\pm$
$\bar{s}s$	η_s	ϕ
Quarks	Octet baryon	Decouplte baryon
uuu		Δ^{++}
uud	P	Δ^+
udd	N	Δ^0
ddd		Δ^-
uus	Σ^+	$(\Sigma^*)^+$
uds	Σ^0, Λ	$(\Sigma^*)^0$
dds	Σ^-	$(\Sigma^*)^-$
uss	Ξ^0	$(\Xi^*)^0$
dss	Ξ^-	$(\Xi^*)^-$
sss		Ω

10 Path integral formula

$$
\begin{aligned}
S &= S_G(U) + S_F(\psi, \bar{\psi}, U), \qquad S_F = \bar{\psi} D(U) \psi \\
\langle \mathcal{O}(\bar{\psi}.\psi, U) \rangle &= \frac{\int dU \; d\bar{\psi} \; d\psi \; \mathcal{O} e^{-S}}{\int dU \; d\bar{\psi} \; d\psi \; e^{-S}}
\end{aligned}
$$

Fermion integral

$$
\begin{aligned}
&\int \; dU \;\; d\bar{\psi} \; d\psi \; \mathcal{O}(\bar{\psi}.\psi, U) e^{-S} \\
&= \int dU \; e^{-S_G(U)} \mathcal{O}(-\frac{\delta}{\delta\eta}, \frac{\delta}{\delta\bar{\eta}}, U) \int d\bar{\psi} \; d\psi e^{-\bar{\psi} D \psi + \bar{\psi}\eta + \bar{\eta}\psi}|_{\eta=\bar{\eta}=0} \\
&= \int dU \; e^{-S_G(U)} \mathcal{O}(-\frac{\delta}{\delta\eta}, \frac{\delta}{\delta\bar{\eta}}, U) \det \; D(U) e^{\bar{\eta} D^{-1}(u) \eta}|_{\eta=\bar{\eta}=0}
\end{aligned}
$$

Example: π meson 2-pt. function

$$
\begin{aligned}
&\langle \bar{\psi}(x) \gamma_5 \tau^a \psi(x) \bar{\psi}(y) \gamma_5 \tau^b \psi(y) \rangle \\
&= \int dU \; \det \; D(U) e^{-S_G(U)} \Big[-\mathrm{tr} \; \gamma_5 \tau^a D^{-1}(U)_{xy} \gamma_5 \tau^b D^{-1}(U)_{yx} \\
&\quad + \mathrm{tr} \; \gamma_5 \tau^a D^{-1}(U)_{xx} \mathrm{tr} \; \gamma_5 \tau^b D^{-1}(U)_{yy} \Big] \\
(D(U) &= D(U)^{AB,gf}_{xy,\alpha\beta} = \delta_{gf} D(U)^{AB}_{xy,\alpha\beta} \Big) \\
&= \int dU \; \det \; D(U) e^{-S_G(U)} \delta_{ab} \Big[-\mathrm{tr} \; \gamma_5 D^{-1}(U)_{xy} \gamma_5 D^{-1}(U)_{yx} \\
&\quad + \mathrm{tr} \; \gamma_5 D^{-1}(U)_{xx} \mathrm{tr} \; \gamma_5 D^{-1}(U)_{yy} \delta_{a0} \Big]
\end{aligned}
$$

Explicit form of $D(U)$ for Wilson fermion ($r = 1$)

$$
D_{n,m} = \delta_{n,m} - K \sum_{\mu} \Big[\delta_{m,n+\mu} U_{n,\mu} (1 - \gamma_\mu) + \delta_{m,n-\mu} U^{\dagger}_{m,\mu} (1 + \gamma_\mu) \Big]
$$

計算の手順

1. Generate U with probability $P(U) \propto \det\ D(U)e^{-S_G(U)}$
2. Calculate $D^{-1}(U)_{y,x_0}$ for $^{\exists}x_0$, $^{\forall}y$

$$\begin{aligned} D_{xy}\phi_y &= \delta_{x,x_0} \qquad \text{linear equation} \\ \phi_y &= D^{-1}_{yx}\delta_{x,x_0} = D^{-1}_{y,x_0} \end{aligned}$$

3. $\gamma_5 D^\dagger \gamma_5 = D \to D^{-1}_{XY} = \gamma_5 (D^\dagger)^{-1}_{XY}\gamma_5 = \gamma_5(\bar{D}^{-1})_{YX}\gamma_5$
4. construct operater $\mathcal{O}$

$$\begin{aligned} \text{tr } \gamma_5 D^{-1}_{x_0 y}\gamma_5 D^{-1}_{yx_0} &= \text{tr } (D^\dagger)^{-1}_{x_0 y} D^{-1}_{yx_0} \\ &= (\bar{D}^{-1})^{BA_0}_{yx_0,\beta\alpha_0}(D^{-1})^{BA_0}_{yx_0,\beta\alpha_0} = \sum_{y,\beta,B} \bar{\phi}^B_{y,\beta}\phi^B_{y,\beta} \\ \left(D^{AB}_{xy,\alpha\beta}\phi^B_{y,\beta} &= \delta_{x,x_0}\delta^{A,A_0}\delta_{\beta,\beta_0}\right) \end{aligned}$$

5. Go to 1 and repeat
6. $U_1 \to U_2 \to \cdots U_N$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \mathcal{O}(U_i) \longrightarrow \int\ dU\ \det\ D(U)e^{-S(U)}\mathcal{O}(U)$$

Remark

calculation of det $D(U)$ is time-consuming

det $D(U) \Rightarrow 1$ "quenched approximation"

neglect creation of $q\bar{q}$ from vacuum

$$P(U) \propto e^{-S_G(U)}$$

11 Generation of U (Monte-Carlo simulation)

11.1 Pseudo Heat-bath for quenched QCD

11.1.1 原理

$$\begin{aligned}\langle\mathcal{O}\rangle &= \int_a^b dx\ P(x)\mathcal{O}(x), \quad \int_a^b dx\ P(x) = 1\\ y &= \int_a^x dz\ P(z), \quad dy = dxP(x)\\ \langle\mathcal{O}\rangle &= \int_0^1 dy\ \hat{\mathcal{O}}(y), \quad \hat{\mathcal{O}}(y) = \mathcal{O}(x(y))\end{aligned}$$

1. generate $0 \le y \le 1$ uniformly
2. $y \to x$ by solving $y = \int_a^x dz\ P(z)$ analytically
3.

$$\langle\mathcal{O}\rangle = \frac{1}{N}\sum_{i=1}^{N}\hat{\mathcal{O}}(y_i) = \frac{1}{N}\sum_{i=1}^{N}\mathcal{O}(x(y_i))$$

11.1.2 SU(2) lattice gauge theory

$$-S_G(U) = \beta\sum_{n,\mu\nu}U_{\mu\nu}(n)$$

Consider link $(n, n+\mu)$

$$\begin{aligned}\Delta &= -S_G(n,\mu) = \beta[\mathrm{tr}\ U_{n,\mu}V_\mu + \mathrm{tr}\ U_{n,\mu}^\dagger V_\mu^\dagger]\\ V_\mu &= \sum_{\nu\neq\mu}[U_{n+\mu,\nu}U_{n+\nu,\mu}^\dagger U_{n,\nu}^\dagger + U_{n+\mu-\nu,\nu}^\dagger U_{n-\nu,\mu}^\dagger U_{n-\nu,\nu}]\\ V_\mu &= b_0\mathbf{1} + ib_j\tau^j = RV_0, \quad V_0 \in \mathrm{SU}(2), R^2 = b_0^2 + b_j^2\\ W &= U_{n,\mu}V_0 = a_0\mathbf{1} + ia_j\tau^j, \quad a_0^2 + a_j^2 = 1, \Delta = 2\beta R a_0\end{aligned}$$

$$\begin{aligned}&\int dU_{n,\mu}\ e^{\beta\mathrm{tr}(U_{n,\mu}V_\mu + U_{n,\mu}^\dagger V_\mu^\dagger)} = \int d(U_{n,\mu}V_0)\ e^{2\beta R a_0} = \int \delta(a^2-1)d^4a\ e^{2\beta R a_0}\\ &= \int da_0\ r^2dr\ \sin\theta d\theta\ d\phi\delta(a_0^2 + r^2 - 1)e^{2\beta R a_0}\\ &= \int \sin\theta d\theta\ d\phi\int_{-1}^{1} da_0\sqrt{1-a_0^2}e^{2\beta R a_0} = \int \sin^2\theta d\theta\ d\phi\int_{e^{-2\beta R}}^{e^{2\beta R}}\frac{de^{2\beta R a_0}}{2\beta R}\sqrt{1-a_0^2}\end{aligned}$$

$0 \leq y \leq 1$ uniform random number

$$\begin{aligned} e^{2\beta R a_0} &= z = Ay + B, \quad A = 2\sinh(2\beta R), B = e^{-2\beta R} \\ a_0 &= \frac{1}{2\beta R}\log(Ay + B) \end{aligned}$$

Algorithym

1. generate $0 \leq y \leq 1$ randomly
2. $a_0 = \frac{1}{2\beta R}\log(Ay + B)$
3. accept a_0 with probability $\sqrt{1 - a_0^2}$
   (a) generate $0 < \xi < 1$ randomly
   (b) accept if $\xi \leq \sqrt{1 - a_0^2}$, stop
   (c) reject if $\xi > \sqrt{1 - a_0^2}$, go to (a) (max N times)
4. generate $0 \leq \phi \leq 2\pi$ and $-1 \leq \cos\theta \leq 1$
5. construct $a_1 = (1 - a_0^2)^{1/2}\cos\theta$, $a_2 = (1 - a_0^2)^{1/2}\sin\theta\cos\phi$, $a_3 = (1 - a_0^2)^{1/2}\sin\theta\sin\phi$
6. $W^{\text{new}} = UV_0$
7. new link $U_{n,\mu}^{\text{new}} = W^{\text{new}}V_0^{-1}$ (updation)

" N hits pesudo heat-bath algorithym"

Remark

- Over-relaxization (OR)

  $$\begin{aligned} U_{\text{new}} &= V_0 U_{\text{old}} V_0^\dagger \\ S^{\text{new}} &= \beta\text{tr}(U_{\text{new}}V + V^\dagger U_{\text{new}}^\dagger) = \beta R\text{tr}(U_{\text{old}}V_0 + V_0^\dagger U_{\text{old}}^\dagger) = S^{\text{old}} \end{aligned}$$

  Enerergy is conserved $\Rightarrow$ accepted (by detailed valance)
  1 iteration = 1 HB + 4 OR

- SU(3)
  SU(2) subgroup update

$$U_1 = \begin{pmatrix} a & b & \\ c & d & \\ & & \end{pmatrix} U_2 = \begin{pmatrix} & & \\ & a & b \\ & c & d \end{pmatrix} U_3 = \begin{pmatrix} a & & b \\ & & \\ c & & d \end{pmatrix}$$

  - choose randomly U_i
  - SU(2) heat-bath for subgroup U_i
  - 3 SU(2) updations = 1 SU(3) updation

11.2 Fundation for Monte-Carlo method

$\{C_i\} = \{C_1, C_2, \cdots\}$, set of configurations/states

Markov process

$P(C_i \to C_j) \equiv P_{ij}$: transition probability, P_{ij} only depends on C_i, C_j

Markov chain

$$C_{\tau_1} \xrightarrow{P} C_{\tau_2} \xrightarrow{P} C_{\tau_3} \xrightarrow{P} \cdots$$

P only depends on the previous state

Definition

1. irreducible

$$^{\forall}C_i, ^{\forall}C_j \Rightarrow ^{\exists}N < \infty, P_{ij}^N \equiv \sum_{i_1,\cdots,i_{N-1}} P_{ii_1} P_{i_1 i_2} \cdots P_{i_{N-1} j} \neq 0$$

2. aperiodic: $P_{ii}^{(N)} \neq 0$ for $\forall N$
3. positive: $\tau_i = \Sigma_{n=1}^{\infty} n P_{ii}^{(n)} < \infty$
   $P_{ii}^{(n)}$: probability of $i \to i$ first time in n step

Theorem 1

Markov chain is irreducible, aperiodic, states are positive, then
$N \to \infty$ limit exists and unique: $\lim_{N\to\infty} P_{ij}^{(N)} = \pi_j$

$$\pi_j \;>\; 0, \quad \sum_j \pi_j = 1, \quad \pi_j = \sum_i \pi_i P_{ij}$$

$$\left(\sum_i \pi_i P_{ij} \;=\; \lim_{N\to\infty} \sum_i P_{ki}^{(N)} P_{ij} = \lim_{N\to\infty} P_{kj}^{(N+1)} = \pi_j\right)$$

Theorem 2

chain is irreducible, states are positive, and

$$\tau_i^{(2)} \;=\; \sum_{n=1}^{\infty} n^2 P_{ii}^{(n)} < \infty$$

then, average of observable becomes

$$\langle \mathcal{O} \rangle \;\equiv\; Z^{-1} \sum_i \mathcal{O}(C_i) e^{-S(C_i)} = \sum_i \pi_i \mathcal{O}(C_i) = \sum_i P_{ji}^{(N)} \mathcal{O}(C_i) + O(1/\sqrt{N})$$

Use the continuum notation C instead of C_i

$$\begin{aligned} P_{eq}(C) \;&=\; \sum_{C'} P_{eq}(C') P(C' \to C) \qquad (11.1) \\ \langle \mathcal{O} \rangle \;&=\; \sum_C P_{eq}(C) \mathcal{O}(C) \end{aligned}$$

Theorem 3

Markov process satisfise

$$\begin{aligned} e^{-S(C)} P(C \to C') \;&=\; e^{-S(C')} P(C' \to C) \qquad \text{detailed balance} \\ \sum_{C'} P(C \to C') \;&=\; 1 \end{aligned}$$

then

$$P_{eq}(C) = \frac{e^{-S(C)}}{\Sigma_C \, e^{-S(C)}}$$

Proof

Assume eq. (11.2), then

$$\sum_C P_{eq}(C) \;=\; 1$$

$$\begin{aligned}\sum_{C'} P_{eq}(C')P(C' \to C) &= \frac{1}{\Sigma_C\, e^{-S(C)}} \sum_{C'} e^{-S(C')} P(C' \to C) \\ &= \frac{e^{-S(C)}}{\Sigma_C\, e^{-S(C)}} \overbrace{\sum_{C'} P(C \to C')}^{=1} = P_{eq}(C)\end{aligned}$$

Eq. (11.1) is satisfied. Since the solution to eq. (11.1) is unique, eq. (11.2) is correct.

11.3 Hybrid Monte Carlo (HMC) for full QCD

11.3.1 General case

$S(\phi)$ is given

1. coordinate ϕ (initial configuration)
2. choose momenta π with probability $P(\pi) \propto e^{-\pi^2/2}$
3. Molecular Dynamics (MD) step (time length = 1)

$$\begin{aligned}\dot{\phi} &= \frac{\partial H[\phi,\pi]}{\partial \pi} = \pi \\ \dot{\pi} &= -\frac{\partial H[\phi,\pi]}{\partial \phi} = -\frac{\partial S(\phi)}{\partial \phi} \\ H[\phi,\pi] &= \frac{1}{2}\pi^2 + S(\phi)\end{aligned}$$

: discrete time step $\epsilon = 1/N$

Leap-frog method

$$\text{initial} \qquad \begin{cases} \phi(0) = \phi \\ \pi(0) = \pi \end{cases}$$

$$\pi(1/2) = \pi(0) - \frac{1}{2}\epsilon \frac{\partial S(\phi)}{\partial \phi(0)}$$

$$\text{main step} \qquad \begin{cases} \phi(n+1) = \phi(n) + \epsilon\pi(n+1/2) \\ \pi(n+3/2) = \pi(n+1/2) - \epsilon \dfrac{\partial S(\phi)}{\partial \phi(n+1)} \end{cases}$$

$$\phi' = \phi(N) \;=\; \phi(N-1) + \epsilon\pi(N-1/2)$$
$$\pi' = \pi(N) \;=\; \pi(N-1/2) - \frac{1}{2}\epsilon\frac{\partial S(\phi)}{\partial\phi(N)}$$

4. accept-reject for ϕ', π'

$$P(\{\phi,\pi\} \to \{\phi',\pi'\}) = \min\{1, e^{-\Delta H}\}$$
$$\Delta H \;=\; H(\phi',\pi') - H(\phi,\pi) \begin{cases} < 0 & \text{accpet} \\ > 0 & \text{accept with probability } e^{-\Delta H} \end{cases}$$
$$\Delta H \propto \epsilon^2\sqrt{V} \Rightarrow \epsilon \sim V^{-1/4}$$

5. if accepted $\{\phi,\pi\} \longleftarrow \{\phi',\pi'\}$
   if rejected, keep $\{\phi,\pi\}$
   Go to 2

Proof of detailed valance

1. $P_G(\pi) = N_0 e^{-\pi^2/2}$
2. reversivility
   MD $\{\phi,\pi\} \longrightarrow \{\phi',\pi'\}$ then
   $\{\phi',-\pi'\} \longrightarrow \{\phi,-\pi\}$ then
$$P_M(\{\phi,\pi\} \to \{\phi',\pi'\}) = P_M(\{\phi',-\pi'\} \to \{\phi,-\pi\})$$
3. Metropolis test
$$P_A(\{\phi,\pi\} \to \{\phi',\pi'\}) = \min\{1, e^{-\Delta H}\}$$
4. Denote $\{\phi,\pi\} = \Phi$
$$e^{-H} P_A(\Phi \to \Phi') = e^{-H'} P_A(\Phi' \to \Phi)$$
   Proof
$$H - H' \;>\; 0 \begin{cases} e^{-H} P_A(\Phi \to \Phi') \;=\; e^{-H} \\ e^{-H'} P_A(\Phi' \to \Phi) \;=\; e^{-H'} e^{-H} / e^{-H'} = e^{-H} \end{cases}$$

$$H - H' \ < \ 0 \begin{cases} e^{-H} P_A(\Phi \to \Phi') \ = \ e^{-H} e^{-H'} / e^{-H} = e^{-H'} \\ e^{-H'} P_A(\Phi' \to \Phi) \ = \ e^{-H'} \end{cases}$$

Total probability

$$\begin{aligned} P(\phi \to \phi') \ &= \ \int d\pi d\pi' P_G(\pi) P_M(\Phi \to \Phi') P_A(\Phi \to \Phi') \\ & \quad e^{-S(\phi)} P(\phi \to \phi') = \int d\pi d\pi' e^{-H} P_M(\Phi \to \Phi') P_A(\Phi \to \Phi') \\ &= \ \int d\pi d\pi' e^{-H'} P_A(\Phi' \to \Phi) P_M(\Phi \to \Phi') \\ &= \ e^{-S(\phi')} \int d\pi d\pi' P_G(\pi') P_M(\{\phi', -\pi'\} \to \{\phi, -\pi\}) P_A(\Phi' \to \Phi) \\ &= \ e^{-S(\phi')} \int d\pi d\pi' P_G(\pi') P_M(\{\phi', \pi'\} \to \{\phi, \pi\}) P_A(\Phi' \to \Phi) \\ &= \ e^{-S(\phi')} P(\phi' \to \phi). \end{aligned}$$

11.3.2 HMC for QCD

$$P(U) = \det \ D(U) e^{-S_G(U)}$$

N_f=even degenerate flavors

$$\det \ D(U) = [\det \ D_0(U)]^{N_f} = [det D_0^\dagger D_0]^{N_f/2} \qquad (D_0^\dagger = \gamma_5 D_0 \gamma_5)$$

$$\begin{aligned} P(U) \ &= \ e^{-S_G(U)} \int d\phi^* d\phi e^{-\phi^* (D_0^\dagger D_0)^{-1} \phi} \\ H \ &= \ \frac{1}{2} \sum_l P_l^2 + S_G(U) + \phi^* (D_0^\dagger D_0)^{-1} \phi \\ Z \ &= \ \int \mathcal{D}U \ \mathcal{D}\phi \ \mathcal{D}\phi^* \mathcal{D} P e^{-H} \end{aligned}$$

ϕ: $N_f/2$ pseudo fermion(boson), P_l: moneta for U_l

Algorithym

1. initial U
2. P_l from $P(P_l) \propto e^{-P_l^2/2}$
3. ξ from $P(\xi) \propto e^{-\xi^*\xi/2}$
4. $\phi = D_0^\dagger(U)\xi$ fixed

$$e^{-\xi^*\xi/2} = \exp[-\frac{1}{2}\phi^*(D_0^\dagger D_0)^{-1}\phi]$$

5. MD (leap-frog)

$$\begin{aligned}
\eta &= (D_0^\dagger D_0)^{-1}(U)\phi \quad \phi : \text{fix} \\
\dot{U}_l &= iP_l U_l, \qquad U_l = e^{iA_l} \\
\dot{P}_l &= -\frac{\partial S_G}{\partial A_l} - \phi^* \frac{\partial (D_0^\dagger D_0)^{-1}}{\partial A_l}\phi \\
&= -\frac{\partial S_G}{\partial A_l} + \eta^* \frac{\partial D_0^\dagger D_0}{\partial A_l}\eta
\end{aligned}$$

6. Metropolis test $P = \min\{1, e^{-H'}/e^{-H}\}$
7. Go to 2

leap-frog

$$\begin{aligned}
\eta &= (D_0^\dagger D_0)^{-1}(U)\phi \quad \phi : \text{fix} \\
\dot{U}_l &= iP_l U_l, \qquad U_l = e^{iA_l} \\
P_l(\tau + \epsilon/2) &= P_l(\tau - \epsilon/2) - \epsilon \left[\frac{\partial S_G}{\partial A_l(\tau)} + \eta^*(\tau) \frac{\partial D_0^\dagger D_0}{\partial A_l(\tau)} \eta(\tau) \right] \\
U_l(\tau) &= e^{i\epsilon P_l(\tau - \epsilon/2)} U_l(\tau - \epsilon) \\
\eta(\tau) &= (D_0^* D_0)^{-1}(U_l(\tau))\phi (\leftarrow \text{fixed})
\end{aligned}$$

12 Construction of propagator

12.1 Calculation of propagator ($(D^{-1})^{AB}_{xy,\alpha\beta}$)

Linear equation

$$\begin{aligned} D(X,Y)\phi(Y) &= \delta_{X,X_0} \qquad X=(x,A,\alpha) \\ \phi(X) &= D^{-1}(X,X_0) \end{aligned}$$

general method for solving $D \cdot x = b$

12.1.1 Gradiant method

Minimize

$$S(x) = \frac{1}{2}|b - Dx|^2$$

(i) $x_k \to p_k$: correction vector

(ii) $x_{k+1} = x_k + \lambda_k p_k$ st. $S(x_{k+1})$ is minimized

$$\begin{aligned} S(x_{k+1}) &= \frac{1}{2}|b - Dx_{k+1}|^2 = \frac{1}{2}|b - Dx_k - \lambda_k Dp_k|^2 \\ 0 &= \frac{\delta S(x_{k+1})}{\delta \lambda_k} = \lambda_k (Dp_k, Dp_k) - (Dp_k, b - Dx_k) \\ \Rightarrow \lambda_k &= \frac{(Dp_k, b - Dx_k)}{|Dp_k|^2} = \frac{(p_k, s_k)}{|Dp_k|^2} \\ s_k &= D^\dagger r_k, \qquad r_k = b - Dx_k \text{ residual vector} \\ \nabla_x S(x_k) &= -D^\dagger (b - Dx_k) = -s_k \quad \text{gradiant} \end{aligned}$$

$(p_k, s_{k+1}) = (p_k, D^\dagger(b - Dx_k - \lambda_k Dp_k)) = (p_k, s_k) - \lambda (Dp_k, Dp_k) = 0$

12.1.2 Conjugate Gradiant (CG) method

D: $n \times n$ matrix, $p_0, \cdot p_{n-1}$ linearly independent

$$x = \underbrace{x_0 + \alpha_0 p_0 + \cdots \alpha_j p_j}_{x_{j+1}} + \cdots + \alpha_{n-1} p_{n-1}$$

is a solution
if we take $(p_j, D^\dagger D p_k) = 0$ $(j \neq k)$, then

$$(p_i, s_{k+1}) = 0 \quad (i = 0, 1, \cdots, k)$$

Proof

$$\begin{aligned} x_{k+1} &= x_{i+1} + \sum_{j=i+1}^{k} \lambda_j p_j \\ s_{k+1} &= D^\dagger(b - D x_{k+1}) = s_{i+1} - \sum_{j=i+1}^{k} \lambda_j D^\dagger D p_j \\ (p_i, s_{k+1}) &= (p_i, s_{i+1}) - \sum_{j=i+1}^{k} \lambda_j (p_i, D^\dagger D p_j) = 0 \end{aligned}$$

Choice of p_k

$$\begin{aligned} p_k &= s_k + \mu_{k-1} p_{k-1} \\ (p_{k-1}, D^\dagger D p_k) &= (p_{k-1}, D^\dagger D s_k) + \mu_{k-1}(p_{k-1}, D^\dagger D P_{k-1}) = 0 \\ \mu_{k-1} &= -\frac{(p_{k-1}, D^\dagger D s_k)}{|D p_{k-1}|^2} \end{aligned}$$

then, for $i \neq j$

$$(s_i, s_j) = 0, \qquad (p_i, D^\dagger D p_j) = 0$$

Proof: If correct for $i \leq k$, $j \leq k$

$$\begin{aligned} 0 &= (p_i, s_{k+1}) = (s_i + \mu_{i-1} p_{i-1}, s_{k+1}) = (s_i, s_{k+1}) \\ &\Rightarrow (s_i, s_j) = 0 \quad (i \neq j) \\ (p_i, D^\dagger D p_{k+1}) &= (p_i, D^\dagger D(s_{k+1} + \mu_k p_k)) = (p_i, D^\dagger D s_{k+1}) \\ &= \frac{1}{\lambda_i}(x_{i+1} - x_i, D^\dagger D s_{k+1}) = -\frac{1}{\lambda_i}(s_{i+1} - s_i, s_{k+1}) = 0 \\ &\Leftarrow D(x_{i+1} - x_i) = (b - D x_i) - (b - D x_{i+1}) = r_i - r_{i+1} \\ (p_k, D^\dagger D p_{k+1}) &= (p_k, D^\dagger D(s_{k+1} + \mu_k p_k)) = (p_k, D^\dagger D s_{k+1}) + \mu_k(D p_k, D p_k) = \end{aligned}$$

Form these, at worst

$$s_n = D^\dagger(b - Dx_n) = 0 \Longrightarrow b - Dx_n = 0$$

12.1.3 CG algorithym

1. initial x_0

$$r_0 = b - Dx_0, \quad p_0 = s_0 = D^\dagger r_0$$

2. repeat $k = 0, 1, 2, \cdots$

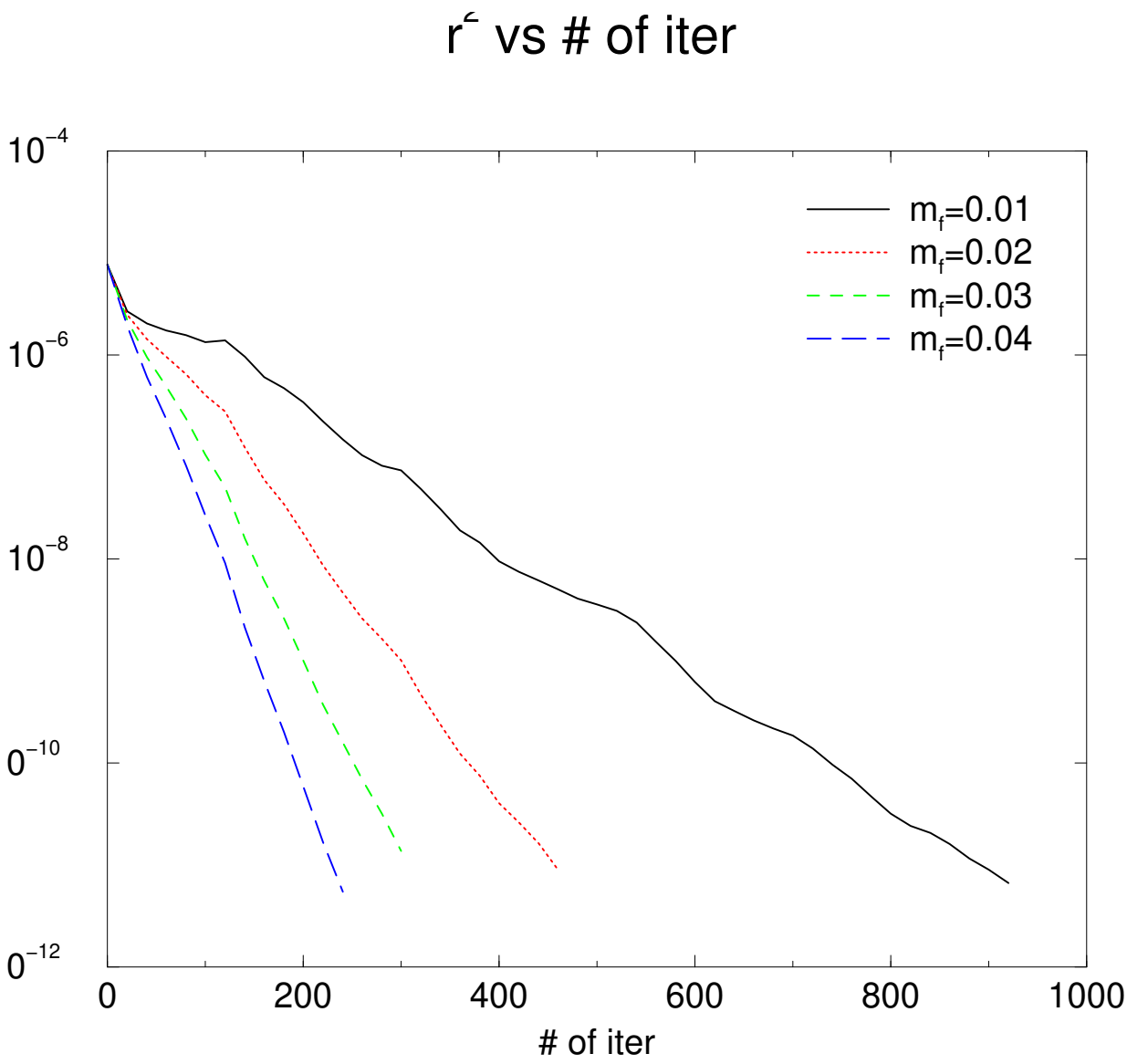
$$\begin{aligned}
\lambda_k &= \frac{(p_k, s_k)}{|Dp_k|^2} \\
x_{k+1} &= x_k + \lambda_k p_k \\
r_{k+1} &= b - Dx_{k+1} = r_k - \lambda_k D p_k \\
&\text{if} \quad |r_{k+1}|^2 < \epsilon \quad \text{stop} \\
s_{k+1} &= D^\dagger r_{k+1} = s_k - \lambda_k D^\dagger D p_k \\
\mu_k &= -\frac{(p_k, D^\dagger D s_k)}{|Dp_k|^2} \\
p_{k+1} &= s_{k+1} + \mu_k p_k
\end{aligned}$$

12.2 Even-odd preconditioning for Wilson fermion

$$\begin{aligned}
D_W x &= b, \quad D_W = 1 - KT \\
x &= x_o + x_e, \qquad b = b_e + b_o \\
(1 + KT)(1 - KT)x &= (1 - K^2T^2)x = (1 + KT)b \\
(1 - K^2T^2)x_e &= b_e + KT_{eo}b_o \equiv Be
\end{aligned}$$

Obtain x_e then $x_o = KT_{oe}x_e + b_o$

Small computational cost, better for smaller K


r^ vs # of iter
10^-4
10^-6
10^-8
0^-10
0^-12
0
200
400
600
800
1000
# of iter
m_f=0.01
m_f=0.02
m_f=0.03
m_f=0.04

13 Extraction of mass from propagator

13.1 Form of propagator

$$\begin{aligned} G(t) &= \sum_{\vec{x}} \langle \pi(\vec{0},0)\pi(\vec{x},t)\rangle = \sum_{\vec{x}} \langle 0|\pi(0)e^{iPx}\pi(0)e^{-iPx}|0\rangle \\ &= \sum_{\vec{x}}\sum_{n} \langle 0|\pi(0)|n\rangle\langle n|\pi(0)|0\rangle e^{ip_n x} \\ &= \sum_{n} |Z_n|^2 e^{-m_n t}, \quad Z_n = \langle 0|\pi(0)|n\rangle,\ (p_n)_0 = im_n \\ &= |Z_0|^2 e^{-m_0 t} + O(e^{-\Delta m_n t}), \quad \Delta m_n = m_n - m_0 \end{aligned}$$

For periodic boundary condition for $0 \le t \le T$

$$\begin{aligned} \langle \pi(0)\pi(t)\rangle &= \langle e^{-HT}\pi(0)e^{-Ht}\pi(0)e^{Ht}\rangle \\ &= \langle 0|\pi|n\rangle\langle n|e^{-m_n t}\pi|0\rangle + \langle n|e^{-m_n T}\pi|0\rangle\langle 0|\pi e^{m_n t}|n\rangle \\ &= |Z_n|^2\left(e^{-m_n t} + e^{-m_n(T-t)}\right) \\ &= |Z_0|^2 e^{-m_0 T/2} 2\cosh m_0(t - T/2) + O(e^{-\Delta m}) \end{aligned}$$

13.2 Effective mass

$$m(t) = -\log \frac{G(t+1)}{G(t)} \overset{t\to\infty}{\longrightarrow} m_0$$

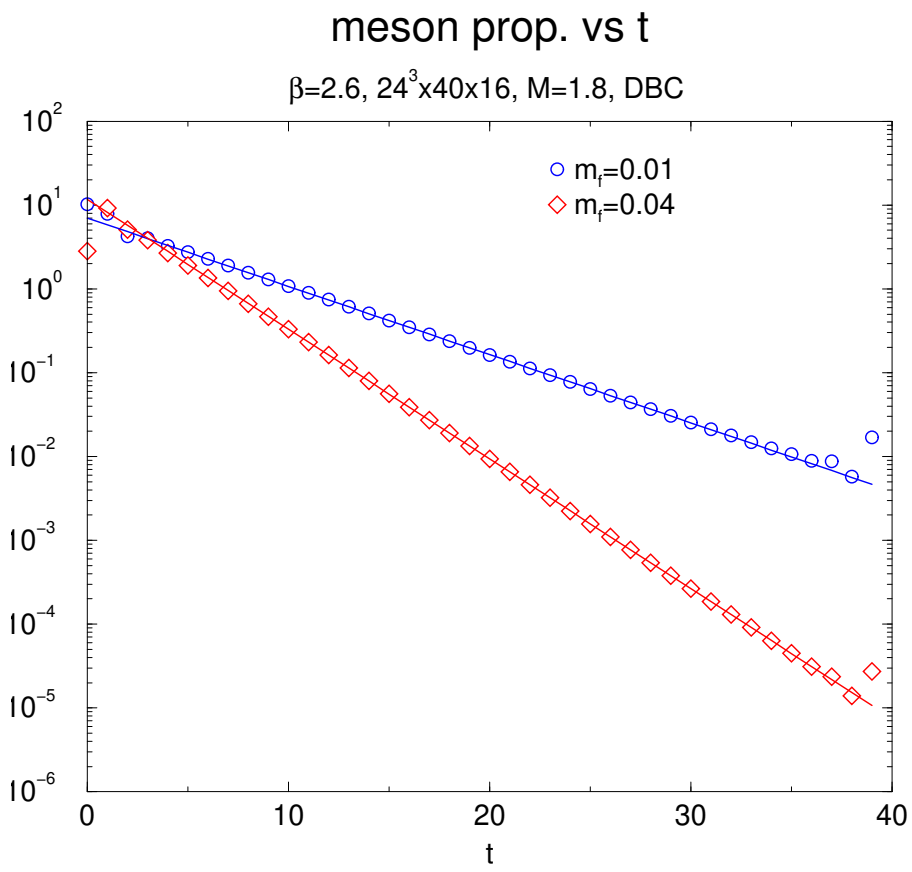
if $m(t)$ becomes independent on t at some $t \ge t_{min}$, the effect of excited states (m_n with $n > 0$) is negligible

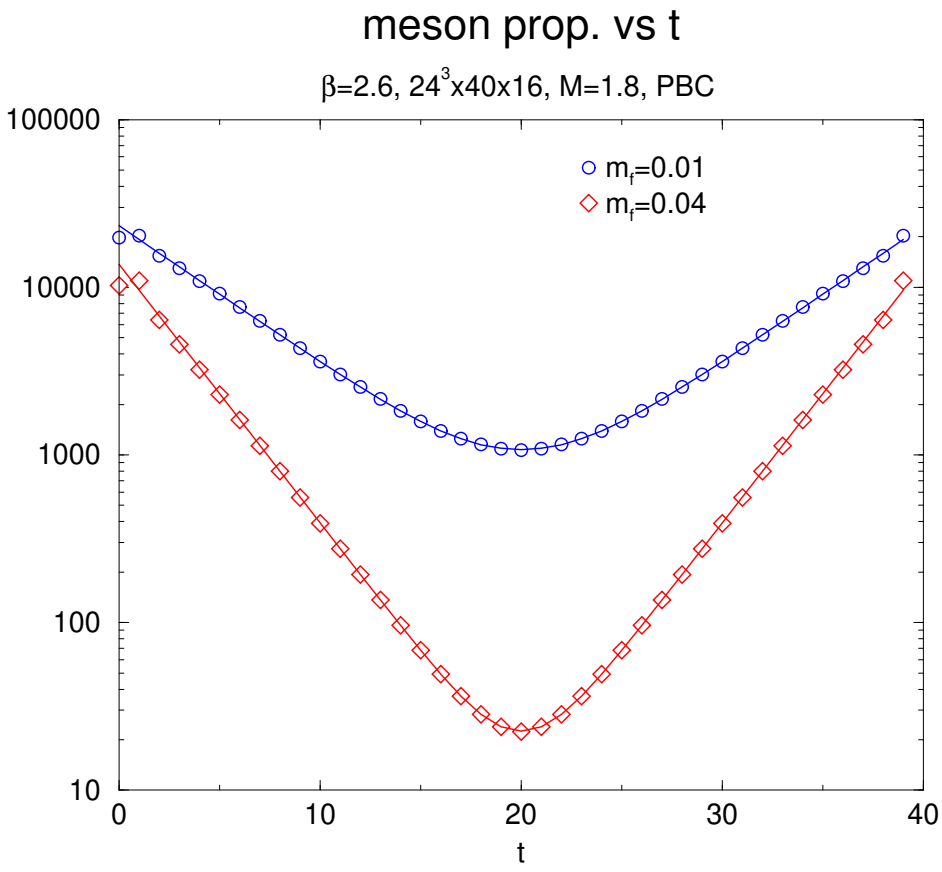
fitting range is $t_{min} \le t (\le t_{max})$

Smearing: Instead of source at $(\vec{x_0}, t_0)$, we may use smearing source such as

$$D_{xy}\phi_y = \delta_{t,t_0} A e^{-B|\vec{x}-\vec{x_0}|},$$

inorder to reduce the efect from excited states, so that t_{min} becomes smaller.


meson prop. vs t
β=2.6, 24^3x40x16, M=1.8, DBC
m_f=0.01
m_f=0.04
10^2
10^1
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
0
10
20
30
40
t


meson prop. vs t
β=2.6, 24^3x40x16, M=1.8, PBC
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m_f=0.04
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10000
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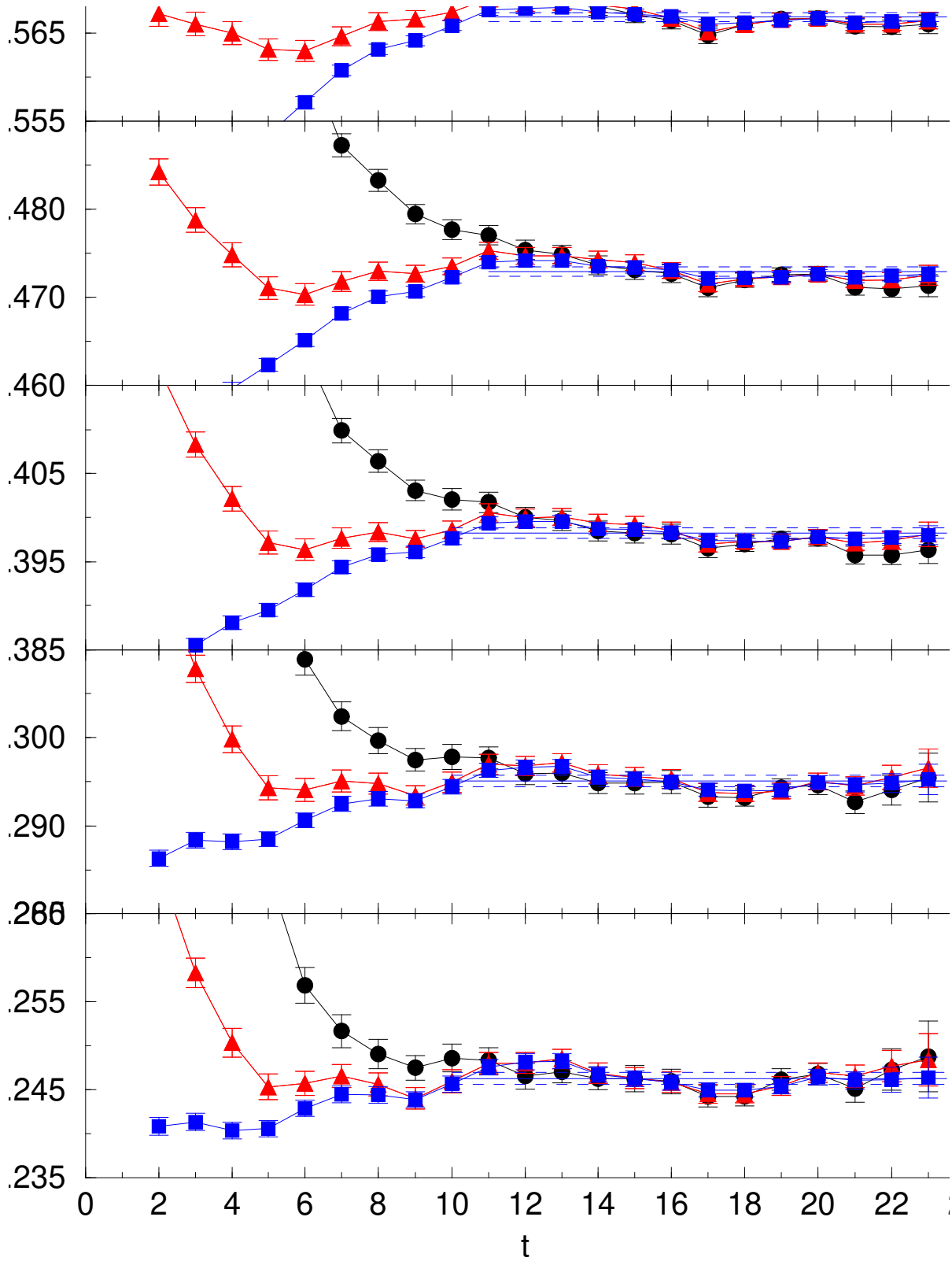
0
10
20
30
40
t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.460
.405
.395
.385
.300
.290
.255
.245
.235
0
2
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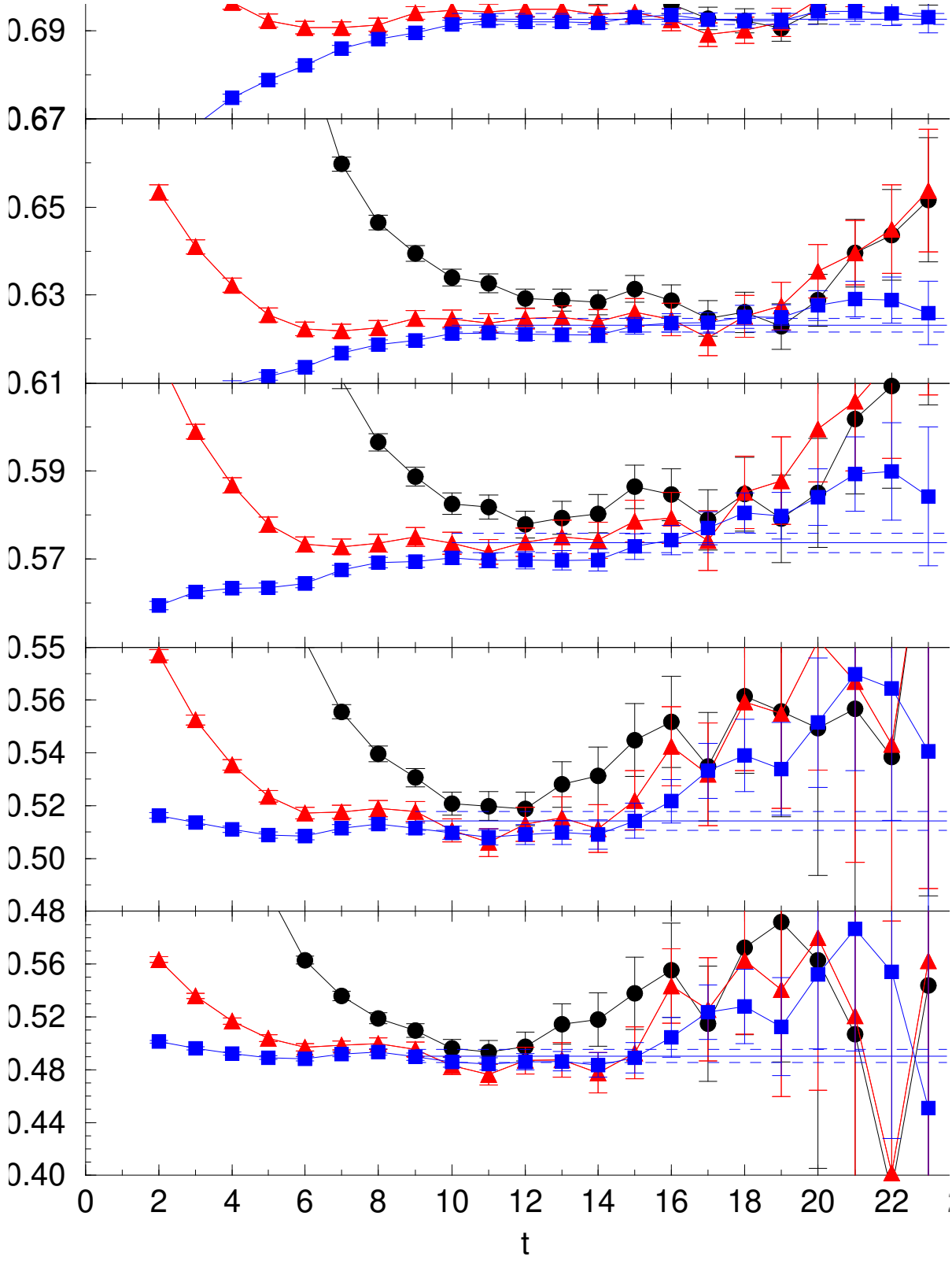
14
16
18
20
22
t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0.67
0.65
0.63
0.61
0.59
0.57
0.55
0.56
0.54
0.52
0.50
0.48
0.56
0.52
0.48
0.44
0.40
0
2
4
6
8
10
12
14
16
18
20
22
t

13.3 Fit

fitting function: $f(t) = Ze^{-mt}$, 2 parameters Z and m
minimize χ^2

$$\chi^2 = \sum_{t=t_{min}}^{t_{max}} \frac{(f(t) - G(t))^2}{(\delta G(t))^2}$$

$\delta G(t)$ is a statistical error of $G(t)$

14 Error analusis and fit

14.1 Statistical error

$$\begin{aligned} \langle \mathcal{O} \rangle &= \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}_i \\ \delta \langle \mathcal{O} \rangle &= \sqrt{\frac{\langle \mathcal{O}^2 \rangle - \langle \mathcal{O} \rangle^2}{N-1}} \end{aligned}$$

14.2 Jack-Knife error

bin-size 1

$$\begin{aligned} \langle \mathcal{O} \rangle_i &= \frac{1}{N-1} \sum_{k \neq i} \mathcal{O}_k \\ \langle f(\mathcal{O}) \rangle &= \frac{1}{N} \sum_{i=1}^{N} f(\mathcal{O}_i) \\ \delta \langle f(\mathcal{O}) \rangle &= \sqrt{(N-1)\{\langle f(\mathcal{O})^2 \rangle - \langle f(\mathcal{O}) \rangle^2\}} \end{aligned}$$

$f(X)$ is an arbitrary function of X (mass, etc)
Advantage

- an error of an arbitrary qunatity ($f(\mathcal{O})$) can be calculated
- correlation between data can be taken into account
- auto-correlation can be estimated by increasing bin size

bin-size n ($b = (i_1, \cdot . i_n)$)

$$\begin{aligned} \langle \mathcal{O} \rangle_b &= \frac{1}{N-n} \sum_{k \notin b} \mathcal{O}_k \\ \langle f(\mathcal{O}) \rangle &= \frac{1}{N} \sum_{b=1}^{N/n} f(\mathcal{O}_b) \\ \delta \langle f(\mathcal{O}) \rangle &= \sqrt{(N/n-1)\{\langle f(\mathcal{O})^2 \rangle - \langle f(\mathcal{O}) \rangle^2\}} \end{aligned}$$

14.3 Fit

Data $g(x_i)$ $(i = 1, \cdot, K)$ with a statistical error $\delta g(x_i)$
Fit function $f(x)$, to minimize

$$\chi^2 = \sum_{i=1}^{K} \frac{(f(x_i) - g(x_i))^2}{(\delta g(x_i))^2}$$

Fit error: change of parameters within $\chi^2 - \chi^2_{\min} \leq 1$
This error becomes inaccurate if $g(x_i)$ and $g(x_j)$ are correlated.
$\Longrightarrow$ Jack-Knife error or correlated fit

14.4 Correlated fit

correlation matrix for $g(x_i) \equiv \langle G_i \rangle$

$$C_{ij} = \frac{1}{N-1} \left(\langle G_i G_j \rangle - \langle G_i \rangle \langle G_j \rangle \right)$$

No correlation $\Rightarrow C_{ij} = \delta_{i,j} (\delta g(x_i))^2$
Correct χ^2

$$\chi^2 = \sum_{i,j} (f(x_i) - g(x_i)) C^{-1}_{ij} (f(x_j) - g(x_j))$$

14.5 Auto-correlation

Data $\{\mathcal{O}_i\}$ with $i = 1, \cdots, N$

Auto-correlation function

$$\begin{aligned} A(t) &= \frac{1}{N-t} \sum_{i=1}^{N-t} \mathcal{O}_i \mathcal{O}_{i+t} - \langle \mathcal{O} \rangle^2 \\ C(t) &= A(t)/A(0) \simeq e^{-t/\tau}, \quad \tau : \text{ auto-correlation time} \end{aligned}$$

τ gives iterations needed for thermalization.

Integrated auto-correlation time

$$\tau_{\text{int}} = \frac{1}{2} + \lim_{N \to \infty} \sum_{i=1}^{N} C(i)$$

τ_{int} gives iterations needed to generate independent configurations

correct error estimation

$$\delta \langle \mathcal{O} \rangle = \sqrt{\frac{\langle \mathcal{O}^2 \rangle - \langle \mathcal{O} \rangle^2}{(N-1)/\tau_{\text{int}}}}$$

14.6 Jack-Knife and auto-correlation

In JK analyssis, error increases as bin-size n increases. error becomes constant against n at $n_{\text{min}} \Rightarrow$ correct error

$n_{\text{min}} \simeq \tau_{\text{int}}$

14.7 Quark mass dependence of hadron masses

14.8 π meson

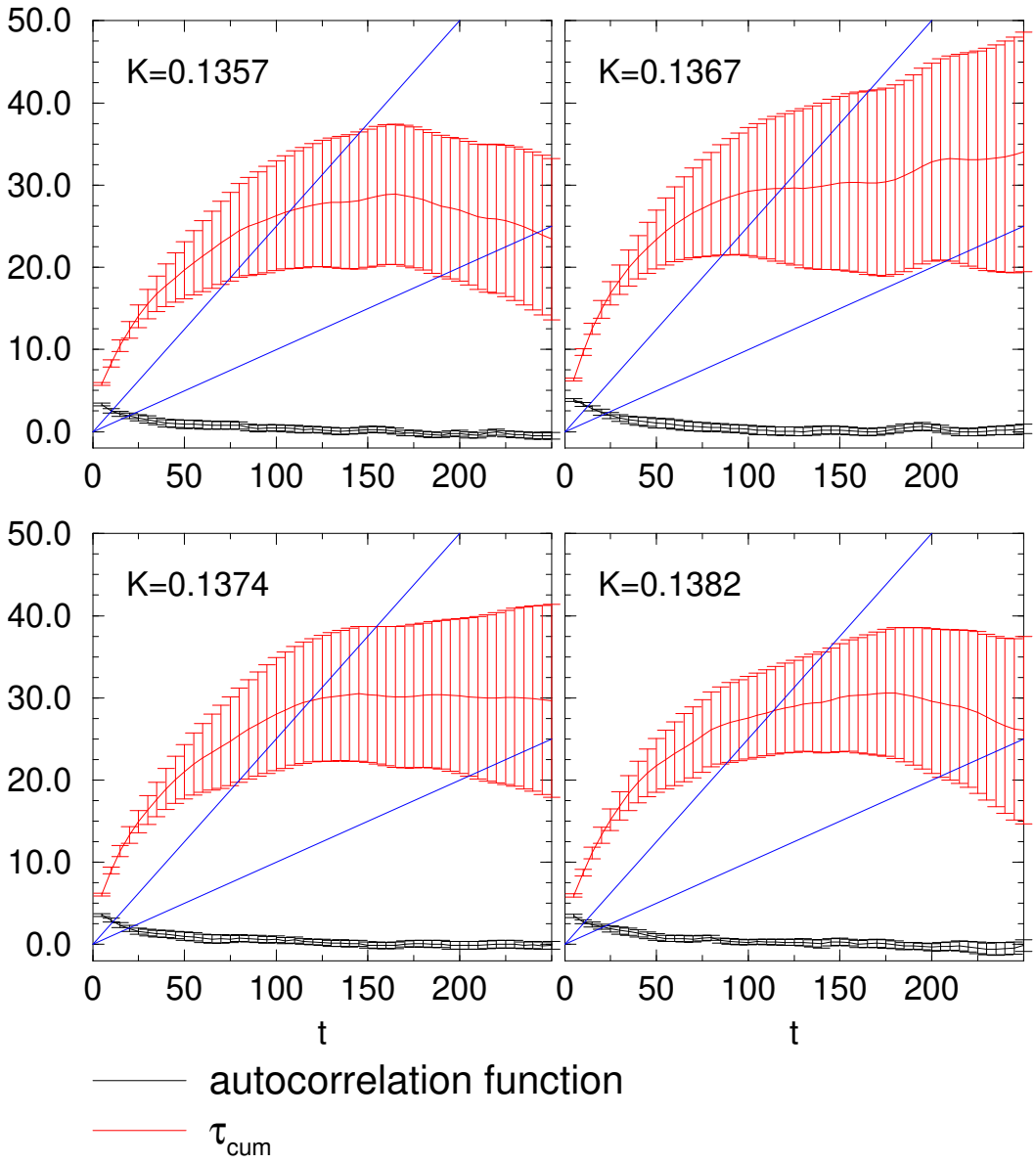
Wilson fermion

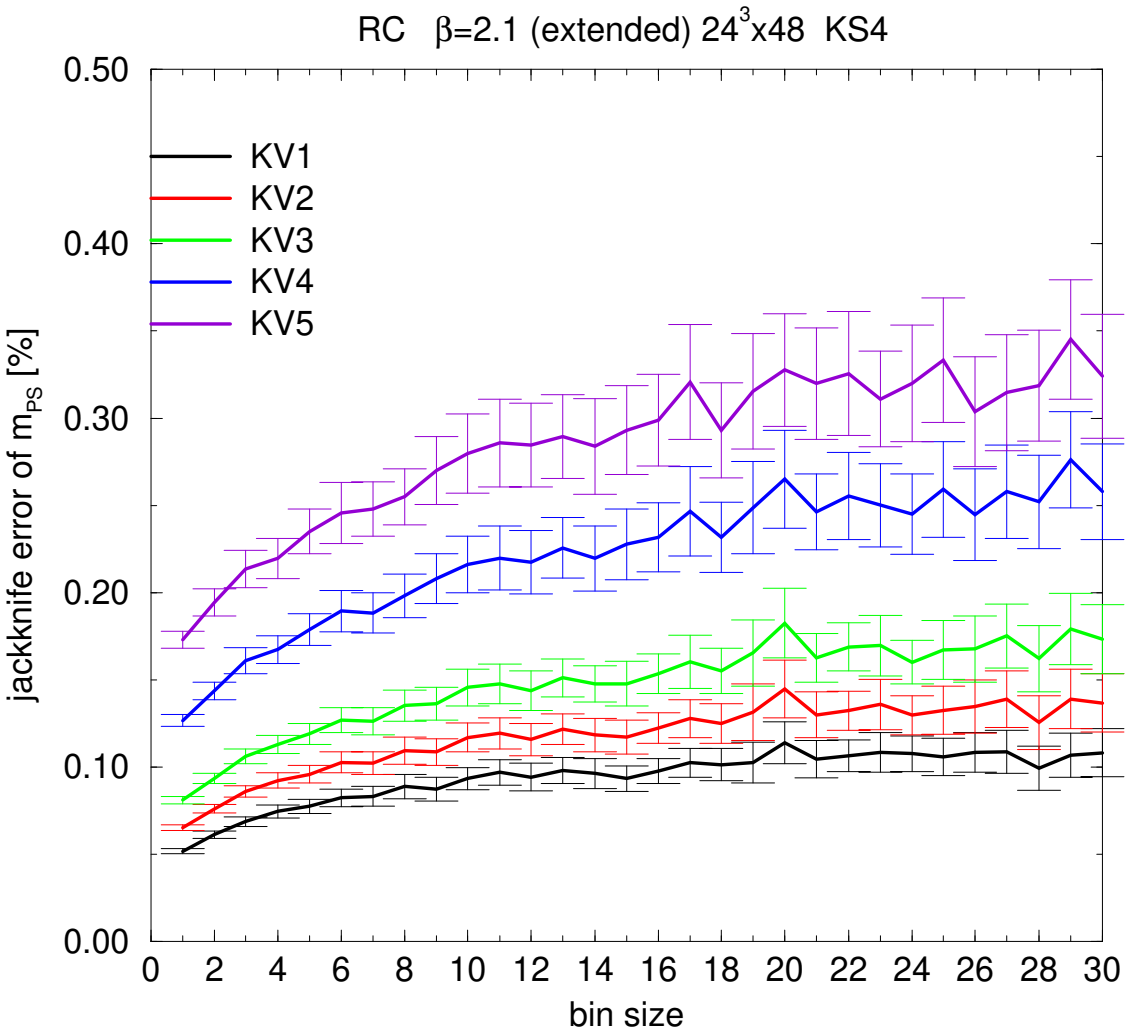
$$m_\pi^2 = A(1/K - 1/K_c) + B(1/K - 1/K_c)^2 + \cdots$$

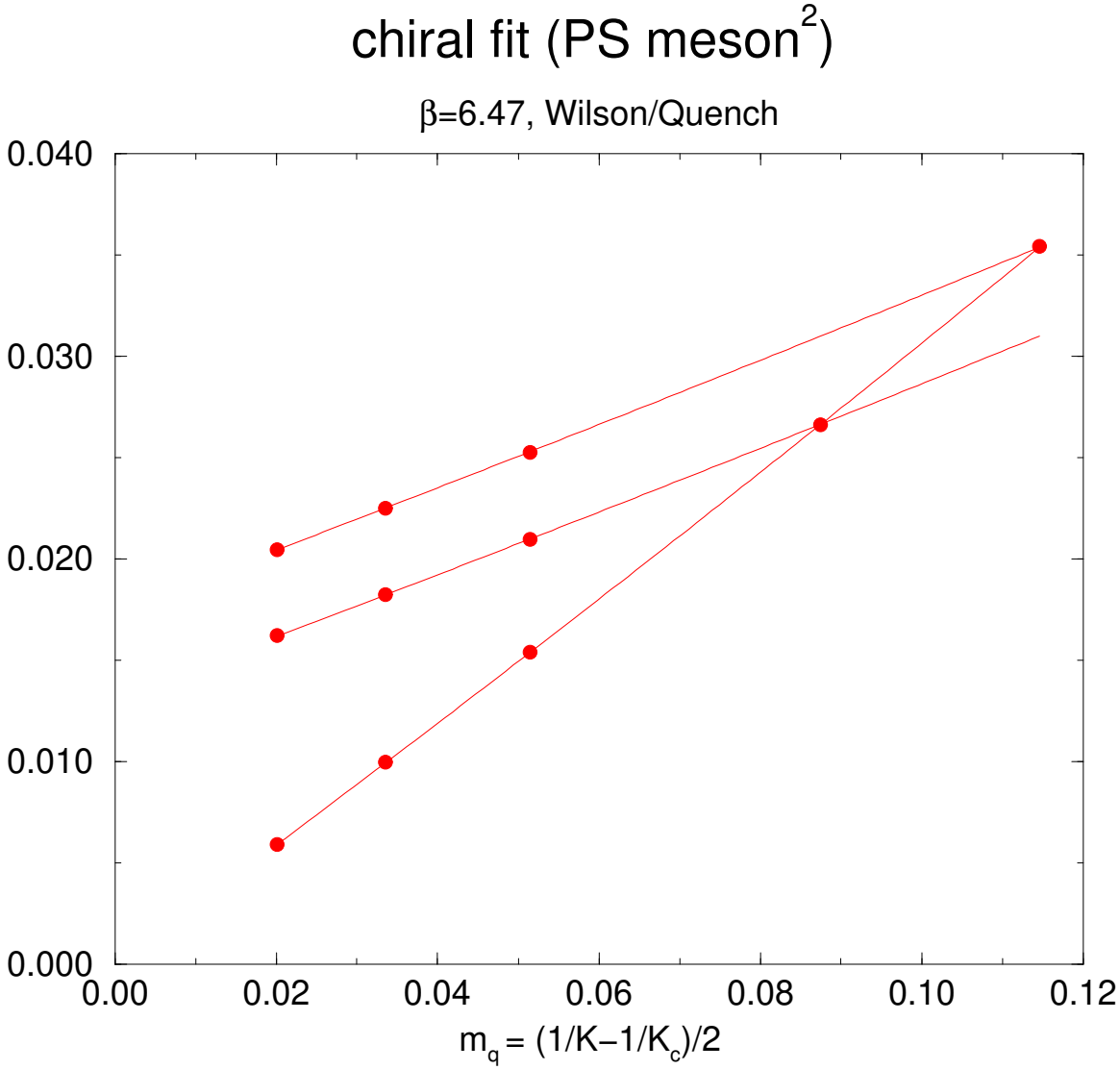
DWQCD

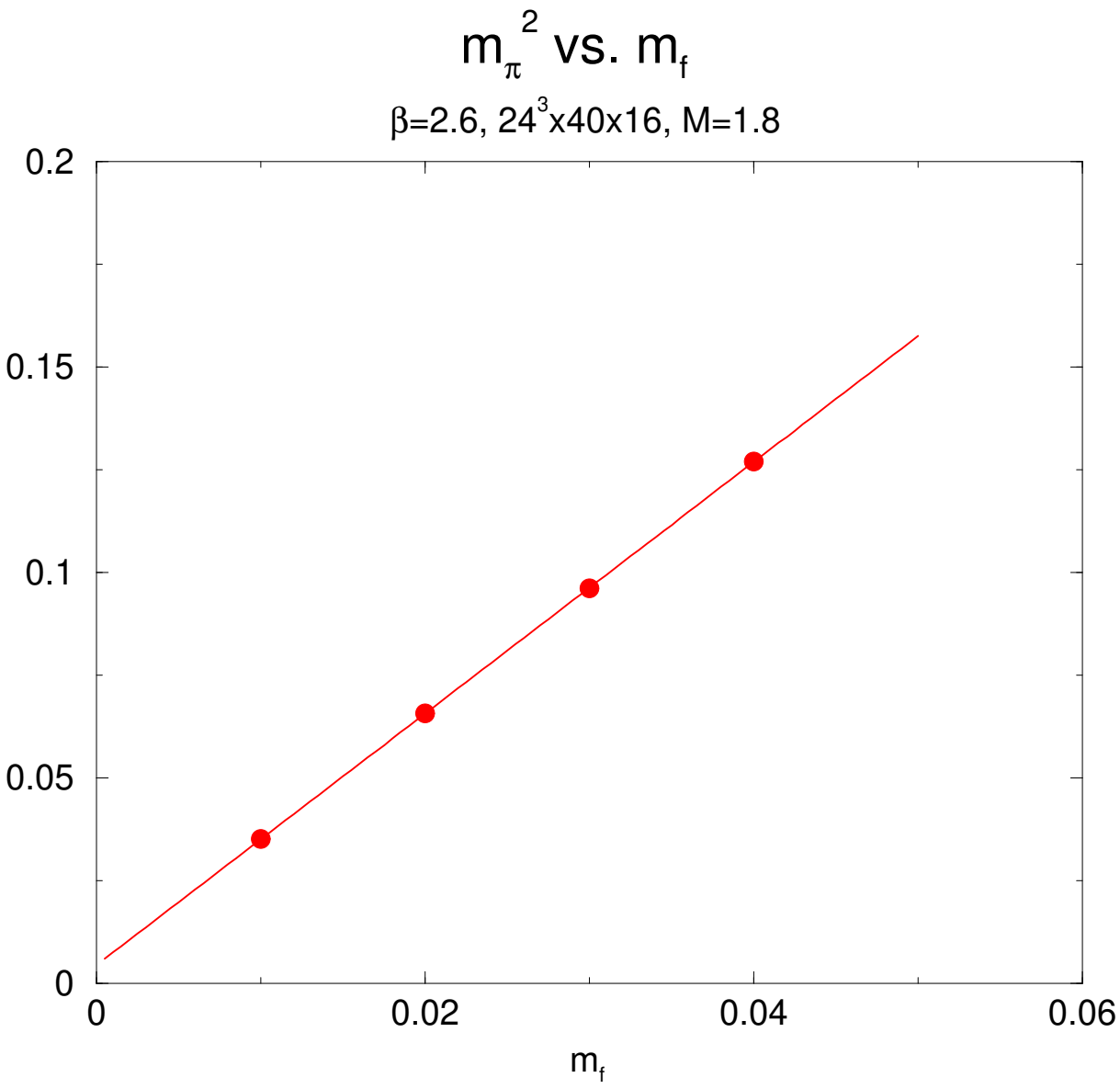
$$m_\pi^2 = A m_f + B m_f^2 + \cdots$$

図 1: auto cprrelation function of Q. $Q = \frac{1}{32\pi^2}\int F \cdot F^* d^4x$




RC β=2.1 (extended) 24^3x48 KS4
KV1
KV2
KV3
KV4
KV5
jackknife error of m_PS [%]
bin size
0.00
0.10
0.20
0.30
0.40
0.50
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30


chiral fit (PS meson^2)
β=6.47, Wilson/Quench
0.040
0.030
0.020
0.010
0.000
0.00
0.02
0.04
0.06
0.08
0.10
0.12
$m_q = (1/K - 1/K_c)/2$


m_π^2 vs. m_f
β=2.6, 24^3x40x16, M=1.8
0.2
0.15
0.1
0.05
0
0
0.02
0.04
0.06
m_f

14.9 Other hadrons

$$
\begin{aligned}
m_\rho &= A_\rho + B_\rho m_f + \cdots \\
m_N &= A_N + B_N m_f + \cdots
\end{aligned}
$$

14.10 Finite size effect

$$
\frac{m_\pi^2}{m_f} = A + B m_f + \cdots
$$

14.11 Chiral fit and quark masses

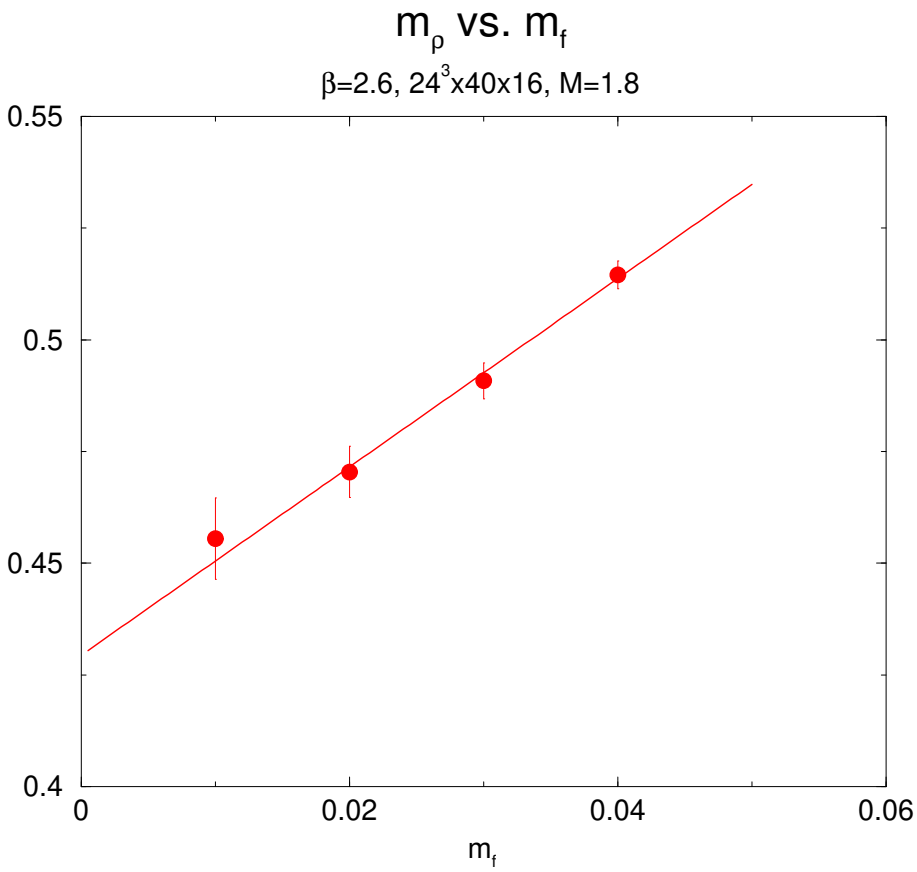
$$
\begin{aligned}
m_{PS}^2(m_1, m_2)a^2 &= A_P(m_1 + m_2) + B_P(m_1 + m_2)^2 + C_P(m_1 - m_2)^2 + \cdots \\
m_V(m_1, m_2)a &= A_V + B_V(m_1 + m_2) + C_V(m_1 + m_2)^2 + D_V(m_1 - m_2)^2 +
\end{aligned}
$$

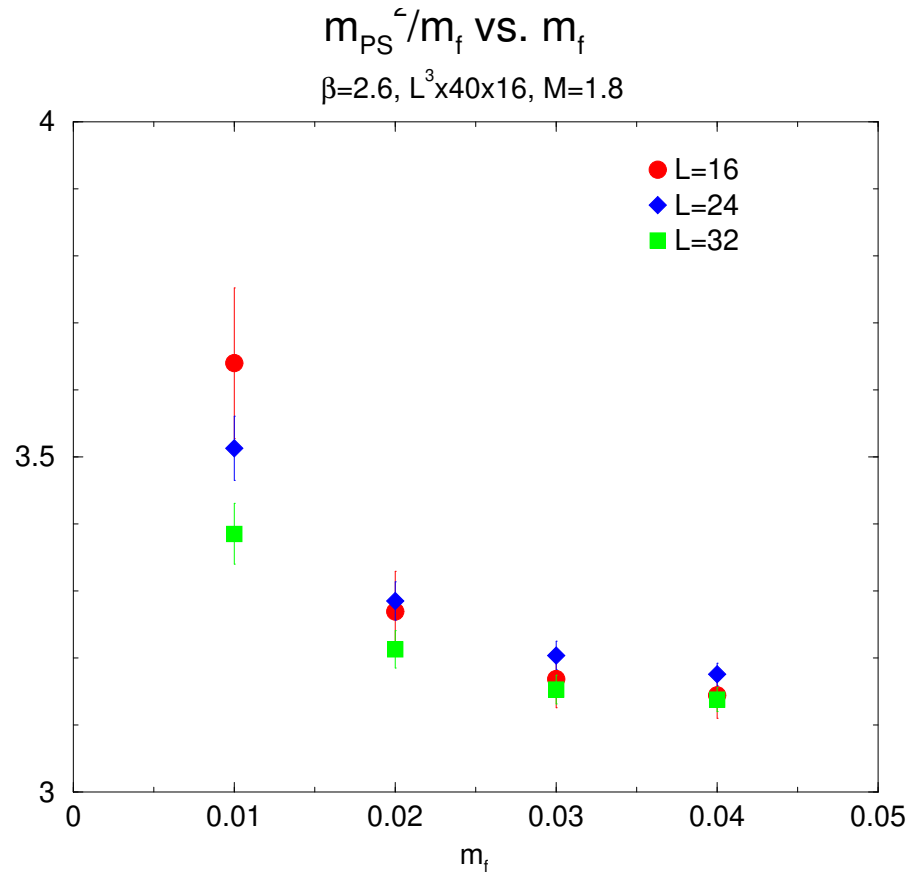
At the lowest order,

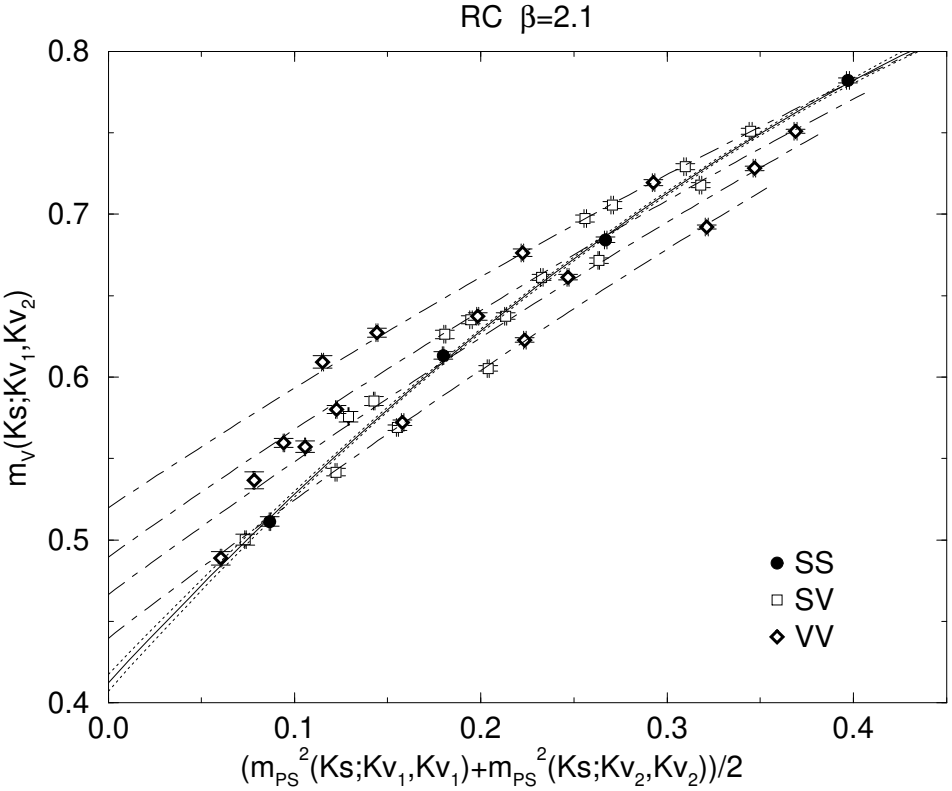
$$
\begin{aligned}
\frac{m_\pi^2}{m_\rho^2} &= \frac{(135)^2}{(770)^2} = \frac{A_P 2m_l}{(A_V + B_V 2m_l)^2}, && m_l \text{ light quark mass} \\
\frac{m_K^2}{m_\rho^2} &= \frac{(497)^2}{(770)^2} = \frac{A_P(m_l + m_s)}{(A_V + B_V 2m_l)^2}, && m_s \text{ strange quark mass } (K\text{-input}) \\
\frac{m_{K^*}}{m_\rho} &= \frac{864}{770} = \frac{A_V + B_V(m_l + m_s)}{A_V + B_V 2m_l}, && m_s \text{ strange quark mass } (K^*\text{-input})
\end{aligned}
$$

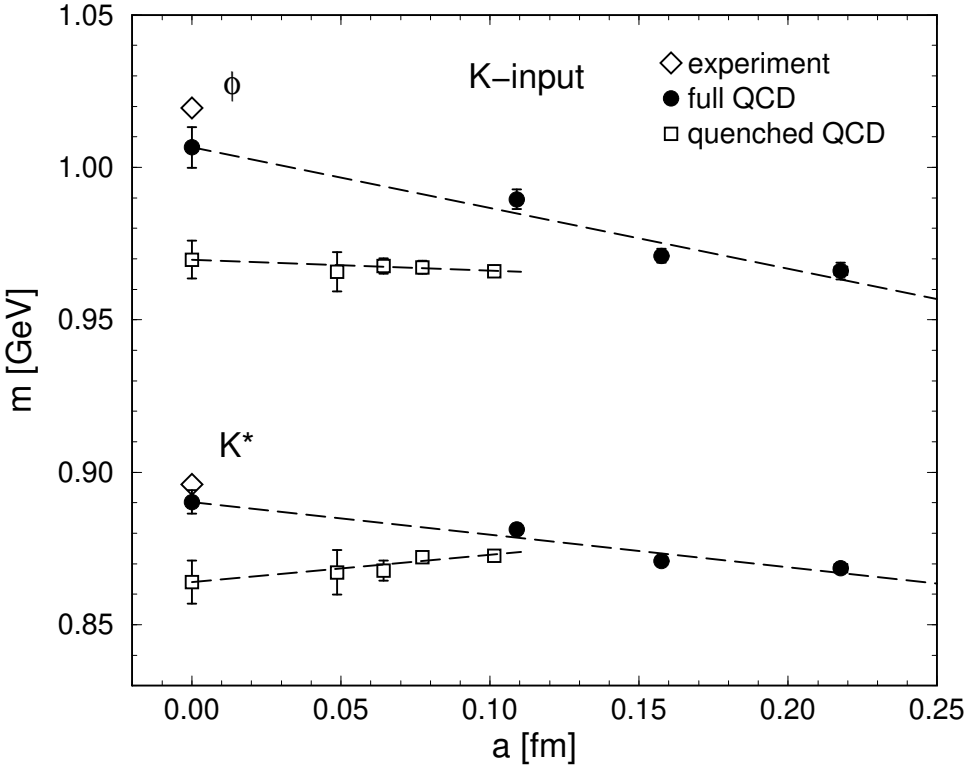
Scale (lattice spacing a) is given by

$$
\tilde{m_\rho} a = 770(\text{MeV}) \cdot a = A_V + B_V 2m_l
$$


m_ρ vs. m_f
β=2.6, 24^3x40x16, M=1.8
0.55
0.5
0.45
0.4
0
0.02
0.04
0.06
m_f


m_{PS}^2/m_f vs. m_f
β=2.6, L^3x40x16, M=1.8
L=16
L=24
L=32
4
3.5
3
0
0.01
0.02
0.03
0.04
0.05
m_f


RC β=2.1
0.8
0.7
0.6
0.5
0.4
0.0
0.1
0.2
0.3
0.4
m_V(Ks;Kv_1,Kv_2)
(m_PS^2(Ks;Kv_1,Kv_1)+m_PS^2(Ks;Kv_2,Kv_2))/2
SS
SV
VV


1.05
1.00
0.95
0.90
0.85
0.00
0.05
0.10
0.15
0.20
0.25
m [GeV]
a [fm]
φ
K*
K−input
experiment
full QCD
quenched QCD

15 Renormalization

15.1 Lattice quark mass and $\overline{\rm MS}$ quark mass

$$
\begin{aligned}
\Sigma^{(1)}_{lat} &= i\not{p}A_{lat} + mB_{lat} \\
S_F^{-1}(p) &= i\not{p}[1 - \frac{g^2}{16\pi^2}C_F\{\log(\lambda^2 a^2) + \Sigma_1\}] \\
&+ m[1 - \frac{g^2}{16\pi^2}C_F\{4\log(\lambda^2 a^2) + \Sigma_2\}] \\
S^{-1}_{F,cont}(p) &= i\not{p}[1 - \frac{g^2}{16\pi^2}C_F\{\log(\lambda^2/\mu^2) + \Sigma_1^{\overline{\rm MS}}\}] \\
&+ m_{\overline{\rm MS}}[1 - \frac{g^2}{16\pi^2}C_F\{4\log(\lambda^2/\mu^2) + \Sigma_2^{\overline{\rm MS}}\}]
\end{aligned}
$$

Matching

$$
\begin{aligned}
Z_\psi\langle\psi^{lat}\bar\psi^{lat}\rangle &= \langle\psi^{\overline{\rm MS}}\bar\psi^{\overline{\rm MS}}\rangle \\
Z_\psi(\mu a) &= 1 + \frac{g^2}{16\pi^2}C_F\{-\log(\mu^2 a^2) + \Sigma_1^{\overline{\rm MS}} - \Sigma_1\}] \\
Z_m \cdot m &= m_{\overline{\rm MS}} \\
Z_m(\mu a) &= 1 + \frac{g^2}{16\pi^2}C_F\{-3\log(\mu^2 a^2) + \Sigma_2^{\overline{\rm MS}} - \Sigma_2 - \Sigma_1^{\overline{\rm MS}} + \Sigma_1\}]
\end{aligned}
$$

$$
m_{\overline{\rm MS}}(\mu) = Z_m(\mu a) m_{lat} a\frac{1}{a} + O(g^4)
$$

15.2 Horizontal matching

$$
\begin{aligned}
m_{\overline{\rm MS}}(1/a) &= Z_m(1) m_{lat} a\frac{1}{a} + O(g^4(1/a)) \\
\frac{1}{g^2_{\overline{\rm MS}}(1/a)} &= \frac{1}{g^2} + c_g = \frac{P}{g_0^2} + \tilde{c_g}, \quad P: \text{ average Plaquette} \\
& m_{\overline{\rm MS}}(1/a) \overset{\text{continuum RG}}{\longrightarrow} m_{\overline{\rm MS}}(\mu)
\end{aligned}
$$